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GROUPS OF $C^{r,s}$-DIFFEOMORPHISMS RELATED TO A FOLIATION

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Abstract. The notion of a $C^{r,s}$-diffeomorphism related to a foliation is introduced. A perfectness theorem for the group of $C^{r,s}$-diffeomorphisms is proved. A remark on C^{n+1}-diffeomorphisms is given.

1. Introduction. The goal of this note is to show that some automorphism groups of a foliated manifold are perfect. Let us recall that a group G is called perfect if $G = [G, G]$, where the commutator subgroup is generated by all commutators $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$, $g_1, g_2 \in G$. In terms of homology of groups this means that $H_1(G) = G/[G, G] = 0$.

The following fundamental result is well-known. Throughout the subscript c indicates the compactly supported subgroup, and the subscript 0 indicates the identity component.

THEOREM 1.1 (Herman, Thurston, Mather). *Let M be a smooth manifold, and let $n = \dim(M)$. If $r = 1, 2, \dots, \infty$, $r \neq n+1$, then $\mathrm{Diff}^r_c(M)_0$ is perfect. Consequently, this group is simple as well.*

The case $r = \infty$ and $M = T^n$ is due to Herman [3] who applied a difficult small denominator theory argument. Next, Thurston [8] used the result of Herman to obtain that $\mathrm{Diff}^\infty_c(M)_0$ is perfect for an arbitrary manifold M (cf. [1] for the proof). By a completely different method Mather [4] proved the first assertion for $r \neq n + 1$, r finite. Finally, the second assertion follows from Epstein [2].

Given a foliated manifold $(M, \mathcal{F})$ a diffeomorphism $f : M \to M$ is said to be leaf preserving if $f(L_x) = L_x$ for all $x \in M$, where L_x is the leaf of $\mathcal{F}$ passing through x.

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THEOREM 1.2. *Let $(M, \mathcal{F})$ be a foliated smooth manifold with $k = \dim \mathcal{F}$, and let $\mathrm{Diff}^r_c(M, \mathcal{F})$ be the group of leaf preserving diffeomorphisms. Then $\mathrm{Diff}^r_c(M, \mathcal{F})_0$ is perfect, provided $r \leq k$, or $r = \infty$.*

The proof of Theorem 1.2 for $r = \infty$ modifies arguments of Herman and Thurston (cf. Rybicki [6]). In the case $r \leq k$ it is easily checked that the proof of Mather [4], II, applies to leaf preserving diffeomorphisms thanks to 'foliated properties' of Mather's operators $P_{i,A}$.

Observe that the group $\mathrm{Diff}^r_c(M, \mathcal{F})$ is locally contractible (cf. [7]) and, consequently, the identity component of it coincides with the totality of its elements that can be joined to the identity by an isotopy in $\mathrm{Diff}^r_c(M, \mathcal{F})$.

Our aim is to study here the remaining case of $r > k$, r finite. In the final section we shall show how a possible analogue of Theorem 1.2 for $r > k+1$ is related to the simplicity of $\mathrm{Diff}^{n+1}_c(M)_0$. Observe that there exist some strong arguments suggesting that $\mathrm{Diff}^{n+1}_c(M)_0$ is not simple (Mather [4], III, [5]). This suggests, in turn, that the assertion of Theorem 1.2 for $r > k$ might not be true. However, if we consider groups of leaf preserving diffeomorphisms with some 'loss of smoothness' in the transversal direction then we are able to obtain a positive result.

Given a foliated manifold $(M, \mathcal{F})$ with $k = \dim \mathcal{F}$, let $\mathrm{Diff}^{r,s}(M, \mathcal{F})$ denote the group of leaf preserving C^1-diffeomorphisms which are of class C^r in the tangent direction, and of class C^s in the transversal direction, where $1 \leq s \leq r$. (See section 2.)

THEOREM 1.3. *The group $\mathrm{Diff}^{r,s}_c(M)_0$ is perfect, provided $r - s > k + 1$.*

In the whole paper we exploit the techniques from Mather's fundamental paper [4]. We retain the notation of that paper as far as possible and we recall definitions and facts from it. In particular, the construction of the rolling-up operators is adopted to $C^{r,s}$-diffeomorphisms (section 4).

2. Preliminaries. Let $r, s \geq 1$ and $k \geq 1$ be fixed integers. Let $f(x, y) = (f_1(x, y), y)$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$.

DEFINITION 2.1. A partial derivative of order $p \geq 1$ of f is called *s-admissible* if it contains at most s derivatives in the direction of the last $n - k$ coordinates. We say that f is *of class $C^{r,s}$* if it has all the s-admissible partial derivatives up to order r and they are continuous. For f of class $C^{r,s}$ and $1 \leq p \leq r$ we denote by $D^{p,s} f : \mathbb{R}^n \to L^p(\mathbb{R}^n, \mathbb{R}^n)$ the mapping, called the *p-th derivative* of f, consisting of s-admissible partial derivatives of f of order p, and of zeros in place of partial derivatives of f of order p which are not s-admissible. In other words, we may say that f is of class $C^{r,s}$ if $D^{r,s} f$ exists and is continuous. The symbol $C^{r,s}(n, k)$ will stand for the space of all mappings of the form $f(x, y) = (f_1(x, y), y)$, which are of class $C^{r,s}$.

It is clear that $D^{r,s} f = D^r f$ if $r \leq s$. In particular we have then the standard derivative formulas for composed mappings

$$D(f \circ g) = (Df \circ g) \cdot Dg \tag{2.1}$$

and

$$D^{r,s}(f\circ g) = (D^{r,s}f\circ g)\cdot(Dg\times\ldots\times Dg) + (Df\circ g)\cdot D^{r,s}g + \sum C_{i,j_1,\ldots,j_i}(D^{i,s}f\circ g)\cdot(D^{j_1,s}g\times\ldots\times D^{j_i,s}g), \tag{2.2}$$

where the sum is over $1 < i < r$, $1 \le j_l$, $j_1+\ldots+j_i = r$ and $C_{i,j_1,\ldots,j_i}$ are positive integers independent of f and g.

Notice that for $r > s$ the above formula (2.2) is no longer valid. The following fact is simple but clue.

Proposition 2.2. *Let $f, g \in C^{r,s}(n,k)$. If an entry of the matrix $D^{r,s}(f\circ g)$ on the l.h.s. of (2.2) is an s-admissible partial derivative of $f\circ g$ then the corresponding entry on the r.h.s. is expressed by means of s-admissible partial derivatives of orders $\le r$ of f and g.*

Proof. It suffices to make the following observation. Any partial derivative in the direction of x_i, $i = 1,\ldots,k$, of $f\circ g$ cannot produce a partial derivative of f or g in the direction of y_j, $j = 1,\ldots,n-k$. In case of g this is obvious, in case of f this follows from the fact that $g = (g_1, g_2) \in C^{r,s}(n,k)$ satisfies $g_2(x,y) = y$ and, consequently, $\frac{\partial g_2}{\partial x_i} = 0$. ∎

Definition 2.3. A *modulus of continuity* is a continuous, strictly increasing function $\alpha : [0,\infty) \to \mathbb{R}$, such that $\alpha(0) = 0$ and $\alpha(tx) \le t\alpha(x)$ for every $x \in [0,\infty)$ and $t \ge 1$.

