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CHARACTERISTIC NUMBERS, BORDISM THEORY AND THE NOVIKOV CONJECTURE FOR OPEN MANIFOLDS

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1. Introduction. Let (M^n, g) be closed, oriented, with triangulation K. Then the characteristic numbers

$$\langle \omega_1^{i_1} \dots \omega_n^{i_n}, [M] \rangle = \omega_{i_1 \dots i_n}, \quad 1i_1 + \dots + ni_n = n \tag{1.1}$$

and for $n = 4k$

$$\langle p_1^{i_1} \dots p_k^{i_k}, [M] \rangle = p_{i_1 \dots i_k}, \quad 1i_1 + \dots + ki_k = k, \tag{1.2}$$

are well defined. Here ω_i are the Stiefel–Whitney and p_i the Pontrjagin classes.

They characterize the elements of Ω_n^{SO}. More generally, the numbers

$$\langle p_1^{i_1} \dots p_k^{i_k} f^* h^{(m)}, [M] \rangle, \tag{1.3}$$

where $m = n - 4k$, $h^{(m)} \in H^m(X, Z)$, X a finite CW-complex and $f : M^n \to X$ characterize the singular bordism class of $[(M, f)] \in \Omega_n^{SO}(X)$ modulo torsion.

If M^n is open (hence K infinite) then $\omega_{i_1 \dots i_n}$ and $p_{i_1 \dots i_k}$ are in generally not defined. More generally, we have the following simple

Proposition 1.1. *There does not exist any nontrivial number valued (vector valued) invariant which is defined for all connected oriented manifolds and which is additive w.r.t. connected sums.* ∎

There are several ways out from this situation.

1) One should admit more general ranges of definition, e.g. K-groups (Mishchenko, Roe et al.) and give up the condition of additivity.
2) One could impose certain restrictions, i.e. define invariants not for "all" manifolds.
3) One could work with other definitions of characteristic numbers, e.g. more analytical ones.
4) One could introduce relative characteristic numbers.

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The fundamental criterion for establishing such numbers should be their geometrical meaning and the applicability.

We define in section 2 analytical characteristic numbers, study their invariance properties and applications. Section 3 is devoted to combinatorial characteristic numbers. In section 4, we define relative characteristic numbers and apply them to bordism theory and we study bordism theory of manifolds with nonexpanding ends and relate it with the growth of the signature. Finally, section 5 is devoted to suitable versions of the Novikov conjecture.

2. Absolute characteristic numbers for open manifolds. Let (M^{4k}, g) be closed, oriented, g an arbitrary Riemannian metric, $p_i(M,g)$ the associated by Chern–Weil construction Pontrjagin classes, $e(M,g)$ the Euler class, L_k the Hirzebruch polynomial. Then there are the well known equations

$$\sigma(M^{4k}) = \int L_k(M,g) = \int L_k(p_1(M,g),\dots,p_k(M,g)) = \sigma(M,g) \tag{2.1}$$

and for (M^n, g) oriented

$$\chi(M^n) = \int e(M,g) = \chi(M,g). \tag{2.2}$$

These equations express in particular that the r.h.s. are in fact independent of g and are homotopy invariants. We proved that the space of Riemannian metrics on a manifold splits w.r.t. a canonical uniform structure into "many" components and that on a compact manifold there is only one component (cf. e.g. [7]). The independence of g can be reformulated as the r.h.s. depend only on $\mathrm{comp}(g)$, since the space of Riemannian metrics on a closed manifold consists only of one component. We will extend the definitions of the l.h.s. and the r.h.s. to certain classes of open manifolds. In some cases there even holds equality. The main questions connected with such an extension are

1) the invariance properties,

2) applications, the geometrical meaning.

It is clear that the definition of characteristic numbers via Chern–Weil construction can be extended to an open manifold if the Chern–Weil integrand is $\in L_1$, as a very special case if this integrand is bounded and (M^n, g) has finite volume.

Let (M^n, g) be an open complete manifold, G a compact Lie group with Lie algebra $\mathfrak{G}$, $\varrho : G \to U_N$ or $\varrho : G \to SO_N$ a faithful representation, $P = P(M,G)$ a principal fibre bundle and $E = P \times_G E_N$ the associated vector bundle which is endowed with a Hermitean or Riemannian metric. According to the faithfulness of ϱ, the connections on P and E are in a one-to-one correspondence, $\omega \leftrightarrow \nabla^\omega = \nabla$. Denote by $\mathcal{C}(P, B_0, f, p) = \mathcal{C}(E, B_0, f, p)$ the set of all connections $\omega \leftrightarrow \nabla^\omega = \nabla$ with bounded curvature, i.e. satisfying (B_0): $|R| \le C$, where R denotes the curvature and $|\,|$ the pointwise norm, and having finite p-action

$$\int |R^{\nabla^\omega}|^p \,\mathrm{dvol}_x(g) < \infty.$$

We fix P and E and write therefore simply $\mathcal{C}(B_0, f, p)$. Let $\delta > 0$ and set

$$\begin{aligned} V_\delta = \{(\nabla, \nabla') \in \mathcal{C}(B_0, f, p)^2 | {}^{b,1}|\nabla - \nabla'|_{\nabla,p,1} = {}^b|\nabla - \nabla'| + {}^b|\nabla(\nabla - \nabla')| \\ +|\nabla - \nabla'|_p + |\nabla(\nabla - \nabla')|_p < \delta\} \end{aligned}$$

LEMMA 2.1. $\mathfrak{B} = \{V_\delta\}_{\delta > 0}$ *is a basis for a metrizable uniform structure* ${}^{b,1}\mathfrak{U}^{p,1}(\mathcal{C}(B_0, f, p))$.

Proof. We start with (U_2'): For each $V \in \mathfrak{B}$ there exists $V' \in \mathfrak{B}$ such that $V' \subseteq V^{-1}$.

$$^{b,1}|\nabla' - \nabla|_{\nabla',p,1} = {}^b|\nabla' - \nabla| + {}^b|\nabla'(\nabla' - \nabla)| + |\nabla' - \nabla|_p + |\nabla'(\nabla' - \nabla)|_p.$$

Hence we have to estimate only

$$^b|\nabla'(\nabla' - \nabla)| \leq {}^b|(\nabla' - \nabla)(\nabla' - \nabla)| + {}^b|\nabla(\nabla' - \nabla)| \leq C\, {}^{b,1}|\nabla' - \nabla|^2 + {}^{b,1}|\nabla' - \nabla|$$

and

$$\begin{aligned} |\nabla'(\nabla' - \nabla)|_p &\leq |(\nabla' - \nabla)(\nabla' - \nabla)|_p + |\nabla(\nabla' - \nabla)|_p \\ &\leq C_2\, {}^b|\nabla' - \nabla||\nabla' - \nabla|_p + |\nabla(\nabla' - \nabla)|_p, \end{aligned}$$

i.e.

$$^{b,1}|\nabla' - \nabla|_{\nabla',p,1} \leq P_1({}^{b,1}|\nabla' - \nabla|_{\nabla,p,1}),$$

where P_1 is a polynomial without constant term. (U_2') is done.

For (U_3'): For each $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ such that $W \circ W \subseteq V$ we have to estimate in

$$^{b,1}|\nabla_1 - \nabla_2|_{\nabla_1,p,1} \leq {}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1} + {}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1} \tag{2.3}$$

only the term ${}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1}$. But

$$\begin{aligned} {}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1} &= {}^b|\nabla - \nabla_2| + {}^b|\nabla_1(\nabla - \nabla_2)| + |\nabla - \nabla_2|_p + |\nabla_1(\nabla - \nabla_2)|_p \\ &\leq {}^b|\nabla - \nabla_2| + {}^b|(\nabla_1 - \nabla)(\nabla - \nabla_2)| + {}^b|\nabla(\nabla - \nabla_2)| + |\nabla - \nabla_2|_p \\ &\quad +|(\nabla_1 - \nabla)(\nabla - \nabla_2)|_p + |\nabla(\nabla - \nabla_2)|_p \\ &\leq {}^{b,1}|\nabla - \nabla_2|_{\nabla,p,1} + 2\, {}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1} \cdot {}^{b,1}|\nabla - \nabla_2|_{\nabla,p,1}, \end{aligned}$$

together with (2.3)

$$^{b,1}|\nabla_1 - \nabla_2|_{\nabla_1,p,1} \leq P_2({}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1}, |\nabla - \nabla_2|_{\nabla,p,1}),$$

where P_2 is a polynomial without constant term. (U_3') is done. ■

Denote by ${}^{b,m}\Omega^q(\mathfrak{G}_E)$ or $\Omega^{q,p,r}(\mathfrak{G}_E)$ or ${}^{b,m}\Omega^{q,p,r}(\mathfrak{G}_E)$ the completion of

$$^b_m\Omega^q(\mathfrak{G}_E) = \left\{\eta \in \Omega^q(\mathfrak{G}_E) \,|\, {}^{b,m}|\eta| := \sum_{\mu=0}^m \sup_x |\nabla^\mu \eta|_x < \infty\right\}$$

or

$$\Omega_r^{q,p}(\mathfrak{G}_E) := \left\{\eta \in \Omega^q(\mathfrak{G}_E) \,|\, |\eta|_{p,r} := \left(\int \sum_{i=0}^r |\nabla^i \eta|_x^p \, \mathrm{dvol}_x(g)\right)^{\frac{1}{p}} < \infty\right\}$$

$$^b_m\Omega_r^{q,p}(\mathfrak{G}_E) = {}^b_m\Omega^q(\mathfrak{G}_E) \wedge \Omega_r^{q,p}(\mathfrak{G}_E)$$

with respect to ${}^{b,m}|\,|$ or $|\,|_{p,r}$ or ${}^{b,m}|\,|_{p,r} = {}^{b,m}|\,| + |\,|_{p,r}$, respectively. We obtain $\Omega^{q,p,d}$ etc. by replacing $\nabla \to d$.

Denote by ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ the completion w.r.t. ${}^{b,1}\mathfrak{U}^{p,1}$.

THEOREM 2.2. **a)** *${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ is locally arcwise connected.*
b) *In ${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ components and arc components coincide.*
c) *${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ has a decomposition as a topological sum*

$$ {}^{b,1}\mathcal{C}^{p,1}(B_0,f,p) = \sum_{i\in I} {}^{b,1}\mathrm{comp}^{p,1}(\nabla_i). $$

d)
$$ \begin{aligned} {}^{b,1}\mathrm{comp}^{p,1}(\nabla) &= \{\nabla' \in {}^{b,1}\mathcal{C}^{p,1}(B_0,f,p) \mid {}^{b,1}|\nabla-\nabla'|_{\nabla,p,1} < \infty\} \\ &= \nabla + (\text{completion of } {}^{b}_{1}\Omega^1(\mathfrak{G}_E,\nabla)\cap\Omega^{1,p}_1(\mathfrak{G}_E,\nabla) \\ &\qquad \text{w.r.t.} {}^{b,1}|\;|_{\nabla,p,1}) = \nabla + {}^{b,1}\Omega^{1,p,1}(\mathfrak{G}_E,\nabla). \end{aligned} $$

Proof. The only fact to prove is a). b) and c) are consequences of a) and d) follows from $\nabla' = \nabla + (\nabla'-\nabla)$. Let $\varepsilon > 0$ be so small that in $U_\varepsilon(\nabla)$ ${}^{b,1}|\cdot - \cdot|_{\nabla,p,1}$ and the metric of ${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ are equivalent. Put for $\nabla'\in U_\varepsilon(\nabla)$, ${}^{b,1}|\nabla-\nabla'|_{\nabla,p,1}<\varepsilon$, $\nabla_t := (1-t)\nabla + t\nabla' = \nabla + t(\nabla'-\nabla)$. If $\nabla_\nu \in {}^{b}_{1}\Omega(\mathfrak{G}_E,\nabla)\cap\Omega^{1,p}_1(\mathfrak{G}_E,\nabla)$ and ${}^{b,1}|\nabla_\nu-\nabla|_{\nabla,p,1} \underset{\nu\to\infty}{\longrightarrow} 0$ then $\nabla_{\nu,t} = \nabla + t(\nabla_\nu-\nabla) \to \nabla + t(\nabla'-\nabla) = \nabla_t$, i.e. $\nabla_t \in {}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$. Moreover, ${}^{b,1}|\nabla_{t_1}-\nabla_{t_2}|_{\nabla,p,1} = |t_1-t_2|\cdot {}^{b,1}|\nabla'-\nabla|_{\nabla,p,1} \underset{t_1\to t_2}{\longrightarrow} 0$. ∎

LEMMA 2.3. *The elements ∇ of ${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ satisfy (B_0) and*

$$ \int |R^\nabla|^p_x \,\mathrm{dvol}_x(g) < \infty. $$

Proof. By the definition of ${}^{b,1}\mathcal{C}^{p,1}$ its elements are C^1 (since they arise by uniform convergence of 0-th and 1rst derivatives) hence R^∇ is defined. If $\nabla_\nu\to\nabla$, $\nabla_\nu\in\mathcal{C}(B_0,f,p)$, $\nabla = \nabla_\nu + (\nabla-\nabla_\nu)$, then, for fixed ν,

$$ R^\nabla = R^{\nabla_\nu+(\nabla-\nabla_\nu)} = R^{\nabla_\nu} + d^\nabla(\nabla-\nabla_\nu) + \frac{1}{2}[\nabla-\nabla_\nu,\nabla-\nabla_\nu]. \tag{2.4} $$

Each term of the r.h.s. of (2.4) is bounded, hence R^∇. Moreover $|R^{\nabla_\nu}|\in L_p$, $d^\nabla(\nabla-\nabla_\nu)|$ $\in L_p$ and $[\nabla-\nabla_\nu,\nabla-\nabla_\nu] \le C\cdot {}^{b}|\nabla-\nabla_\nu|\cdot|\nabla-\nabla_\nu| \in L_p$. ∎

Now let $\omega \leftrightarrow \nabla^\omega = \nabla$ be given. After choice of a bundle chart with local base $s_1,\ldots,s_N : U \to E|_U$ the curvature Ω will be described as $\Omega s_i = \sum_j \Omega_{ij}\otimes s_j$, where (Ω_{ij}) is a matrix of 2-forms on U, $\Omega_{ij}(s_k,s_l) = \Omega_{ij,kl} = R_{ij,kl}$. An invariant polynomial $P : \mathrm{Mat}_N \to \mathbb{C}$ defines in the well known manner a closed graded differential form $P = P(\Omega) = P_0 + P_1 + \cdots$, where P_ν is a homogeneous polynomial, $P_r(\Omega) = 0$ for $2r > n$. The determinant is an example for P. If ω is not smooth then $P(\Omega)$ is closed in the distributional sense. Let $\sigma_r(\Omega)$ be the $2r$-homogeneous part (in the sense of forms) of $\det(1+\Omega_{ij})$.

LEMMA 2.4. *Each invariant polynomial is a polynomial in $\sigma_1,\ldots,\sigma_N$.*

LEMMA 2.5. *If $\omega\in {}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ and $r\ge 1$ then*

$$ \int |\sigma_r(\Omega)|^p_x \,\mathrm{dvol}_x(g) < \infty. \tag{2.5} $$

Proof. For the pointwise norm $|\;|_x$ we have $|\Omega|^2_x = \frac{1}{2}\sum_{i,j}\sum_{k<l}|\Omega_{ij,kl}|^2_x$, where $\Omega_{ij,kl} = \Omega_{ij(e_k,e_l)}$ and $e_1,\ldots,e_n$ is an orthogonal base of T_xM. According to our assumption we

have $|\Omega|_p^p = \int |\Omega|_x^p \,\mathrm{dvol} < \infty$ and $|\Omega|_x \le b$ for all $x \in M$. The proof is done if we could estimate $|\sigma_r(\Omega)|_x$ from above by $|\Omega|_x$. By definition

$$\sigma_r(\Omega) = \frac{1}{r!} \sum \varepsilon^{i_1 \dots i_r}_{j_1 \dots j_r} \Omega_{i_1 j_1} \wedge \dots \wedge \Omega_{i_r j_r}, \tag{2.6}$$

where summation runs over all $1 \le i_1 < \dots < i_r \le N$ and all permutations $(i_1 \dots i_r) \to (j_1 \dots j_r)$. ε denotes the sign of this permutation. We perform induction. For $r = 1$ $\sigma_1(\Omega) = \sum \Omega_{ii}$. The inequality

$$|\Omega_{ij}|_x^2 \le \sum_{s,t} |\Omega_{st}|_x^2 = 2|\Omega|_x^2 \tag{2.7}$$

implies in particular $|\sigma_1(\Omega)|_x^2 \le 2N|\Omega|_x^2$. For arbitrary forms φ, ψ we have

$$|\varphi \wedge \psi|_x \le |\varphi|_x \cdot |\psi|_x. \tag{2.8}$$

For forms with values in a vector bundle we have to multiply the r.h.s. of (2.8) with a constant. (2.6), (2.7), (2.8) and the induction assumption thus give

$$|\sigma_r(\Omega)|_x^2 \le a \cdot |\Omega|_x^{2r}, \tag{2.9}$$

together with $|\Omega|_x^2 \le b^2$ finally $|\sigma_r(\Omega)|_x = c \cdot |\Omega|_x$. ∎

Corollary 2.6. *Let P be an invariant polynomial, $\omega \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$, $r \ge 1$. Then each form $P_r(\Omega)$ is an element of ${}^{b,1}\Omega^{2,rp,1}$.*

Proof. This follows from 2.4, 2.5 and (2.8). ∎

Denote by $H^{*,p}$ or ${}^bH^*$ the L_p or bounded cohomology, respectively.

Corollary 2.7. *Under the assumptions of 2.6, P and ω define well defined classes $[P_\varrho(\Omega^\omega)] \in H^{2\varrho,p}(M)$, $[P_\varrho(\Omega^\omega)] \in {}^bH^{2\varrho}(M)$.* ∎

Now the natural question arises: how does $[P_\varrho(\Omega^\omega)]$ depend on ω? We denote $I = [0,1]$, $i_t : M \to I \times M$ the imbedding $i_t(x) = (t,x)$ and furnish $I \times M$ with the product metric $\left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$. Here we write $H^{q,p,\{d\}} \equiv H^{q,p}$ etc.

Lemma 2.8. *For every $q \ge 0$ there exists a linear bounded mapping $K : \Omega^{q+1,p,d}(I \times M) \to \Omega^{q,p,d}(M)$ resp. $K : {}^b\Omega^{q+1,d}(I \times M) \to {}^b\Omega^{q,d}(M)$ such that $dK + Kd = i_1^* - i^* - 0$.*

Proof. Since $g_{I\times M} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$ is an isometric imbedding and i_t^* is bounded. i_t^* maps into $\Omega^{q,p,d}(M)$ because $|di^*\varphi|_x = |i^* d\varphi|_x \le C \cdot |d\varphi|_x$. Denote $X_0 = \frac{\partial}{\partial t}$ and for $\varphi \in \Omega^{q+1,p,d}(I \times M)$ $\varphi_0(X_1, \dots, X_q) := \varphi(X_0, X_1, \dots, X_q)$. Then $\varphi_0 \in \Omega^{q,p,d}(I \times M)$, $|\varphi_0|_{(t,x)} \le |\varphi|_{(t,x)}$, and we define

$$K\varphi(X_1, \dots, X_n) := \int_0^1 i_t^* \varphi_0(X_1, \dots, X_q) dt.$$

Thus K is bounded too. The equation $dK + Kd = i_1^* - t_0^*$ is standard. Replacing $\Omega^{q,p,d}$ by ${}^b\Omega^{q,d}$ gives the same conclusions. ∎

Lemma 2.9. *Let $f, g : M \to N$ be smooth mappings, $F : I \times M \to N$ a smooth homotopy, $f^*, g^* : \Omega^{q,p,d}(N) \to \Omega^{q,p,d}(M)$, $F^* : \Omega^{q,p,d}(N) \to \Omega^{q,pd}(I \times M)$ resp. $f^*, g^* : {}^b\Omega^{q,d}(N) \to {}^b\Omega^{q,d}(M)$, $F^* : {}^b\Omega^{q,d}(N) \to {}^b\Omega^{q,d}(I \times M)$ bounded and $\varphi \in \Omega^{q,p,d}(N)$ resp. $\varphi \in {}^b\Omega^{q,d}(N)$ closed, i.e. $\varphi \in Z^{q,p}(N)$ resp. $\varphi \in {}^bZ^q(N)$. Then we have $(g^* - f^*)\varphi \in B^{q,p}(M)$ resp. $(g^* - f^*)\varphi \in {}^bB^q(M)$.*

Proof. We consider the case $\Omega^{q,p,d}$. According to our assumption we have $K\varphi := KF^*\varphi \in \Omega^{q-1,p,d}(M)$ and $(g^*-f^*)\varphi = ((F\circ t_1)^* - (F\circ i_0)^*)\varphi = (i_1^*F^* - i_0^*F^*)\varphi = (i_1^* - i_0^*)F^*\varphi = (dK+Kd)F^*\varphi = dKF^*\varphi = dK\varphi$. The case of bounded forms will be treated by the same equation. ∎

Now we are able to prove one of our main theorems.

THEOREM 2.10. *Let $Q : M_N(C) \to C$ be an invariant polynomial, $r \geq 1$, $p = 1$ or 2. Then each component U of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ determines uniquely a cohomology class $[Q_r(\Omega^U)] \in H^{p,2r}(M)$ resp. $[Q_r(\Omega^U)] \in {}^bH^{2r}(M)$.*

Proof. Assume $\omega_0, \omega_1 \in U$. Then, according to theorem 2.2, d) $\eta := \omega_1 - \omega_0 \in {}^{b,1}\Omega^{1,p,1}(\mathfrak{G}_E, \omega_0)$ and $\omega_t = \omega_0 + t_\eta$, $-\delta < t < 1+\delta$, is contained in U. We have to show $[Q_r(\Omega^{\omega_0})] = [Q_r(\Omega^{\omega_1})]$. Consider

$$\Omega_t := \Omega^\omega, \quad \Omega_t = \Omega_0 + td^{\omega_0}\eta + \frac{1}{2}t^2[\eta,\eta]. \tag{2.10}$$

For all $t \in]-\delta, \delta+1[$ we have $\int |\Omega_t|^p \,\mathrm{dvol} < \infty$ and $|\Omega_t|_x$ is bounded at M. This follows from (2.10) and the assumption $\omega_0, \omega_1 \in U$. If $\overline{p} :]-\delta, 1+\delta[\times M \to M$ denotes the projection $(t,x) \to x$, $P' = \overline{p}^*P$ resp. $E' = \overline{p}^*E$ the liftings of the bundles to $]-\delta, 1+\delta[\times M$, p (which covers $\overline{p}$) the associated mapping of the bundle spaces, then $p^*\omega_0$, $p^*\omega_1$ are connections for the lifted bundles.

