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CONTACT AND EQUIVALENCE OF SUBMANIFOLDS OF HOMOGENEOUS SPACES

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1. Introduction. The problem of equivalence of submanifolds of homogeneous spaces of Lie groups was extensively treated by F. Cartan by his method of moving frames [2]. A basic idea of Cartan's method is that for sufficiently high k, G-contact of order k (see §4) implies G-equivalence. In other words, for each homogeneous space M there exists an integer k, depending on the dimension p, such that if two submanifolds S and $\overline{S}$ of same dimension p have G-contact of order k, then there exists $g \in G$ such that $gS = \overline{S}$. Cartan treated several important geometrical examples and proved in each case the existence of k.

Essentially, Cartan's method of proving the existence of the element $g \in G$ consists in using the uniqueness of solution of a system of first order differential equations as in Frobenius theorem. Cartan's theory has been the subject of interest of a great number of authors (see for example [4], [5]). However, they all reduce the proof of the existence of the element $g \in G$ to the uniqueness of solution of a first order differential system whereas it seems more natural and geometrical to deal directly with a higher order differential system.

The notion of contact element as defined by Ehresmann [3] allows a geometrical formulation of the theorem of existence and uniqueness of solution of higher order completely integrable differential systems which is a straight forward generalization of Frobenius theorem (theorem 1). It is the uniqueness of this theorem that we use to solve the problem of G-equivalence. As a result, the regularity conditions on the submanifolds S and $\overline{S}$, which are necessary for the theorem of equivalence to hold (theorem 3), can be given a simple and geometrical definition, valid in any homogeneous space M. Also, in the method of moving frames, the invariants of a submanifold S of M are defined attaching special higher order frames to the points of S, [2], [5]. These frames are constructed by subtle geometrical arguments valid for a fixed homogeneous space whereas we construct

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the invariants of S as the elements of a complete set of invariants of the orbits of G acting on a manifold of higher order contact elements.

The equivalence problem may be posed for two immersions $f, h : S \to M$ of a differentiable manifold S. f and g are equivalent if there exists $g \in G$ such that $h = L_g \circ f$ where $L_g(x) = gx$, $x \in S$. This fixed parametrization theorem has been treated by J. A. Verderesi [7] by means of a higher order differential system defined in a manifold of jets.

The paper ends with a necessary and sufficient condition for a submanifold S of M to be an open set of an orbit of a Lie subgroup K of G.

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2. Contact elements. All manifolds and maps considered in this paper are assumed to be differentiable of class C^∞. If M and N are manifolds and $f : M \to N$ is a map, the induced map on tangent spaces at points $a \in M$ and $b = f(a) \in N$ will be denoted by $f_a : T_aM \to T_bN$. Given integers $p, k \geq 0$, $p \leq \dim M$, $J^{k,p}M$ denotes the manifold of all k-jets of rank p whose source is the origin of $\mathbb{R}^p$ and whose target is any point of M. Let $GL^k\mathbb{R}^p$ be the Lie group of invertible k-jets whose source and target are at the origin of $\mathbb{R}^p$. By definition, a contact element of order k and dimension p of M is an equivalence class of $J^{k,p}M$ under the equivalence relation: for $X, Y \in J^{k,p}M$, $X \sim Y$ if there exists $Z \in GL^k(R^p)$ such that $Y = X \circ Z$. The set of contact elements of order k and dimension p of M is a differentiable manifold denoted by $C^{k,p}(M)$. $C^{0,p}(M)$ identifies naturally with M [3].

For $0 \leq k' \leq k$ there is a natural projection $\pi^k_{k'} : C^{k,p}(M) \to C^{k',p}(M)$. If $k' = 0$, we write $\pi^k : C^{k,p}M \to M$ instead of π^k_0. The fiber of $C^{k,p}M$ over $a \in M$ is denoted by $C^{k,p}_aM$. If p is the dimension of M, $C^{k,p}_aM$ has only one element which is denoted by C^k_aM and is called the contact element of order k of M at the point $a \in M$.

Given a submanifold S of M, $S \subset M$, and an integer p, $0 \leq p \leq \dim S$, there is a natural injection of $C^{k,p}S$ into $C^{k,p}M$. If p is the dimension of S, composing the map $a \in S \to C^{k,p}_aS \in C^{k,p}M$ with the injection $C^{k,p}S \to C^{k,p}M$, we define an injection $C^k : a \in S \to C^k_aS \in C^{k,p}M$. The image of this injection is denoted by $C^kS \subset C^{k,p}M$. Two submanifolds S and $\overline{S}$ of M of the same dimension p have contact of order k at a common point a if $C^k_aS = C^k_a\overline{S}$.