Let X, Y be two metric spaces, and let α be a modulus of continuity. We say that $f : X \to Y$ is *α-continuous* if there exist $C > 0$ and $\varepsilon > 0$ such that for every $x_1, x_2 \in X$ and $d_X(x_1,x_2) \le \varepsilon$ we have $d_Y(f(x_1), f(x_2)) \le C\alpha(d_X(x_1,x_2))$. f is called *locally α-continuous* if each point has a neighborhood U such that $f|_U$ is α-continuous. Obviously these concepts depend on equivalence classes of metrics only.

It is clear that every $f : X \to Y$ that is Lipschitz, is α-continuous for all moduli of continuity α. In particular a C^1-mapping $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$, is locally α-continuous for all moduli of continuity α.

The following fact is well-known.

Lemma 2.4. *Let $f : X \to Y$ be a continuous mapping from a compact, convex subset of a normed vector space to a metric space. Then there exists a modulus of continuity α such that f is α-continuous.*

We say that f is of class $C^{r,s,\alpha}$ if it is $C^{r,s}$ and $D^{r,s}f$ is locally α-continuous. Clearly this notion does not depend on the choice of a norm on $L^r(\mathbb{R}^n,\mathbb{R}^n)$. In the sequel the symbol $C^{r,s,[\alpha]}$ will stand for $C^{r,s}$ or $C^{r,s,\alpha}$. We denote by $\mathcal{D}^{r,s,[\alpha]}(n,k)$ the group of leaf preserving diffeomorphisms of class $C^{r,s,[\alpha]}$ on $\mathbb{R}^n$ with compact support which are isotopic to the identity through compactly supported $C^{r,s,[\alpha]}$-isotopies, and by $\mathcal{D}^{r,s,[\alpha]}_K(n,k)$ the subgroup of $\mathcal{D}^{r,s,[\alpha]}(n,k)$ of diffeomorphisms supported in K.

Proposition 2.5. *If $f, g \in C^{r,s,[\alpha]}(n,k)$ then $f\circ g \in C^{r,s,[\alpha]}(n,k)$.*

Proof. In fact, this is a consequence of Proposition 2.2. ∎

Proposition 2.6. *If $f \in C^{r,s,[\alpha]}(n,k)$ and f has a C^1-inverse, then $f^{-1} \in C^{r,s,[\alpha]}(n,k)$.*

Proof. We have the formula

$$D(f^{-1}) = \operatorname{inv}\circ Df\circ f^{-1},$$

where inv is the inversion in $L(\mathbb{R}^n,\mathbb{R}^n)$. It is well-known that inv is of class C^∞. $D(f^{-1})$ is of class $C^{r-1,s-1}$. Considering each entry in matrix $D(f^{-1})$ it is easy to see that $D^{r,s}(f^{-1})$ exists and is continuous. ■

A leaf preserving mapping of a smooth foliated manifold $f:(M,\mathcal{F})\to(M,\mathcal{F})$ is of class $C^{r,s}$ if for every $x\in M$ and for every distinguished chart (V,v) on $(M,\mathcal{F})$ with $f(x)\in V$, there exists a distinguished chart (U,u) on $(M,\mathcal{F})$ with $x\in U$, $f(U)\subset V$ and $v\circ f\circ u^{-1}$ is $C^{r,s}$.

We define

$$C^{r,s}(M,\mathcal{F})=\{f:(M,\mathcal{F})\to(M,\mathcal{F})\mid f \text{ is } C^{r,s} \text{ and } f(L_x)\subset L_x,\ \forall\, x\in M\}.$$

It is obvious that $C^{p+1,s+1}(M,\mathcal{F})\subset C^{p+1,s}(M,\mathcal{F})\subset C^{p,s}(M,\mathcal{F})$, for $1\le p<r$. By $\mathrm{Diff}^{r,s}(M,\mathcal{F})_0$ we denote the group of all leaf preserving C^1-diffeomorphisms on M of class $C^{r,s}$ which can be joined to the identity by a compactly supported $C^{r,s}$-isotopy.

The following fact can be proved as usual (cf. [1]).

Lemma 2.7. *Let $g\in\mathrm{Diff}^{r,s}_c(M,\mathcal{F})_0$. Then there exist open balls U_i and $g_i\in\mathrm{Diff}^{r,s}_c(M,\mathcal{F})_0$, $i=1,\dots,l$, such that* $\mathrm{supp}(g_i)\subset U_i$ *and $g=g_1\dots g_l$.*

This fragmentation property enables us to reduce the proof of Theorem 1.3 to the case of $(M,\mathcal{F})=(\mathbb{R}^n,\mathcal{F}_k)$, where $\mathcal{F}_k=\{\mathbb{R}^k\times\{\mathrm{pt}\}\}$.

As a consequence of Lemmas 2.4 and 2.7 we have the following

Lemma 2.8. *One has*

$$\mathcal{D}^{r,s}(n,k)=\bigcup\mathcal{D}^{r,s,\alpha}(n,k),$$

where the union is taken over all moduli of continuity α.

3. Basic estimates. Let $s\ge1$ and $0\le p\le r$. For $f\in C^{r,s}(\mathbb{R}^n,\mathbb{R}^n)$ we define

$$\|f\|_{p,s}=\sup_{x\in\mathbb{R}^n}\|D^{p,s}f(x)\|\le\infty,$$

and

$$\|f\|_{p,s,\alpha}=\sup_{x\ne y\in\mathbb{R}^n}\frac{\|D^{p,s}f(x)-D^{p,s}f(y)\|}{\alpha(\|x-y\|)}\le\infty,$$

where $\|\cdot\|$ denotes the usual norm in the space of p-linear mappings. Further we put $\mu_{p,s}(f)=\|f-\mathrm{id}\,\|_{p,s}$ and $\mu_{p,s,\alpha}(f)=\|f-\mathrm{id}\,\|_{p,s,\alpha}$. Moreover, we denote

$$M_{p,s,\alpha}(f)=\sup\{\mu_{1,s,\alpha}(f),\dots,\mu_{p,s,\alpha}(f)\}.$$

By simple computation we see that $\mu_{1,s}(f)\le\|f\|_{1,s}+1$ and $\|f\|_{1,s}\le\mu_{1,s}(f)+1$. Further, we have $\mu_{p,s}(f)=\|f\|_{p,s}$ for $p\ge2$, and $\mu_{p,s,\alpha}(f)=\|f\|_{p,s,\alpha}$ for $p\ge1$.

Let K be a closed subset of $\mathbb{R}^n$. We define

$$R_K=\sup\{\mathrm{dist}(q,\overline{\mathbb{R}^n\setminus K}) : q\in\mathbb{R}^n\}\le\infty$$

and

$$R^v_K=\sup\{\mathrm{dist}(q,\overline{\mathbb{R}^n\setminus K}\cap L_q) : q\in\mathbb{R}^n\}\le\infty.$$

Here $q\in L_q\in\mathcal{F}$. Clearly $R_K\le R^v_K$.

Proposition 3.1. *Let K be a closed interval of $\mathbb{R}^n$.*

(1) *Assume that* $R_K < \infty$. *Then there exists a constant* $C > 0$, *depending on* R_K *and* α *such that for all* $1 \le p \le r$

$$\mu_{p,s}(f) \le C\mu_{p,s,\alpha}(f),$$

whenever $f \in \mathcal{D}_K^{r,s,\alpha}(n,k)$.

(2) *If* $R_K^v < \infty$ *then there exists a constant* $C > 0$, *depending on* R_K^v *and* α *such that for all* $1 \le p < r$

$$\mu_{p,s,\alpha}(f) \le C\mu_{p+1,s,\alpha}(f),$$

for any $f \in \mathcal{D}_K^{r,s,\alpha}(n,k)$.