$tp^*\omega_1 + (1-t)p^*\omega_0 = p^*\omega_0 + tp^*\eta$ is again a connection ω'. According to (2.10), we have $\Omega^{\omega'} = p^*\Omega_0 + t\partial^{p^*\omega_0}p^*\eta + \frac{t^2}{2}[p^*\eta, p^*\eta]$. p^* is bounded. Thus $\Omega^{\omega'}$ is surely p-integrable and bounded if this holds for $d^{p^*\omega_0}p^*\eta$. But this follows from the equation $(p^*\omega_0)_{ij} = p^*(\omega_{0,ij})$ for the connected matrix, $\eta \in {}^{b,1}\Omega^{1,p,d}(\mathfrak{G}_E, \omega_0)$ and from the boundedness of p^*. ω', $\Omega^{\omega'}$ define well determined p-integrable resp. bounded cocycles at $]-\delta, 1+\delta[\times M$. Let i_t again be the mapping $x \to (t,x)$. Then $i_0^*(E',\omega')$ resp. $i_1^*(E',\omega')$ can be identified with (E,ω_0) resp. (E,ω_1). i_t, $0 \leq t \leq 1$, is a smooth bounded homotopy between i_0 and i_1. According to 2.9, $i_0^*\Omega_r(\Omega')$ and $i_1^*\Omega_r(\Omega')$ are cohomologous in $H^{2r,p}$ resp. ${}^bH^{2r}$, i.e. $Q_r(\Omega_0)$ and $Q_r(\Omega_1)$ are cohomologous in $H^{2r,p}$ resp. ${}^bH^{2r}$. ∎

DEFINITION. For a component U of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ we define the r-th *Chern class* $c_r(PU, p)$ by

$$c_r(E, U, p) = c_r(P, U, p) := \frac{1}{(2\pi)^r}[\sigma_r(\Omega^U)].$$

Then we have $c_r \in H^{2r,p}$, $c_r \in {}^bH^{2r}$.

REMARK 2.11. For $\omega_0, \omega_1 \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ the cocycles $\sigma_r(\Omega^{\omega_0}/(2\pi i)^r$, $\sigma_r(\Omega^{\omega_1}/(2\pi i)^r$ are contained in the Chern class $c_r(E)$ and therefore they are cohomologous, but they do not need to be cohomologous in $H^{2r,p}$. Take for example an $\omega \in \Omega^{p,1}(B_0, f, p)$ and apply a gauge transformation g with $\omega - g^*\omega \notin {}^{b,1}\Omega^{p,1,d}(\mathfrak{G}_E, \omega)$. Then $|\Omega^\omega|_x = |\Omega^{g^*\omega}|_x$. An explicit example is given by $M = R^2$, $P = M \times U_N$, ω the canonical flat connection, the gauge transformation g at the point (x,y) given by $e^{i(x^2+y^2)} \cdot \mathrm{id}$, where id denotes the unit matrix. Then $|\omega - g^*\omega|_{(x,y)} = |g^{-1}dg|_{(x,y)} = |i(x\,dx - y\,dy)\cdot \mathrm{id}\,|_{(x,y)} = |N(x^2+y^2)|^{\frac{1}{2}}$ is neither bounded nor p-integrable. For this reason our approach above seems to be suitable for the general situation on noncompact Riemannian manifolds. ∎

DEFINITION. For $\varrho : G \to O_N$, $E = P \times_G \mathbb{R}^N$ denote by E^c or P^c the complexification of E or P, respectively. Any connection ω on E resp. P extends in a canonical manner to a connection on E^c resp.P^c and we have an inclusion of the components U of ${}^{b,1}\mathcal{C}^{p,1}(P, B_0, f, p)$ into the components U^c of ${}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$. Then we define the k-th *Pontrjagin class* $p_k(P, U, p)$ by

$$p_k(P, U, p) = p_k(E, U, p) := (-1)^k c_{2k}(P^c, U^c, p).$$

Let P be the Pfaff polynomial for skew symmetric $2N$-matrices, $\varrho : G \to SO_{2N}$, $E = P \times_G \mathbb{R}^{2N}$. Then for a component U of ${}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$ we call

$$e(E, U, p) := \frac{1}{(2\pi)^N} Pf(\Omega^U)$$

the *Euler class* of U. Then $e \in H^{2N,p}(M)$, $e \in {}^bH^{2N}(M)$.

Now come in characteristic numbers. Consider $\varrho : G \to U_N$, let $\dim M = 2k$ and Q an invariant polynomial, $\omega \in {}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$, $Q(\Omega^\omega) = a + Q_1(\Omega) + \cdots + Q_k(\Omega)$. Then $Q_{i_1 \ldots i_k} := Q_{i_1} \wedge \cdots \wedge Q_{i_k}$ with $i_1 + \cdots + i_k = k$ defines a characteristic $2k$-form and a characteristic number $\int Q_{i_1 \ldots i_k} = Q_{i_1 \ldots i_k}(P, \omega)(M)$ if the latter integral exists. In particular we consider the classes $c_{i_1 \ldots i_k} := c_{i_1} \wedge \cdots \wedge c_{i_k}$ and have to ensure the existence of the corresponding integral.

LEMMA 2.12. **a)** *If $k = 1$ and $\omega \in {}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$, then $\int_M c_1$ converges.*
b) *If $k > 1$ and $\omega \in {}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$ or $\omega \in {}^{b,1}\mathcal{C}^{2,1}(B_0, f, 2)$ then $\int_M c_{i_1 \ldots i_k}$ converges.*

Proof. a) is clear. We have to prove b). At each $x \in M$, $c_{i_1 \ldots i_k}$ is a sum of monomials $a.\ \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_k j_k}$. According to lemma 2.5 $|\Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_k j_k}|_x \leq D_1 \cdot |\Omega|_x$ resp. $\leq D_2 \cdot |\Omega|_x^2$ if $p = 1$ resp. $p = 2$. ∎

COROLLARY 2.13. *Under the assumption of 2.12, for any invariant polynomial Q, the integral $\int_M Q_{i_1 \ldots i_k}$ converges.*

Proof. This follows from 2.4 and the proof of 2.12. ∎

Lemma 2.12 b) is also valid in the case $\varrho : G \to O_N$, $\dim M = 4k$ for $p_{i_1 \ldots i_k}$, $i_1 + \cdots + i_k = k$, $k \geq 1$, resp. in the case $\varrho : G \to SO_{2N}$, $\dim M = N$ for the Euler form $e(E, \omega, 1, g)$ (2.12 a) for $N = 1$, 2.12 b) for $N > 1$).

The above characteristic numbers until now are defined only for a chosen connection ω. One would like that the characteristic numbers are constant on least on the components of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. This is in fact the case for $p = 1$.

THEOREM 2.14. *The characteristic numbers are constant on the components of the space ${}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$.*

Proof. If ω, ω' are contained in the same component U, then according to 2.10, $Q_{i_1 \ldots i_k}(\omega)$ and $Q_{i_1 \ldots i_k}(\omega')$ define the same cohomology class in $H^{2k,1}(M)$ resp. $H^{4k,1}(M)$, i.e. there exists an absolutely integrable φ with $d\varphi = Q_{i_1 \ldots i_k}(\omega) - Q_{i_1 \ldots i_k}(\omega')$. A fundamental result of Gaffney then says $\int_M d\varphi = 0$ for (M, g) complete and $d\varphi$ itself absolutely integrable ([12]). ∎

Thus one gets characteristic numbers $Q_{i_1 \ldots i_k}(P, U)(M)$.

REMARK 2.15. For $\omega, \omega' \in {}^{b,1}\mathcal{C}^{2,1}(B_0, f, 2)$ and $\deg(Q_{i_1\ldots i_k}) \geq 4$ the characteristic numbers $Q_{i_1\ldots i_k}(\omega)(M)$, $Q_{i_1\ldots i_k}(\omega')(M)$ are defined. If $Q_{i_1\ldots i_k}(\omega)$, $Q_{i_1\ldots i_k}(\omega')$ define the same cohomology class in $H^{2k,1}(M^{2k})$ resp. $H^{4k,1}$ resp. $H^{2N,1}$, then the characteristic numbers coincide. ∎

A very special but interesting case in our considerations is the case $\mathrm{vol}(M) < \infty$. Consider ${}^{b,1}\mathcal{C}(B_0)$. It is defined by means of $\mathfrak{B} = \{V_\delta\}_{\delta>0}$, where $V_\delta = \{(\nabla, \nabla') \in \mathcal{C}(B_0)^2 \mid {}^{b,1}|\nabla - \nabla'|_\nabla < \delta\}$.

THEOREM 2.16. *If* $\mathrm{vol}(M) < \infty$ *then characteristic numbers are constant on each component of* ${}^{b,1}\mathcal{C}(B_0)$.

Proof. According to 2.10 each component U of ${}^{b,1}\mathcal{C}(B_0)$ determines uniquely a cohomology class $[Q_{i_1\ldots i_k}]$ in ${}^bH^{2k}(M^{2k})$ or ${}^bH^{4k}$ or ${}^bH^{2N}$ respectively. Taking two cocycles of this class, there exists a bounded C^1-form φ such that there difference equals to $d\varphi$. φ, $d\varphi$ are bounded, $\mathrm{vol}(M) < \infty$, thus φ, $d\varphi$ are absolutely integrable and the theorem of Gaffney gives the desired result. ∎

REMARK 2.17. $\mathrm{vol}(M) < \infty$ implies ${}^{b,1}\mathcal{C}(B_0) = {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. Thus the conclusion of 2.16 also holds for the components of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. ∎

We call the quasi isometry class of g the uniform structure $US(g)$ generated by g. For all metrics of $US(g)$ the cohomology spaces $H^{*,p}(M^n, g')$ coincide. The same holds for ${}^bH^*(M^n, g')$. This leads immediately to

THEOREM 2.18. *The cohomology classes* $Q_{i_1\ldots i_k}(\omega)$ *resp. the characteristic numbers* $Q_{i_1\ldots i_k}(\omega)(M)$ *in 2.10 respectively 2.3, 2.5 are the same for all metrics* $g' \in US(g)$. ∎

The situation completely changes if ω itself depends on g. Then it is not true in general that for $g' \in US(g)$, $c(\omega(g)) \sim c(\omega(g'))$. The case $\omega = \omega(g)$ is essentially the case $P =$ bundle of orthogonal frames of (M^n, g), $\nabla =$ Levi-Civita connection ∇^g. Therefore we briefly describe the metrics which come into question and describe their admitted variation (for fixed M).

Let

$$\mathcal{M}(B_0, p, f) = \Big\{ g \,|\, g \text{ complete, satisfies } (B_0) \text{ and } \int |R^g|_x^p \,\mathrm{dvol}_x(g) < \infty \Big\},$$

$$\begin{aligned} {}^{b,2}|g - g'|_{g,p,2} &= {}^{b,2}|g - g'|_g + |g - g'|_{g,p,2} \\ &= {}^b|g - g'|_g + {}^b|\nabla^g - \nabla^{g'}|_g + {}^b|\nabla^g(\nabla^g - \nabla^{g'})| \\ &\quad + \Big(\int \Big(|g - g'|_{g,x}^p + \sum_{i=0}^{1} |(\nabla^g)^i(\nabla^g - \nabla^{g'})|_{g,x}^p \Big) \mathrm{dvol}_x(g) \Big)^{\frac{1}{p}} \end{aligned}$$

and set

$$V_\delta = \{(g, g') \in \mathcal{M}(B_0, p, f)^2 \,|\, C(n, \delta)^{-1} g \leq g' \leq C(n, \delta) g \text{ and } {}^{b,2}|g - g'|_{g,p,2} < \delta\}.$$

Here $C(n, \delta) = 1 + \delta + \delta\sqrt{2n(n-1)}$.

LEMMA 2.19. $\mathfrak{B} = \{V_\delta\}_{\delta>0}$ *is a basis for a metrizable uniform structure.* ∎

Denote by ${}^{b,2}\mathcal{M}^{p,2}(B_0, p, f)$ its completion.

PROPOSITION 2.20. **a)** ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *is locally arcwise connected.*
b) *In* ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *coincide components with arccomponents.*
c) ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *has a representation as a topological sum*

$$ {}^{b,2}\mathcal{M}^{p,2}(B_0,p,f) = \sum_{i\in I} {}^{b,2}\mathrm{comp}^{p,2}(g_i). $$

d) $\mathrm{comp}(g) = \{g' \in {}^{b,2}\mathcal{M}^{p,2}(B_0,p,f) \mid {}^{b,2}|g-g'|_{g,p,2} < \infty\}$. ∎

PROPOSITION 2.21. *If* $g' \in \mathrm{comp}(g)$ *then* $\nabla^{g'} \in \mathrm{comp}(\nabla^g)$ *is the sense of theorem 2.2, d).*

Hence we obtain well defined characteristic classes $C(\nabla^g) = C(g)$ and characteristic numbers $C\ldots(\nabla^g)(M) = C\ldots(g)(M)$ as above. The main important cases are the Euler form $e = E(g)$,

$$ \chi(M^n,g) := \int_M E(g) $$

and the signature case

$$ \sigma(M^n,g) := \int_M L(g), $$

where $L(g)$ is the Hirzebruch genus.

The following natural questions arise.

1) How does $E(g)$ depend on g?
2) What is the topological meaning of $\chi(M^n,g)$?
3) Under which conditions, $\chi(M^n,g) = \chi(M^n)$, i.e. the Gauß–Bonnet formula holds?

The same questions should be put for $\sigma(M^n,g)$, $\sigma(M^n)$. To the first question we have a partial answer.

PROPOSITION 2.22. *If* $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ *then*

$$ \chi(M^n,g) = \chi(M^n,g') $$

and

$$ \sigma(M^n,g) = \sigma(M^n,g'). \ \blacksquare $$

In the case $g' \notin {}^{b,2}\mathrm{comp}^{1,2}(g)$ we can't say anything. The examples in [3] for $\chi(M^n,g) \neq \chi(M^n,g')$, $\sigma(M^n,g) \neq \sigma(M^n,g')$ are of this kind, i.e. g' does not lie in the component of g.

Concerning the second question, we start with a simple case in dimension two which has been discussed by Cohn–Vossen [5] and Huber [16] and has been endowed with particular short proofs by Rosenberg [22], which we present below for completeness.

THEOREM 2.23. *Let* (M^n,g) *be a finitely connected complete noncompact Riemannian surface with curvature* K.

a) *If* $K \in L_1$ *then* $\chi(M) \geq \int_M K\,\mathrm{dvol}_x(g)$.
b) *If* $\mathrm{vol}(M^2,g) < \infty$ *and* $K \in L_1$ *then*

$$ \chi(M) = \int_M K\,\mathrm{dvol}_x(g) = \chi(M,g). $$

Proof. M^2 is diffeomorphic to a compact surface with p points deleted. A neighborhood of each point is diffeomorphic to $S^1 \times R_+$ and the metric can be put in the form $g_{11}(\theta,t)d\theta^2 + dt^2$. Set $M_k = M \setminus \bigcup_1^p S^1 \times]k,\infty[$. The Gauß–Bonnet theorem for surfaces with boundary yields $\chi(M_k) = \int_{M_k} K \,\mathrm{dvol}_x(g) + \int_{\partial M_k} \omega_{12}$, where ω_{12} is the connection 1-form associated to an orthonormal frame on M. $\chi(M) = \chi(M_k)$, hence one has to show $\lim_{k\to\infty} \int_{\partial M_k} \omega_{12} \geq 0$ for a) and $\lim_{k\to\infty} \int_{\partial M_k} \omega_{12} = 0$ for b). W.r.t. the orthonormal frame $\theta^1 = \sqrt{g_{11}}d\theta$ and $\theta^2 = dt$ the first structure equation $d\theta^1 = \omega_{12} \cap \theta^2$ gives $\omega_{12} = \frac{d}{dt}(\sqrt{g_{11}})d\theta$ and the second one gives $K\,\mathrm{dvol}_x(g) = \Omega_{12} = d\omega_{12} = \frac{d^2}{dt^2}(\sqrt{g_{11}})d\theta dt$. $\int_M K\,\mathrm{dvol}_x(g) < \infty$ implies $\lim_{k\to\infty} \int_{\partial M_k} \frac{d^2}{dt^2}\sqrt{g_{11}}d\theta = 0$ or $\lim_{k\to\infty} \int_{\partial M_k} \frac{d^2}{dt^2}\sqrt{g_{11}}d\theta = const = C$. In the case b), $\mathrm{vol}(M,g) < \infty$, i.e. $\int_M \sqrt{g_{11}}d\theta dt < \infty$ which implies $\lim_{k\to\infty} \int_{\partial M_k} \sqrt{g_{11}}d\theta = 0$, hence $\lim_{k\to\infty} \int_{\partial M_k} \omega_{12} = \lim_{k\to\infty} \frac{d}{dt} \int \sqrt{g_{11}}d\theta = C$, $C = 0$. In the case a), $\int_{\partial M_k} \sqrt{g_{11}}d\theta \sim C \cdot k + D$ as $k \to \infty$. $C < 0$ would imply $\int_{\partial M_k} \sqrt{g_{11}}d\theta = 0$ for k sufficiently large. But this is impossible for a positive integrand. ∎

In the case of arbitrary n, there are many approaches to study the equation $\chi(M,g) = \chi(M)$. To have $\chi(M)$ defined, one must require that each homology group over $\mathbb{R}$ is finitely generated. Sufficient for this is that M has finite topological type, i.e. it has only finitely many ends $\varepsilon_1, \ldots, \varepsilon_s$, each of them collared, $U(\varepsilon_i) \cong \partial U_i \times [0,\infty[$. Then M can be given a boundary ∂M to get a compact manifold $\overline{M}$. The case of n odd is absolutely trivial.

Proposition 2.24. *Assume (M^{2k+1}, g) is of finite topological type, g arbitrary. Then*

$$\chi(M) = \int_M E(g) = \chi(M,g) \text{ if and only if } \chi(\partial M) = 0.$$

Proof. For $n = 2k+1$, the Euler form $E(g)$ vanishes since the Pfaffian of an odd dimensional skew symmetric matrix is zero, $\int E(g) = \chi(M,g) = 0$. On the other hand, $0 = \chi(\overline{M} \underset{\partial M}{\cup} \overline{M}) = 2\chi(\overline{M}) - \chi(\partial M) = 2\chi(M) - \chi(\partial M)$. ∎

The more interesting case are even dimensional manifolds. We recall some definitions.

For a local orthonormal frame $\theta^1, \ldots, \theta^n$ the connection 1-forms ω_{ij} satisfy the equations

$$d\theta^i = \sum_j \omega_{ij} \wedge \theta^j \text{ and } \omega_{ji} = -\omega_{ij}.$$

They are related with the curvature 2-forms Ω_{ij} by

$$\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Denote by $S(M)$ the tangent sphere bundle which is a $(2n-1)$-dimensional manifold. For a point $(x,\xi) \in S(M)$ let $\theta^1, \ldots, \theta^n$ be a frame such that θ^1 is dual to ξ. We put

$$(2.11) \qquad II(g) := \sum_{0 \leq k < n} c_k \sum_\alpha \mathrm{sign}(\alpha) \Omega_{\alpha(2)\alpha(3)} \wedge \cdots \wedge \Omega_{\alpha(2k)\alpha(2k+1)} \wedge \omega_{\alpha(2k+2)1} \wedge \cdots \wedge \omega_{\alpha(n)1},$$

where we will not specify the c_k and $\sum_\alpha$ means the sum over all permutations α of $\{2, \ldots, n\}$. $II(g)$ can be understand as pull back on an $(n-1)$-form on M to $S(M)$ by means of $pr : S(M) \to M$. If $X \in TS(M)$ at (x,ξ), $X = X_1 + X_2$ with X_1 tangent to

M and X_2 tangent to S_x^{n-1} then $\Omega_{ij}(X) = \Omega_{ij}(X_1)$ and similarly for $\omega_{i1}(X)$. If M is compact with boundary ∂M and ϱ is the section of $S(M)$ over ∂M given by the outward normal vector, then $\varrho^*\Omega_{ij}(X) = \Omega_{ij}(X_1)$, the same for ω_{i1}. Then, according to Chern,

$$\chi(M) = \int_M E(g) + \int_{\partial M} \varrho^* II(g) = \int_M E(g) + \int_{\partial M} II(g). \tag{2.12}$$

Assume now that $(M^n, g) = (M^{2m}, g)$ is even-dimensional and of finite topological type. By gradient flow of an appropriate Morse function we can introduce coordinates $(x_1, \ldots, x_{n-1}, x_n = r)$ at each end such that $0 \le r < \infty$, $g_{in} = 0$, $1 \le i \le n-1$, $g_{nn} = 1$. Let as above M_k be characterized by $x_n = r \le k$. Then

$$\chi(M) = \chi(M_k) = \int_{M_k} E(g) + \int_{\partial M_k} II(g). \tag{2.13}$$

At each end ε $TM|_\varepsilon$ splits as $TM = W \oplus \mathbb{R}$. Suppose additionally that W splits as

$$W = W_2 \oplus \cdots \oplus W_{r_\varepsilon}, \quad r_\varepsilon \ge 2, \quad [W_i, W_j] = 0 \quad \text{if } i \ne j, \tag{2.14}$$

and that with respect to this splitting g has the form

$$g = f_2^2(r) g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r) g_{r_\varepsilon} + dr^2. \tag{2.15}$$

Then S. Rosenberg calculated in [23] the expression (2.11) at each end can could show if $f_j(r) \underset{r\to\infty}{\longrightarrow} 0$, $f_j'(r) \underset{r\to\infty}{\longrightarrow} 0$, then $\int_{M_k} E(g) \to \int_M E(g)$ and $\int_{\partial M_k} II(g) \to 0$. We will not repeat the really simple calculations but state Rosenberg's

THEOREM 2.25. *Let (M^n, g) be open, complete and of finite topological type. Assume that in an open neighborhood of each end ε M splits as a product manifold $N_2 \times \cdots \times N_{r_\varepsilon} \times \mathbb{R}$ with the metric $f_2^2(r) g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r) g_{r_\varepsilon} + dr^2$, where g_j is a metric on N_j. If $f_j(r) \underset{r\to\infty}{\longrightarrow} 0$ and $f_j'(r) \underset{r\to\infty}{\longrightarrow} 0$, then $\chi(M) = \int_M E(g) \equiv \chi(M, g)$. In particular, any even-dimensional manifold of finite topological type admits complete warped product metrics satisfying Gauß–Bonnet (setting $N_2 = \partial M$).* ■

COROLLARY 2.26. *Assume the hypothesis of 2.25 and additionally $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, f, 1)$. If $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ then $\chi(M) = \int_M E(g') \equiv \chi(M, g')$.* ■

REMARK 2.27. We see in 2.26 a considerable improvement of 2.25 since now the admitted class of metrics is much larger. ■

If one gives up the integrability of the Ws in (2.14), i.e. the product structure of the εs then one must strengthen the conditions to the f_j. This has been done by Rosenberg too.