3. Completely integrable differential systems of higher order. A differential system of order $k \geq 1$ and dimension p defined over a manifold M is a submanifold Ω^k of $C^{k,p}M$ such that the projection $\pi^k : \Omega^k \to M$ is of rank equal to the dimension of M. An integral manifold of Ω^k is a submanifold S of M of dimension p such that $C^k_xS \in \Omega^k$ for all $x \in S$. For $X \in C^{k,p}M$, let F_X be the fiber of X by the projection $\pi^k_{k-1} : C^{k,p}M \to C^{k-1,p}M$. The symbol $\sigma(X)$ of Ω^k at the point $X \in \Omega^k$ is by, definition, the vector space

$$\sigma(X) = T_X\Omega^k \cap T_XF_X.$$

Let $X^{k+1} \in C^{k+1,p}M$, $X^k = \pi^{k+1}_k(X)$, and let S be a submanifold of M such that $X^{k+1} = C^{k+1}_aS$, $a \in S$. Then, $C^1_{X^k}(C^kS)$ depends only on X^{k+1} and not on the choice of S. Hence, there is a natural imbedding

$$\overline{\Lambda}^{k,1} : C^{k+1,p}M \to C^{1,p}(C^{k,p}M)$$

which maps X^{k+1} into $C^1_{X^k}(C^kS)$. By definition, the first prolongation of the differential system Ω^k is the subset $\Omega^{k,1}$ of $C^{k+1,p}M$ defined by

$$\Omega^{k,1} = (\overline{\Lambda}^{k,1})^{-1}[C^{1,p}(\Omega^k) \cap \overline{\Lambda}^{k,1}(C^{k+1,p}M)].$$

Since $\pi_k^{k+1} = \pi_0^1 \circ \Lambda^{k,1}$, it follows that π_k^{k+1} maps $\Omega^{k,1}$ into Ω^k. If S is an integral manifold of Ω^k then, $C_x^{k+1}S \in \Omega^{k,1}$ for every $x \in S$. Hence, a necessary condition for the existence of an integral manifold of Ω^k going through every point of Ω^k is that the projection $\pi_k^{k+1} : \Omega^{k,1} \to \Omega^k$ be surjective.

THEOREM 1. *Let $\Omega^k \subset C^{k,p}M$ be a differential system of order $k \geq 1$ and let $X \in \Omega^k$ be a contact element such that*

1) $\sigma(X) = \{0\}$;

2) The image of $\Omega^{k,1}$ by the projection $\pi_k^{k+1} : \Omega^{k,1} \to \Omega^k$ is a neighborhood of X in Ω^k.

Then, there exists an integral manifold S of Ω^k such that $X \in C^kS$. Moreover, if S and S' are integral manifolds of Ω^k such that $X \in C^kS \cap C^kS'$, there exists a set W which is an open neighborhood of X in C^kS and C^kS'.

Theorem 1 is a geometrical version of the theorem of existence and uniqueness of solutions of completely integrable systems of partial differential equations of order $k \geq 1$. Taking suitable coordinates in $C^{k+1,p}M$ and $C^{k,p}M$, the existence of integral manifolds of Ω^k reduces to the existence of solutions of a completely integrable system of partial differential equations [6].

4. Contact of submanifolds. Let G be a Lie group acting transitively on the manifold M. Two submanifolds S and $\overline{S}$ of M of same dimension p, have G-contact of order p at points $a \in S$ and $\overline{a} \in \overline{S}$ if there exists $g \in G$ such that $ga = \overline{a}$ and gS and $\overline{S}$ have contact of order k at the point $\overline{a}$. S and $\overline{S}$ have G-contact of order $k \geq 0$ if there exists a diffeomorphism $\phi : S \to \overline{S}$ such that for all $x \in S$, S and $\overline{S}$ have contact of order k at points x and $\phi(x) = g(x)x$. We say in this case that ϕ makes contact of order k of S onto $\overline{S}$. S and $\overline{S}$ are G-equivalent if there exists $g \in G$ such that $gS = \overline{S}$. S and $\overline{S}$ are locally G-equivalent at points $a \in S$ and $\overline{a} \in \overline{S}$ if there are open neighborhoods of a and $\overline{a}$ in S and $\overline{S}$ which are G-equivalent.