Proof. The inequality in (1) follows by properties of moduli of continuity.

The proof of (2) consists of three steps. First, let us take (x, y^1), $(x, y^2) \in \mathbb{R}^n$, where $x \in \mathbb{R}^k$ and $y^1, y^2 \in \mathbb{R}^{n-k}$. As K is an interval we can choose $x_0 \in \mathbb{R}^k$ such that $(x_0, y^1), (x_0, y^2) \in \overline{\mathbb{R}^n \setminus K}$ and $\|x - x_0\| = \|(x, y^j) - (x_0, y^j)\| \le R_K^v$, $j = 1, 2$.

We denote $I_t(a, b) = ta + (1 - t)b$. Then we have

$$\begin{aligned}
&\|D^{p,s}f(x,y^1) - D^{p,s}f(x,y^2)\| \\
&\quad= \|D^{p,s}f(x,y^1) - D^{p,s}f(x_0,y^1) + D^{p,s}f(x_0,y^2) - D^{p,s}f(x,y^2)\| \\
&\quad= \left\| \int_0^1 (D^{p+1,s}f(I_t(x,x_0),y^1) - D^{p+1,s}f(I_t(x,x_0),y^2))(x - x_0, 0)\, dt \right\| \\
&\quad\le \sup_{t\in[0,1]} \|D^{p+1,s}f(I_t(x,x_0),y^1) - D^{p+1,s}f(I_t(x,x_0),y^2)\| \|x - x_0\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\|D^{p,s}f(x,y^1) - D^{p,s}f(x,y^2)\|}{\alpha(\|(x,y^1) - (x,y^2)\|)} \\
&\qquad\le \sup_{z\in\mathbb{R}^k} \frac{\|D^{p+1,s}f(z,y^1) - D^{p+1,s}f(z,y^2)\|}{\alpha(\|y^1 - y^2\|)} \|x - x_0\| \\
&\qquad\le R_K^v \mu_{p+1,s,\alpha}(f).
\end{aligned}$$

In the next step we take $(x^1, y), (x^2, y) \in \mathbb{R}^n$, where $x^1, x^2 \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$. If $\|x^1 - x^2\| > 1$, we choose $x_0^1, x_0^2 \in \mathbb{R}^k$ such that $(x_0^1, y), (x_0^2, y) \in \overline{\mathbb{R}^n \setminus K}$, $\|x^1 - x_0^1\| \le R_K^v$ and $\|x^2 - x_0^2\| \le R_K^v$. We obtain

$$\begin{aligned}
&\frac{\|D^{p,s}f(x^1,y) - D^{p,s}f(x^2,y)\|}{\alpha(\|x^1 - x^2\|)} \\
&\qquad\le \frac{\|D^{p,s}f(x^1,y) - D^{p,s}f(x_0^1,y)\| + \|D^{p,s}f(x_0^2,y) - D^{p,s}f(x^2,y)\|}{\alpha(1)} \\
&\qquad\le \frac{2}{\alpha(1)} \|f\|_{p+1,s} (\|x^1 - x_0^1\| + \|x^2 - x_0^2\|).
\end{aligned}$$

If $\|x^1 - x^2\| \le 1$ then

$$\frac{\|D^{p,s}f(x^1,y) - D^{p,s}f(x^2,y)\|}{\alpha(\|x^1 - x^2\|)} \le \frac{\|f\|_{p+1,s}\|x^1 - x^2\|}{\alpha(\|x^1 - x^2\|)} \le \frac{\|f\|_{p+1,s}}{\alpha(1)}$$

as $\frac{t}{\alpha(t)}$ is an increasing function. In view of (1) one has the inequality in (2).

Finally, for arbitrary $q, q' \in \mathbb{R}^n$ we take $q_0 \in \mathbb{R}^n$ with $q - q_0 \in \mathbb{R}^k \times \{0\}$ and $q' - q_0 \in \{0\} \times \mathbb{R}^{n-k}$, and we use the preceding steps of the proof. ∎

Lemma 3.2 ([4]). *Let f be a C^1-diffeomorphism and $\mu_{1,s}(f) \le \frac{1}{2}$. Then*

$$\mu_{1,s}(f^{-1}) \le 2\mu_{1,s}(f).$$

Definition 3.3. We say that a polynomial F is *admissible* if it has no constant term and its coefficients are nonnegative.

Lemma 3.4. *Let $1 \le p \le r$, let α be a modulus of continuity, and let K be a closed interval of $\mathbb{R}^n$ such that $R_K^v < \infty$.*

(1) There exist $\delta_1 > 0$ and $C_1 > 0$ depending on n, p, α and R_K^v such that

$$\mu_{p,s,\alpha}(f \circ g) \le \mu_{p,s,\alpha}(f) + \mu_{p,s,\alpha}(g) + C_1\mu_{p,s,\alpha}(f)\mu_{p,s,\alpha}(g)$$

whenever $f, g \in \mathcal{D}_K^{r,s,\alpha}(n,k)$ and $\mu_{p,s,\alpha}(f), \mu_{p,s,\alpha}(g) \le \delta_1$.

(2) For every $\lambda > 1$ there exists $\delta_2 > 0$ depending on n, p, α and R_K^v such that

$$\mu_{p,s,\alpha}(f^{-1}) \le \lambda\mu_{p,s,\alpha}(f)$$

provided $f \in \mathcal{D}_K^{r,s,\alpha}(n,k)$ with $\mu_{p,s,\alpha}(f) \le \delta_2$.

Notice that Lemma 3.4 is formulated only for K such that $R_K^v < \infty$. But some parts of its proof are valid for the weaker assumption $R_K < \infty$. Moreover, the above inequalities are valid for $R_K < \infty$, if only $p \le s$.

Proof. We have for any $q, q' \in \mathbb{R}^n$

$$\frac{\|(D^{i,s}f \circ g)(D^{j_1,s}g \times \ldots \times D^{j_i,s}g)(q) - (D^{i,s}f \circ g)(D^{j_1,s}g \times \ldots \times D^{j_i,s}g)(q')\|}{\alpha(\|q-q'\|)}$$
$$\le \mu_{i,s,\alpha}(f)(1+\mu_{1,s}(g))\|g\|_{j_1,s}\ldots\|g\|_{j_i,s} + 2^{i-1}\|f\|_{i,s}\mu_{j_1,s,\alpha}(g)\|g\|_{j_2,s}\ldots\|g\|_{j_i,s}.$$

Then from (2.2), Proposition 2.2 and Proposition 3.1 (1) we have

$$\begin{aligned}\mu_{p,s,\alpha}(f \circ g) &\le \mu_{p,s,\alpha}(f)(1+\mu_{1,s}(g))\|g\|_{1,s}^p + 2^{p-1}\|f\|_{p,s}\mu_{1,s,\alpha}(g)\|g\|_{1,s}^{p-1}\\ &\quad + \mu_{1,s,\alpha}(f)(1+\mu_{1,s}(g))\|g\|_{p,s} + \|f\|_{1,s}\mu_{p,s,\alpha}(g)\\ &\quad + \sum C_{i,j_1,\ldots,j_i}\Big(\mu_{i,s,\alpha}(f)(1+\mu_{1,s}(g))\|g\|_{j_1,s}\ldots\|g\|_{j_i,s}\\ &\qquad\qquad + 2^{i-1}\|f\|_{i,s}\mu_{j_1,s,\alpha}(g)\|g\|_{j_2,s}\ldots\|g\|_{j_i,s}\Big)\\ &\le \mu_{p,s,\alpha}(f) + \mu_{p,s,\alpha}(g) + M_{p,s,\alpha}(f)F(M_{p,s,\alpha}(g))\end{aligned} \tag{3.1}$$

for arbitrary $f, g \in \mathcal{D}_K^{p,s,\alpha}(n,k)$ with $R_K < \infty$. Here F is an admissible polynomial independent of f and g, and $F = 0$ for $p = 1$.