THEOREM 2.28. *Let (M^n, g) be open, complete and of finite topological type. Assume that in an open neighborhood of each end ε, $TM|_\varepsilon = W_2 \oplus \cdots \oplus W_{r_\varepsilon} \oplus \mathbb{R}$ and the metric is of the form $f_2^2(r) g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r) g_{r_\varepsilon} + dr^2$ with g_i a metric on W_i. If $f_i(r) \underset{r\to\infty}{\longrightarrow} 0$, $f_i'(r) \underset{r\to\infty}{\longrightarrow} 0$ and $f_j f_i^{-1}$ and $(f_j f_i^{-1})'$ are bounded for all r, i, j then*

$$\chi(M) = \int E(g). \ ■$$

Example. Let $M\backslash G/K$ be an arithmetic quotient of an even-dimensional split rank-one symmetric space. Then at each component ∂M_i of ∂M, ∂M is the total space of a fibration over a torus T_1 with a torus T_2 as fiber. We have $TM|_{V\times\mathbb{R}} = W_1 \oplus W_2 \oplus \mathbb{R}$ for open $V \subset \partial M$ where the fibration restricted to V is trivial. W_i is the tangent space to the torus T_i. But in general the G-invariant metric g does not respect this splitting. Donnelly has shown in [6] that each end ε has the structure $N \times \mathbb{R}$, N at most two-step nilpotent. The Lie algebra $\mathfrak{n}$ of N splits as a sum $\mathfrak{n} = V_2 \oplus V_3$ of root spaces, $V_3 = Z(\mathfrak{n})$, and the invariant metric at the identity of N has the form

$$e^{-2r} g_2 + e^{-4r} g_3 + dr^2, \tag{2.16}$$

where g_2 is a metric on V_2, g_3 a metric on V_3. $[\mathfrak{n},\mathfrak{n}] \subset Z(\mathfrak{n})$ and the G-invariant distribution V_2 is not integrable. Hence theorem 2.25 is not applicable in general. In the hyperbolic case $G/K = SO(n,1)/SO(n)$, one has $V_2 = \mathfrak{n}$, which yields Gauß–Bonnet. ∎

Corollary 2.29. *Assume that the hypotheses of theorem 2.28 hold and additionally* $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, f, 1)$. *If* $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ *then* $\chi(M) = \int E(g') \equiv \chi(M, g')$. ∎

There is another Gauß–Bonnet case which does not fall under 2.2–2.29.

Proposition 2.30. *Let* (M^{2m}, g) *be open, complete, of finite topological type and the metric at* ∞ *constant with respect to* r, *i.e. there exists an* $r_0 \ge 0$ *such that* $g(r_1, x) = g(r_2, x)$ *for all* $x \in \partial M$ *and* $r_1, r_2 > r_0$. *Then*

$$\chi(M) = \int_M E(g) \equiv \chi(M, g).$$

Proof. Let $k > r_0 + \delta$. Then $M_k \cup M_k$ yields a smooth closed manifold. Hence

$$\begin{aligned} \chi(M_k \cup M_k) &= \int_{M_k \cup M_k} E(g_{M_k \cup M_k}) = 2\int_{M_k} E(g_{M_k}), \\ \chi(M_k \cup M_k) &= 2\chi(M_k) - \chi(\partial M_k) = 2\chi(M) \\ \chi(M) &= \int_{M_k} E(g_{M_k}). \end{aligned} \tag{2.17}$$

Forming $\lim_{k\to\infty}$ in (2.17) gives the desired result. ∎

A special case of 2.28 would be a metric cylinder at infinity, $g|_{U(\infty)} = g_{\partial M} \otimes + dr^2$. This is simultaneously a warped product with warping function $f(r) = 1$. $f(r) = 1$ does not satisfy $f(r) \underset{r\to\infty}{\longrightarrow} 0$, 2.25 is not applicable. Clearly, such an (M^{2m}, g) satisfies (B_0) but either $\int_{U(\infty)} |R|^p \,\mathrm{dvol}_x(g) = 0$ or $\int_{U(\infty)} |R|^p \,\mathrm{dvol}_x(g) = \infty$, similarly either $\int_{U(\infty)} |E(g)| \,\mathrm{dvol}_x(g) = 0$ or $\int_{U(\infty)} |E(g)| \,\mathrm{dvol}_x(g) = \infty$. In the second case $\int E(g)$ exists but $|E(g)| \notin L_p$, $p \ge 1$.

Another class of examples which yields very useful insights are surfaces of revolution. We state from [23] without proof

Proposition 2.31. *Let* $f : \,]0, \infty[\, \to \mathbb{R}$ *be smooth,* $f(0) = f'(0) = 0$ *and* ($M^2 = \{z = f(x^2 + y^2)\}$, *induced metric from* $\mathbb{R}^3$) *be the associated surface of revolution. Then*

$$\chi(M) = \frac{1}{2\pi} \int_M K \,\mathrm{dvol}_x(g) = \chi(M, g) \tag{2.18}$$

if and only if

$$r^{\frac{1}{2}} f'(r) \underset{r\to\infty}{\longrightarrow} \pm\infty. \blacksquare$$

Hence, if f is for all $r > 0$ strongly convex or concave, (2.18) holds. In both cases M has for $r > 0$ positive curvature and infinite volume. On the other hand, we have 2.15 in the case of 2.23 (b) in the finite volume case, i.e. one can have $\chi(M) = \chi(M, g)$ as in the finite volume case. For this reason we should find additional conditions which assure in the finite volume case or the infinite volume case, respectively, that

1) $\chi(M, g)$ is a (proper) homotopy invariant,
2) $\chi(M, g) = \chi(M)$ if M has finite topological type.

We start with $\mathrm{vol}(M^n, g) < \infty$ and $|K| \le 1$ where the latter (after rescaling) is equivalent to (B_0). Then

$$\chi(M, g) = \int_M E(g)$$

is well defined and for $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$

$$\chi(M, g) = \chi(M, g'). \tag{2.19}$$

LEMMA 2.32. *Let* (M^n, g) *be complete,* $\mathrm{vol}(M, g) < \infty$ *and* $|K| \le 1$*. Then* M^n *admits an exhaustion by compact manifolds with smooth boundary,* $M_1^n \subset M_2^n \subset \cdots$*,* $\bigcup_k M_k^n = M$*, such that* $\mathrm{vol}(\partial M_k^n) \to 0$ *and for which the second fundamental forms* $II(\partial M_k^n)$ *are uniformly bounded.*

Proof. This is just a corollary of theorem 2.33 below. $\blacksquare$

If we take such an exhaustion as just described then

$$\chi(M_k^n) = \chi(M_k^n, g) + \int_{\partial M_k^n} II(\partial M_k^n). \tag{2.20}$$

$\int_{\partial M_k^n} II(\partial M, g) \underset{k\to\infty}{\longrightarrow} 0$, $\chi(M_k^n) \in \mathbb{Z}$, hence for k sufficiently large $\chi(M_k^n, g) \in \mathbb{Z}$, but we are far from a certain convergence of $(\chi(M_k^n, g))_k$ and don't know anything about the topological properties of such a limit if it exists. To obtain more insight and definite results we follow [3] and consider the following additional hypothesis.

For some neighborhood $U(\infty) \subset M$, some profinite or normal covering space $\tilde U(\infty)$ has the injectivity radius at least (say) 1 for the pull back metric,

$$r_{\mathrm{inj}}(\tilde U_\infty)) \ge 1. \tag{2.21}$$

Together with $|K| \le 1$ on $\tilde U(\infty)$ we write $\mathrm{geo}_{\underset{\infty}{\sim}}(M) \le 1$. If $U = M$ then we denote $\mathrm{geo}(\tilde M) \le 1$. In any case we assume in this hypothesis that $\tilde U$ or $\tilde M$ are profinite or normal coverings.

Here $\tilde M \to M$ is profinite if there exists a decreasing sequence $\{\Gamma_j\}_j$ of subgroups of finite index, $\Gamma_j \subset \pi_1(M)$, such that $\bigcap \Gamma_j = \pi_1(\tilde M)$.

The key for everything is the following very general theorem which assures the existence of sufficiently "smooth" exhaustions and which yields 2.32 in the case of $\mathrm{vol}(M, g) < \infty$.

THEOREM 2.33 (Neighborhoods of bounded geometry). *Let (M^n, g) be complete, $X \subset M^n$ a closed subset and $0 < r \leq 1$. Then there is a submanifold U^n with smooth boundary ∂U^n such that for some constant $c(n)$ depending only on n*

$$\textbf{a)}\ X \subset U \subset T_r(x) = r\text{-tubular neighborhood of } X,$$

$$\textbf{b)}\ \mathrm{vol}(\partial U) \leq c(n) \cdot \mathrm{vol}(T_r(X) \setminus X) \cdot r^{-1}, \tag{2.22}$$

$$\textbf{c)}\ |II(\partial U)| \leq c(n) \cdot r^{-1}. \tag{2.23}$$

We refer to [4] for the proof. ∎

Now we will discuss $\chi(M, g)$ in the profinite or normal case, $\mathrm{geo}(\tilde{M}) \leq 1$. Here we follow [3]. Put for $j : A_1 \subset A_2$ and real coefficients $\beta^i(A_1, A_2) = \dim\{j^*(H^i(A_2)) \subset H^i(A_1)\}$ and $\beta^i(A) = \dim\{j^*(H^i(A, \partial A)) \subset H^i(A)\}$. b^i shall denote the usual Betti number. Then for $A_1 \subset A_2 \subset A_3 \subset A_4$ and $A \subset Y$ a finite closed and $f : Y \to Z$, $g : Z \to Y$ simplicial, determining a homotopy equivalence,

$$\beta^i(A_1) \subseteq \beta^i(A_2) \leq \beta^i(A_2, A_4) \leq \beta^i(A_3, A_4) \tag{2.24}$$

and

$$\beta^i(A, Y) \leq \beta^i(f(A), Z) \leq \beta^i(g \circ f(A), Y). \tag{2.25}$$

Put for $p : \tilde{Y}^n \to Y^n$ profinite with $\mathrm{ind}(\Gamma_j) = d_j$ and corresponding covering spaces $p_j : \tilde{Y}^n_j \to Y^n$

$$\sup \tilde{\chi}(Y^n) := \overline{\lim_{A\to\infty}}\, \overline{\lim_{j\to\infty}} \sum_{i=1}^{n} (-1)^i \frac{1}{d_j} \beta^i(P_j^{-1}(A), \tilde{Y}^n_j) \leq \infty \tag{2.26}$$

and define $\inf \tilde{\chi}(Y^n)$ similarly. $A \to \infty$ is defined by partial ordering of finite subcomplexes induced by inclusion. Using (2.24) and a diagonal argument, there are subsequences $S = \tilde{Y}^n_{j(e)}$ s.t.

$$\begin{aligned} \infty \geq \tilde{\beta}^i(Y^n, S) &:= \lim_{A\to\infty} \overline{\lim_{e\to\infty}} \frac{1}{d_{j(e)}} \beta^i(P^{-1}_{j(e)}(A), \tilde{Y}^n_{j(e)}) \\ &= \lim_{A\to\infty} \underline{\lim_{e\to\infty}} \frac{1}{d_{j(e)}} \beta^i(P^{-1}_{j(e)}(A), \tilde{Y}^n_{j(e)}) \end{aligned} \tag{2.27}$$

exists. From (2.25) we infer immediately that $\tilde{\beta}^i(Y^n, S)$ is a homotopy invariant. Suppose $\tilde{\beta}^i(Y^n, S) < \infty$, $i = 0, \ldots, n$ and $\sup \tilde{\chi}(Y^n) = \inf \tilde{\chi}(Y^n)$, then the latter number is also a homotopy invariant.

THEOREM 2.34. *Suppose (M^n, g) complete, $\mathrm{vol}(M^n, g) < \infty$, $\tilde{M}$ either profinite or normal and $\mathrm{geo}(\tilde{M}) \leq 1$.*

a) *Then $\chi(M^n, g)$ is a proper homotopy invariant,*

b) *in the case $\tilde{M}$ profinite*

$$\chi(M, g) = \sup \tilde{\chi}(M) = \inf \tilde{\chi}(M),$$

c) *if additionally M has finite topological type,*

$$\chi(M, g) = \chi(M).$$

Proof. Assume $\tilde{M} \to M$ profinite, let $M_1 \subset M_2 \subset \cdots$, $\bigcup_k M_k = M$ be an exhaustion of M by compact submanifolds with boundary and denote $M_k - R = \{x \in M_k | \mathrm{dist}(x, \partial M_k) = R\}$. For j sufficiently large, theorem 2.33 is applicable and we apply it to $p_j^{-1}(M_k) - 1, p_j^{-1}(M_k)$ with $\varepsilon = \frac{1}{2}$. This yields submanifolds $A_{jk} \subset p_j^{-1}(M_k) \subset B_{jk}$. Given $\varepsilon > 0$ arbitrary, there exist k_0, $N(k)$ such that for $k > k_0$, $j > N(k)$

$$\left|\chi(M^n, g) - \frac{1}{d_j}\chi(B_{jk})\right| \le \left|\chi(M^n, g) - \frac{1}{d_j}\int_{B_{jk}} E(g)\right| + \left|\frac{1}{d_j}\int_{B_{jk}} E(g) - \frac{1}{d_j}\chi(B_{jk})\right| < \varepsilon. \tag{2.28}$$

We see this immediately from (2.12) and (2.22), (2.23): $\chi(M^n, g) = \chi(M_k^n, g) + \chi(M^n \setminus M_k^n, g)$, here $|\chi(M^n \setminus M_k^n, g)|$ becomes arbitrarily small for k sufficiently large.

$$\left|\chi(M^n, g) - \frac{1}{d_j}\int_{B_{jk}} E(g)\right| \le |\chi(M^n, g) - \chi(M_k^n, g)| + \left|\chi(M_k^n, g) - \frac{1}{d_j}\int_{B_{jk}} E(g)\right|$$

$$\begin{aligned}\left|\chi(M_k^n, g) - \frac{1}{d_j}\int_{B_{jk}} E(g)\right| &\le \left|\chi(M_k^n, g) - \frac{1}{d_j}\int_{P_j^{-1}(M_k^n)} E(g)\right| \\ &\quad + \left|\frac{1}{d_j}\int_{P_j^{-1}(M_k^n)} E(g) - \frac{1}{d_j}\int_{B_{jk}} E(g)\right| \\ &= \left|\frac{1}{d_j}\int_{B_{jk}\setminus P_j^{-1}(M_k^n)} E(g)\right|,\end{aligned}$$

but this becomes arbitrarily small for j and k sufficiently large. Finally

$$\left|\frac{1}{d_j}\int_{B_{jk}} E(g) - \frac{1}{d_j}\chi(B_{jk})\right| = \left|\frac{1}{d_j}\int_{\partial B_{jk}} II(\partial B_{jk})\right| \underset{j,k\to\infty}{\longrightarrow} 0$$

according to (2.22). (2.28) is proven.

We obtain from (2.24)

$$\beta^i(A_{jk}) \le \beta^i(p_j^{-1}(M_k)) \le \beta^i(p_j^{-1}(M_k), \tilde{M}_j) \le b^i(B_{jk}) \tag{2.29}$$

and from the exact cohomology sequence of the pair $(B_{jk}, \overline{B_{jk} \setminus A_{jk}})$ together with the excision property

$$|\beta^i(A_{jk}) - b^i(B_{jk})| \le b^{i-1}(\overline{B_{jk} \setminus A_{jk}}) + b^i(\overline{B_{jk} \setminus A_{jk}}):$$

$$\begin{aligned}\cdots \to H^{i-1}(\overline{B_{jk} \setminus A_{jk}}) \to H^i(B_{jk}, &\overline{B_{jk} \setminus A_{jk}}) \to H^i(B_{jk}) \to H^i(\overline{B_{jk} \setminus A_{jk}}) \to \cdots \\ &\cong H^i(A_{jk}, \partial A_{jk})\end{aligned}$$

The manifold $\overline{B_{jk} \setminus A_{jk}}$ satisfies (B_0), (I) for $j > N(k)$ and for k sufficiently large,

$$\mathrm{vol}(\overline{B_{jk} \setminus A_{jk}}) \le d_j \varepsilon. \tag{2.30}$$

According to a theorem of Gromov,

$$\sum_i \beta^i(\overline{B_{jk} \setminus A_{jk}}) \le c(n) \cdot \mathrm{vol}(\overline{B_{jk} \setminus A_{jk}}). \tag{2.31}$$

We infer from (2.29)–(2.32) that we can replace in (2.28) $\chi(B_{jk})$ by $\chi(p_j^{-1}M_k, \tilde{M}_j)$, hence

$$\left|\chi(M^n, g) - \frac{1}{d_j}\chi(p_k^{-1}(M_k), \tilde{M}_j)\right|$$

becomes arbitrarily small, any proper homotopy equivalence preserves a subsequence of $\left(\frac{1}{d_j}\chi(p_j^{-1}(M_k), \tilde{M}_j)\right)_{j,k}$, $\chi(M^n, g)$ is a proper homotopy invariant. By the same argument we conclude in the profinite case assertion b). If M has finite topological type then for k sufficiently large $\beta^i(p_j^{-1}(M_k), \tilde{M}_j) = \beta^i(\tilde{M}_j)$ and

$$\chi(p_j^{-1}(M_k), \tilde{M}_j) = \chi(\tilde{M}_j) \cdot \frac{1}{d_j}\chi(\tilde{M}_j) = \chi(M_j) = \chi(M)$$

yields assertion c). ■

The case of a normal covering $\tilde{M} \to M$ will be discussed in theorem 2.38.

The second characteristic number of particular importance is given by $\sigma(M, g) = \int_M L(M, g)$, where $L(M, g)$ is the Hirzebruch genus. For closed M it is the topological index of the signature operator, i.e. it coincides with the topological signature. For simple open manifolds this equality does not longer hold in general, as we see in section 5. Nevertheless, we could ask for $\sigma(M, g)$ the same questions as for $\chi(M, g)$, the question for the invariance properties and the topological significance of $\sigma(M, g)$. Concerning the invariance, a first answer is given by proposition 2.22.

But we consider also other derivations of g. A key role plays again the formula for the compact case with boundary, $\partial M = N$,

$$\sigma(M, g) + \eta(N, g) + \int_N II_\sigma(N, g) = \sigma(M), \tag{2.32}$$

where $II_\sigma(N, g)$ essentially involves the second fundamental form and $\eta(N, g)$ is the eta invariant. If M^n is open and $M_1 \subset M_2 \subset \cdots$, $\bigcup_k M_k = M$, an appropriate exhaustion such that $\int II_\sigma(\partial M_k) \to 0$ and $\eta(\partial M_k) \to 0$ then we would have in fact $\sigma(M_k, g) \to \sigma(M)$. Hence we should ask for conditions which assure $\eta(\partial M_k) \to 0$. There is a clear (and for our case complete) answer.

THEOREM 2.35. *Let (N^{4l-1}, g) be compact satisfying* $\mathrm{geo}(N) \leq 1$. *Then there is a constant $c = c(4l-1)$ such that*

$$|\eta(N^{4l-1})| \leq c(4l-1) \cdot \mathrm{vol}(N^{4l-1}, g). \tag{2.33}$$

We refer to [3], [7] for the proof. ■

Now we define $\sup \tilde{\sigma}(M)$, $\inf \tilde{\sigma}(M)$ quite analogous to the Euler characteristic as follows. Let M^{4l} be complete, $\tilde{M}^{4l} \to M$ profinite and $M_k^{4l} \subset M^{4l}$ a compact submanifold with boundary. Put

$$\sup \tilde{\sigma}(M_k) := \lim_j \sup \frac{1}{d_j}\sigma(P_j^{-1}(M_k)),$$
$$\sup \tilde{\sigma}(M) := \lim_{M_k} \sup \sup \tilde{\sigma}(M_k))$$

and similarly $\inf \tilde{\sigma}(M_k)$, $\inf \tilde{\sigma}(M)$. Here as always $\sigma(M_k)$ is defined as the signature of the cup product pairing on $j^*H^{2l}(M_k^{4l}, \partial M_k^{4l}) \subset H^{2l}(M_k^{4l})$.

THEOREM 2.36. *Let* (M^{4l}, g) *be complete,* $\mathrm{vol}(M, g) < \infty$ *and suppose* $\tilde{M}$ *either profinite or normal and* $\mathrm{geo}(\tilde{M}) \leq 1$*. Then we have*

a) *Assume* $\tilde{M}$ *normal. Then* $\sigma(M, g)$ *is a proper homotopy invariant of* M*.*
b) *In the case* $\tilde{M} \to M$ *profinite, for any exhaustion* $M_1 \subset M_2 \subset \cdots$*,* $\bigcup_k M_k = M$*, by compact manifolds,*

$$\sigma(M, g) = \sup \tilde{\sigma}(M) = \inf \tilde{\sigma}(M).$$

c) *If, additionally,* M *has finite topological type,*

$$\sigma(M, g) = \lim_{j \to \infty} \frac{1}{d_j} \sigma(\tilde{M}_j).$$

Proof. In the normal case $\tilde{M} \to M$ below a) follows from theorem 2.38. The proof of b) is quite analogous to that of theorem 2.34 b), using a chopping of M according to theorem 2.33, (2.32) and theorem 2.35. c) then follows from b) and the fact that for sufficiently large k, $\frac{1}{d_j}\sigma(p_j^{-1}(M_k)) = \frac{1}{d_j}\sigma(\tilde{M}_j)$. ∎

We now turn to the normal case $\tilde{M} \to M$, being even more explicit than in the profinite case. The first key here is the extension of Atiyah's L_2-index theorem for normal coverings $\tilde{M} \to M$ of closed M to normal coverings $\tilde{M} \to M$, $M = \tilde{M}/\Gamma$, $r_{\mathrm{inj}}(\tilde{M}) \geq 1$, (M^n, g) complete, $\mathrm{vol}(M^n, g) < \infty$, $|K| \leq 1$. We denote by $\mathcal{H}^{q,2}(\tilde{M})$ the space of L_2-harmonic q-forms, by $P_{\mathcal{H}^{q,2}} : L_2(\Lambda^q T^* A) = \Omega^{q,2} \to \mathcal{H}^{q,2}$ the orthogonal projection. $P_{\mathcal{H}}$ has Schwartz kernel $\tilde{h}^q(x, y)$ which is a symmetric C^∞ double form whose pointwise norm satisfies

$$|h^q(x, y)| \leq c(n). \tag{2.34}$$

(2.34) comes from $\mathrm{geo}(\tilde{M}) \leq 1$ and the elliptic estimate for the Laplacian. $\tilde{h}^q(x, y)$ is invariant under the isomtries Γ, hence the pointwise trace $\mathrm{tr}\, \tilde{h}^q(x, x)$ can be understood as function on M and we put as usual

$$\tilde{b}^{q,2}(M) := \mathrm{tr}_\Gamma\, P_{\mathcal{H}^{q,2}(\tilde{M})} = \int_M \mathrm{tr}\, \tilde{h}^q(x, x)\, \mathrm{dvol}_x(g) < \infty.$$

$\tilde{b}^{q,2}(M)$ is just the von Neumann dimension $\dim_\Gamma \overline{H}^{q,2}(\tilde{M})$ of the Γ-module $\overline{H}^{q,2}(\tilde{M})$. Now we define the L_2-Euler characteristic and L_2-signature by

$$\tilde{\chi}_{(2)}(M) := \sum_{q=0}^{n} (-1)^q \tilde{b}^{q,2}(M) \quad \text{and} \quad \tilde{\sigma}_{(2)}(M) := \mathrm{tr}_\Gamma(* P_{\mathcal{H}^{2k,2}(\tilde{M}^{4k})}).$$

Now we state the L_2-index theorem for open manifolds with finite volume and bounded curvature.

