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THE REDUCTION OF THE STANDARD k-COSYMPLECTIC MANIFOLD ASSOCIATED TO A REGULAR LAGRANGIAN

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Abstract. The aim of the paper is to define a k-cosymplectic structure on the standard k-cosymplectic manifold associated to a regular Lagrangian and to reduce it via Marsden-Weinstein reduction.

1. Introduction. Considering a Hamiltonian action of a Lie group on a k-cosymplectic manifold [3], one divides a level set of a momentum map by an action of a subgroup, in order to form a new k-cosymplectic manifold.

Let M be an n-dimensional smooth differential manifold and $\tau_M^* : T^*M \to M$ the cotangent bundle. Denote by $(T_k^1)^*M = T^*M \oplus \ldots \oplus T^*M$ the Whitney sum of k copies of T^*M, with the canonical projection $\tau^* : (T_k^1)^*M \to M, \tau^*(\alpha_{1q}, \ldots, \alpha_{kq}) = q$, that is canonically identified with the cotangent bundle of k^1-covelocities of the manifold M [2]. Let $\tau_M : TM \to M$ be the tangent bundle. Denote by $T_k^1 M = TM \oplus \ldots \oplus TM$ the Whitney sum of k copies of TM, with the canonical projection $\tau : T_k^1 M \to M, \tau(v_{1q}, \ldots, v_{kq}) = q$, that is canonically identified with the tangent bundle of k^1-velocities of the manifold M [2].

Using the Legendre transformation corresponding to a regular Lagrangian [3], we want to transfer the k-cosymplectic structure from $\mathbb{R}^k \times (T_k^1)^*M$ on $\mathbb{R}^k \times T_k^1 M$. Since all the considerations will be local, without loosing generality, we'll consider the n-dimensional manifold M as being $\mathbb{R}^n$.

2. The standard k-cosymplectic structure associated to a regular Lagrangian. Consider $T_k^1\mathbb{R}^n = T\mathbb{R}^n \oplus \ldots \oplus T\mathbb{R}^n$ the Whitney sum of k-copies of $T\mathbb{R}^n$, and denote by

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$(t^1, \dots, t^k, q^1, \dots, q^n, v_1^1, \dots, v_k^1, \dots, v_1^n, \dots, v_k^n)$ the coordinate functions on the manifold $\mathbb{R}^k \times T_k^1\mathbb{R}^n$.

DEFINITION 1. The map $L : \mathbb{R}^k \times T_k^1\mathbb{R}^n \to \mathbb{R}$ is a *Lagrangian* if

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left(\frac{\partial L}{\partial v_A^i}\right) - \frac{\partial L}{\partial q^i} = 0,$$

where $v_A^i = \partial q^i/\partial t^A$, for any $1 \leq A \leq k, 1 \leq i \leq n$.

Define the Legendre transformation $LT : \mathbb{R}^k \times T_k^1\mathbb{R}^n \to \mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ associated to the Lagrangian $L : \mathbb{R}^k \times T_k^1\mathbb{R}^n \to \mathbb{R}$, by

$$(LT(t^1, \dots, t^k, v_{1q}, \dots, v_{kq}))^A(w_q) := \frac{d}{ds} |_{s=0} L(t^1, \dots, t^k, v_{1q}, \dots, v_{Aq} + sw_q, \dots, v_{kq}),$$

for any $1 \leq A \leq k$.

The Lagrangian $L : \mathbb{R}^k \times T_k^1\mathbb{R}^n \to \mathbb{R}$ is said to be regular if the Jacobian matrix $(\partial^2 L/\partial v_A^i \partial v_B^j)_{1\leq A,B\leq k, 1\leq i,j\leq n}$ of L is nonsingular.