Hence, from (3.1) and Proposition 3.1 (2) we obtain (1) for sufficiently small $\mu_{p,s,\alpha}(f)$ and $\mu_{p,s,\alpha}(g)$.

To show (2) we proceed by induction on p. First assume that $p = 1$ and $R_K < \infty$. For $\mu_{1,s}(f) < 1$ we have the formula

$$(Df)^{-1} = \sum_{m=1}^{\infty}(-(Df - \mathrm{Id}))^m. \tag{3.2}$$

Then from (3.2) we get

$$\mu_{1,s,\alpha}(f^{-1}) \le \sup_{q\ne q'} \frac{\|(Df)^{-1}(f^{-1}(q)) - (Df)^{-1}(f^{-1}(q'))\|}{\alpha(\|f^{-1}(q)-f^{-1}(q')\|)} \frac{\alpha(\|f^{-1}\|_{1,s}\|q-q'\|)}{\alpha(\|q-q'\|)}$$
$$\le \sqrt{\lambda}\sum_{m=1}^{\infty}\sup\frac{\|(Df-\mathrm{Id})^m(q)-(Df-\mathrm{Id})^m(q')\|}{\alpha(\|q-q'\|)},$$

since from Proposition 3.1 (1) and Lemma 3.2

$$\|f^{-1}\|_{1,s}\le 1+\mu_{1,s}(f^{-1})\le 1+2C\mu_{1,s,\alpha}(f)\le\sqrt{\lambda},$$

provided $\mu_{1,s,\alpha}(f)$ is small. By using Lemma 3.1 (1) we get

$$\mu_{1,s,\alpha}(f^{-1})\le\sqrt{\lambda}\mu_{1,s,\alpha}(f)\sum_{m=1}^{\infty}m\mu_{1,s}(f)^{m-1}$$
$$=\sqrt{\lambda}\mu_{1,s,\alpha}(f)\frac{1}{(1-\mu_{1,s}(f))^2}\le\lambda\mu_{1,s,\alpha}(f)$$

for sufficiently small $\mu_{1,s,\alpha}(f)$.

For $p\le s$ we obtain from (2.2)

$$D^{p,s}(f^{-1}) = D(f^{-1})(D^{p,s}f\circ f^{-1})(D(f^{-1})\times\ldots\times D(f^{-1}))$$
$$+ D(f^{-1})\sum C_{i,j_1,\ldots,j_i}(D^{i,s}f\circ f^{-1})(D^{j_1,s}(f^{-1})\times\ldots\times D^{j_i,s}(f^{-1})). \tag{3.3}$$

Now let $p\ge 2$ and $R_K<\infty$. Using (3.3), Propositions 2.2 and 3.1 (1) we have

$$\frac{\|\big(D(f^{-1})(D^{p,s}f\circ f^{-1})(D(f^{-1}))^p\big)(q)-\big(D(f^{-1})(D^{p,s}f\circ f^{-1})(D(f^{-1}))^p\big)(q')\|}{\alpha(\|q-q'\|)} \tag{3.4}$$
$$\le\mu_{1,s,\alpha}(f^{-1})\mu_{p,s}(f)(1+\mu_{1,s}(f^{-1}))^p+\mu_{p,s,\alpha}(f)(1+\mu_{1,s}(f^{-1}))^{p+2}$$
$$+2^{p-1}\mu_{p,s}(f)\mu_{1,s,\alpha}(f^{-1})(1+\mu_{1,s}(f^{-1}))^p$$
$$\le\mu_{p,s,\alpha}(f)+\mu_{p,s,\alpha}(f)F(\mu_{1,s,\alpha}(f^{-1})),$$

where F is an admissible polynomial.

Similarly, we can estimate the second summand of (3.3) and then

$$\frac{\|(D(f^{-1})\sum C_{i,j_1,\ldots,j_i}(D^{i,s}f\circ f^{-1})(D^{j_1,s}(f^{-1})\times\ldots\times(D^{j_i,s}(f^{-1})))\big|_{q'}^{q}\|}{\alpha(\|q-q'\|)} \tag{3.5}$$
$$\le\mu_{1,s,\alpha}(f^{-1})\sum C_{i,j_1,\ldots,j_i}\mu_{i,s}(f)\prod_{l=1}^{i}(1+\mu_{j_l,s}(f^{-1}))$$
$$+(1+\mu_{1,s}(f^{-1}))\sum C_{i,j_1,\ldots,j_i}\mu_{i,s,\alpha}(f)(1+\mu_{1,s}(f^{-1}))$$
$$\cdot\,\mu_{j_{i_0},s}(f^{-1})\prod_{l\ne i_0}(1+\mu_{j_l,s}(f^{-1}))$$
$$+(1+\mu_{1,s}(f^{-1}))\sum C_{i,j_1,\ldots,j_i}\mu_{i,s}(f)\mu_{j_1,s,\alpha}(f^{-1})2^{i-1}\prod_{l=2}^{i}(1+\mu_{j_l,s}(f^{-1}))$$
$$\le M_{p-1,s,\alpha}(f)F(M_{p-1,s,\alpha}(f^{-1})).$$

Now, suppose $R^v_K < \infty$. From (3.4) and (3.5), by using Proposition 3.1 (2) and induction on p, we obtain

$$\mu_{p,s,\alpha}(f^{-1}) \le \mu_{p,s,\alpha}(f)(1 + F(\mu_{p,s,\alpha}(f))) \le \lambda\mu_{p,s,\alpha}(f),$$

provided $\mu_{p,s,\alpha}(f)$ is sufficiently small. ∎

From the above lemma we obtain by standard arguments the following

COROLLARY 3.5. *Let $K \subset \mathbb{R}^n$ be a compact interval, let $p \ge 1$ and let α be a modulus of continuity. Then $(f,g) \mapsto \|f-g\|_{p,s,[\alpha]}$ is a metric on $\mathcal{D}^{p,s,[\alpha]}_K(n,k)$. The induced topology is called the $C^{p,s,[\alpha]}$-topology. $\mathcal{D}^{p,s,[\alpha]}_K(n,k)$ equipped with the $C^{p,s,[\alpha]}$-topology is a connected topological group.*

4. Rolling-up operators $\Psi_{i,A}$. Following Mather [4], I, we let $C_i := \mathbb{R}^{i-1} \times S^1 \times \mathbb{R}^{n-i}$, where $S^1 \cong \mathbb{R}/\mathbb{Z}$, $i = 1,\dots,k$. Let $\pi_i : \mathbb{R}^n \to C_i$ be the covering projection, and let $\tilde{p}_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $p_i : C_i \to \mathbb{R}^{n-1}$ be the projections, which omit the i-th coordinate. Clearly $p_i \circ \pi_i = \tilde{p}_i$.

The mapping $\pi_i : \mathbb{R}^n \to C_i$ gives us a system of coordinates in a neighborhood of any point of C_i, compatible with the foliation $\mathcal{F}_{k,i} = \{\mathbb{R}^{i-1} \times S^1 \times \mathbb{R}^{k-i} \times \{\mathrm{pt}\}\}$ on C_i. Notice that the seminorms introduced above do make also sense on C_i, and the group $\mathcal{D}^{r,s,[\alpha]}(C_i,k)$ is defined analogously as before.