THEOREM 2.37. *Suppose* (M, g) *complete with* $\mathrm{vol}(M^n, g) < \infty$*,* $|K| \leq 1$ *and* $\tilde{M} \to M$ *normal with* $\mathrm{geo}(\tilde{M}) \leq 1$*. Then*

$$\chi(M, g) = \tilde{\chi}_{(2)}(M) \tag{2.35}$$

and

$$\sigma(M, g) = \tilde{\sigma}_{(2)}(M). \tag{2.36}$$

We refer to [2], [7] for the proof. ∎

We recall the existence of good chopping sequences $M_1 \subset M_2 \subset \cdots$, $\bigcup_1^\infty M_k = M$, $\mathrm{vol}(\partial M_k) \to 0$, $|II(\partial M_k)| \le c$, $|\tilde{h}_k^q(x,y) \le c(n)|$, where $\tilde{h}_k^q$ denotes the kernel corresponding to projection on the harmonic q-forms for $p^{-1}(M_k) \subset \tilde{M}$. Then we obtain

$$\lim_{k\to\infty} \tilde{b}^{q,2}(\partial M_k) = 0 \tag{2.37}$$

and

$$\lim_{k\to\infty} \tilde{b}^{q,2}(M \setminus M_k, \partial(M \setminus M_k)) = 0. \tag{2.38}$$

Define $\tilde{\beta}^{q,2}(B)$ by

$$\tilde{\beta}^{q,2}(B) := \dim_\Gamma \mathrm{im}(H^{q,2}(p^{-1}(B), p^{-1}(\partial B)) \subset \overline{H}^{q,2}(p^{-1}(B))) \tag{2.39}$$

and for $A \subset B$

$$\tilde{\beta}^{q,2}(A,B) := \dim_\Gamma \mathrm{im}(\overline{H}^{q,2}(p^{-1}(B)) \subset \overline{H}^{q,2}(p^{-1}(A))). \tag{2.40}$$

It follows from the properties of $\dim_\Gamma$ that

$$\tilde{\beta}^{q,2}(A) \le \tilde{\beta}^{q,2}(B) \tag{2.41}$$

and

$$\tilde{\beta}^{q,2}(A) \le \tilde{\beta}^{q,2}(A,B) \le \tilde{b}^{2,q}(A). \tag{2.42}$$

We remark that (2.41) and (2.42) are the reformulation of (2.24), (2.29) in the language of $\dim_\Gamma$. We established in theorem 2.37 the equations $\chi(M,g) = \tilde{\chi}_{(2)}(M)$, $\sigma(M,g) = \tilde{\sigma}_2(M)$. Now we discuss the invariance properties of the right hand sides. This is the content of

THEOREM 2.38. *Let* (M^n, g) *be complete,* $|K| \le 1$, $\mathrm{vol}(M,g) < \infty$ *and assume for some normal covering* $\mathrm{geo}(\tilde{M}) \le 1$.

a) *If* $M_1 \subset M_2 \subset \cdots$, $\bigcup M_k = M$ *is an exhaustion then*

$$\lim_{k\to\infty} \tilde{\beta}^{q,2}(M_k) = \lim_{k\to\infty}\lim_{l\to\infty} \tilde{\beta}^{q,2}(M_k, M_l) = \tilde{b}^{q,2}(M). \tag{2.43}$$

This implies the homotopy invariance of the $\tilde{b}^{q,2}(M)$.
b) $\chi(M,g)$ *resp.* $\sigma(M,g)$ *is a homotopy invariant resp. proper homotopy invariant of* M.
c) *If* M *has the topological type of some* $M_k \subset M$, *then*

$$\tilde{b}^{q,2}(M_k) = \tilde{b}^{q,2}(M) \tag{2.44}$$

and

$$\chi(M,g) = \chi(M_k). \tag{2.45}$$

Proof. b) follows immediately from theorem 2.37 and a). For c) suppose that M has finite topological type. Then there exists an exhaustion $M_1 \subset M_2 \subset \cdots$ s.t. each inclusion $M_k \to M$ is a homotopy equivalence. This implies

$$\tilde{\beta}^{q,2}(M_k, M_k) = \tilde{b}^{q,2}(M_k)$$

and we obtain (2.44) from (2.43) and moreover $\chi(M,g) = \chi(M_k)$. Hence there remains to show a). For this we must refer to [2]. ∎

We apply these results on characteristic numbers to 4-manifolds.

Let (M^4, g) be open, complete and oriented, $* : \Lambda^2 M \to \Lambda^2 M$ the Hodge operator, $*^2 = 1$, $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. The special orthogonal group acts on the space of algebraic curvature tensors $\mathcal{C}^2_b$ (cf. [21]). Let $\mathcal{C}^2_b = \mathcal{U} + \mathcal{S} + \mathcal{W}$ be the corresponding (fiberwise) decomposition into irreducible subspaces. Then this induces for the curvature tensor $R = R^g$ a decomposition $R = U + S + W$. For $R = R^g = R_+ + R_-$, we denote as in I 1 by $\operatorname{Ric} = \operatorname{Ric}^g$ the Ricci tensor, by $\tau = \tau^g$ the scalar curvature, by $K = K^g$ the sectional curvature and by $W = W^g = W_+ + W_-$ the Weyl tensor. There are decompositions for the pointwise norms $|\,|_x$ as follows:

$$|R|^2 = |R_+|^2 + |R_-|^2 = |U|^2 + |S|^2 + |W|^2 = 4|W_+|^2 + |W_-|^2 + 2|\operatorname{Ric}|^2 - \frac{1}{3}\tau^2, \tag{2.46}$$

$$|\operatorname{Ric}|^2 = 6|U|^2 + 2|S|^2, \tag{2.47}$$

$$\tau^2 = 24|U|^2. \tag{2.48}$$

We obtain other decompositions if we consider the curvature operator R as acting from $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ to $\Lambda^2_+ \oplus \Lambda^2_-$, for an orthonormal basis e_1, e_2, e_3, e_4

$$R(e_i \wedge e_j) = \frac{1}{2} \sum R_{ijkl} e_k \wedge e_l = \Omega_{ij},$$

$\Omega = (\Omega_{ij})$ = matrix of curvature forms, $\Omega_{ij}(e_k, e_l) = R_{ijkl}$. We can write R with respect to the orthogonal basis $e_1 \wedge e_2 + e_3 \wedge e_4$, $e_1 \wedge e_4 + e_2 \wedge e_3$, $e_1 \wedge e_3 + e_2 \wedge e_4$ in Λ^2_+, $e_1 \wedge e_3 + e_2 \wedge e_4$, $e_1 \wedge e_2 - e_3 \wedge e_4$, $e_1 \wedge e_4 - e_2 \wedge e_3$ in Λ^2_-, as

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A = A^*$, $C = B^*$, $D = D^*$, $\operatorname{tr} A = \operatorname{tr} D = \frac{\tau}{4}$, $B = \operatorname{Ric} - \frac{1}{4}\tau g$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) - \frac{\tau}{12} = W$, $W^+ = A - \frac{\tau}{12}$, $W^- = D - \frac{\tau}{12}$. We obtain for the first Pontrjagin form p_1

$$\begin{aligned} p_1 &= -\frac{1}{8\pi^2} \operatorname{tr}(R \wedge R) = -\frac{1}{8\pi^2} \operatorname{tr}(A \wedge A) + \operatorname{tr}(D \wedge D) \\ &= -\frac{1}{8\pi^2}(-2)(|W_+|^2 - |W_-|^2) \operatorname{dvol} = \frac{1}{4\pi^2}(|W_+|^2 - |W_-|^2) \operatorname{dvol} \\ &= \frac{1}{12\pi^2}(|R_+|^2 - |R_-|^2) \operatorname{dvol} \end{aligned}$$

and for $\sigma(M^4, g) = \int L(g) = \int \frac{1}{3} p_1 = \frac{1}{12\pi^2} \int (|W_+|^2 - |W_-|^2) \operatorname{dvol}$. Assuming $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, 1, f)$, $\sigma(M^4, g)$ is well defined. The Euler form $E(g)$ has the representation

$$\begin{aligned} E(g) &= \frac{1}{8\pi^2} \operatorname{tr}(*R)^2 \operatorname{dvol} = \frac{1}{8\pi^2}(|U|^2 - |S|^2 + |W|^2) \operatorname{dvol} \\ &= \frac{1}{8\pi^2} \operatorname{tr}(A^2 - 2BB^* + D^2) \operatorname{dvol} = \frac{1}{32\pi^2}(|R|^2 - 4|\operatorname{Ric}|^2 + \tau^2) \operatorname{dvol}. \end{aligned}$$

For $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, 1, f)$, $\int E(g) = \chi(M, g)$ is well defined. Hence we obtain

PROPOSITION 2.39. *Let (M^4, g) be open, complete, oriented and $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, 1, f)$. Then $\sigma(M, g)$ and $\chi(M, g)$ are well defined and an invariant of* comp(g). ∎

REMARK 2.40. According to (2.46)–(2.48), $\int |R^g|^2 \operatorname{dvol} < \infty$ would be sufficient for the existence of $\sigma(M, g)$ and $\chi(M, g)$. But this condition would not establish a uniform strucutre, we would not have components and invariance properties (where we used in particular Gaffney's theorem). Moreover, we need the bounded curvature property for the connection with the theorems 2.37, 2.38. ∎

We obtain from proposition 2.39 and its proof the simple

Corollary 2.41. *If (M^4,g) is additionally Einstein then $\chi(M,g)\geq 0$ and $|\sigma(M^4,g)| \leq \frac{2}{3}\chi(M^4,g)$. Moreover, $\chi(M^4,g)=0$ if and only if (M^4,g) is flat.*

Proof. If (M^4,g) is Einstein then $S\equiv 0$, $B\equiv 0$ and $\frac{1}{12\pi^2}(|W_+|^2-|W_-|^2)\leq \frac{2}{3}\frac{1}{8\pi^2}(|U|^2+|W_+|^2-|W_-|^2)$. Hence $\sigma(M^4,g)\leq\frac{2}{3}\chi(M,g)$. Changing the orientation replaces $\sigma(M^4,g)$ by $-\sigma(M^4,g)$ and we get altogether $|\sigma(M^4,g)|\leq\frac{2}{3}\chi(M^4,g)$ ∎

The same estimate holds for $\frac{2}{3}$-pinched Ricci curvature.

Proposition 2.42. *Suppose the hypotheses of 2.39 and additionally that the Ricci curvature of (M^4,g) is negative and $\frac{2}{3}$-pinched, i.e. there exists $A>0$ s.t.*

$$-Ag\leq \mathrm{Ric}\leq -\frac{2}{3}Ag. \tag{2.49}$$

Then for all $g'\in\mathrm{comp}(g)\subset {}^{b,2}\mathcal{M}^{1,2}(B_0,1,f)$,

$$|\sigma(M^4,g')|\leq\frac{2}{3}\chi(M^4,g'). \tag{2.50}$$

Proof. We have

$$|\sigma(M^4,g)|=\left|\int L(g)\right|\leq\int|L(g)|\,\mathrm{dvol}=\frac{1}{12\pi^2}\int(|W_+|^2+|W_-|^2)\,\mathrm{dvol}$$

and

$$\chi(M^4,g)=\int E(g)=\frac{1}{8\pi^2}\int(|U|^2-|S|^2+|W|^2)\,\mathrm{dvol}\,.$$

Sufficient for (2.50) would be $|S|^2\leq|U|^2$ and sufficient for this is (2.49) as pointed out by [21]. ∎

Examples 2.43. **1)** Examples for 2.39 with infinite volume are e.g. manifolds M^4 of the smooth type $M^4=M_0^4\cup\partial M_0^4\times[0,\infty[$ where the curvature at the cylinder $\partial M_0^4\times[0,\infty[$ is bounded and asymptotically flat in the sense $\int_{\partial M_0^4\times[0,\infty[}|R|\,\mathrm{dvol}<\infty$. This can be easily realized by warped product metrics.
2) Examples for 2.39, 2.41, 2.42 with finite volume are given by hyperbolic 4-manifolds of finite volume.
3) Generalizations of these examples are given by variation of g inside $\mathrm{comp}(g)$. ∎

Theorem 2.44. *Let (M^4,g) be open, complete, $\mathrm{vol}(M^4,g)<\infty$, $|K|\leq 1$ and suppose that (M^4,g) admits a normal covering $(\tilde{M},g)$ satisfying $\mathrm{geo}(\tilde{M})\leq 1$.*

a) *If $\chi(M^4,g)<0$ then M^4 does not admit a complete Einstein metric g' satisfying $\mathrm{vol}(M^4,g')<\infty$, $|K_{g'}|\leq 1$, $\mathrm{geo}(\tilde{M^4},g')\leq 1$ for some normal covering.*
b) *If $\chi(M^4,g)>0$ and $|\sigma(M^4)|>\frac{2}{3}\chi(M^4,g)$ then M^4 does not admit a complete Einstein metric g', s.t. $\mathrm{vol}(M^4,g')<\infty$, $|Kg'|\leq 1$, $\mathrm{geo}(\tilde{M^4},g')\leq 1$. Moreover, there does not exist a complete metric g' satisfying*

$$-Ag'\leq\mathrm{Ric}(g')\leq-\frac{2}{3}Ag'$$

and $|Kg'|\leq 1$, $\mathrm{vol}(M^4,g')<\infty$ and $\mathrm{geo}(\tilde{M^4},g')\leq 1$ for some normal covering.

Proof. a) Suppose the existence of an Einstein metric g' with the required properties. Then $\chi(M^4,g)$, $\chi(M^4,g')$ are well defined. $\chi(M^4,g) = \chi(M^4,g')$, according to theorem 2.38 b). But this contradicts $\chi(M^4,g') = \frac{1}{8\pi^2}\int(|U|^2 + |W|^2)\,\mathrm{dvol} \geq 0$. b) and c): Quite analogously we derive by means of theorem 2.38 b), corollary 2.41 and proposition 2.42 a contradiction. ∎

Until now we defined characteristic numbers in the following cases:

1) $R \in L_1$ and bounded, $\mathrm{vol}(M)$ arbitrary,
2) R bounded, $\mathrm{vol}(M) < \infty$.

There remains the case R bounded, $\mathrm{vol}(M) = \infty$. It is clear that in this case we will not get characteristic numbers by integration. (M^n,g) is called closed at infinity if for any $\varphi \in C(M)$, $0 < A^{-1} < \varphi < A$, $A > 0$ some constant, the form $\varphi \cdot \mathrm{dvol}$ generates a nontrivial cohomology class in ${}^bH^n(M^n,g)$. A fundamental class for M is a positive continuous linear function $\mathfrak{m} : {}^b\Omega^n(M) \to \mathbb{R}$ such that $\langle \mathfrak{m}, \mathrm{dvol}\rangle \neq 0$ and $\langle \mathfrak{m}, d\psi\rangle = 0$.

Proposition 2.45. *M has a fundamental class if and only if M is closed at infinity.*

Proof. Write $\mathcal{L}(\mathrm{dvol})$ for the linear hull of dvol, $0 \notin [\mathrm{dvol}] \in {}^b\overline{H}^n(M)$ and set $\langle \mathfrak{m}, \mathrm{dvol}\rangle = 1$, $\mathfrak{m}|_{{}^bB^n} \equiv 0$. Then we obtain by linear extension $\mathfrak{m}$ on $\mathcal{L}(\mathrm{dvol}) \oplus {}^b\overline{B}^n$ as positive continuous linear functional. The Hahn–Banach theorem for the extension of such functionals yields the desired $\mathfrak{m}$. The other direction is absolutely trivial. ∎

Define the penumbra for $K \subset M$ by

$$\mathrm{Pen}^+(K,r) = \mathrm{CL}\Big(\bigcup_{x\in K} B_r(x)\Big), \quad \mathrm{Pen}^-(K,r) = \mathrm{CL}(M \setminus \mathrm{Pen}^+(M \setminus K, r)).$$

We call an exhaustion $M_1 \subset M_2 \subset \cdots$, $\bigcup_i M_i = M$, by compact submanifolds a regular exhaustion if for each $r \geq 0$

$$\lim_{i\to\infty} \mathrm{vol}(\mathrm{Pen}^+(M_i,r))/\,\mathrm{vol}(\mathrm{Pen}^-(M_i,r)) = 1.$$

It is clear that then automatically

$$\lim_{i\to\infty} \mathrm{vol}(\mathrm{Pen}^+(M_i,r))/\,\mathrm{vol}(M_i) = 1, \quad \lim_{i\to\infty} \mathrm{vol}(M_i)/\,\mathrm{vol}(\mathrm{Pen}^-(M_i,r)) = 1.$$

Examples 2.46. **1)** $(M^n,g) = (\mathbb{R}^n, g_{standard})$ admits a regular exhaustion.
2) Any (M^n,g) with subexponential growth admits a regular exhaustion.
3) The hyperbolic space admits no regular exhaustion. ∎

Let $\{M_i\}_{i\geq 1}$ be a regular exhaustion and set for $\omega \in {}^b\Omega^n$

$$\langle \mathfrak{m}_i, \omega\rangle := \frac{1}{\mathrm{vol}(M_i)} \int_{M_i} \omega.$$

Then $|\langle \mathfrak{m}_i, \omega\rangle| \leq \sup_x |\omega|_x = {}^b|\omega|$, i.e. $|\mathfrak{m}_i| \leq 1$, the $\mathfrak{m}_i$ belong to the unit ball in $({}^b\Omega^n)^*$. This unit ball is compact in the weak star topology, according to the Banach–Alaoglu theorem, hence the sequence $\{\mathfrak{m}_i\}_i$ has a weak star limit point $\mathfrak{m}$. $\mathfrak{m}$ is then called associated to the regular exhaustion $\{M_i\}_i$.

Proposition 2.47. *Let $\mathfrak{m}$ be associated to a regular exhaustion $\{M_i\}_i$. Then $\mathfrak{m}$ is a fundamental class for M.*

Proof. There remains only to show $\langle \mathfrak{m}, d\psi\rangle = 0$. Let $\Phi_i \in C^\infty(M)$ such that $0 \le \Phi_i(x) \le 1$, $\Phi_i = 1$ on M_i, $\Phi_i = 0$ outside $\mathrm{Pen}^+(M_i, 1)$, $|\nabla\Phi_i| \le 2$. We obtain for $\omega \in {}^b\Omega^n$

$$\left| \int_{M_i} \omega - \int_M \Phi_i \omega \right| \le (\mathrm{vol}(\mathrm{Pen}^+(M_i,1)) - \mathrm{vol}(M_i))\, {}^b|\omega|,$$

hence

$$\lim_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} \left(\int_{M_i} \omega - \int_M \Phi_i\omega \right) = 0.$$

Therefore we would be done if we could show

$$\lim_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} \int_M \Phi_i d\psi = 0.$$

Integration by parts yields

$$\int \Phi_i d\psi = -\int d\Phi_i \wedge \psi,$$

$$\left| \int \Phi_i d\psi \right| = \left| \int d\Phi_i \wedge \psi \right| \le 2(\mathrm{vol}(\mathrm{Pen}^+(M_i,1)) - \mathrm{vol}(M_i))\, {}^b|\psi|,$$

which implies the assertion. ∎

Define for $\omega \in {}^{b,1}\mathcal{C}_p(B_0)$, $[Q_{i_1\dots i_k}(\omega)] \in {}^bH^n(M)$ a (bounded) characteristic class and a regular exhaustion $\{M_i\}_i$ with associated fundamental class $\mathfrak{m}$ the characteristic number

$$Q_{i_1\dots i_k}(P, \mathrm{comp}(\omega))[\mathfrak{m}] := \langle \mathfrak{m}, [Q_{i_1\dots i_k}]\rangle := \lim_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} (Q_{i_1\dots i_k}).$$

Then, according to proposition 2.47, $Q_{i_1\dots i_k}(P, \mathrm{comp}(\omega))[\mathfrak{m}]$ is well defined. In particular we obtain in this case average Euler numbers, average signatures, which are special cases of Roe's (average) topological index. Average characteristic numbers are also considered in [17], [18], [15]. Some simple geometric examples are calculated in [17].

In all cases discussed until now, we restricted to the case of connections (or metrics) with finite p-action or bounded curvature or both. The next proposition shows that this is in fact a restriction.

Proposition 2.48. *Let (M^n, g) be open, complete, satisfying (I), G a compact Lie group, $P = P(M, G)$ a G-principal fibre bundle, $\varrho : G \to U(N)$ resp. $O(N)$ a faithful representation, E the associated vector bundle, $p \le 1$. Then there exist G-connections ω such that their p-action is infinite or the curvature is unbounded or both, respectively.*

Proof. Consider the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^n$ and set up in $\overline{B}_1(0)$ constant 1-forms ω_{ij}, $\omega_{ij} = -\overline{\omega}_{ji}$ or $\omega_{ij} = -\omega_{ji}$, $1 \le i, j \le N$, respectively, such that some $\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}$ are $\ne 0$. Now consider an infinite sequence $U_\nu = U_{\varepsilon_\nu}(x_\nu)$ of closed geodesic balls with pairwise distance $\ge d > 0$, introduce in each geodesic ball normal coordinates $u^1, \dots, u^n$, $\sum_i (u^i)^2 \le \varepsilon_\nu$, choose over U_ν orthonormal bases $e_{1,\nu}, \dots, e_{N,\nu}$ and define with respect to these bases local connection matrices $\omega'_{ij,\nu}$ by $\omega'_{ij,\nu}(u_1, \dots, u_n) := \omega_{ij}$. If $\int_{u_\nu} |\Omega'_{ij,\nu}|_x^p \,\mathrm{dvol}_x(g) = a_\nu \ne 0$, set $\omega''_{ij,\nu} = (a_\nu + \frac{1}{a_\nu})^{\frac{1}{2}p} \omega'_{ij,\nu}$. This connection over $\bigcup_\nu U_\nu$ is smoothly extendable over the whole of M and gives a connection with $\int_M |\Omega''|_x^p \,\mathrm{dvol}_x(g) \ge$

$\sum_\nu \int_{U_\nu} |\Omega''|_x^p \,\mathrm{dvol}_x(g) \geq \sum_\nu 1 = \infty$. Setting $\omega''_{ij,\nu} = \nu \cdot (a_\nu + \frac{1}{a_\nu})^{\frac{1}{2}p} \cdot \omega''_{ij,\nu}$ yields examples for the other cases. ■

The conditions of finite p-action or boundedness can be reformulated in the language of classifying spaces and classifying mappings.