The action of G on M extends to an equivariant action on the manifold $C^{k,p}M$ of contact elements of order k and dimension p of M. For a point $x \in M$, let C_x^kS, G_x^k and $d^k(x)$ denote respectively the contact element of order k of S at the point x, the isotropy subgroup of G at the point C_x^kS and the dimension of G_x^k. We call G_x^k the isotropy subgroup of order k of the point x of S. Put $X = C_x^kS$ and let $h^k(x)$ be the dimension of the vector space $T_X(GX) \cap T_XC^kS$ where C^kS is the submanifold of $C^{k,p}M$ of all contact elements of order k of S and $T_X(GX)$ and T_XC^kS are the tangent spaces of the orbit GX and of C^kS at the point X.

For $k' \leq k$, $d^k(x) \leq d^{k'}(x)$ and $h^k(x) \leq h^{k'}(x)$. Hence, there exists an integer $k \geq 1$ such that $d^k(x)d^{k-1}(x)$ and $h^k(x) = h^{k-1}(x)$. We say that $a \in S$ is a k-regular point of

S under the action of G if there exists $k \geq 1$ such that

1) $d^k(a) = d^{k-1}(a)$ and $h^k(a) = h^{k-1}(a)$;
2) $d^k(x)$ and $h^k(x)$ are constant for x varying in a neighborhood of a in S.

The order of a is the least integer satisfying conditions above. If a is a k-regular point of S then ga is a k-regular point of gS.

THEOREM 2. *Let S, $\overline{S}$ be two submanifolds of M of same dimension p. Let $a \in S$ and $\overline{a} \in \overline{S}$ be two points. Assume that $\overline{a}$ is a k-regular point of $\overline{S}$ and that there exists a continuous map $\varphi : V \to G$, defined in a neighborhood V of a in S, such that $\varphi(a).a = \overline{a}$, $\varphi(x).x \in \overline{S}$ and $\varphi(x).C^k_x S = C^k_{\varphi(x)}\overline{S}$ for all $x \in V$. Then, there exist open neighborhoods W and $\overline{W}$ of a and $\overline{a}$ in S and $\overline{S}$ which are G-equivalent.*

The proof of theorem 2 is based on the uniqueness statement of theorem 1.

We assume in theorems 3, 4, 5, 6, 8 that the action of G on M is proper and that H is a closed subgroup of G. Let L be the union of all G-orbits of $C^{k,p}M$ of type H that is, orbits whose isotropy subgroups are conjugate to H. Denote by L/G the quotient space of L by the orbits and by $\pi : L \to L/G$ the natural projection. It is known [1] that L and L/G are differentiable manifolds and that $(L, L/G, \pi)$ is a locally trivial fiber bundle.

Let $f : S \to \overline{S}$ be a diffeomorphism such that S and $\overline{S}$ have G-contact of order $k \geq 1$ at corresponding points $x \in S$ and $\overline{x} = f(x) \in \overline{S}$ and let $a \in S$ and $\overline{a} = f(a) \in \overline{S}$ be two points. Considering suitable cross sections of the fiber bundle $(L, L/G, \pi)$, one can prove the existence of a neighborhood V of a in S and of a differentiable map $\varphi : V \to G$ such that $\varphi(x).x = f(x)$ and $\varphi(x).C^k_x S = C^k_{\overline{x}}\overline{S}$. Hence, theorem 2 can be restated as follows:

THEOREM 3. *Assume that the action of G on M is proper and that there exists $k \geq 1$ such that*

1. *$\overline{a} \in \overline{S}$ is a k-regular point.*
2. *The isotropy subgroups of $C^k_{\overline{x}}\overline{S}$ are conjugate in G for all $\overline{x} \in \overline{S}$.*
3. *There exists a diffeomorphism $f : S \to \overline{S}$ such that S and $\overline{S}$ have G-contact of order k at corresponding points.*

Let $a \in S$ be such that $f(a) = \overline{a}$. Then S and $\overline{S}$ are locally G-equivalent at points a and $\overline{a}$.

THEOREM 4. *Assume that S and $\overline{S}$ are connected and that there exists an integer $k \geq 1$ such that:*

1. *$\overline{x} \in \overline{S}$ is a k-regular point of $\overline{S}$ and $h^k(\overline{x}) = 0$ for all $\overline{x} \in \overline{S}$.*
2. *The isotropy subgroups of $C^k_{\overline{x}}\overline{S}$ are conjugate in G for all $\overline{x} \in \overline{S}$.*
3. *There exists a diffeomorphism $f : S \to \overline{S}$ such that S and $\overline{S}$ have G-contact of order k at corresponding points.*

Then, f is the restriction to S of the translation by an element g of $G : f = L_g|S$.