Let $A \ge 1$ and $K_i = [-2,2]^i \times [-2A,2A]^{k-i} \times [-2,2]^{n-k}$, $i = 0,\dots,k$. We have that $K_0 = [-2A,2A]^k \times [-2,2]^{n-k} \supset K_1 \supset \dots \supset K_k = [-2,2]^n$. Next, let $K_i' = [-2A,2A]^{i-1} \times S^1 \times [-2A,2A]^{k-i} \times [-2,2]^{n-k}$.

Choose $\tilde{\rho}_A \in C^\infty(\mathbb{R},[0,1])$ with $\mathrm{supp}(\tilde{\rho}_A) = [-2A-1,2A+1]$ and $\tilde{\rho}_A = 1$ on $[-2A,2A]$. We define $\rho_A \in C^\infty(\mathbb{R}^n,[0,1])$ by $\rho_A(x,y) = \tilde{\rho}_A(x_1)\dots\tilde{\rho}_A(x_k)$, where $x = (x_1,\dots,x_k), y = (y_1,\dots,y_{n-k})$. Then $\mathrm{supp}(\rho_A) = [-2A-1,2A+1]^k \times \mathbb{R}^{n-k}$ and $\rho_A|_{[-2A,2A]^k\times\mathbb{R}^{n-k}} \equiv 1$. Let $\tau_{i,A} = \mathrm{Fl}_1^{\rho_A\partial_i} \in \mathrm{Diff}^\infty(\mathbb{R}^n,\mathcal{F}_k)_0$, where ∂_i denotes the unit vector field on $\mathbb{R}^n$ in the direction of the i-th coordinate, and Fl_t^X denotes the flow of the vector field X. Further we denote by T_i the unit translation in the direction of the i-th coordinate, i.e. $T_i = \mathrm{Fl}_1^{\partial_i}$.

Let $f \in \mathcal{D}^{r,s,\alpha}(n,k)$ with $\mathrm{supp}(f) \subset K_0$ and $\mu_{0,s}(f) \le \frac{1}{2}$. For $\theta \in C_i$ we choose $(x,y) \in \mathbb{R}^n$ with $\pi_i(x,y) = \theta$ and $x_i < -2A$. Then we choose $N \in \mathbb{N}$ such that $((T_if)^N(x,y))_i > 2A$. We define $\Gamma_{i,A}(f) : C_i \to C_i$ as

$$\Gamma_{i,A}(f)(\theta) = \pi_i((T_if)^N(x,y)),$$

which is independent of the choice of x and N.

It is obvious that $\Gamma_{i,A}$ preserves the identity. There exists a neighbourhood U of $\mathrm{Id} \in \mathcal{D}^{1,s}(n,k)$ such that

$$\Gamma_{i,A} : \mathcal{D}^{r,s,\alpha}_{K_0}(n,k) \cap U \to \mathcal{D}^{r,s,\alpha}_{K_i'}(C_i,k)_0$$

is continuous with respect to the $C^{r,s}$-topology. Moreover we have the following

LEMMA 4.1. *There exists $\delta > 0$ depending on n, r, α and A such that*

$$\mu_{r,s,\alpha}(\Gamma_{i,A}(f)) \le 9A\mu_{r,s,\alpha}(f)$$

for $f \in \mathcal{D}^{r,s,\alpha}_{K_0}(n,k) \cap U$ *with* $\mu_{r,s,\alpha}(f) \le \delta$, *and*

$$\mu_{r,s,\alpha}(\Gamma_{i,A}(f)) \le 9\mu_{r,s,\alpha}(f)$$

for $f \in \mathcal{D}^{r,s,\alpha}_{K_i}(n,k) \cap U$, *where* $i > 0$, *with* $\mu_{r,s,\alpha}(f) \le \delta$.

Proof. Let $N \in \mathbb{N}$ and choose $\varepsilon > 0$ such that $\sum_{j=0}^{N-1}(1+\varepsilon)^j \le N+1$. We will show that

$$\mu_{r,s,\alpha}((T_i f)^N) \le (1 + (1+\varepsilon) + \ldots + (1+\varepsilon)^{N-1})\mu_{r,s,\alpha}(f) \tag{4.1}$$

for $\mu_{r,s,\alpha}(f)$ sufficiently small.

By simple computation we have $\mu_{r,s,\alpha}(T_i f) = \mu_{r,s,\alpha}(f)$. Then for $1 < m < N$ from (3.1) (which is valid for $R_K < \infty$), Proposition 3.1 (2), and Lemma 3.4 (1) we obtain arguing by induction

$$\begin{aligned}
&\mu_{r,s,\alpha}((T_i f)^m) \\
&\quad \le \mu_{r,s,\alpha}(T_i f) + \mu_{r,s,\alpha}((T_i f)^{m-1}) + M_{r,s,\alpha}(T_i f) F(M_{r,s,\alpha}((T_i f)^{m-1})) \\
&\quad \le \mu_{r,s,\alpha}(f) + (1 + \ldots + (1+\varepsilon)^{m-2})\mu_{r,s,\alpha}(f) \\
&\quad + M_{r,s,\alpha}(f) F((1 + \ldots + (1+\varepsilon)^{m-2}) M_{r,s,\alpha}(f)) \\
&\quad \le \mu_{r,s,\alpha}(f) + (1 + \ldots + (1+\varepsilon)^{m-2})\mu_{r,s,\alpha}(f) \\
&\quad + (R^v_{K_0})^{r-1}\mu_{r,s,\alpha}(f) F((1 + \ldots + (1+\varepsilon)^{m-2})(R^v_{K_0})^{r-1}\mu_{r,s,\alpha}(f)) \\
&\quad \le \mu_{r,s,\alpha}(f) + (1 + \ldots + (1+\varepsilon)^{m-2})\mu_{r,s,\alpha}(f) + \varepsilon(1 + \ldots + (1+\varepsilon)^{m-2})\mu_{r,s,\alpha}(f) \\
&\quad \le \mu_{r,s,\alpha}(f) + (1+\varepsilon)(1 + \ldots + (1+\varepsilon)^{m-2})\mu_{r,s,\alpha}(f) \\
&\quad = (1 + \ldots + (1+\varepsilon)^{m-1})\mu_{r,s,\alpha}(f)
\end{aligned}$$

whenever $\mu_{r,s,\alpha}(f)$ is sufficiently small.

Next we choose $N \in \mathbb{N}$ with $8A + 1 < N < 8A + 3$. Then from (4.1)

$$\mu_{r,s,\alpha}(\Gamma_{i,A}(f)) = \mu_{r,s,\alpha}((T_i f)^N) \le (N+1)\mu_{r,s,\alpha}(f) \le 9A\mu_{r,s,\alpha}(f)$$

for $f \in \mathcal{D}^{r,s,\alpha}_{K_0}(n,k) \cap U$ with $\mu_{r,s,\alpha}(f) \le \delta$. Analogously

$$\mu_{r,s,\alpha}(\Gamma_{i,A}(f)) = \mu_{r,s,\alpha}((T_i f)^N) \le \mu_{r,s,\alpha}((T_i f)^8) \le 9\mu_{r,s,\alpha}(f)$$

for $f \in \mathcal{D}^{r,s,\alpha}_{K_i}(n,k) \cap U$ with $\mu_{r,s,\alpha}(f) \le \delta$, where $i > 0$. ∎