We start with $G = U(N)$. Let $V_{N,k} \stackrel{U(k)}{\longrightarrow} G_{N,k}$ be the Stiefel bundle over the complex Grassmann manifold $G_{N,k}$ of all k-subspaces $\subset \mathbb{C}^N$ and S the matrix valued function on $V_{N,k}$ defined by $S(v_1, \ldots, v_k) = a_{ij} := (b_{ij})^t$, where $v_1, \ldots, v_k$ is a unitary k-frame, $e_1, \ldots, e_N$ the standard base in $\mathbb{C}^N$ and $v_i = \sum_{j=1}^N b_{ij} e_j$.

PROPOSITION 2.49. **a)** *$\gamma_U = S^* dS$ is a $U(N)$-invariant connection form at $V_{N,k}$.*
b) *Let be $m = (n+1)(2n+1)k^3$. If P is a $U(k)$-principal fibre bundle over a manifold of dimension $\leq n$ and ω a connection form for P, then there exists a smooth bundle morphism $f_P : P \to V_{m,k} = P_{n,U(k)}$ such that $f_P^* \gamma = \omega$.*

We refer to [20], p. 564, 568 for the proof. ■

γ_0 is called an n-universal connection for $U(k)$. In a similar manner one defines on the real Stiefel bundle $V_{m,k}^r \stackrel{O(k)}{\longrightarrow} G_{m,r}^r$ an n-universal $O(k)$-connection γ_0^r.

For an arbitrary compact Lie group G one constructs by means of a faithful representation $G \to O(k)$ an n-universal connection γ_G on the n-universal bundle $P_{n,G} \to B_{n,G}$ (cf. [20], p. 570).

According to proposition 2.49, we refine the bundle concept and consider instead of a bundle P pairs $(P, f_P), f_P : P \to P_{n,G}$ a C^1-classifying bundle map.

(P, f_P) is called a (p, f)-bundle if $f_p^* \gamma_G \in C^1\mathcal{C}_p(f, p) = \{\omega \text{ a } C^1\text{-connection}| \int |\Omega^\omega|_x^p \,\mathrm{dvol}_x(g) < \infty\}$, i.e. $\int |\Omega^{f_p^* \gamma_G}|_x^p \,\mathrm{dvol}_x(g) < \infty$. In the same manner we define (P, f_P) to be a b-bundle if $f_p^* \gamma_G \in C^1\mathcal{C}_p(B_0)$, i.e. ${}^b|\Omega^{f_p^* \gamma_G}| < \infty$.

Most interesting for applications is the case assuming (B_0) and finite p-action. Hence we assume (B_0) for (M^n, g). (P, f_P) is a (b, p, f)-bundle, if $f_p^* \gamma_G \in {}^{b,1}\mathcal{C}_p^{p,1}(B_0, f, p)$. Two (b, p, f)-bundles (P, f_P), (P, f_P') are called equivalent if $f_P^* \gamma_G$, ${f'}_P^* \gamma_G$ are contained in the same component of ${}^{b,1}\mathcal{C}_P^{p,1}(B_0, f, p)$. Assume G to be a subgroup of $U(N)$, $\dim M^n = 2k$. At the level of base spaces we consider classifying maps $f_M : M \to B_{n,G}$. A pair (M, f_M) is called a (p, c)-bundle if all classes $f_M^* c_{i_1 \ldots i_k}$, $i_1 + \cdots + i_k = k$, are elements of $H^{2k,p}(M)$. (M, f_M) is called a (b, c)-bundle if all classes $f_M^* c_{i_1 \ldots i_k}$ are elements of ${}^bH^{2k}(M)$. (M, f_M) is called a (b, p, c)-bundle if all classes $f_M^* c_{i_1 \ldots i_k}$ are elements of ${}^bH^{2k,p}(M)$. It is clear that a given $f_P : P \to P_{n,G}$ uniquely determines $f_M : M \to B_{n,G}$.

The case $G \subseteq O(N)$, $\dim M = 4k$, is quite parallel. Then we consider the $p_{i_1 \ldots i_k}$, $i_1 + \cdots + i_k = k$ and define (M, f_M) to be a (p, po)-bundle if all classes $f_M^* p_{i_1 \ldots i_k}$, $i_1 + \cdots + i_k = k$ are elements of $H^{4k,p}(M)$. Analogously for (b, po)- and (b, p, po)-bundles (M, f_M).

If we replace $p_{i_1 \ldots i_k}$ by the class of Hirzebruch genus L_k then we get the notion of a (p, L_k)-, (b, L_k)- or (b, p, L_k)-bundle (M, f_M), respectively.

THEOREM 2.50. **a)** *Suppose $G \subset U(N)$, $\dim M = 2k$. (M, g) satisfying (B_0), $p \geq 1$. A (b, p, f)-bundle (P, f_P) defines a unique (b, p)-bundle (M, f_M). If (P, f_P), (P, f_P') are*

equivalent then $f_M^* c_{i_1 \dots i_k} = {f'}_M^* c_{i_1 \dots i_k}$ *for all* $c_{i_1 \dots i_k}$, $i_1 + \dots + i_k = k$. *If additionally* $p = 1$ *and* (M, g) *is complete then even the corresponding characteristic numbers coincide.*
b) *Suppose* $G \subseteq O(N)$, $\dim M = 4k$, (M, g) *satisfying* (B_0), $p \geq 1$. *A* (b, p, f)*-bundle* (P, f_P) *defines a unique* (b, p, po)*-bundle* (M, f_M) *which is simultaneously a* (b, p, L_k)*-bundle. If* (P, f_P), (P, f'_P) *are equivalent then* $f_M^* p_{i_1 \dots i_k} = {f'}_M^* p_{i_1 \dots i_k}$ *and* $f_M^* L_k = {f'}_M^* L_k$. *If additionally* $p = 1$ *and* (M, g) *is complete then the corresponding characteristic numbers coincide.*

The proof follows immediately from the definitions and theorem 2.14. ∎

Example 2.51. It is possible that ${}^{b,1}\mathcal{C}_p^{1,1}(B_0, 1, f) = \emptyset$. Let (M^2, g) be an infinitely connected open complete Riemannian manifold with bounded sectional curvature K, $K = K_+ - K_-$,

$$K_+ = \begin{cases} K, & K \geq 0, \\ 0, & K < 0, \end{cases} \qquad K_- = \begin{cases} -K, & K \leq 0, \\ 0, & K > 0. \end{cases}$$

Then $\int K_- \,\mathrm{dvol} = \infty$ (cf. [16], theorem 13). In particular $\int |K| \,\mathrm{dvol} = \infty$ which implies $\int |\Omega^\omega(g)| \,\mathrm{dvol} = \infty$. The proof essentially relies on the Gauß–Bonnet theorem (as one would expect) for compact surfaces. But this theorem holds for any metrizable connection in the orthogonal 2-frame bundle $P(M^2, O(2))$ over M^2 ([19], p. 305/306). The sectional curvature K is defined by $\Omega_{1,2} = K \,\mathrm{dvol}$. As conclusion we obtain ${}^{b,1}\mathcal{C}_p(B_0, 1, f) = \emptyset$. ∎

3. Combinatorial characteristic numbers.

Let (M^n, g) be open, complete, oriented. We consider triangulations $T : |K| \xrightarrow{\cong} M$. Let σ^n be a curved n-simplex in M^n. We define the fullness $\theta(\sigma)$ by $\theta(\sigma) := \mathrm{vol}(\sigma)/(\mathrm{diam}(\sigma))^n$, where $\mathrm{vol}(\sigma)$, $\mathrm{diam}(\sigma)$ means volume and diameter with respect to g. $T : |K| \xrightarrow{\cong} M$ will be called uniform if it satisfies the following conditions:

a) There exists a $\theta_0 > 0$ such that for every curved simplex σ^n the fullness satisfies the inequality $\theta(\sigma) \geq \theta_0$.
b) There exist constants $c_1 > c_2 > 0$ such that for every σ^n we have

$$c_2 \leq \mathrm{vol}(\sigma) \leq c_1.$$

c) There exists a constant $c > 0$ such that for every vertex $\nu \in K$ the barycentric coordinate function $\varphi_\nu : M \to R$ satisfies the condition $|\nabla \varphi_\nu| \leq c$.

If one assumes a), then b) is equivalent to the existence of bounds $d_1 > d_2 > 0$ with $d_2 \leq \mathrm{diam}(\sigma) \leq d_1$ for all $\sigma \in K$. a) and b) are equivalent to the boundedness of the volumes from below and the diameters from above.

Consider the Whitney transformation W,

$$\sigma^q \to W(\sigma^q) = \omega_\sigma := q! \sum_{i=0}^{q} (-1)^i \varphi_i d\varphi_0 \wedge \dots \wedge \widehat{d\varphi_i} \wedge \dots \wedge d\varphi_q,$$

φ_i the barycentric coordinates.

Theorem 3.1. *If* $T : |K| \to (M^n, g)$ *is uniform then* W *induces topological isomorphisms*

$$H^{*,p}(K) \to H^{*,p}_{dR}(M), \qquad \overline{H}^{*,p}(K) \to \overline{H}^{*,p}_{dR}(M).$$

We refer to [13] for the proof. ∎

The proof of 3.1 in [13] is performed even under weaker assumptions, K uniformly locally finite and $|dT|, |dT^{-1}| \le C$.

The point is the multiplicativity which is settled by

THEOREM 3.2. *Let* $[z^q] \in H^{q,p}(K)$, $[z'^{n-q}] \in H^{n-q,r}(K)$, $\frac{1}{p} + \frac{1}{r} = 1$. *Then*

$$\int_M W(z) \wedge W(z') = C(n,q)\langle [z^q] \cup [z'^{n-q}], [K]\rangle. \tag{3.1}$$

We refer to [8], [10] for the meaning of the r.h.s. of (3.1) and the proof of 3.2. ∎

In the case of $n = 4k$, $p = r = 2$, iteration of (3.1) yields

COROLLARY 3.3. *If* (M^{4k}, g) *is open, oriented,* $g \in {}^{b,2}\mathcal{M}^{2,2}(B_0, 2, f)$ *and satisfies the condition* $r_{\mathrm{inj}}(M,g) > 0(I)$, *then* (M^n, g) *admits uniform triangulations* $T : |K| \xrightarrow{\cong} M$ *and*

$$p_{i_1 \dots i_k, c} = C(4k) p_{i_1 \dots i_k, a}, \tag{3.2}$$

where $p_{i_1 \dots i_k, c}$ *or* $p_{i_1 \dots i_k, a}$ *are the combinatorial or analytical Pontrjagin numbers, respectively.* ∎

4. Bordism and relative characteristic numbers. We consider oriented open manifolds (M^n, g) satisfying

$$|\nabla^i R| \le C_i, \qquad i = 0, 1, 2, \dots, k \qquad (B_k)$$

and

$$r_{\mathrm{inj}}(M,g) \equiv \inf r_{\mathrm{inj}}(g,x) > 0. \qquad (I)$$

(B^{n+1}, g_B) is a bordism between (M_1^n, g_1) and (M_2^n, g_2) if it satisfies the following conditions.

1) $(\partial B, g_B|_{\partial B}) \cong (M_1, g_1) \cup (-M_2, g_2)$,
2) there exists $\delta > 0$ such that $g_B|_{U_\delta(\partial B)} \cong g_{\partial B} + dt^2$,
3) (B, g_B) satisfies (B_k) and $\inf_{x \in B \setminus U_\delta(\partial B)} r_{\mathrm{inj}}(g_B, x) > 0$,
4) there exists $R > 0$ such that $B \subset U_R(M_1)$, $B \subset U_R(M_2)$.

We denote $(M_1, g_1) \underset{b}{\sim} (M_2, g_2)$. (B^{n+1}, g_B) is called a bordism. Sometimes we denote additionally $\underset{b, b_g}{\sim}$, b_g stands for bounded geometry, i.e. (I) and (B_k).

LEMMA 4.1. **a)** $\underset{b}{\sim}$ *is an equivalence relation. Denote by* $[M^n, g]$ *the bordism class.*
b) $[M \cup M', g \cup g'] = [M \# M', g \# g']$.
c) *Set* $[M, g] + [M', g'] := [M \cup M', g \cup g'] = [M \# M', g \# g']$. *Then* $+$ *is well defined and the set of all* $[M^n, g]$ *becomes an abelian semigroup.* ∎

Denote by Ω_n^{nc} the corresponding Grothendieck group. Similarly one defines $\Omega_n^{nc}(X)$ generated by pairs $((M^n, g), f : M^n \to X)$, f bounded and uniformly proper.

REMARKS 4.2. **1)** Condition 4) above looks like $d_{GH}(M, M') \le R$, where d_{GH} is the Gromov–Hausdorff distance (cf. [9]). But this is wrong.
2) There is no chance to calculate Ω_n^{nc}.
3) One would like to have a geometric representative for 0 and for $-[M, g]$. ∎

The way out from this is to establish bordism theory for special classes of open manifolds or/and further restrictions on bordism.

Our first example is bordism with compact support. Here condition 1) above remains but one replaces 2)–4) by the condition

$$\begin{aligned}&\text{There exists a compact submanifold } C^{n+1} \subset B^{n+1}\\ &\text{such that } (\overline{B \setminus C}, g_B|_{\overline{B\setminus C}}) \text{ is a product bordism, i.e.}\\ &(\overline{B \setminus C}, g_{\overline{B\setminus C}}) \cong (\overline{M \setminus C} \times [0,1], g_{\overline{M\setminus C}} + dt^2). \qquad (cs)\end{aligned}$$

We write $\underset{b,cs}{\sim}$. Then one gets a bordism group $\Omega_n^{nc}(cs)$ (= Grothendieck group).

At a first glance, the calculation of $\Omega_n^{nc}(cs)$ or at least the characterization of the bordism classes seems to be very difficult. But we will see that this is not the case. For this, we introduce still some uniform, structures. Denote by $\mathfrak{M}^n(mf) := \mathfrak{M}^n(mf, nc) \subset \mathfrak{M}_L$ the set of isometry classes of complete, open, oriented Riemannian manifolds. Consider pairs $(M_1^n, g_1), (M_2^n, g_2) \in \mathfrak{M}^n(mf)$ with the following property:

$$\begin{aligned}&\text{There exist compact submanifolds } K_1^n \subset M_1^n \text{ and } K_2^n \subset M_2^n\\ &\text{and an isometry } M_1 \setminus K_1 \xrightarrow{\Phi} M_2 \setminus K_2. \qquad (4.1)\end{aligned}$$

For such pairs, we define

$$\begin{aligned}&{}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) :=\\ &\inf\{\max\{0, \log {}^b|df|\} + \max\{0, \log {}^b|dh|\} + \sup_{x\in M_1} \operatorname{dist}(x, hfx) + \sup_{y\in M_2} \operatorname{dist}(y, fhy) |\\ &f \in C^\infty(M_1, M_2), g \in C^\infty(M_2, M_1), \text{ and}\\ &\text{for some } K_1 \subset K, f|_{M_1\setminus K_1} \text{ is an isometry and } g|_{f(M_1\setminus K} = f^{-1}\}.\end{aligned}$$

If (M_1, g_1) and (M_2, g_2) do not satisfy (4.1), then we define ${}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) = \infty$. We have ${}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) = 0$ if (M_1, g_1) and (M_2, g_2) are isometric.

Remarks 4.3. **1)** The notions of Riemannian isometry and distance isometry coincide for Riemannian manifolds. Furthermore, if f is an isometry f, then we have ${}^b|df| = 1$.
2) Any f that occurs in the definition of $d_{L,iso,rel}$ is automatically an element of $C^{\infty,m}(M_1, M_2)$ for all m. The same holds true for g. ∎

We write $\mathfrak{M}^n_{L,iso,rel}(mf) = \mathfrak{M}^n(mf)/\sim$ where by definition $(M_1, g_1) \sim (M_2, g_2)$ if ${}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) = 0$. Set

$$V_\delta = \{((M_1, g_1), (M_2, g_2)) \in (\mathfrak{M})^2_{L,iso,rel}(mf))^2 \mid {}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) < \delta\}.$$

Proposition 4.4. *$\mathcal{L} = \{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure ${}^b\mathcal{U}_{L,iso,rel}$.* ∎

Denote by ${}^b\mathfrak{M}^n_{L,iso,rel}(mf)$ the corresponding uniform space.

Proposition 4.5. *If $r_{inj}(M_i, g_i) = r_i > 0$, $r = \min\{r_1, r_2\}$ and ${}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) < r$ then M_1 and M_2 are (uniformly proper) bi-Lipschitz homotopy equivalent.* ∎

Corollary 4.6. *If we restrict ourselves to open manifolds with injectivity radius $\geq r$, then manifolds (M_1, g_1) and (M_2, g_2) with ${}^b d_{L,iso,rel}$-distance less than r are automatically (uniformly proper) bi-Lipschitz homotopy equivalent.* ∎

REMARK 4.7. If (M_1, g_1) satisfies (I) or (I) and (B_k) and ${}^b d_{L,iso,rel}(M_1, g_1), (M_2, g_2))$ $< \infty$ then (M_2, g_2) also satifies (I) or (I) and (B_k). ■

We cannot show that ${}^b\mathfrak{M}^n_{L,iso,rel}$ is locally arcwise connected, that components are arc components and ${}^b\mathrm{comp}_{L,iso,rel}(M, g) = \{(M', g') | {}^b d_{L,iso,rel}((M, g), (M', g')) < \infty\}$ is wrong. The reason is that we cannot connect non-homotopy-equivalent manifolds by a continuous family of manifolds. A parametrization of nontrivial surgery always contains bifurcation levels where we leave the category of manifolds. A very simple case comes from corollary 4.6.

COROLLARY 4.8. *If we restrict ${}^b\mathcal{U}_{L,iso,rel}$ to open manifolds with injectivity radius $\geq r > 0$, then the manifolds in each arc component of this subspace are bi-Lipschitz homotopy equivalent.*

Proof. This subspace is locally arcwise connected and components are arc components. Consider an (arc) component and two elements (M_1, g_1) and (M_2, g_2) of it, connect them by an arc, cover this arc by sufficiently small balls, and apply 4.6. ■

By definition, we have

$$ {}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) < \infty \Rightarrow d_L((M_1, g_1), (M_2, g_2)) < \infty, $$

where d_L is the Lipschitz distance of [9]. Hence, $(M_2, g_2) \in \mathrm{comp}_L(M_1, g_1)$, i.e.

$$ \{(M_2, g_2) \in \mathfrak{M}^n(mf) | {}^b d_{L,iso,rel}((M_1, g_1), (M_2, g_2)) < \infty\} \subseteq \mathrm{comp}_L(M_1, g_1). \qquad (4.2) $$

For this reason, we denote the left hand side $\{\dots\}$ of (4.2) by gen ${}^b\mathrm{comp}_{L,iso,rel}(M_1, g_1) = \{\dots\} = \{\dots\} \cap \mathrm{comp}_L(M_1, g_1)$, keeping in mind that this is not an arc component, but a subset of (manifolds in) a Lipschitz arc component.

If we fix (M_1, g_1), then in a special case, we have a good overview of the elements in gen ${}^b\mathrm{comp}_{L,iso,rel}(M_1, g_1)$.

EXAMPLE 4.9. Let $(M_1, g_1) = (\mathbb{R}^n, g_{\mathrm{standard}})$. Then gen ${}^b\mathrm{comp}_{L,iso,rel}(M_1, g_1)$ is in a 1-1 correspondence with $\{(M^n, g) | M^n$ is a closed manifold and g is flat in an annulus contained in a disk neighborhood of a point $\}$. ■

This can be generalized as follows.

THEOREM 4.10. *Any component* gen ${}^b\mathrm{comp}_{L,iso,rel}(M, g)$ *contains at most countably many diffeomorphism types.*

Proof. Fix $(M, g) \in$ gen ${}^b\mathrm{comp}_{L,iso,rel}(M, g)$ and an exhaustion $K_1 \subset K_2 \subset \dots$, $\bigcup K_1 = M$, of M by compact submanifolds, and let $(M', g') \in {}^b\mathrm{comp}_{L,iso,rel}(M, g)$. Then there are $K' \subset M'$ and $K_i \subset M$ such that $M \setminus K_i$ and $M' \setminus K'$ are isometric. The diffeomorphism type of M' is completely determined by that of the pair $(K_1 \bigcup_{\partial K_1 \cong \partial K'} K', K_1)$, but the set of types of such pairs (after fixing M and $K_1 \subset K_2 \subset \dots$) is at most countable. ■

Thus, after fixing (M, g), the diffeomorphism classification of the elements in ${}^b\mathrm{comp}_{L,iso,rel}$ (M, g) seems to be reduced to a "handy" countable discrete problem. This is in fact the case in a sense which is parallel to the classification of compact manifolds.

Now we connect the calculation of $\Omega_n^{nc}(cs)$ with the generalized components gen ${}^b\mathrm{comp}_{L,iso,rel}(\cdot) \subset {}^b\mathfrak{M}^n_{L,iso,rel}(mf)$.

REMARK 4.11. If $(M_1, g_1), (M_2, g_2) \in$ gen ${}^b\mathrm{comp}_{L,iso,rel}(M, g)$, then, in general, (M_1, g_1) $\#(M_2, g_2) \notin$ gen ${}^b\mathrm{comp}_{L,iso,rel}(M, g)$. ■

Let $\Omega_n^{nc}(cs, \mathrm{gen}\ {}^b\mathrm{comp}_{L,iso,rel}(M, g)) \subset \Omega_n^{nc}(cs)$ be the subgroup generated by

$$\{[M', g']_{cs} | (M', g') \in \mathrm{gen}\ {}^b\mathrm{comp}_{L,iso,rel}(M, g)\}.$$

We know $\Omega_n^{nc}(cs)$ completely if we know all $\Omega_n^{nc}(cs, \mathrm{gen}\ {}^b\mathrm{comp}_{L,iso,rel}(M, g))$, and we know

$$\Omega_n^{nc}(cs,\ {}^b\mathrm{comp}_{L,iso,rel}(M, g))$$

completely if we know a corresponding generating set. However, the elements of such a set are completely determined by their (relative) characteristic numbers.