Using the k-cosymplectic structure $(\eta_A, \omega_A, V)_{1\leq A\leq k}$ on the standard k-cosymplectic manifold $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ [4], we can define a k-cosymplectic structure $((\eta_L)_A, (\omega_L)_A, V_L)_{1\leq A\leq k}$ on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$, by means of the Legendre transformation LT associated to a regular Lagrangian L, as follows:

1. $(\eta_L)_A = (LT)^*\eta_A$;
2. $(\omega_L)_A = (LT)^*\omega_A$;
3. $V_L = ker(\pi_L)_*$,

where $\pi_L : \mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n, \pi_L(t^1, \dots, t^k, v_{1q}, \dots, v_{kq}) = (t^1, \dots, t^k, q)$, for any $1 \leq A \leq k$.

Notice that the projection π_L that defines the distribution V_L is the pull-back by the Legendre transformation LT of the projection π that defines the distribution V [4]. Therefore, the two distributions V and V_L are related by the relation $V = (LT)_*V_L$.

PROPOSITION 1 ([3]). *Using the notations above, the following assertions are equivalent:*

1. *L is regular;*
2. *LT is a local diffeomorphism;*
3. *$(\mathbb{R}^k \times T_k^1\mathbb{R}^n, (\eta_L)_A, (\omega_L)_A, V_L)_{1\leq A\leq k}$ is a k-cosymplectic manifold.*

Note that, locally, by pushing-forward the Reeb vector fields on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ associated to $((\eta_L)_A, (\omega_L)_A), 1 \leq A \leq k$ through the Legendre transformation LT, one obtains the Reeb vector fields on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ associated to $(\eta_A, \omega_A), 1 \leq A \leq k$, i.e. $R_A = (LT)_*(R_L)_A, 1 \leq A \leq k$. Indeed, one can easily check that the vector fields $(LT)_*(R_L)_A, 1 \leq A \leq k$ satisfy the two conditions that uniquely characterize the Reeb vector fields associated to $(\eta_A, \omega_A), 1 \leq A \leq k$.

Consider the bundle morphism

$$\Omega_L^\sharp : T_k^1(\mathbb{R}^k \times T_k^1\mathbb{R}^n) \to T^*(\mathbb{R}^k \times T_k^1\mathbb{R}^n),$$

$$\Omega_L^\sharp(X_1, \dots, X_k) := \sum_{A=1}^{k}(i_{X_A}(\omega_L)_A + (i_{X_A}(\eta_L)_A)(\eta_L)_A).$$

Note that the two bundle morphisms $\Omega^\sharp$ [4] and $\Omega_L^\sharp$ defined on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ and respectively, on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$, are related by the relation $\Omega_L^\sharp = (LT)^* \circ \Omega^\sharp \circ (LT)_*$. Therefore, the Hamiltonian systems of k-vector fields on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ can be obtained from the Hamiltonian systems of k-vector fields on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$, using the Legendre transformation LT associated to the regular Lagrangian L. Notice that, locally, $(X_1, \ldots, X_k)$ is an H-Hamiltonian system of k-vector fields on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ (i.e. a solution of the equation $\Omega^\sharp(X_1, \ldots, X_k) = dH$) if and only if $((LT)_*^{-1}X_1, \ldots, (LT)_*^{-1}X_k)$ is an $H \circ LT$-Hamiltonian system of k-vector fields on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$. In particular, the fundamental vector fields on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ are related to the fundamental vector fields on $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ by the relation $(\xi_A)_{\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n} = (LT)_*(\xi_A)_{\mathbb{R}^k \times T_k^1\mathbb{R}^n}, 1 \leq A \leq k$.

Using the Liouville 1-forms $\theta_A, 1 \leq A \leq k$ on the standard k-cosymplectic manifold $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ [4], we can define the Liouville 1-forms $(\theta_L)_A, 1 \leq A \leq k$ on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$, by means of the Legendre transformation LT:

$$(\theta_L)_A = (LT)^*\theta_A,$$

for any $1 \leq A \leq k$. Indeed, $d(\theta_L)_A = d((LT)^*\theta_A) = (LT)^*(d\theta_A)(LT)^*(-\omega_A) = -(LT)^*\omega_A = -(\omega_L)_A$.