Consider the S^1-action $S^1 \times C_i \to C_i$ given by

$$\beta \cdot (x_1, \ldots, \theta, \ldots, x_k, y) = (x_1, \ldots, \beta + \theta, \ldots, x_k, y),$$

where y stands for $y_1, \ldots, y_{n-k}$. Let $G^{r,s,\alpha}_i$ denote the group of equivariant $C^{r,s,\alpha}$-diffeomorphisms of C_i,

$$G^{r,s,\alpha}_i = \{f \in \mathcal{D}^{r,s,\alpha}(C_i, k) : f(\beta \cdot \theta) = \beta \cdot f(\theta)\ \forall\, \beta \in S^1\ \forall\, \theta \in C_i\}.$$

PROPOSITION 4.2. *Let* α *be a modulus of continuity and* $A \ge 1$. *Then there exists a neighborhood* U_A *of* $\mathrm{id} \in \mathrm{Diff}^1_c(\mathbb{R}^n)$ *depending on* n, r, α *and* A *such that for* $f, g \in \mathcal{D}^{r,s,\alpha}_{K_0}(n,k) \cap U_A$ *with* $\Gamma_{i,A}(f)\Gamma_{i,A}(g)^{-1} \in G^{r,s,\alpha}_i$, *the mappings* $\tau_{i,A} f$ *and* $\tau_{i,A} g$ *are conjugate in* $\mathcal{D}^{r,s,\alpha}(n,k)$.

For the proof, see [4], I.

Now we will follow Mather [4], I, to define rolling-up operators $\Psi_{i,A}$. It is important that all steps of definition $\Psi_{i,A}$ are 'leaf preserving'.

Let $f \in U_A \cap \mathcal{D}^{r,s,\alpha}_{K_0}(n,k)$, where U_A is as in Proposition 4.2. We can find $g \in C^1(C_i, C_i)$ such that $g \in G_i^{1,s,\alpha}$ and $g = \Gamma_{i,A}(f)$ on $\{\theta \in C_i : \theta_i = 0\}$ by the formula

$$g(x_1,\dots,\theta_i,\dots,x_k,y) = \Gamma_{i,A}(f)(x_1,\dots,0,\dots,x_k,y) + (0,\dots,\theta_i,\dots,0).$$

Moreover g depends continuously on f, so we can shrink U_A such that $g \in \mathcal{D}^{1,s}_{K_i'}(C_i,k)_0$. It is easily seen that

$$\mu_{r,s,\alpha}(g) \le \mu_{r,s,\alpha}(\Gamma_{i,A}(f)).$$

Let $h = g^{-1}\Gamma_{i,A}(f) \in \mathcal{D}^{1,s}_{K_i'}(C_i,k)_0$.

We identify g, h and $\Gamma_{i,A}(f) = gh$ with periodic diffeomorphisms of $\mathbb{R}^n$ supported in $K_i'' = [-2A,2A]^{i-1} \times \mathbb{R} \times [-2A,2A]^{k-i} \times [-2,2]^{n-k}$. By abuse these periodic diffeomorphisms will be denoted by the same letters. It is easily seen that the lifted diffeomorphisms have the same norms $\mu_{r,s,[\alpha]}$ as the initial ones.

In order to perform further steps of the construction we need the following analog of Proposition 3.1 for periodic diffeomorphisms.

Proposition 4.3. *Let f be a periodic $C^{r,s}$-diffeomorphisms of $\mathbb{R}^n$ with period 1 with respect to the variable x_i for some $i = 1,\dots,k$, and let $f|_{\{x\in\mathbb{R}^n : x_i=0\}} = \mathrm{Id}$. Then for every $p = 0,\dots,r$*

(1) $\mu_{p,s}(f) \le \mu_{r,s}(f)$, and
(2) $\mu_{p,s,\alpha}(f) \le C^r \mu_{r,s,\alpha}(f)$, where $C > 0$ depends on α.

Proof. Note that f is periodic with period 1 if and only if $(f-\mathrm{Id})(x_1,\dots,x_i+1,\dots,x_k,y) = (f-\mathrm{id})(x,y)$. In order to show (1) we just integrate partial derivatives of $f-\mathrm{Id}$ with respect to x_i, and this procedure does not change s in $\mu_{j,s}$. Here we use the periodicity of $f-\mathrm{Id}$ and of its derivatives, and the condition $f|_{\{x\in\mathbb{R}^n : x_i=0\}} = \mathrm{Id}$ for $p=0$.

Now we prove (2). Let $\beta \in \mathbb{N}^n$, $|\beta| = p$, $\beta = (\beta',\beta'')$, where $\beta' \in \mathbb{R}^k$ and $\beta'' \in \mathbb{R}^{n-k}$ and $|\beta''| \le s$. First, let us take (x,y^1), $(x,y^2) \in \mathbb{R}^n$, where $x \in \mathbb{R}^k$ and $y^1, y^2 \in \mathbb{R}^{n-k}$. We may write

$$D^\beta f(x,y_j) = \frac{\partial^{|\beta'|}}{\partial x^{\beta'}}(D^{\beta''} f(x,y_j))$$

and $D^{\beta''} f(x,y_j)$ viewed as a function of x, is a periodic function with period 1 with respect to x_i which is equal to 0 on $\{x \in \mathbb{R}^n : x_i = 0\}$, for $j = 1,2$. Then

$$\begin{aligned}
\|D^\beta f(x,y_1) - D^\beta f(x,y_2)\| &= \left\| \frac{\partial^{|\beta'|}}{\partial x^{\beta'}}(D^{\beta''} f(x,y_1) - D^{\beta''} f(x,y_2)) \right\| \\
&\le \sup_{x\in\mathbb{R}^k} \left\| \frac{\partial^{|\beta_i'|}}{\partial x^{\beta_i'}}(D^{\beta''} f(x,y_1) - D^{\beta''} f(x,y_2)) \right\| \\
&= \sup_{x\in\mathbb{R}^k} \|D^{\beta_i} f(x,y_1) - D^{\beta_i} f(x,y_2)\| \\
&\le \sup_{x\in\mathbb{R}^k} \|D^{p+1,s} f(x,y_1) - D^{p+1,s} f(x,y_2)\|,
\end{aligned}$$

where $\beta_i' = (\beta_1,\dots,\beta_i+1,\dots,\beta_k)$ and $\beta_i = (\beta_i',\beta'')$, after integration along the x_i-axis.

Therefore

$$\frac{\|D^{p,s}f(x,y^1)-D^{p,s}f(x,y^2)\|}{\alpha(\|(x,y^1)-(x,y^2)\|)}\le\frac{\|D^{p+1,s}f(x,y^1)-D^{p+1,s}f(x,y^2)\|}{\alpha(\|y^1-y^2\|)}$$
$$\le\mu_{p+1,s,\alpha}(f).$$

In the second case, where $(x^1,y),(x^2,y)\in\mathbb{R}^n$ with $x^1,x^2\in\mathbb{R}^k$ and $y\in\mathbb{R}^{n-k}$, we proceed as in the proof of Proposition 3.1, by using the periodicity of f. The general case follows from the first two cases. ∎