Fix $(M^n, g) \in$ gen ${}^b\mathrm{comp}_{L,iso,rel}(M^n, g)$, where M is oriented. Assume that $(M_1, g_1) \in$ gen ${}^b\mathrm{comp}_{L,iso,rel}(M^n, g)$, and let $\Phi : M \setminus K \to M_1 \setminus K_1$ be an orientation preserving isometry. Define the (relative) Stiefel–Whitney numbers of the pair (M_1, M) by

$$w_1^{r_1} \dots w_n^{r_n}(M_1, M) := \langle w_1^{r_1} \dots w_n^{r_n}, [K_1]\rangle + \langle w_1^{r_1} \dots w_n^{r_n}, [K]\rangle. \tag{4.3}$$

Similarly, for (M_1, M) and $n = 4k$, we define the (relative) Pontrjagin numbers

$$p_1^{r_1} \dots p_k^{r_k}(M_1, M) := \int_{K_1} p_1^{r_1} \dots p_k^{r_k}(M_1) - \int_K p_1^{r_1} \dots p_k^{r_k}(M) \tag{4.4}$$

and the (relative) signature by

$$\sigma(M_1, M) := \sigma(K_1) + \sigma(-K). \tag{4.5}$$

LEMMA 4.12. *The numbers* $w_1^{r_1} \dots w_n^{r_n}(M_1, M)$, $p_1^{r_1} \dots p_k^{r_k}(M_1, M)$, *and* $\sigma(M_1, M)$ *are well defined, and we have*

$$w_1^{r_1} \dots w_n^{r_n}(M_1, M) = \langle w_1^{r_1} \dots w_n^{r_n}(K_1 \cup K), [K_1 \cup K]\rangle, \tag{4.6}$$

$$p_1^{r_1} \dots p_k^{r_k}(M_1, M) = \langle p_1^{r_1} \dots p_k^{r_k}(K_1 \cup -K), [K_1 \cup -K]\rangle, \tag{4.7}$$

and

$$\sigma(M_1, M) = \sigma(K_1 \cup -K). \tag{4.8}$$

Proof. The equations (4.3), (4.4) are clear. (4.5) comes from Novikov additivity of σ. Hence we have only to show the well definedness, i.e. the independence of the choice of $K \subset M$, $K_1 \subset M_1$. Start with (4.4). If $K_1' \supset K_1$, $K' \supset K$, $\Phi|_{M \setminus K'} : M \setminus K' \xrightarrow{\cong} M_1 \setminus K_1'$ orientation preserving isometric, then

$$\int_{K_1'} \dots - \int_{K'} \dots = \int_{K_1' \setminus \mathring{K}_1} \dots + \int_{K_1} \dots - \left(\int_{K' \setminus \mathring{K}} \dots + \int_K \dots \right). \tag{4.9}$$

But $\int_{K_1' \setminus \mathring{K}_1} \dots - \int_{K' \setminus \mathring{K}} \dots = 0$ since $K_1' \setminus \mathring{K}_1$ and $K' \setminus \mathring{K}$ are isometric under Φ by assumption. The analogous conclusion can be done for $K_1'' \subset K_1$, $K'' \subset K$, $\Phi : M \setminus \mathring{K}'' \to M_1 \setminus \mathring{K}_1''$ already an isometry. In the general case K_1, K_1', K, K' one considers $K_1' \cap K_1$, $K \cap K'$ and reduces to the first two considerations after smoothing out $K_1' \cap K_1$, $K' \cap K$ by arbitrary small perturbations. The proof for (4.3) is quite similar replacing integrations in (4.9) by application of cocycles to cycles. The independence of (4.5) comes again from Novikov additivity, applying it several times. ∎

THEOREM 4.13. *Fix* $(M_1, g_1), (M_2, g_2) \in \text{gen}\,{}^b\text{comp}_{L,iso,rel}(M, g)$. *Then* $(M_1, g_1) \underset{b,cs}{\sim} (M_2, g_2)$ *if and only if all characteristic numbers of* (M_1, M) *coincide with the corresponding characteristic numbers of* (M_2, M).

Proof. Assume $(M_1, g_1) \underset{b,cs}{\sim} (M_2, g_2)$. Choose $C^{n+1} \subset B^{n+1}$ large enough such that with $\partial C^{n+1} = \partial_1 C \cup \partial_2 C \cup \partial_3 C$, $\partial_1 C = K_1 \subset M_1$, $\partial_2 C = K_2 \subset M_2$, $\partial_3 C \cong \partial K_1 \times [0,1]$ we have $M_1 \setminus \overset{\circ}{K}_1 \underset{\Phi_1}{\cong} M \setminus \overset{\circ}{K}$, $M_2 \setminus \overset{\circ}{K}_2 \underset{\Phi_2}{\cong} M \setminus \overset{\circ}{K}$. Then after smoothing out by arbitrary small perturbations, ∂C is diffeomorphic ot $K_1 \underset{\Phi}{\cup} -K_2$, $\Phi = \partial(\Phi_2^{-1}\Phi_1)$. Hence $\text{char.n.}(K_1 \underset{\Phi}{\cup} -K_2) = \text{char.n.}(K_1 \underset{\Phi_1}{\cup} -K) + \text{char.n.}(K \underset{\Phi_2^{-1}}{\cup} -K_2) = \text{char.n.}(M_1, M) + \text{char.n.}(M, M_2) = \text{char.n.}(M_1, M) - \text{char.n.}(M_2, M)$, i.e. $\text{char.n.}(M_1, M) = \text{char.n.}(M_2, M)$. Conversely, if $\text{char.n.}(M_1, M) = \text{char.n.}(M_2, M)$ then $\text{char.n.}(K_1 \underset{\Phi}{\cup} -K_2) = 0$, $K_1 \underset{\Phi}{\cup} -K_2$ is 0-bordant, $K_1 \underset{\Phi}{\cup} -K_2 = \partial C^{n+1}$. Form $K_1 \cup (\partial K_1 \times [0,1]) \underset{\Phi}{\cup} -K_2)$ which equals ∂C^{n+1}, glue $(M_1 \setminus \overset{\circ}{K}_1) \times [0,1] \cong (M \setminus \overset{\circ}{K}) \times [0,1] \cong (M_2 \setminus \overset{\circ}{K}_2) \times [0,1]$ and smooth out (the topology and the metrics). The result is a bordism (B^{n+1}, g_B) with compact support between (M_1, g) and (M_2, g). ∎

COROLLARY 4.14. *Description of all elements of* $\Omega_n^{nc}(cs)$ *reduces to "counting" the generalized components of* ${}^b\mathfrak{M}^n_{L,iso,rel}(mf)$. ∎

EXAMPLE 4.15. Consider $M_i = (M_i' \cup \partial M_i' \times [0, \infty[, g_i)$, $i = 1, 2$, M_i' compact, $\partial M_1' = \partial M_2'$, $(\partial M_1' \times [0, \infty[, g_{1,a,\infty})$ isometric to $(\partial M_2' \times [0, \infty[, g_{2,a,\infty})$, $g_{i,a,\infty} = g_i|_{\partial M_i' \times [a,\infty[}$. Let $M = (D^n \cup S^n \times [0, \infty[, g_{standard})$. If $\sigma(M_1') \neq \sigma(M_2')$ then $(M_1, g), (M_2, g) \in \text{gen}\,{}^b\text{comp}_{L,iso,rel}(M, g)$ are not cs–bordant. ∎

There is a simple approach to calculate the local algebraic structure of $\Omega_n^{nc}(cs)$. Consider as above $\Omega_n^{nc}(cs, M) := \Omega_n^{nc}(cs, \text{gen}\,{}^b\text{comp}_{L,iso,rel}(M))$ and let Ω_n be the usual bordism group of closed oriented n-manifolds. Then there exists a map $\Phi = \Phi_M : \Omega_n \to \Omega_n^{nc}(cs, M)$, $\Phi([N]) := [M \# N, g_{M\#N}]_{cs}$. Here the bordism class in $\Omega_n^{nc}(cs, M)$ is independent of the metric of N. Moreover, we have a map $\Psi = \Psi_M : \Omega_n^{nc}(cs, M) \to \Omega_n$, $\Psi([M', g']) := [K' \cup -K]$, where $(M' \setminus \overset{\circ}{K}{}', g'|_{M' \setminus \overset{\circ}{K}{}'})$ is isometric to $(M \setminus \overset{\circ}{K}, g|_{M \setminus \overset{\circ}{K}})$. It is very easy to see that Ψ_M is well defined: Let $(M'', g'') \in [M', g']_{cs}$. Then there exist $K_1'' \subset M''$, $K_1' \subset M'$, $K_1 \subset M$ such that $M'' \setminus \dot{K}_1''$, $M' \setminus \dot{K}_1'$, $M \setminus \dot{K}_1$ are isometric. By assumption and according to theorem 4.12 we have $\text{char.n.}(M'', M) = \text{char.n.}(M', M)$, i.e. $\text{char.n.}(K_1'' \cup -K_1) = \text{char.n.}(K_1' \cup -K_1) = \text{char.n.}(K' \cup -K)$, $[K_1'' \cup -K_1]_{\Omega_n} = [K' \cup -K]_{\Omega_n}$. We have $\Psi_M \Phi_M = \text{id} : (\Psi\Phi)[N] = \Psi([M \# N]) = [N]$ and $\Phi_M \Psi_M = \text{id} : (\Phi\Psi)([M', g']) = \Phi([K' \cup K]) = [M \# (K' \cup -K), g_{M\#(K' \cup -K)}] = [M', g']$ since $\text{char.n.}(M', M) = \text{char.n.}(M \# (K' \cup -K), M)$. Here Φ_M and Ψ_M are 1-1 maps. Moreover we have maps

$$\Phi_M \times \Phi_{M'} : \Omega_n \times \Omega_n \to \Omega_n^{nc}(cs, M) \times \Omega_n^{nc}(cs, M'),$$
$$\Phi_M \times \Phi_{M'}([N_1],[N_2]) = ([M\#N_1, g_{M\#N_1}],[M'\#N_2, g_{M'\#N_2}]),$$
$$\Phi_{M\#M'} : \Omega_n \to \Omega_n^{nc}(cs, M\#M'),$$
$$[N]_{\Omega_n} \to [M\#M'\#N, g_{M\#M'\#N}]_{cs},$$

$$\begin{array}{ccc} \Omega_n \times \Omega_n, & & ([N_1],[N_2]) \\ \downarrow & & \downarrow \\ \Omega_n & & [N_1\#N_2] \end{array}$$

and

$$\begin{array}{ccc} \Omega_n^{nc}(cs, M) \times \Omega_n^{nc}(cs, M'), & & ([M_1, g_1],[M_1', g_1']) \\ \downarrow & & \downarrow \\ \Omega_n^{nc}(cs, M\#M') & & [M_1\#M_1', g_1\#g_1'] \end{array}$$

Proposition 4.16. *The diagrams*

$$\begin{array}{ccc} \Omega_n \times \Omega_n & \overset{\Phi_M\times\Phi_{M'}}{\longrightarrow} & \Omega_n^{nc}(cs, M) \times \Omega_n^{nc}(cs, M') \\ \downarrow & & \downarrow \\ \Omega_n & \overset{\Phi_{M\#M'}}{\longrightarrow} & \Omega_n^{nc}(cs, M\#M') \end{array} \tag{4.10}$$

and

$$\begin{array}{ccc} \Omega_n^{nc}(cs, M) \times \Omega_n^{nc}(cs, M') & \overset{\Phi_M\times\Phi_{M'}}{\longrightarrow} & \Omega_n \times \Omega_n \\ \downarrow & & \downarrow \\ \Omega_n^{nc}(cs, M\#M') & \overset{\Phi_{M\#M'}}{\longrightarrow} & \Omega_n \end{array} \tag{4.11}$$

commute. ■

Remark 4.17. It is important that we consider (4.10), (4.11) at bordism class level. In (4.11) e.g. $(K_1 \cup -K)\#(K_1' \cup -K') \neq K_1\#K_1' \cup -(K\#K')$, but their bordism classes coincide. ■

The 1-1 property of Φ_M, Ψ_M moreover implies

Proposition 4.18. *Modulo torsion (which is well defined at the Ω_n-level) we have $\Omega_n^{nc}(cs, M^n) = 0$ for $n \neq 4k$, for $n = 4k$, $[M\#P^{2i_1}(\mathbb{C})\times\cdots\times P^{2i_k}(\mathbb{C}), g_{M\#P^{2i_1}(\mathbb{C})\times\cdots\times P^{2i_k}(\mathbb{C})}]$ are independent generators for $\Omega_n^{nc}(cs, M^n)$ over $\mathbb{Q}$, $i_1 + \cdots + i_k = k$.* ■

Remarks 4.19. **1)** 4.13, 4.14, 4.18 provide sufficient means to characterize cs-bordism classes and to calculate $\Omega_n^{nc}(cs)$.
2) An analogous procedure can be applied to calculate e.g. $\Omega_n^{nc,spin}(cs, M)$, where ${}^b\text{comp}_{L,iso,rel}(M)$ is now a component consisting of Spin-manifolds. ■

As we pointed out, a contentful theory should be developed under three aspects.

1) A convenient characterization of bordism classes is desirable.
2) It should be possible to exhibit sets of independent generators, at least for the intersections with gen-components.
3) A geometric realization of zero and the inverse are desirable.

The general bordism group Ω_n^{nc} did not satisfy any of these three wishes. $\Omega_n^{nc}(cs)$ satisfies the first two wishes. We develop below a bordism theory which satisfies the second and the third wish. This will be the bordism theory for manifolds with a finite number of ends, each of them nonexpanding.

Let ε be an isolated end of (M^n, g). A ray in ε is a geodesic γ defined on $[0,\infty[$ which is a shortest geodesic between any two of its points and such that some neighborhood of ε contains up to a finite segment the whole of $|\gamma|$. Then the latter holds for any neighborhood of ε.

LEMMA 4.20. *Let ε be an isolated end of (M^n, g).*

a) *Then there exists a ray in ε.*
b) *If (M^n, g) additionally satisfies (I) then there exists a ray in ε with a uniformly thick neighborhood.*

Proof. A proof of a) is e.g. contained in [11], p. 43. b) follows immediately from a) and (I). ∎

We call an end ε of (M^n, g) nonexpanding if there exist a ray γ in ε and an $R = R_M > 0$ and an element $G \in \varepsilon$ such that $G \subseteq U_R(|\gamma|)$, roughly written $\varepsilon \subseteq U_R(|\gamma|)$.

In the sequel we restrict to open manifolds satisfying (I), $B(_\infty)$, with finitely many ends, each of them nonexpanding.

EXAMPLES 4.21. **1)** Consider the sphere $S_r^{n-1} \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ of radius r and

$$\mathrm{chc}^n(r) := (S_r^{n-1} \times [0,\infty[\cup D_r^n, g_{st}),$$

i.e. the closed half cylinder of radius r with a standard metric g_{st} which should be the product metric of $S_r^{n-1} \times [0,\infty[$ smoothly extended to the glued bottom D_r^n and with standard orientation. Then $\mathrm{chc}^n(r)$ is an open manifold with one nonexpanding end, satisfying (I), (B_∞).
2) $\#_{i=1}^k \mathrm{chc}(r_i)$ has finitely many nonexpanding ends.
3) Any manifold $(M^n = M' \cup \partial M' \times [0,\infty[, g_M)$ where M' is compact and $g_M|_{\partial M' \times [a,\infty[} = dt^2 + g_{\partial M'}$ satisfying (I), (B_∞) and has finitely many nonexpanding ends.
4) The same is true if we allow g_M of 3) to vary in $\mathrm{comp}^{p,r}(g_M) \cap C^\infty$.
5) If we consider M of smooth type of 3) and $g_M|_{\partial M' \times [a,\infty[} = dt^2 + f(t)^2 g_{\partial M'}$ with $c_2 \geq f(t) \geq c_1 > 0$, $\frac{f^{(\nu)}}{f}$ bounded for all ν, $t \geq a$ then M has finitely many nonexpanding ends. ∎

We define now a slightly sharpened bordism relation.

Let (M^n, g), (M'^n, g') be as above, each with finitely many nonexpanding ends $\varepsilon_1, \ldots, \varepsilon_s$ or $\varepsilon'_1, \ldots, \varepsilon'_{s'}$, respectively. Let $\gamma_{M,1}, \ldots, \gamma_{M,s}$ or $\gamma_{M',1}, \ldots, \gamma_{M',s'}$ corresponding rays as above. From $(M, g) \underset{bg}{\sim} (M', g')$ and all ends nonexpanding follows in particular that for all sufficiently large compact $C^{n+1} \subset B^{n+1}$ there exists $R = R_B > 0$ s.t.

$$B^{n+1} \setminus C^{n+1} \subset \bigcup_1^s U_R(|\gamma_{M,\sigma}|), \qquad B^{n+1} \setminus C^{n+1} \subset \bigcup_1^{s'} U_R(|\gamma_{M',\sigma'}|).$$

We require additionally the additive compatibility of the inner γ-distance and the $(B\backslash C)$-distance for points x_γ, y_γ on the γ's.

There exist $C^{n+1} \subset B^{n+1}$ and $c' > 0$ s.t. for $x_\gamma, y_\gamma \in |\gamma| \setminus C$,

$$d_\gamma(x_\gamma, y_\gamma) - c' \le d_{B\backslash C}(x_\gamma, y_\gamma) \le d_\gamma(x_\gamma, y_\gamma) + c'. \qquad (GH)$$

Here γ stands for $\gamma_{M,1}, \dots, \gamma_{M,s}$, $\gamma_{M',1}, \dots, \gamma_{M',s'}$, respectively and $d(\cdot,\cdot) \equiv \mathrm{dist}(\cdot,\cdot)$.

We denote $(M, g) \underset{ne}{\sim} (M', g')$ if they are bg-bordant by means of (B, g_B) satisfying (GH).

Remarks 4.22. **1)** The right hand inequality of (GH) trivially holds. We added it only for symmetry reasons.
2) It was essentially Thomas Schick who pointed out to the author the meaning of the condition (GH) or (GH_1) and who proposed to include them into the definition of bordism. ■

We consider instead of (GH) the condition:

There exist $C^{n+1} \subset B^{n+1}$ and $c' > 0$ s.t. for all $x, y \in U(\varepsilon)$,

$$d_{U_\varepsilon}(x, y) - c \le d_{B\backslash C}(x, y) \le d_{U(\varepsilon)}(x, y) + c. \qquad (GH_1)$$

Here ε stands for $\varepsilon_1, \dots, \varepsilon_s$, $\varepsilon'_1, \dots, \varepsilon'_{s'}$ and $U(\varepsilon)$ for a neighborhood of ε, $U(\varepsilon) \cap C = \emptyset$.

Lemma 4.23. *(GH) and (GH_1) are equivalent.*

Proof. Assume (GH_1). Then (GH) holds since for $x_\gamma, y_\gamma \in |\gamma| \subset U(\varepsilon)$, $U(\varepsilon) \cap C = \emptyset$, $d_{U(\varepsilon)}(x_\gamma, y_\gamma) = d_\gamma(x_\gamma, y_\gamma)$. If conversely $x, y \in U(\varepsilon)$ then there exist $x_\gamma, y_\gamma \in |\gamma| \subset U(\varepsilon)$ s.t. $d_{U(\varepsilon)}(x, x_\gamma) \le R_M$, $d_{U(\varepsilon)}(y, y_\gamma) \le R_M$. Then the assertion follows from

$$\begin{aligned}
&d_{U(\varepsilon)}(x, y) - d_{U(\varepsilon)}(x_\gamma, y_\gamma) \le d_{U(\varepsilon)}(x, x_\gamma) + d_{U(\varepsilon)}(y, y_\gamma),\\
&d_{U(\varepsilon)}(x, y) - d_{U(\varepsilon)}(x, x_\gamma) - d_{U(\varepsilon)}(y, y_\gamma) \le d_\gamma(x_\gamma, y_\gamma)\\
&= d_\gamma(x_\gamma, y_\gamma) - c' + c' \le d_{B\backslash C}(x_\gamma, y_\gamma) + c',\\
&d_{U(\varepsilon)} - 2R_M - c' \le d_{B\backslash C}(x_\gamma, y_\gamma),\\
&d_{U(\varepsilon)} - 4R_M - c' \le d_{B\backslash C}(x, y). \ ■
\end{aligned}$$

Remark 4.24. (GH_1) immediately implies that $d_{GH}(\overline{B \setminus C}, \overline{U(\varepsilon)}) < \infty$, where $d_{G,H}(\cdot,\cdot)$ is the Gromov–Hausdorff distance between proper metric spaces. This follows from the following facts. $d_{GH}(\overline{B \setminus C}, \overline{U(\varepsilon)}) < \infty$ if we endow $\overline{U(\varepsilon)}$ with the induced length metric and use $\overline{B \setminus C} \subset U_R(U(\varepsilon))$. Then we use $d_{GH}(U(\varepsilon))$, its own length metric, $U(\varepsilon)$, induced length metric $< \infty$, which follows from (GH_1). As a matter of fact, we introduced (GH) to assure $d_{GH}(\overline{B \setminus C}, \overline{U(\varepsilon)}) < \infty$. ■

Proposition 4.25. *$\underset{ne}{\sim}$ is an equivalence relation.*

Proof. Reflexivity $((B = m \times [0,1], dt^2 + g_M))$ and symmetry $((-B, g_B))$ are immediately clear. For transitivity, the only point in the proof is (GH) or (GH_1). Let (B_{12}, g_{12}) and (B_{23}, g_{23}) be fe, ne-bordisms between (M_1, g_1), (M_2, g_2) and (M_2, g_2), (M_3, g_3), respectively and set $(B_{13}, g_{13}) = (B_{12} \cup B_{23}, g_{12} \cup g_{23})$. We assume w.l.o.g. global Riemannian collars at the boundaries. Let $C_{12}^{n+1} \subset B_{12}$, $C_{23} \subset B_{23}$ as required in (GH_1) and choose

$C = C_{13}^{n+1} \supset C_{12} \cup C_{23}$. Let ε be one of the ends of M_1, $U(\varepsilon) \subset M_1$. Then we have to show

$$d_{U(\varepsilon)}(x,y) - c \le d_{B_{13}\setminus C}(x,y). \tag{4.12}$$

We write in the sequel $d_i = d_{M_i\setminus C}$, $i = 1,2$, $d_{12} = d_{B_{12}\setminus C}$, $d_{23} = d_{B_{23}\setminus C}$, $d_{13} = d_{B_{13}\setminus C}$.

The required c in (4.12) exists for all pairs $x, y \in M_1 \setminus C$ s.t. $d_1(x,y)$ can be realized by a curve in $B_{12} \setminus C$. Let $x, y \in M_1 \setminus C$ be pair which does not have this property and let $z(t)$ be a curve in $B_{13} \setminus C$ which realizes $d_{13}(x,y)$, $z(0) = x = \alpha_1$, $z(1) = y \equiv y_1$. Then there exists a first point $x_2 \in M_2 \setminus C$ on $\{z_t\}_t$ and a last point $y_2 \in M_2 \setminus C$ on z.

Let moreover $x_2' \in M_2 \setminus C$ and $y_2' \in M_2 \setminus C$ be points which realize the distances $d_{13}(x_1, M_2 \setminus \mathring{C})$ and $d_{13}(y_1, M_2 \setminus \mathring{C})$, respectively. Then

$$\begin{aligned} &d_{13}(x_2', y_2') - d_{13}(x_1, y_1) \le d_{13}(x_1, x_2') + d_{13}(y_1, y_2'),\\ &d_{13}(x_2', y_2') - 2R_{12} \le d_{13}(x_1, y_1), \end{aligned} \tag{4.13}$$

Claim. *There exists a $c' > 0$ s.t.*

$$d_2(x_2', y_2') - c' \le d_{13}(x_2', y_2') \tag{Cl}$$

for all $x_2', y_2' \in M_2 \setminus C$.