Let $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ be an action of a Lie group G (with its Lie algebra $\mathcal{G}$) on $\mathbb{R}^n$. Define the canonical lifted actions to $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ and respectively, to $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ by

$$\Phi^{T_k^*} : G \times (\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n) \to \mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n,$$

$$\Phi^{T_k^*}(g, t^1, \ldots, t^k, \alpha_{1q}, \ldots, \alpha_{kq}) := (t^1, \ldots, t^k, \alpha_{1q} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)}, \ldots, \alpha_{kq} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)})$$

and

$$\Phi^{T_k} : G \times (\mathbb{R}^k \times T_k^1\mathbb{R}^n) \to \mathbb{R}^k \times T_k^1\mathbb{R}^n,$$

$$\Phi^{T_k}(g, t^1, \ldots, t^k, v_{1q}, \ldots, v_{1q}) := (t^1, \ldots, t^k, (\Phi_g)_{*q}v_{1q}, \ldots, (\Phi_g)_{*q}v_{1q}).$$

If L is a regular Lagrangian, invariant with respect to Φ (i.e. $({\Phi_g}^{T_k})^*L = L$, for any $g \in G$), then one can check that the two lifted actions of G to $\mathbb{R}^k \times (T_k^1)^*\mathbb{R}^n$ and respectively, to $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ are related by the relation $\Phi_g^{T_k^*} \circ LT = LT \circ \Phi_g^{T_k}$, for any $g \in G$.

We obtain that the Liouville 1-forms $(\theta_L)_A, 1 \leq A \leq k$ are invariant with respect to Φ^{T_k}, i.e. $(\Phi_g^{T_k})^*(\theta_L)_A = (\theta_L)_A$, for any $g \in G, 1 \leq A \leq k$. Indeed, as the Liouville 1-forms $\theta_A, 1 \leq A \leq k$ are invariant with respect to $\Phi^{T_k^*}$, we get $(\Phi_g^{T_k})^*(\theta_L)_A = (\Phi_g^{T_k})^*((LT)^*\theta_A) = (LT \circ \Phi_g^{T_k})^*\theta_A = (\Phi_g^{T_k^*} \circ LT)^*\theta_A = (LT)^*((\Phi_g^{T_k^*})^*\theta_A) = (LT)^*(\theta_A)(\theta_L)_A$.

Using this fact, one can easily check that Φ^{T_k} is a k-cosymplectic action (i.e. it preserves the k-cosymplectic structure), taking into account that $\Phi^{T_k^*}$ is a k-cosymplectic action.

3. Marsden-Weinstein reduction of the standard k-cosymplectic manifold associated to a regular Lagrangian. Consider $\mu \in \mathcal{G}^{k*}$ a regular value of J_L. This implies that ${J_L}^{-1}(\mu)$ is a manifold. Denote by $G_\mu : \{g \in G : {Ad_{g^{-1}}}^{k*}(\mu) = \mu\}$ the isotropy group of μ with respect to Φ. Note that it acts on ${J_L}^{-1}(\mu)$. Indeed, let $g \in G$ and $x \in {J_L}^{-1}(\mu)$, i.e. ${Ad_{g^{-1}}}^{k*}(\mu) = \mu$ and $J_L(x)\mu$. Then $J_L(\Phi(g, x)) = J_L(\Phi_g(x)){Ad_{g^{-1}}}^{k*}(\mu) = \mu$, which implies that $\Phi(g,x) \in {J_L}^{-1}(\mu)$. Suppose that G_μ acts freely and properly on ${J_L}^{-1}(\mu)$. This implies that $(\mathbb{R}^k \times T_k^1\mathbb{R}^n)_\mu = {J_L}^{-1}(\mu)/G_\mu$ is a manifold.