We have $\mu_{1,s,\alpha}(g^{-1})\le 2\mu_{1,s,\alpha}(g)$. From (3.4), (3.5) (bearing in mind that $R_{K_i''}=2A<\infty$), the above inequalities and Propositions 3.1 (1) and 4.3 we have by induction on p

$$\begin{aligned}\mu_{p,s,\alpha}(g^{-1})&\le\mu_{p,s,\alpha}(g)+M_{p,s,\alpha}(g)F(M_{p-1,s,\alpha}(g^{-1}))\\&\le\mu_{p,s,\alpha}(\Gamma_{i,A}(f))+M_{p,s,\alpha}(\Gamma_{i,A}(f))F(M_{p-1,s,\alpha}(\Gamma_{i,A}(f)))\\&\le 9A\mu_{p,s,\alpha}(f)+C^p9A\mu_{p,s,\alpha}(f)F(C^{p-1}9A\mu_{p-1,s,\alpha}(f))\\&\le C^pA\mu_{p,s,\alpha}(f)\end{aligned}$$

if $\mu_{p,s,\alpha}(f)$ is sufficiently small. Here C is independent of A. Therefore

$$\begin{aligned}\mu_{r,s,\alpha}(h)&\le\mu_{r,s,\alpha}(g^{-1})+\mu_{r,s,\alpha}(\Gamma_{i,A}(f))+M_{r,s,\alpha}(g^{-1})F(M_{r,s,\alpha}(\Gamma_{i,A}(f)))\\&\le C^rA\mu_{r,s,\alpha}(f)\end{aligned}$$

for $f\in U_A\cap\mathcal{D}^{r,s,\alpha}_{K_0}(n,k)$ with $\mu_{r,s,\alpha}(f)\le\delta_1$.

Fix a function $\xi\in C^\infty(\mathbb{R},[0,1])$ of period 1, which equals 0 near m and equals 1 near $m+\frac{1}{2}$, where $m\in\mathbb{Z}$. We define functions

$$h_-=(\xi\circ\mathrm{pr}_i)\cdot(h-\mathrm{Id})+\mathrm{Id},\qquad h_+=h_-^{-1}h.$$

Shrinking U_A if necessary, $h_-,h_+\in\mathrm{Diff}^{1,s}_{K_i''}(\mathbb{R}^n)_0$.

Then we have

$$\begin{aligned}\mu_{r,s,\alpha}(h_-)&\le\sup_{q\ne q'\in\mathbb{R}^n}\frac{\|(D^{r,s}((\xi\circ\mathrm{pr}_i)(h-\mathrm{Id})))|^q_{q'}\|}{\alpha(\|q-q'\|)}\\&\le\sum_{j=0}^r\binom{r}{j}\Big[\sup_{q\ne q'}\frac{\|(D^{j,s}(\xi\circ\mathrm{pr}_i))|^q_{q'}\|\|D^{r-j,s}(h-\mathrm{Id})(q)\|}{\alpha(\|q-q'\|)}\\&\quad+\sup_{q\ne q'}\frac{\|D^{j,s}(\xi\circ\mathrm{pr}_i)(q')\|\|(D^{r-j,s}(h-\mathrm{Id}))|^q_{q'}\|}{\alpha(\|q-q'\|)}\Big]\\&\le\sum_{j=0}^r\binom{r}{j}(\|\xi\circ\mathrm{pr}_i\|_{j,s,\alpha}\mu_{r-j,s}(h)+\|\xi\circ\mathrm{pr}_i\|_{j,s}\mu_{r-j,s,\alpha}(h))\\&\le C_1\mu_{r,s,\alpha}(h).\end{aligned}$$

By Lemma 3.4 and Proposition 4.3 there exists $C_2>0$ independent of A such that

$$\mu_{r,s,\alpha}(h_+)\le C_2A\mu_{r,s,\alpha}(f)$$

for small $\mu_{r,s,\alpha}(f)$. Let us take

$$E_- = \{(x,y) \in \mathbb{R}^n : -1 \leq x_i \leq 0\}, \qquad E_+ = \left\{(x,y) \in \mathbb{R}^n : \frac{1}{2} \leq x_i \leq \frac{3}{2}\right\},$$

and define $\Psi_{i,A}(f)$ by

$$\Psi_{i,A}(f)|_{E_+} = h_+|_{E_+}, \quad \Psi_{i,A}(f)|_{E_-} = h_-|_{E_-},$$

and $\Psi_{i,A}(f)|_{\mathbb{R}^n \setminus (E_- \cup E_+)} = \mathrm{Id}$.

Then we have

$$\Gamma_{i,A}(f)\Gamma_{i,A}(\Psi_{i,A}(f))^{-1} = \Gamma_{i,A}(f)h^{-1} = g \in G_i^{r,s,\alpha}.$$

Shrinking U_A if necessary and using Proposition 4.2 we see that $\tau_{i,A}f$ and $\tau_{i,A}\Psi_{i,A}(f)$ are conjugate.

Summing up the above considerations we have the following

Proposition 4.4. *There exist a neighborhood U_A of $\mathrm{Id} \in \mathrm{Diff}^1_{K_0}(\mathbb{R}^n)_0$ and the operators*

$$\Psi_{i,A} : U_A \to \mathrm{Diff}^1_{K_0}(n,k), \qquad 1 \leq i \leq k,$$

with the following properties.

1. *$\Psi_{i,A}$ preserves the identity.*
2. *$\Psi_{i,A} : U_A \cap \mathcal{D}^{r,s,\alpha}_{K_{i-1}}(n,k) \to \mathcal{D}^{r,s,\alpha}_{K_i}(n,k)$ is continuous with respect to the $C^{r,s}$-topology.*
3. *For every $f \in U_A \cap \mathcal{D}^{r,s,\alpha}(n,k)$ we have*
$$[f] = [\Psi_{i,A}(f)] \in H_1(\mathcal{D}^{r,s,\alpha}(n,k)).$$
4. *There exists $\delta > 0$ depending on n, r, α, A, and $C > 1$ depending on n, r, α but independent of A with*
$$\mu_{r,s,\alpha}(\Psi_{i,A}(f)) \leq CA\mu_{r,s,\alpha}(f)$$
for all $f \in U_A \cap \mathcal{D}^{r,s,\alpha}(n,k)$ with $\mu_{r,s,\alpha}(f) \leq \delta$.

5. Proof of the Theorem 1.3. Let $f \in \mathcal{D}^{r,s}(n,k)$. In view of Lemmas 2.7 and 2.8 we may assume that $f \in \mathcal{D}^{r,s,\alpha}_{[-2,2]^n}(n,k)$. Moreover, f can be chosen sufficiently close to the identity in the $C^{r,s,\alpha}$-topology due to Corollary 3.5. We have to show that f belongs to the commutator subgroup $[\mathcal{D}^{r,s,\alpha}(n,k), \mathcal{D}^{r,s,\alpha}(n,k)]$.

Let us take $\chi_A \in \mathrm{Diff}^\infty_c(\mathbb{R}^n)_0$ such that for any $(x,y) \in [-2,2]^k \times \mathbb{R}^{n-k}$ one has $\chi_A(x,y) = (Ax,y)$. Then for any $g \in \mathcal{D}^{r,s,\alpha}(n,k)$ we define $g_0 = \chi_A fg\chi_A^{-1}$, and $g_i = \Psi_{i,A}(g_{i-1})$ for $i = 1, \ldots, k$. It is obvious by Proposition 4.4 that $[g_k] = [fg]$.