Lemma 4.26. *(Cl) implies* (4.12).

Proof. We infer from (4.13) and (Cl)

$$d_2(x_2', y_2') - c' - 2R_{12} \le d_{13}(x_1, y_1). \tag{4.14}$$

Moreover

$$d_{12}(x_2', y_2') \le d_2(x_2', y_2'), \tag{4.15}$$

$$d_{12}(x_1, y_1) - 2R_{12} \le d_{12}(x_2', y_2'), \tag{4.16}$$

$$d_1(x_1, y_1) - c_{12} \le d_{12}(x_1, y_1). \tag{4.17}$$

Here (4.17) is the condition (GH_1) for B_{12}. (4.13)–(4.17) yield $d_{13}(x_1,y_1) \ge d_2(x_2', y_2') - c' - 2R_{12} \ge d_{12}(x_2', y_2') - c' - 2R_{12} \ge d_{12}(x_1, x_2) - 4R_{12} - c' \ge d_1(x_1, y_1) - 4R_{12} - c' - c_{12}$, i.e. we obtain (4.12) with $c = 4R_{12} + c' + c_{12}$. ∎

There remains to establish (Cl), i.e.

$$d_2(x_2', y_2') - c' \le d_{13}(x_2', y_2'). \tag{Cl}$$

Unfortunately (Cl) is wrong and hence the whole approach to prove transitivity. Uwe Abresch has constructed an explicit ingenious counterexample. Nevertheless, we performed the proof of transitivity until (Cl) to indicate the crucial point. It is in fact possible that an appropriate winding around the middle M_2, M_2 again and again penetrating curve has a much shorter B_{13}-length than the M_2-distance of the initial and the other intersection points in M_2. From this it is clear that one should forbid such a B_{13}-distance diminishing curve. But this can be achieved by forbidding a distance realizing curve to move out from the collar. That is if $x, y \in U(\varepsilon) \subset M_1$ or M_3 (or $x, y \in$ the corresponding geodesic ray, which is equivalent) then a B_{13}-distance realizing curve should remain in the collar of M_1 or M_3, respectively. For this we perform a conformal change in the metric $g_{12} \cup g_{23}$ of B_{13}.

We consider in $U_\delta(\partial B_{13})$ Gaussian normal coordinates which give for boundary points $\in \partial B_{13}$ coordinates only for a half ball. Nevertheless we have still about $x_0 \in U_\delta(\partial B_{13})$

$$a_{13}^{\partial B}(x_0)|\xi|^2 \le g_{13,ij}(x)\xi^i\xi^j \le b_{13}^{\partial B}(x_0)|\xi|^2, \quad \operatorname{dist}(x_0,\partial B) \le \operatorname{dist}(x,x_0) \le \delta/2$$

and about $x_0 \in B_{13} \setminus U_{\frac{3}{4}\delta}(\partial B_{13})$

$$a_{13}^{\partial B}(x_0)|\xi|^2 \le g_{13,ij}(x)\xi^i\xi^j \le b_{13}^{\partial B}(x_0)|\xi|^2, \quad \operatorname{dist}(x_0,x) \le \delta/2.$$

Set $a_{13}^{\partial B} = \sup_{x_0 \in U_\delta(\partial B)} a_{13}^{\partial B}(x_0)$, $b_{13}^{\partial B} = \inf_{x_0 \in U_\delta(\partial B)} b_{13}^{\partial B}(x_0)$, $a_{13}^{B} = \sup_{x_0 \text{ as above}} a_{13}^{B}(x_0)$, $b_{13}^{B} = \inf_{x_0 \text{ as above}} b_{13}^{B}(x_0)$. All these numbers are > 0. Then we have for $x \in U_\delta(\partial B_{13})$, $y \in B_{13} \setminus U_{\frac{3}{4}\delta}(\partial B_{13})$ and $0 \ne \xi \in \mathbb{R}^{n+1}$

$$\begin{aligned} g_{13,ij}(x)\xi^i\xi^j \le b_{13}^{\partial B} \le b_{13}^{\partial B}|\xi|^2 = \frac{b_{13}^{\partial B}}{a_{13}^{\partial B}} a_{13}^{B}|\xi|^2 \le \frac{b_{13}^{\partial B}}{a_{13}^{\partial B}} g_{13,ij}(y)\xi^i\xi^j \\ < \left(\frac{b_{13}^{\partial B}}{a_{13}^{\partial B}} + 1\right) g_{13,ij}(y)\xi^i\xi^j. \end{aligned} \tag{4.18}$$

Define now $\varphi \in C^\infty(B_{13})$ as follows:

$$\varphi = 1 \text{ on } U_{\frac{1}{2}\delta}(\partial B).$$

$$\varphi = \left(\frac{b_{13}^{\partial B}}{a_{13}^{B}} + 1\right) \text{ on } B_{13} \setminus U_{\frac{3}{4}\delta}(\partial B).$$

φ is increasing in inward normal direction in $U_\delta(M_1)$ and decreasing in outward normal direction in $U_\delta(M_3)$,

$$|\nabla^i\varphi| \le D_i, \quad i = 0,1,2,\ldots.$$

The existence of such a φ in the case of bounded geometry is standard (cf. [1]).

LEMMA 4.27. **a)** *$\tilde g_{13} = \varphi \cdot g_{13}$ is a metric of bounded geometry.*
b) *$(B_{12}, \tilde g_{13})$ is a bordism between (M_1, g_1) and (M_3, g_3) in the bounded geometry nonexpanding sense.*

Proof. a) Inside the $\frac{\delta}{2}$-collar nothing changes. g and g' are quasi isometric. Moreover for $i \ge 1$ $|\nabla^i(\tilde g - g)| = |(\nabla^i\varphi \cdot g)| \le c_i$ for all i, i.e. $\tilde g \in {}^{b,\infty}\mathrm{comp}(g)$. The assertion now immediately follows.

b) By assumption $B_{12} \subset U_{R_{12}}(M_i) = \bigcup_{x\in M_i} U_{R_{12}}(x)$, $i = 1,2$, $B_{23} \subset U_{R_{23}}(M_i) = \bigcup_{x\in M_i} U_{R_{23}}(x)$, $i = 2,3$, hence $B_{13} = B_{12} \cup B_{23} \subset \bigcup_{x\in M_i} U_{R_{12}+R_{23}}(x) = U_{R_{13}}(M_i)$, $i = 1,3$. g and g' are quasi isometric, hence $B_{13} \subset U_{\tilde R_{13}}(M_i)$, $i = 1,3$. $(B_{13}, \tilde g_{13})$ satisfies (GH) since for $x_\gamma, y_\gamma \in |\gamma| \setminus C$ in M_i no distance realizing or approximating curve in B_{13} will leave $U_{\frac{1}{2}\delta}(M_i)$, $i = 1$ or 3. This follows from indroducing normal charts and applying (4.18). But in the collar $U_{\frac{1}{2}\delta}(M_i)$, (GH) holds. According to the Pythagoraen principle the curve remains in M_i, it is even γ. This finishes the proof of proposition 4.25. ■

$\Omega_n^{nc}(ne) \equiv \Omega^{nc}(f_e, ne, b_g)$ is again defined as Grothendieck group. Next we develop geometric realizations for 0 and $-[M,g]_{ne}$ in $\Omega_n^{nc}(ne)$.

Let (M^n, g) be as above, i.e. oriented, with (I), (B_∞), finitely many ends $\varepsilon_1, \ldots, \varepsilon_s$, each of them nonexpanding. Let ε be one of them, $C \subset M$ compact and so large that ε is defined by one of the components of $M \setminus C$, $U_\varepsilon \subset M \setminus C$ a neighborhood, γ a ray in $U(\varepsilon)$. γ admits a tubular neighborhood of radius $\delta_3 > 0$. Consider $(B, g_B) = (M \times I, g_M + dr^2)$. Then $\varepsilon \times I = \{U_j(\varepsilon) \times I\}_{j\in J}$ is an end of $M \times I$, $U(\varepsilon \times I) = U(\varepsilon) \times I$ a neighborhood disjoint from $C_{M\times I} = C \times I$, and for $0 < \delta_1 < 1$, the curve $\gamma_{\delta_1} = \gamma \times \{\delta_1\} = (\gamma, \delta_1)$ is a ray in $U(\varepsilon \times I)$. $\varepsilon \times I$ is nonexpanding. γ_{δ_1} admits a tubular neighborhood with a radius $\delta_2 > 0$, $T_{\delta_2}(\gamma_{\delta_1})$.

THEOREM 4.28. *$\partial T_{\delta_2}(\gamma_{\delta_1})$ has bounded geometry, one nonexpanding end and*

$$\partial T_{\delta_2}(\gamma_{\delta_1}) \underset{ne}{\sim} \mathrm{chc}^n(\delta_2), \quad \delta_2 > 0.$$

Proof. First we show that $\partial T = \partial T_{\delta_2}(\gamma_{\delta_1})$ has bounded geometry. Consider the point $x = \exp_{\gamma_{\delta_1}(t)}(\delta_1 \cdot u)$, $\delta_1 u \perp \dot\gamma_{\delta_1}(t)$, $x \in \partial T$, and the equation $B_U'' + R^{M\times I} B_U = 0$ along $\sigma(\tau) = \exp_{\gamma_{\delta_1}(t)} \tau(\delta_1 u)$ for the endomorphism valued function $B_U : \tau \to B_U(\tau)$, $B_U(\tau) : (\dot\gamma)^\perp \to (\dot\gamma)^\perp$, with the initial conditions $B_U(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B_U'(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$. Then $|\nabla^i B_U|(\delta_1) \le C_i$, C_i independent of u and t since the curvature tensor $R^{M\times I}$ satisfies (B_∞). According to [24], p. 57, the second fundamental form $S_{\gamma_{\delta_1}}(x)$ at $x = \exp_{\gamma_{\delta_1}(t)}(\delta_1 u)$ is given by

$$S_{\delta_1}(x) = (B_U' B_U^{-1})(\delta_1).$$

Hence S_{δ_1} satisfies (B_∞) since B does. According to Gauss' equations, $g_{\partial T} = g_{M\times I}|_{\partial T}$ satisfies (B_∞). Similarly it follows that ∂T satisfies (I) since γ, γ_{δ_1} and the fibres have bounded from below diameter. We omit the trivial considerations at the bottom. $(\partial T, g_{\partial T})$ has one end and this end is nonexpanding: $\partial T = \bigcup_t \Sigma_t$, where Σ_t is the geodesic δ_2-sphere in $M \times I$ about $\gamma_{\delta_1}(t)$. Each such sphere intersects $\gamma_{\delta_1+\delta_2} = (\gamma, \delta_1 + \delta_2)$ once and there is a common constant K_Σ which uniformly bounds the circumference of all such geodesic spheres. The latter comes from Rauch's comparison theorem. Hence $\partial T \subseteq U_{\frac{K_\Sigma}{2}}(|\gamma_{\delta_1+\delta_2}|)$. Next we construct a bibounded diffeomorphism from $\mathrm{chc}^n(\delta_2)$ onto ∂T.

Consider in $\mathbb{R}^{n+1}$ the standard basis $e_1, \dots, e_{n+1}$, the geodesic $t \cdot e_{n+1}$, the parallel translation of $e_1, \dots, e_n$ along $t \cdot e_{n+1}$, and the map $(x_1, \dots, x_n), x_1^2 + \cdots + x_n^2 = \delta_2 \to x = \exp_{te_{n+1}}(x_1 e_1 + \cdots + x_n e_n)$. Do the same in $M \times I$: Let $E_1, \dots, E_{n+1}$ be orthonormal at the beginning of γ_{δ_1}, $E_{n+1} = \dot\gamma_{\delta_1}$, $E_1 = \partial_t = \frac{\partial}{\partial t}$, translate this parallel along γ_{δ_1}, and consider the map $(y_1, \dots, y_n), y_1^2 + \cdots + y_n^2 = \delta_2 \to y = \exp_{\gamma_{\delta_1}(t)}(y_1 E_1 + \cdots + y_n E_n)$. Then we get a diffeomorphism $\Phi = \exp_{\gamma_{\delta_1}(t)} \circ \Psi \circ \exp_{te_{n+1}}^{-1}$ from $\mathrm{chc}^n(\delta_2)$ onto ∂T, where $\Psi(x_1 e_1 + \cdots + x_n e_n) = x_1 E_1 + \cdots + x_n E_n$. We omit the very simple considerations at the bottoms, respectively. According to [7], the exponential maps are C^∞-bibounded, Ψ too, hence Φ, and $\Phi^* g_{\partial T}$ is a second metric of bounded geometry on $\mathrm{chc}^n(\delta_2)$. $(\partial T, g_{\partial T})$ and $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ are isometric hence ne- (and bg-) bordant. Finally we want to show that $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ and $(\mathrm{chc}^n(\delta_2), g_{standard})$ are ne-bordant. We perform this in two steps. First we show that they are bg-bordant and thereafter we verify after conformal change the conditions (GH_1). $\Phi^* g_{\partial T}$ and g_{st} live in the same ${}^{b,\infty}$component: For $X \in T\,\mathrm{chc}^n(\delta_2)$ we have

$$|X|_{\Phi^* g_{\partial T}} = |\Phi_* X|_{g\partial T} \le |\Phi_*||X|_{g_{st}},$$

$$|X|_{g_{st}} = |\Phi_*^{-1} \Phi_* X|_{g_{st}} \le |\Phi_*^{-1}||\Phi_* X|_{g\partial T} = |\Phi_*^{-1}||X|_{\Phi^* g_{\partial T}},$$

i.e. $\Phi^* g_{\partial T}$ and g_{st} are quasi isometric. There remains to show that

$$|(\nabla^{g_{st}})^i(\Phi^* g_{\partial T} - g_{st}| = |(\nabla^{g_{st}})^i(\Phi^* g_{\partial T})| \tag{4.19}$$

remains bounded for all i. Here we use that the (Fermi) coordinates of $x \in D^n_{\delta_2} \times [0, \infty[$ and of $\Phi(x) \in T_{\delta_2}(\gamma_{\delta_1})$ are the same, where we extended Φ to the full tubes. But then (4.19) follows.

According to theorem 4.28, $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ and $(\mathrm{chc}^n(\delta_2), g_{st})$ are bg-bordant. Let (B^{n+1}, g_B) be a bg-bordism as in the proof of 4.28, i.e. $(B^{n+1}, g_B) = (\mathrm{chc}^n(\delta_2) \times [0,1], \varphi(t) + dt^2)$, $\varphi(t) = \Phi^* g_{\partial T}$, $0 \le t \le \delta$, $\varphi(t) = g_{st}$, $1 - \delta \le t \le 1$, and $\varphi(t) =$ smooth convex combination of $\Phi^* g_{\partial T}$ and g_{st} for $\delta \le t \le 1 - \delta$ (smoothed out at δ, $1 - \delta$). There remains to establish (GH) or (GH_1). We prove (GH) and change for this the metric in $B^{n+1} = \mathrm{chc}^n(\delta_2) \times [0,1]$. Choose smooth functions $\psi_1, \psi_2 : [0,1] \to [0,1]$, $\psi_1 = 1$, $0 \le t \le 1 - \frac{3}{4}\delta$, decreasing on $\left[1 - \frac{3}{4}\delta, 1 - \frac{1}{2}\delta\right]$, $= 0$ on $\left[1 - \frac{1}{2}\delta, 1\right]$, $\psi_2 = 0$, $0 \le t \le \frac{1}{2}\delta$, increasing on $\left[\frac{1}{2}\delta, \frac{3}{4}\delta\right]$, $= 1$ on $\left[\frac{3}{4}\delta, 1\right]$, and set our new $g_B = \psi_1 \Phi^* g_{\partial T} + \psi_2 g_{st} + dt^2$. Then again (B, g_B) is a bg-bordism between $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ and (chc^n, g_{st}). We recall that if γ is a ray in (M^n, g_M) then $\gamma_\lambda = (\gamma, \lambda)$ is a ray in $(M \times [0,1], g_M + dt^2)$, in particular $\gamma_{\delta_1 + \delta_2} \subset \partial T_{\delta_2}(\gamma_{\delta_1})$ is a ray in $(M \times [0,1], g_M + dt^2)$ and hence in $\partial T_{\delta_2}(\gamma_{\delta_1})$. This can easily be proven by distinct methods. Hence $\Phi^{-1}\gamma_{\delta_1+\delta_2}(\delta_2, 0, \dots, 0, \tau)$ is a ray in $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$. Let $x = (\delta_2, 0, \dots, 0, \tau_1)$, $y = (\delta_2, 0, \dots, 0, \tau_2)$ be $\in (\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ and $\{c(s)\}_{0 \le s \le \sigma}$ be a curve in (B, g_B) connecting x and y. The tangent vector $\dot{c}(s)$ of $\dot{c}(s) = (x_\Sigma(s), t(s))$ decomposes as orthogonal sum $\dot{c}(s) = \dot{x}_\Sigma(s), \dot{t}(s) = \dot{t}(s)^\perp + \dot{t}(s)$ and we obtain

$$\begin{aligned} \mathrm{length}(c(s)) &= \int_0^{\sigma_2} |\dot{t}(s)^\perp|_{g_B} ds + \int_0^{\sigma_2} |\dot{t}(s)|_{dt^2} ds \\ &\ge \int_0^{\sigma_2} |\dot{t}(s)^\perp|_{g_B} ds = \int_0^{\sigma_2} |\dot{t}(s)^\perp|_{\Psi_1 \Phi^* g_{\partial T} + \Psi_2 g_{st}} ds. \end{aligned} \tag{4.20}$$

We consider several cases. The first case is that $c(s)$ remains in $\mathrm{chc}^n(\delta_2) \times \left[0, 1 - \frac{3}{4}\delta\right]$ which implies

$$|\dot{t}(s)^\perp|_{\psi_1 \Phi^* g_{\partial T} + \psi_2 g_{st}} = |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}}. \tag{4.21}$$

But $x_\Sigma(s)$, $\dot{x}_\Sigma(s)$, is a curve in $(\mathrm{chc}^n(\delta_2), \Phi^* g_{\partial T})$ and hence

$$\mathrm{length}(x_\Sigma(s)) = \int_0^{\sigma_2} |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}} ds \ge |\tau_1 - \tau_2| \tag{4.22}$$

according to the ray property above. Altogether, in this case

$$\mathrm{length}(c(s)) \ge |\tau_1 - \tau_2| = d_{(\mathrm{chc}^n(\delta_2) \cdot \Phi^* g_{\partial T})}(x, y). \tag{4.23}$$

The second case under consideration is the case that at $s = s_1$, $c(s)$ leaves $\mathrm{chc}^n(\delta_2) \times \left[0, 1 - \frac{3}{4}\delta\right]$, returns to it at $s = s_2$ and then remains in $\mathrm{chc}^n(\delta_2) \times \left[0, 1 - \frac{3}{4}\delta\right]$. In this case

$$\begin{aligned} \int_0^{\sigma_2} |\dot{t}(s)^\perp|_{g_B} ds &= \int_0^{s_1} |\dot{t}(s)^\perp|_{g_B} ds + \int_{s_1}^{s_2} |\dot{t}(s)^\perp|_{g_B} ds + \int_{s_2}^{\sigma_2} |\dot{t}(s)^\perp|_{g_B} ds \\ &\ge \int_0^{s_1} |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}} ds + \int_{s_2}^{\sigma_2} |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}} ds + \int_{s_1}^{s_2} |\dot{t}(s)^\perp|_{g_{st}} ds. \end{aligned} \tag{4.24}$$

Let $x_\Sigma(s_1) = (\delta_2, 0, \ldots, 0, \tau_{11})$, $x_\Sigma(s_1) = (\delta_2, 0, \ldots, 0, \tau_{22})$. Then

$$\int_0^{s_1} |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}} ds \geq |\tau_1 - \tau_{11}|, \tag{4.25}$$

$$\int_{s_2}^{\sigma_2} |\dot{t}(s)^\perp|_{\Phi^* g_{\partial T}} ds \geq |\tau_2 - \tau_{22}|, \tag{4.26}$$

$$\int_{s_1}^{s_2} |\dot{t}(s)^\perp|_{g_{st}} ds \geq |\tau_{11} - \tau_{22}|, \tag{4.27}$$

which immediately implies

$$\text{length}(c(s)) \geq \int_0^{\delta_2} |\dot{t}(s)^\perp|_{g_B} ds \geq |\tau_1 - \tau_2|. \tag{4.28}$$

The most general case would be that $c(s)$ oscillates between $(\text{chc}^n(\delta_2), \left[0, 1 - \frac{3}{4}\delta\right])$ and its complement. In this case we have many times inequalities of type (4.25) – (4.27), sum up, apply the triangle inequality and end again at (4.28). Altogether for $x, y \in \Phi^{-1}\gamma_{\delta_1+\delta_2}$,

$$d_{(B,g_B)}(x,y) \geq d_{(\text{chc}^n(\delta_2), \Phi^* g_{\partial T})}(x,y). \tag{4.29}$$

If $x, y \in \gamma_{\delta_2} \subset (\text{chc}^n(\delta_2), g_{st})$ then we replace in (4.20)–(4.28) $\Phi^* g_{\partial T}$ by g_{st} and calculate as above,

$$d_{(B,g_B)}(x,y) \geq d_{(\text{chc}^n(\delta_2), g_{st})}(x,y). \tag{4.30}$$

(GH) is established. ∎

We shall see $(\text{chc}^n(\delta), g_{st})$ will play the role of our zero in $\Omega_n^{nc}(ne)$.