Consider again the fiber bundle $(L, L/G, \pi)$. There exists a finite number of real valued differentiable functions $\tilde{\rho}_i$, $1 \leq i \leq r$, defined in L, such that two contact elements $X, \overline{X} \in L$ are in the same fiber of L if and only if $\tilde{\rho}_i(X) = \tilde{\rho}_i(\overline{X})$, $1 \leq i \leq r$. Given a submanifold S of M of dimension p, and assuming that the orbits of $C^k_x S$ are of type H for all $x \in S$, one can pull back the functions $\tilde{\rho}_i$ by the map $\sigma^k : x \in S \to C^k_x S \in L$.

The set of functions $\rho_i = \tilde{\rho}_i \circ \sigma^k$, $1 \le i \le r$, is a complete set of G-invariants of order k of the submanifold S of M. Often the invariants can be defined in a natural way and have deep geometrical meaning as for instance, the curvature and torsion of curves and the principal curvatures of surfaces in $\mathbb{R}^3$.

Assuming that the isotropy subgroups of $C^k_x S$ and $C^k_{\overline{x}} \overline{S}$ are of type H for all $x \in S$ and $\overline{x} \in \overline{S}$, complete sets of invariants of order k, ρ_i and $\overline{\rho}_i$ can be defined in S and $\overline{S}$. The condition $h^k(\overline{x}) = 0$ in theorem 4 is then clearly equivalent to stating that the rank of differentials $d\overline{\rho}_i$, $1 \le i \le r$, is p at every point $\overline{x} \in \overline{S}$ One can then restate theorems 3 and 4 in the following way.

THEOREM 5. *Let $\overline{a} \in \overline{S}$ be a k-regular point of $\overline{S}$, $k \ge 1$. Assume the following conditions are satisfied:*

1. *The isotropy subgroups of $C^k_x S$ and $C^k_{\overline{x}} \overline{S}$ are conjugate for all $x \in S$ and $\overline{x} \in S$.*
2. *There exists a diffeomorphism $f : S \to \overline{S}$ such that*

$$\overline{\rho}_i = \rho_i \circ f, \ 1 \le i \le r.$$

Then, S and $\overline{S}$ are locally G-equivalent at points $a = f^{-1}(\overline{a})$ and $\overline{a}$.

THEOREM 6. *Let S, $\overline{S}$ be two connected submanifolds of M and let $k \ge 1$ be such that*

1. *Every point $\overline{x} \in \overline{S}$ is k-regular.*
2. *The isotropy subgroups of $C^k_x S$ and $C^k_{\overline{s}} \overline{S}$ are conjugate for all $x \in S$ and $\overline{x} \in S$.*
3. *There exists a diffeomorphism $f : S \to \overline{S}$ such that*

$$\rho_i = \overline{\rho}_i \circ f, \ 1 \le i \le r.$$

4. *The rank of differentials $d\overline{\rho}_i$, $1 \le i \le r$, is p at every point $\overline{x} \in \overline{S}$.*

Then, f is the restriction to S of the left translation by an element of G: $f = \mathcal{L}_g | S$.

Let us assume that S is an open set of an orbit of a Lie subgroup K of G. Then, $h^k(x) = p$ and the isotropy subgroups of C^k_x are conjugate for all $x \in S$ and $k \ge 0$. Hence there exits $k \ge 1$ such that every $x \in S$ is a k-regular point of S. Conversely,

THEOREM 7. *A necessary and sufficient condition for a connected submanifold S of M to be an open set of an orbit of a Lie subgroup K of G is the existence of $k \ge 1$ such that for all $x \in S$, x is a k-regular point of S and $h^k(x) = p$.*

Assuming that the action of G on M is proper and that the isotropy subgroups of order k of points of S are conjugate, a complete set of invariants of order k can be defined on S. Clearly, $h^k(x) = p$ for every $x \in S$ if and only if the invariants are constant on S. Therefore, the following corollary to theorem 7 holds.

THEOREM 8. *Assume that the action of G on M is proper and that S is connected. Assume also that for some integer $k \ge 1$, every point of S is k-regular and all isotropy subgroups of order k of points of S are conjugate. Then, a necessary and sufficient condition for S to be an open set of an orbit of a Lie subgroup of G, is that the invariants of order k of S be constant.*

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