Lemma 5.1. *Let $r - s > k + 1$. Then there exist $A \geq 1$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and $f, g \in \mathcal{D}^{r,s,\alpha}_{[-2,2]^n}(n,k)$ with $\mu_{r,s,\alpha}(f), \mu_{r,s,\alpha}(g) \leq \varepsilon$ we have $\mu_{r,s,\alpha}(g_k) \leq \varepsilon$.*

Proof. We can choose A so large that $3C^k A^{1-r+s+k} \leq 1$, where C is the constant from Proposition 4.4 (4). There exists $\varepsilon_0 > 0$ such that we have

$$\mu_{r,s,\alpha}(fg) \leq \mu_{r,s,\alpha}(f) + \mu_{r,s,\alpha}(g) + C_1\mu_{r,s,\alpha}(f)\mu_{r,s,\alpha}(g) \leq 3\varepsilon,$$

for every $0 < \varepsilon \le \varepsilon_0$ and f, g as above. In view of definition of $D^{r,s}$ we have

$$\|D^{r,s}(\chi_A f g \chi_A^{-1})(x,y)\| \le \|A^{1-r+s} D^{r,s}(f \circ g)(\frac{1}{A}x, y)\|.$$

Therefore, for $q = (x,y), q' = (x',y')$

$$\begin{aligned}\mu_{r,s,\alpha}(\chi_A f g \chi_A^{-1}) &= \sup_{q \neq q'} \frac{\|D^{r,s}(\chi_A f g \chi_A^{-1})(q) - D^{r,s}(\chi_A f g \chi_A^{-1})(q')\|}{\alpha(\|q-q'\|)} \\ &\le A^{1-r+s} \sup_{q\neq q'} \frac{\|D^{r,s}(fg)(\frac{1}{A}x,y) - D^{r,s}(fg)(\frac{1}{A}x',y')\|}{\alpha(\|q-q'\|)} \\ &\le A^{1-r+s} \sup_{q\neq q'} \frac{\|D^{r,s}(fg)(\frac{1}{A}x,y) - D^{r,s}(fg)(\frac{1}{A}x',y')\|}{\alpha(\|(\frac{1}{A}x,y) - (\frac{1}{A}x',y')\|)} \\ &\le A^{1-r+s}\mu_{r,s,\alpha}(fg) \le 3A^{1-r+s}\varepsilon.\end{aligned}$$

If $\varepsilon_0 \le \delta$, where δ is the constant from Proposition 4.4 (4), we obtain

$$\mu_{r,s,\alpha}(g_k) \le C^k A^k \mu_{r,s,\alpha}(\chi_A f g \chi_A^{-1}) \le 3C^k A^{1-r+s+k}\varepsilon \le \varepsilon. \ \blacksquare$$

LEMMA 5.2. *Let $a > 0$. The set*

$$L = \{h \in \mathcal{D}^{r,s,\alpha}_{[-a,a]^n}(n,k) \ :\ \mu_{r,s,\alpha}(h) \le \varepsilon\}$$

equipped with the $C^{r,s}$-topology has the fixed-point property, i.e. every continuous mapping $L \to L$ has a fixed point.

Proof. Let us consider

$$L' = \{h' : h' + \mathrm{Id} \in L\} \subset (C^{r,s}_{[-a,a]^n}(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{r,s}).$$

Here $h'(x,y) = (h'_1(x,y),0)$ is of class $C^{r,s,\alpha}$.

We have the homeomorphism $L \ni h \mapsto h - \mathrm{Id} \in L'$. L' is closed in $(C^{r,s}_{[-a,a]^n}(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{r,s})$. Let us take

$$T : (C^{r,s}_{[-a,a]^n}(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{r,s}) \ni h \mapsto D^{r,s}h \in (C^{0}_{[-a,a]^n}(\mathbb{R}^n, L^r(\mathbb{R}^n,\mathbb{R}^n)), \|\cdot\|_{\sup}).$$

T is continuous as

$$\|Th\|_{\sup} = \sup_{x\in\mathbb{R}^n} \|D^{r,s}h(x)\| = \|h\|_{r,s}. \tag{5.1}$$

For every $h \in L'$ we have

$$\begin{aligned}\|Th(x) - Th(y)\| &= \|D^{r,s}h(x) - D^{r,s}h(y)\| \\ &\le \frac{\|D^{r,s}h(x) - D^{r,s}h(y)\|}{\alpha(\|x-y\|)}\alpha(\|x-y\|) \le \varepsilon\alpha(\|x-y\|),\end{aligned}$$

so $T(L')$ is equicontinuous, and it is bounded in view of (5.1). By Ascoli-Arzela's theorem, the set $T(L')$ is relative compact in $(C^0_{[-a,a]^n}(\mathbb{R}^n, L^r(\mathbb{R}^n,\mathbb{R}^n)), \|\cdot\|_{\sup})$, so it is compact.

Hence L' and L are compact. Since L is a convex subset of a Fréchet space, by Schauder-Tychonoff's theorem every continuous map $L \to L$ has a fixed point. $\blacksquare$

We choose $\varepsilon > 0$ as in Lemma 5.1. Then L has the fixed-point property, and the mapping

$$\mathcal{D}^{r,s,\alpha}_{[-2,2]^n}(n,k) \ni g \mapsto g_k \in \mathcal{D}^{r,s,\alpha}_{[-2,2]^n}(n,k),$$

is continuous with respect to the $C^{r,s}$-topology. Hence there exists $g \in L$ such that $g = g_k$. Therefore

$$[f][g] = [fg] = [g_k] = [g] \in H_1(\mathcal{D}^{r,s,\alpha}(n,k)).$$

and $[f] = [\mathrm{Id}] \in H_1(\mathcal{D}^{r,s,\alpha}(n,k))$. This completes the proof.

6. Remark on C^{n+1}-diffeomorphisms. It is still not known whether the group $\mathrm{Diff}^{n+1}_c(M)_0$ is perfect and simple. Mather in [5] considered the geometric transfer map and proved that linearized forms of commutator equations are true iff $r \neq n+1$. This result strongly suggests that $\mathrm{Diff}^{n+1}_c(M)_0$ is not perfect.

Observe that the perfectness of $\mathrm{Diff}^{n+1}_c(M)_0$ is strictly related to the perfectness of $\mathrm{Diff}^{r}_c(M,\mathcal{F})_0$ for large r. In fact, let $f \in \mathrm{Diff}^{n+1}_c(M)_0$ be sufficiently close to Id, and let $0 < k < n$. In view of Lemma 2.7 we may assume that $f \in \mathrm{Diff}^{n+1}_c(\mathbb{R}^n)_0$. Then there exist $g \in \mathrm{Diff}^{n+1}_c(\mathbb{R}^n,\mathcal{F}_k)_0$ and $h \in \mathrm{Diff}^{n+1}_c(\mathbb{R}^n,\mathcal{F}'_{n-k})_0$ such that $f = g \circ h$, where $\mathcal{F}_k = \{\mathbb{R}^k \times \{\mathrm{pt}\}\}$ and $\mathcal{F}'_{n-k} = \{\{\mathrm{pt}\} \times \mathbb{R}^{n-k}\}$ are the product foliations of $\mathbb{R}^n$.

In fact, if $f = (f_1, f_2)$ is sufficiently close to the identity then $h = (h_1, h_2)$ given by $h(x,y) = (x, f_2(x,y))$, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$, is a diffeomorphism which belongs to $\mathrm{Diff}^{n+1}_c(\mathbb{R}^n,\mathcal{F}'_{n-k})_0$. Define $g = (g_1, g_2)$ by $g(x,y) = (f_1(h^{-1}(x,y)), y)$. We have that $g \in \mathrm{Diff}^{n+1}_c(\mathbb{R}^n,\mathcal{F}_k)_0$, provided f is sufficiently close to the identity. Then

$$\begin{aligned}(g\circ h)(x,y) &= (g_1(h(x,y)), g_2(h(x,y))) = ((f_1\circ h^{-1})(h(x,y)), h_2(x,y))\\ &= (f_1(x,y), f_2(x,y)) = f(x,y)\end{aligned}$$

is the required decomposition.

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