Lemma 4.29.

$$\textbf{a)}\ \ \textit{For } r_1 < r_2 \textit{ is } \text{chc}^n(r_1) \underset{ne}{\sim} \text{chc}^n(r_2). \tag{4.31}$$

$$\textbf{b)}\ \ [\mathop{\#}_{i=1}^{k} \text{chc}^n(r_i)]_{ne} = [\text{chc}^n(r)]_{ne} \textit{ for } r > r_1 + \cdots + r_k. \tag{4.32}$$

Proof. a) is immediately clear (or follows from b)). Set for b) $r = r_1 + \cdots + r_k + \delta$, place $\text{chc}^n(r_1) \cup \cdots \cup \text{chc}^n(r_k)$ all with parallel $[0, \infty[$ direction into $\text{int}(\text{chc}^n(r))$, where $\text{int}(\text{chc}^n(r))$ corresponds to $\overset{\circ}{D}{}^n_r \times]0, \infty[$. Then $\text{CL}(\text{int}(\text{chc}^n(r)) \setminus \text{int}(\text{chc}^n(r_1) \cup \cdots \cup \text{chc}^n(r_k)))$ defines the desired ne-bordism. ∎

Theorem 4.30. *For any oriented manifold (M^n, g) of bounded geometry and a finite number of ends, each of them nonexpanding, we have*

$$[M^n, g]_{ne} = [(M^n, g) \cup (\text{chc}^n(r), g_{st})]. \tag{4.33}$$

Proof. We must construct a ne-bordism between (M^n, g) and $-((M^n, g) \cup (\text{chc}^n(r), g_{st}))$. Let $(B^{n+1}, g_B) = (M \times [0,1], g_M + dt^2)$, ε be an end of M, γ a ray in ε, form $\gamma_{\delta_1} = (\gamma, \delta_1) \subset M \times [0,1]$, $T_{\delta_2}(\gamma_{\delta_1})$, $\delta_2 < \inf\left\{\frac{\delta_1}{2}, r_{\text{inj}}(M)/2\right\}$ and set $B_\gamma = B^{n+1} \setminus \text{int}T_{\delta_2}(\gamma_{\delta_1})$ with the induced metric. From our assumption $r_{\text{inj}} > 0$ it follows easily that $\partial T_{\delta_2}(\gamma_{\delta_1})$ has a smooth collar $U_\delta(\partial T)$. Endow $U_{\frac{\delta}{2}}$ with the product metric $g_{\frac{\delta}{2}}$ and form on $U_\delta - U_{\frac{\delta}{2}}$ the smooth bg-convex combination of $g_{\frac{\delta}{2}}$ and g_B getting g_{B_γ}. Endow $\partial T_{\gamma_2}(\gamma_{\delta_1})$ with the induced orientation. Then (B_γ, g_{B_γ}) is a bg, ne-bordism between (M^n, g) and $(M^n, g) \cup$

$(\partial T_{\delta_2}(\gamma_{\delta_1}), g_{\partial T})$. Theorem 4.28 yields

$$(M^n, g) \cup (\partial T, g_{\partial T}) \underset{ne}{\sim} (M^n, g) \cup (\mathrm{chc}^n(\delta_2), g_{st}). \ \blacksquare$$

THEOREM 4.31. *$\Omega_n^{nc}(ne) \equiv \Omega_n^{nc}(bg, ne)$ is an abelian group with $-[M^n, g] = [(-M^n, g)]$ and $0 = [\mathrm{chc}^n(r), g_{st}]$.* $\blacksquare$

Our next goal is to produce independent generators of $\Omega_n^{nc}(ne)$. As we shall see in the sequel, infinite connected sums of complex projective spaces (or their cartesian products) supply such elements. We prepare this by several assertions.

LEMMA 4.32. *Let (M_i^n, g_i), $i = 1, 2$, be open, oriented of bounded geometry and with a finite number of ends, each of them nonexpanding. Let further (B^{n+1}, g_B) a ne-bordism between them and $K \subset B$ compact such that the ends of B coincide with the components of $B \setminus K$. Let $C_\varepsilon \subset B \setminus K$ a component of $B \setminus K$ and $x_0 \in C_\varepsilon$. Then there exists a constant $C_1 > 0$ such that the diameter of any metric sphere*

$$S_\varrho(x_0) = \{x \in C_\varepsilon | d_B(x, x_0) = \varrho\}$$

is $\leq C_1$. Here the diameter is with respect to the induced length metric d_B of B.

Proof. We start with the case that $M_i \setminus K$ contains exactly one end of M_i, respectively. Then there are geodesic rays $\gamma_i \subset M_i \setminus K \subset C_\varepsilon \subset B \setminus K$. For sufficiently large ϱ, $S_\varrho(x_0)$ intersects γ_i in some points of a geodesic segment $I_{\varrho, c_1, i} \subset |\gamma_i|$ of length at most $c_{1,i}$, where

$$d_{\gamma_i}(x, y) - c_{1,i} \leq d_B(x, y)$$

for all $x, y \in |\gamma_i|$. Let $x_i = \sup I_{\varrho, c_1, i}$ ($|\gamma_i|$ is totally ordered), $i = 1, 2$. We would be done if we could show that $d_B(x, x_1)$ were uniformly bounded for all $x \in S_\varrho(x_0)$ and for all $\varrho > 0$. Since $C_\varepsilon \subseteq U_R(|\gamma_i|)$ for some $R > 0$, there exists for each $x \in S_\varrho(x_0)$ an $x_{i,\gamma_i} = x_{i,\gamma_i}(x) \in |\gamma_i|$ such that $x_{i,\gamma_i}(x) \in |\gamma_i|$ (which is equivalent to $x \in U_R(x_{i,\gamma_i}(x))$). If $d_B(x_i, x_{i,\gamma_i}(x))$ were uniformly bounded then we would be done:

$$d(x, x_i) \leq d(x_i, x_{i,\gamma_i}(x)) + d(x_{i,\gamma_i}(x), x) \leq d(x_i, x_{i,\gamma_i}(x)) + R.$$

Suppose $x_0 = \gamma_1(0)$. If $x_{1,\gamma_1}(x) < x_1$ then we obtain

$$d_B(x_0, x) \leq d_B(x_{1,\gamma_1}(x), x_0) + d_B(x_{1,\gamma_1}(x), x) \leq d_{\gamma_1}(x_{1,\gamma_1}(x), x_0) + R. \tag{4.34}$$

$$\varrho \leq -d_{\gamma_1}(x_{1,\gamma_1}(x), x_1) + \varrho + R, \tag{4.35}$$

$$d_{\gamma_1}(x_{1,\gamma_1}(x), x_1) \leq R, \quad d_B(x_{1,\gamma_1}(x), x_1) \leq R. \tag{4.36}$$

In the case $x_{1,\gamma_1(x)} > x_1$ we obtain

$$\varrho + d_{\gamma_1}(x_1, x_{1,\gamma_1}(x)) - c_{1,1} \leq d_B(x_0, x) + d_B(x, x_{1,\gamma_1}(x)) \leq \varrho + R, \tag{4.37}$$

$$d_B(x_1, x_{1,\gamma_1}(x)) \leq d_{\gamma_1}(x_1, x_{1,\gamma_1}(x)) \leq R + c_{11}. \tag{4.38}$$

If $x_0 \notin |\gamma_1|$ then we have to add to the right hand sides in (4.36), (4.38) a constant. The same estimates hold if we replace in (4.34)–(4.38) $1 \to 2$. Assume now that $M_i \setminus K$ splits into a finite number of ends $e_{i,1}, \ldots, e_{i,r_i}$, $i = 1, 2$. Then

$$C_\varepsilon = B \setminus K \subset \bigcup_1^{r_1} U_R(|\gamma_{1,i}|), \qquad C_\varepsilon = B \setminus K \subset \bigcup_1^{r_2} U_R(|\gamma_{2,j}|).$$

$S_\varrho(x_0)$ has uniformly bounded diameter if $S_\varrho(x_0) \cap U_R(|\gamma_{i,j}|)$ has uniformly bounded diameter. But the latter case we just solved above. ■

Now we recall the chopping theorem of Cheeger/Gromov (theorem 3.33 in section 3) which is a consequence of Abresch's habilitation.

THEOREM 4.33. *Suppose (M^n, g) open, complete with bounded sectional curvature $|K| \leq C$. Given a closed set $X \subset M^n$ and $0 < r \leq 1$, there is a submanifold, U^n, with smooth boundary, ∂U^n, such that for some constant $c(n, C)$*

$$\begin{aligned} &X \subset U \subset T_r(X), \\ &\mathrm{vol}(\partial U) \leq c(n, C)\,\mathrm{vol}(T_r(X) \setminus X) r^{-1}, \\ &|II(\partial U)| \leq c(n, C) r^{-1}. \end{aligned}$$

Moreover, U can be chosen to be invariant under $\mathcal{I}(r, X)$ = group of isometries of $T_r(X)$ which fix X. ■

In our case, $X = X_\varrho = \overline{B_\varrho(x_0)} \subset B^{n+1}$. To apply 4.33, we form $(V^{n+1}, g_V) = (B^{n+1} \cup B^{n+1}, g_B \cup g_B)$ which is well defined and smooth since we assumed the Riemannian collar $g_B|_{\text{collar}} = g_{\partial B} + dt^2$. Now we set $X_V = X \cup X$ and apply 4.33. Fix $0 < r \leq 1$. Then we get U_V, $H_{\varrho,V,r} = \partial U_V$.

$$X_V \subset U_V \subset T_r(X_V) (= \{x \in V | d_V(x, X_V) \leq r\}), \tag{4.39}$$

$$\mathrm{vol}(H_{\varrho,V,r}) = \mathrm{vol}(\partial U_V) \leq c(n+1, C)\,\mathrm{vol}(T_r(X_V) \setminus X_V) r^{-1} \tag{4.40}$$

$$|II(\partial U_V)| \leq c(n+1, C) r^{-1} \tag{4.41}$$

and U_V is invariant under $\mathcal{I}(r, X_V)$.

The main idea of the proof consists in considering the distance function $F = d(\cdot, X_V)$ where for points $\in V \setminus X_V$, $d(\cdot, X_V) = d(\cdot, X_\varrho) = d(\cdot, S_\varrho)$. Then one applies Yomdin's theorem (cf. [Yomdin]) to F in Abresch's smoothed out metric. All constructions are invariant under the metric involution and this involution remains an isometry also with respect to Abresch's smoothed out metric.

Restricting the obtained U_V, ∂U_V to B, we obtain the desired result for $X = \overline{B_\varrho(x_0)} \subset B$. Restricting for ϱ large to C_ε and using the construction of U as preimage under the smoothed F, we obtain in $\overline{C}_\varepsilon$ a hypersurface $H = H_\varrho$ which decomposes $\overline{C}_\varepsilon$ into a compact and noncompact part $\overline{C}_{\varepsilon,c}$ and $\overline{C}_{\varepsilon,nc}$, respectively. Under our assumptions (∂B is totally geodesic) it is possible to arrange that H^n intersects ∂B transversally under an angle $> \delta$ and that there exists a constant C_1 independent of ϱ such that

$$|II(\partial H^n_\varrho)| \leq C_1. \tag{4.42}$$

We infer from (4.40), bounded curvature and lemma 4.32 that for fixed $0 < r \leq 1$ there is a constant $C_2 > 0$ such that

$$\mathrm{vol}(H^n_\varrho) \leq C_2 \tag{4.43}$$

for all ϱ. Moreover, H^n_ϱ has bounded geometry (at least of order 0) according to (4.41) and to the bounded geometry of B.

Now we are able to present independent generators of $\Omega^{nc}_{4k}(ne)$. Let $P^{2k}(\mathbb{C})$ be the complex projective space with its standard orientation and with its Fubini–Study metric, fix two points z_1, z_2 and form by means of fixed spheres about z_1, z_2 the infinite connected sum

$$M^{4k} = (M^{4k}, g) = \overset{\infty}{\underset{1}{\#}} P^{2k}(\mathbb{C}), \tag{4.44}$$

always with the same glueing metric. Then (M^{4k}, g) is oriented, has bounded geometry, one end which is nonexpanding.

THEOREM 4.34. *$M^{4k} = \#_1^\infty P^{2k}(\mathbb{C})$ defines a non zero bordism class in $\Omega_{4k}^{nc}(ne)$.*

Proof. Suppose $[M^{4k}] = 0$. Then there exists a bordism (B^{n+1}, g_B), $\partial B = M^{4k} \cup -\operatorname{chc}^{4k}(r)$, $g_B|_{U_\delta(\partial B)} = g_{\partial B} + dt^2$, $U_R(M^{4k}) \supseteq B$, $U_R(\operatorname{chc}^{4k}(r)) \supseteq B$ and $d_B \geq d_M - c$, $d_B \geq d_{\mathrm{chc}} - c$. We choose $z_0 \in P_1^{2k}(\mathbb{C})$, $K = \emptyset$ and obtain for any $\varrho > 0$ a compact hypersurface $H_\varrho^{4k} \subset B = B \setminus \Phi = C_\varepsilon$ which decomposes B into a compact and noncompact part $\overline{B}_c$ and $\overline{B}_{nc}$, respectively, and which satisfies (4.42), (4.43) and has bounded geometry at least of order 0 with constants independent of ϱ. Then $\partial\overline{B_c^{4k+1}} = (\partial\overline{B_c^{4k+1}} \cap M^{4k}) \cup H_\varrho \cup (\partial\overline{B_c^{4k+1}} \cap \operatorname{chc}^{4k})$. Here $\sigma(\partial\overline{B_c^{4k+1}} \cap \operatorname{chc}^{4k}) = 0$. $\sigma(\partial\overline{B_c^{4k+1}})$ must be zero since it is 0–bordant (if one wants, after smoothing out). Hence

$$0 = \sigma(\partial\overline{B_c^{4k+1}} \cap M^{4k}) - \sigma(H_\varrho^{4k}). \tag{4.45}$$

But

$$\sigma(H_\varrho^{4k}) = \int_{H_\varrho^{4k}} L + \eta(\partial H_\varrho^{4k}) + \int_{\partial H_\varrho^{4k}} \operatorname{expression}(II(\partial H_\varrho^{4k})). \tag{4.46}$$

The first expression on the r.h.s. of (4.46) is bounded by a bound independent of ϱ according to (4.43) and (B_0) for H_ϱ^{4k}. The same holds for the second expression according to

$$|\eta(\partial H_\varrho^{4k})| \leq C_3 \operatorname{vol}(\partial H_\varrho^{4k})$$

and for the third expression according to (4.42), (4.43). On the other hand, choosing ϱ sufficiently large, $\sigma(\partial\overline{B^{4k+1}} \cap M^{4k})$ can be made arbitrary large. This contradicts (4.45). ∎

Looking at the proof of theorem 4.34, we immediately infer

THEOREM 4.35. *Let (M^{4k}, g) be open, oriented, of bounded geometry and with a finite number of ends, each of them nonexpanding. If for any exhaustion $M_1 \subset M_2 \subset \cdots$ by compact submanifolds, $\bigcup M_i = M$, we have*

$$\lim_{i \to \infty} \sigma(M_i^{4k}) = \infty$$

then $[M^{4k}, g] \neq 0$ in $\Omega_{4k}^{nc}(ne)$. ∎

COROLLARY 4.36. *$\#_1^\infty P^{2k}(\mathbb{C})$, or, more generally, $P^{2i_1}(\mathbb{C}) \times \cdots \times P^{2i_{r_1}} \# P^{2j_1}(\mathbb{C}) \times \cdots \times P^{2j_{r_2}} \# \cdots$, $i_1 + \cdots i_{r_1} = k$, $j_1 + \cdots + j_{r_2} = k, \ldots$ are not torsion elements in $\Omega_{4k}^{nc}(ne)$.* ∎

A special case of theorem 4.35 is given by manifolds M^{4k} of the type

$$M^{4k} = \overset{\infty}{\underset{1}{\#}} M_i^{4k},$$

$\operatorname{vol}(M_i^{4k}) \leq C_1$, $|K(g_i)| \leq C_2$, $r_{\mathrm{inj}}(g_i) \geq C_3 > 0$, $\sigma(M_i^{4k}) \geq 0$ for $i \geq i_0$ and > 0 for infinitely many $i \geq i_0$. Then, in particular, $\mathcal{H}_{2k,2}(M^{4k})$ is infinite-dimensional and $[M^{4k}, g] \neq 0$ in $\Omega_{4k}^{nc}(ne)$, i.e. adding a finite number of closed manifolds with negative signature and an infinite number of closed manifolds with zero signature (such that the bg, ne-end struture remains preserved) does not transform a nonzero element into zero in $\Omega_{4k}^{nc}(ne)$. A finer characterization of nonzero elements in $\Omega_{4k}^{nc}(ne)$ will be presented elsewhere. Moreover there are very interesting specializations of the theory developed until now and generalizations, e.g. the restriction to manifolds with warped product structure at infinity or with prescribed volume growth of the ends etc. This will be the topic of another investigation.

5. The Novikov conjecture for open manifolds. As is very well known, the Novikov conjecture for closed manifolds stimulated many outstanding topologists to prove this and on this road deep results in C^* algebraic topology, C^* K-theory and geometric group theory have been achieved. Hence, the Novikov conjecture has not only its own meaning but even more meaning as a stimulating question.

If M^n is open and we consider the classifying diagram

$$\begin{array}{c}\tilde{M}\\ \downarrow\\ M \underset{f}{\rightarrow} B_\pi\end{array}$$

and $a \in H^*(B_\pi)$ then

$$\langle L(M) \cdot f^*a, [M]\rangle$$

will not be defined in general. For this reason, Gromov proposes to consider

$$\sigma_a(M) = \langle L(M) \cdot f^*a, [M]\rangle$$

for $a \in H_c^*(B_\pi)$. Then the NC for open manifolds would mean the "invariance of $\sigma_a(M)$ under proper homotopy equivalences". Probably much more appropriate would be an approach in the sense of our "open category", i.e.

1) everything is uniformly metrized, we have (I), (B_k), uniform triangulations etc.,
2) maps are bounded and uniformly proper, in particular this holds for homotopy equivalences,
3) one works within functional algebraic topology.

Hence one should consider

$$\langle L(M) \cdot f^*a, [M]\rangle \quad \text{with} \quad L(M) \in L_p, f^*a \in L_q.$$

Of particular meaning would be the cases

$$L(M) \in {}^bH^*(M) \quad \text{and} \quad a \in H^{*,1}(B_\pi) \tag{5.1}$$

or

$$L(M) \in H^{*,2}(M) \quad \text{and} \quad a \in H^{*,2}(B_\pi), \tag{5.2}$$

respectively. If we suppose (M, g) satisfying (B_0) then automatically $L(M) \in {}^bH^*(M)$. (B_0) does not restrict to topological type since any open manifold admits a metric g satisfying even (B_∞) and (I).

In the second case one should additionally assume

$$\inf \sigma_e(\Delta_*(M, g)|_{(\ker \Delta_*)^\perp}) > 0, \tag{5.3}$$

i.e. there is a spectral gap of Δ_* above zero. In this case $H^{*,2} = \mathcal{H}^{*,2} = L_2$-harmonic forms, $C^{*,2}$, $C_{*,2}$ are L_2-complexes and form an L_2-Poincaré complex. Every L_2-(co-)homology class can be represented by an L_2-harmonic (co-)cycle. Bordism of L_2-Poincaré complexes can be defined easily.

We proved in [8] that (5.3) is invariant under bounded uniformly proper homotopy equivalences. W.l.o.g., classifying maps can be assumed to be bounded and uniformly proper,

$$M^n \to B_\pi = M^n \cup \text{cells}.$$

We present now 3 versions of NC (for open manifolds).

First version. In the class of open oriented manifolds (M^n, g), $g \in {}^{b,2}\mathcal{M}^{2,2}(B_0, 2, f)$ with $\inf \sigma_e(\Delta_*(g)|_{(\ker \Delta_*)^\perp}) > 0$ is

$$\begin{aligned}&\langle L(M) f^* a, [M]\rangle, \quad a \in H^{*,2}(B_\pi), \quad f \text{ bounded and}\\ &\text{uniformly proper classifying map, invariant under}\\ &\text{bounded and uniformly proper homotopy equivalences.}\end{aligned} \tag{NCO1}$$

Criticism. This version should hold only in very restricted cases. The starting point in the compact case is the equality

$$\sigma(M^{4k}) = \int L_k(M) \tag{5.4}$$

where the l.h.s. is a priori a homotopy invariant and the r.h.s. is a certain characteristic number. The L_2-version of (5.4) is already wrong in simple open cases. Let (M^{4k}, g) be an open manifold with cylindrical ends, i.e. $(M^{4k}, g) = ({M'}^{4k} \cup \partial {M'}^{4k} \times [0, \infty[, g)$ with $g|_{\partial {M'}^{4k} \times [0,\infty[} \cong g|_{\partial M'} + dt^2$. Then it is well known that

$$\sigma(M^{4k}) = \sigma_{L_2}(M^{4k}) = \int L_k(M) - \eta(\partial {M'}^{4k}),$$

i.e. already the starting point which guarantees the invariance of $L(M)$ in the simplest case is wrong. Hence the first version of NC for open manifolds makes sense only for those classes of manifolds for which

$$\sigma_{L_2}(M^n) = \int L(M)$$

in the case $n = 4k$.

Second version of NC, relative version. Fix (M^n, g) and suppose that $M_1, M_2 \in \text{gen}\ {}^b\text{comp}_{L,iso,rel}(M, g)$,

$$M_1 \setminus K_1 \cong M \setminus K, \qquad M_2 \setminus K_2 \cong M \setminus K$$

with a Riemannian collar at ∂K_1, ∂K_2, ∂K. Then we define

$$\begin{aligned}\sigma(M_i, M) &:= \int_{K_i} L(M_i) - \int_K L(M),\\ \sigma(M_1, M_2) &:= \sigma(M_1, M) - \sigma(M_2, M) = \int_{K_1} L(M_1) - \int_{K_2} L(M_2) = \sigma(K_1 \cup K_2)\\ &= \int_{K_1} L(M_1) - \eta(\partial K_1) - \left(\int_{K_2} L(M_2) - \eta(\partial K_2)\right) = \sigma(K_1) - \sigma(K_2).\end{aligned}$$

The relative NC becomes

$$\int_{K_1} L(M_1) f_1^* a = \int_{K_2} L(M_2) f_2^* a \tag{NCO2}$$

if there exist $\Phi_{12} : M_1 \to M_2$, $\Phi_{21} : M_2 \to M_1$, bounded, uniformly proper, $\Phi_{21}\Phi_{12} \sim \operatorname{id} M_1$, $\Phi_{12}\Phi_{21} \sim \operatorname{id} M_2$ bounded and u.p. and $\Phi_{21}\Phi_{12} = \operatorname{id}$ outside $\tilde{K}_1 \subset M_1$, $\Phi_{12}\Phi_{21} = \operatorname{id}$ outside $\tilde{K}_2 \subset M_2$ and $f_i : M_i \to B_\pi$ are bounded and u.p. classifying maps, $a \in H^*(B_\pi)$.

This relative version has the advantage that we require no conditions for (M^n, g) and NC splits to NC for the generalized Lipschitz components.

Third version of NC. Consider (M^n, g) open, oriented with (B_0), $r_{\text{inj}} > 0$, embeddings $N^{4k} \hookrightarrow M^n \times \mathbb{R}^j$ with trivial normal bundle and bounded second fundamental form such that $PD[N] = f^* a$, $a \in H^{n-4k,1}(B_\pi)$, $f : M^n \to B_\pi$ bounded and uniformly proper classifying map and such that $\sigma_{L_2}(N^{4k})$ is defined (i.e. $\sim \mathcal{H}^{2k,2}(N) < \infty$).

$$\begin{gathered}\text{Then the number } \sigma_a(M) := \sigma_{L_2}(N^{4k}) \text{ is invariant under} \\ \text{bounded and uniformly proper homotopy invariants.}\end{gathered} \tag{NC03}$$

How to attack these conjectures will be the content of a forthcoming investigation.

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