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NON-EXISTENCE OF EXCHANGE TRANSFORMATIONS OF ITERATED JET FUNCTORS

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Abstract. We study the problem of the non-existence of natural transformations $J^rJ^sY \to J^sJ^rY$ of iterated jet functors depending on some geometric object on the base of Y.

1. Introduction. It is well known that for every couple F and G of product preserving bundle functors defined on the category $\mathcal{M}f$ of smooth manifolds and all smooth maps there is an exchange natural equivalence $\kappa^{F,G} : FG \to GF$, [5]. Moreover, denoting by $p^F_M : FM \to M$ and $p^G_M : GM \to M$ the bundle projections, we have $p^G_{FM} \circ \kappa^{F,G}_M = F(p^G_M)$, i.e. $\kappa^{F,G}_M$ interchanges the projections p^G_{FM} and $F(p^G_M)$. We remark that this property generalizes the classical involution $\kappa^{T,T}_M : TTM \to TTM$ of the iterated tangent bundle to every pair F and G of product preserving functors on $\mathcal{M}f$.

In [2] we applied this point of view to natural equivalences

$$A^{F,G} : FG \to GF$$

of iterated bundle functors defined on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and of fibered manifold morphisms covering local diffeomorphisms. In particular, we have extended the concept of the canonical involution in the following way. Given an arbitrary fibered manifold $Y \to M$, we denote by $p^F_Y : FY \to Y$ and $p^G_Y : GY \to Y$ the bundle projections. Then the natural equivalence $A^{F,G}$ is called the

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involution, if $p^G_{FY} \circ A^{F,G}_Y = F(p^G_Y)$ for an arbitrary fibered manifold $Y \to M$. By [2], [6] and [9], involutions can be applied in the prolongation of connections.

An important example of a bundle functor on $\mathcal{FM}_m$ is the r-th jet functor J^r, which associates to an arbitrary fibered manifold $Y \to M$ its r-th jet prolongation $J^rY \to M$. For such functors we have proved that for any r and s there is no involution $J^rJ^s \to J^sJ^r$, [2]. On the other hand, M. Modugno [9] has introduced the involution $\mathrm{ex}_\Lambda : J^1J^1 \to J^1J^1$ depending on a classical linear connection Λ on the base manifold M.

In this paper we study the more general problem on the non-existence of natural transformations (not necessarily involutions)

$$(A_\sigma)_Y : J^rJ^sY \to J^sJ^rY$$

depending on some geometric object σ on the base of Y. The main result will be proved in Section 1. Further, in Section 2 we prove that for $r \neq s$ there is no natural transformation $(A_\omega)_Y : J^rJ^sY \to J^sJ^rY$ depending on a symplectic form ω on the base of Y. As a direct consequence we obtain that for $r \neq s$ there is no natural transformation $J^rJ^sY \to J^sJ^rY$. Finally, Section 3 is devoted to the problem of the non-existence of non-trivial natural transformations $(A_\omega)_Y : J^rJ^rY \to J^rJ^rY$.

We remark that higher order jet functors play an important role in differential geometry, see e.g. [3], [4] and [7]. In what follows we use the terminology and notation from the book [5]. We denote $\mathcal{M}f_m \subset \mathcal{M}f$ the subcategory of m-dimensional manifolds and their local diffeomorphisms and by $\mathcal{FM}_{m,n} \subset \mathcal{FM}_m$ the subcategory of fibered manifolds with n-dimensional fibres and their local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

2. The main result. Let F be a natural bundle on $\mathcal{M}f_m$. Given a manifold M we denote by $\Gamma_{\mathrm{loc}}(FM)$ the set of local smooth sections of FM. Further, suppose that for all $\mathcal{M}f_m$-objects M we have $D(M) \subset \Gamma_{\mathrm{loc}}(FM)$ such that from $\sigma \in D(M_1)$ it follows that $F\psi \circ \sigma \circ \psi^{-1} \in D(M_2)$ for any $\mathcal{M}f_m$-map $\psi : M_1 \to M_2$.