If we denote by G_x the orbit of x with respect to Φ, we have that

$$T_x(Gx) = \{\xi_{\mathbb{R}^k \times T_k^1\mathbb{R}^n}(x), \xi \in \mathcal{G}\}$$

and

$$T_x({J_L}^{-1}(\mu)) = ker(J_L)_{*x},$$

for any $x \in {J_L}^{-1}(\mu)$.

Like in the k-symplectic case [2], one gets an orthogonal decomposition theorem:

Lemma 1. *Under the hypotheses above, for any $x \in {J_L}^{-1}(\mu)$ we have:*

1. $T_x(G_\mu x) = T_x(Gx) \cap T_x({J_L}^{-1}(\mu))$;
2. $T_x({J_L}^{-1}(\mu)) = \{X_x \in T_x(\mathbb{R}^k \times T_k^1\mathbb{R}^n) : \sum_{A=1}^k (\omega_L)_{Ax}((\xi_A)_{\mathbb{R}^k \times T_k^1\mathbb{R}^n}(x), X_x) = 0,$ *for any* $\xi_A \in \mathcal{G}, 1 \leq A \leq k\}$.

Using the orthogonal decomposition theorem we can show that the Reeb vector fields $(R_L)_1, \ldots, (R_L)_k$ are tangent to ${J_L}^{-1}(\mu)$.

Denote by $(i_L)_\mu : {J_L}^{-1}(\mu) \to \mathbb{R}^k \times T_k^1\mathbb{R}^n$ the inclusion map and by $(\pi_L)_\mu : {J_L}^{-1}(\mu) \to (\mathbb{R}^k \times T_k^1\mathbb{R}^n)_\mu$ the projection, the last one being a surjective submersion.

Theorem 1. *Under the hypotheses above, there exists a unique k-cosymplectic structure $(((\eta_L)_\mu)_A, ((\omega_L)_\mu)_A, (V_L)_\mu)_{1 \leq A \leq k}$ on $(\mathbb{R}^k \times T_k^1\mathbb{R}^n)_\mu$ such that*

1. ${(i_L)_\mu}^*(\eta_L)_A = {(\pi_L)_\mu}^*((\eta_L)_\mu)_A$;
2. ${(i_L)_\mu}^*(\omega_L)_A = {(\pi_L)_\mu}^*((\omega_L)_\mu)_A$,

for any $1 \leq A \leq k$.

Moreover, the Reeb vector fields $(R_L)_1, \ldots, (R_L)_k$ on $\mathbb{R}^k \times T_k^1\mathbb{R}^n$ associated to $((\eta_L)_A, (\omega_L)_A), 1 \leq A \leq k$ project to the Reeb vector fields $((R_L)_\mu)_1, \ldots, ((R_L)_\mu)_k$ on $(\mathbb{R}^k \times T_k^1\mathbb{R}^n)_\mu$ associated to $(((\eta_L)_\mu)_A, ((\omega_L)_\mu)_A), 1 \leq A \leq k$.

Proof. Define

$$((\eta_L)_\mu)_{A[x]}([v]) = (\eta_L)_{Ax}(v),$$

$$((\omega_L)_\mu)_{A[x]}([v],[w]) = (\omega_L)_{Ax}(v,w),$$

where $[v] = ((\pi_L)_\mu)_{*x}(v), [w] = ((\pi_L)_\mu)_{*x}(w), [x] = (\pi_L)_\mu(x)$, for any $v, w \in T_x{J_L}^{-1}(\mu)$, $x \in {J_L}^{-1}(\mu)$.

Let $(V_L)_\mu := \bigoplus_{A=1}^k (\cap_{B=1}^k ker((\eta_L)_\mu)_B) \cap (\cap_{B=1, B \neq A}^k ker((\omega_L)_\mu)_B)$. Indeed, the structure defined above is the unique k-cosymplectic structure on $(\mathbb{R}^k \times T_k^1\mathbb{R}^n)_\mu$ with the properties required in the theorem (because of the surjectivity of $(\pi_L)_\mu$ and $((\pi_L)_\mu)_{*x}$, for any $x \in {J_L}^{-1}(\mu)$). $\square$

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