Definition 1. An $\mathcal{FM}_{m,n}$-natural operator $A : D \rightsquigarrow (J^rJ^s, J^sJ^r)$ is an invariant family of functions

$$A : D(M) \to C^\infty_M(J^rJ^sY, J^sJ^rY)$$

into the space $C^\infty_M(J^rJ^sY, J^sJ^rY)$ of all base preserving maps $J^rJ^sY \to J^sJ^rY$ for any $\mathcal{FM}_{m,n}$-object $Y \to M$. The invariance means that for any $\mathcal{FM}_{m,n}$-objects $Y_1 \to M_1$ and $Y_2 \to M_2$ and any $\mathcal{FM}_{m,n}$-map $\Psi : Y_1 \to Y_2$ covering $\psi : M_1 \to M_2$ and any sections $\sigma_1 \in D(M_1)$ and $\sigma_2 \in D(M_2)$ from $\sigma_2 \circ \psi = F\psi \circ \sigma_1$ it follows that $A(\sigma_2) \circ J^rJ^s\Psi = J^sJ^r\Psi \circ A(\sigma_1)$.

In the special case $D(M) = \Gamma_{\mathrm{loc}}(FM)$ we write $A : F \rightsquigarrow (J^rJ^s, J^sJ^r)$ instead of $A : D \rightsquigarrow (J^rJ^s, J^sJ^r)$. The main result of the present paper is the following non-existence theorem.

Theorem 1. *Let r and s be natural numbers such that $r > s$. Let F and D be as above. Suppose that there exists a section $\rho \in D(\mathbf{R}^m)$ and an $\mathcal{M}f_m$-map $\varphi = (\varphi^i) : \mathbf{R}^m \to \mathbf{R}^m$ such that*

(a) $F\varphi \circ \rho \circ \varphi^{-1} = \rho$ *near* $0 \in \mathbf{R}^m$,
(b) $j_0^{r+s-1}\varphi = id$,
(c) φ^1 *depends only on* x^1 *and*
(d) $\frac{d^{r+s}}{d(x^1)^{r+s}}\varphi^1(0) \neq 0$.

Then there is no $\mathcal{FM}_{m,n}$*-natural operator* $A : D \rightsquigarrow (J^r J^s, J^s J^r)$.

Proof. Denoting by $\mathbf{R}^{m,n}$ the product fibered manifold $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$, we identify sections of $\mathbf{R}^{m,n}$ with maps $\mathbf{R}^m \to \mathbf{R}^n$. Further, we use the notation

$$j_0^r j^s(f(x, \underline{x})) = j_0^r(x \to j_x^s(\underline{x} \to f(x, \underline{x}))) \in J_0^r J^s \mathbf{R}^{m,n}.$$

Suppose that there exists an operator A in question. Consider first the restriction $\tilde{A} : J_0^r J^s \mathbf{R}^{m,n} \to J_0^s J^r \mathbf{R}^{m,n}$ of $A(\rho)$ to the fibers over $0 \in \mathbf{R}^m$. Using the invariance of $\tilde{A}$ with respect to the fiber homotheties of $\mathbf{R}^{m,n}$ and the homogeneous function theorem from [5] we see that $\tilde{A}$ is linear. Taking into account the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-maps

$$(x^1, \dots, x^m, y^1, ty^2, \dots, ty^n)$$

for $t \neq 0$ we can write

$$\tilde{A}(j_0^r j^s(x^1, 0, ..., 0)) = j_0^s j^r\Big(\sum_{|\alpha| \le s} \sum_{|\beta| \le r} a_{\alpha\beta} x^\alpha (\underline{x} - x)^\beta, 0, \dots, 0\Big) \tag{1}$$

for some uniquely determined $a_{\alpha\beta} \in \mathbf{R}$ for m-tuples α, β with $|\alpha| \le s$, $|\beta| \le r$. Further, considering the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-map

$$(x^1, \dots, x^m, y^1 - x^1, y^2, \dots, y^n)$$

we get from (1)

$$\tilde{A}(j_0^r j^s(x^1 - \underline{x}^1, 0, \dots, 0)) = j_0^s j^r\Big(\sum_{|\alpha| \le s} \sum_{|\beta| \le r} a_{\alpha\beta} x^\alpha (\underline{x} - x)^\beta - \underline{x}^1, 0, \dots, 0\Big).$$

Then the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{m,n}$-map

$$(x^1, \dots, x^m, y^1 + (y^1)^{s+1}, y^2, \dots, y^n)$$

and the linearity of $\tilde{A}$ yield

$$\begin{aligned} &\tilde{A}(j_0^r j^s((x^1 - \underline{x}^1)^{s+1}, 0, \dots, 0)) \\ &\qquad = j_0^s j^r\Big(\Big(\sum_{|\alpha| \le s} \sum_{|\beta| \le r} a_{\alpha\beta} x^\alpha (\underline{x} - x)^\beta - \underline{x}^1\Big)^{s+1}, 0, \dots, 0\Big). \end{aligned} \tag{2}$$

But $j_0^r j^s(\underline{x}^1 - x^1)^{s+1} = 0$ as $j_x^s(\underline{x}^1 - x^1)^{s+1} = 0$. Then from (2) we get

$$j_0^s j^r\Big(\sum_{|\alpha| \le s} \sum_{|\beta| \le r} a_{\alpha\beta} x^\alpha (\underline{x} - x)^\beta - \underline{x}^1\Big)^{s+1} = 0,$$

which reads $a_{(0)(0)} = 0$ and

$$j_0^s j^r\Big((a_{(0)e_1} - 1)^{s+1}(\underline{x}^1 - x^1)^{s+1} + \sum_{l=1}^m (a_{(0)e_l})^{s+1}(\underline{x}^l - x^l)^{s+1} + \dots\Big) = 0, \tag{3}$$

where the dots denote the linear combination of terms $x^\gamma(\underline{x}-x)^\eta$ for other (γ,η), $|\gamma|\le s$, $|\eta|\le r$. Since $r\ge s+1$ (because of the assumption $r>s$), we have

$$a_{(0)e_1}=1 \text{ and } a_{(0)e_l}=0 \text{ for } l=2,\dots,m, \tag{4}$$

where $e_j=(0,\dots,1,\dots,0)$ is the m-tuple with 1 in the j-th position. Then (by (4) and the assumptions (b) and (c)), the map $\varphi^{-1}\times \mathrm{id}_{\mathbf{R}^n}$ sends

$$j^s_0 j^r\Big(\sum_{|\alpha|\le s}\sum_{|\beta|\le r} a_{\alpha\beta}x^\alpha(\underline{x}-x)^\beta\Big)$$

into

$$j^s_0 j^r\Big(\sum_{|\alpha|\le s}\sum_{|\beta|\le r} a_{\alpha\beta}x^\alpha(\underline{x}-x)^\beta+\frac{1}{(r+s)!}\frac{d^{r+s}}{d(x^1)^{r+s}}\varphi^1(0)(\underline{x}^1)^{r+s}+\dots\Big), \tag{5}$$

where the dots denote an expansion of terms $(x^1)^{r+s}$ and $x^\gamma\underline{x}^\eta$ with $|\gamma+\eta|>r+s$. Of course,

$$j^s_0 j^r(\underline{x}^1)^{r+s}\ne 0 \text{ and } j^s_0 j^r(x^1)^{r+s}=0.$$

By the Newton formula, $j^s_0j^r(x^\gamma\underline{x}^\eta)=j^s_0j^r(x^\gamma(\underline{x}-x+x)^\eta)$ is the linear combination of terms $j^s_0j^r(x^{\underline{\gamma}}(\underline{x}-x)^{\underline{\eta}})$ with $|\underline{\gamma}+\underline{\eta}|=|\gamma+\eta|$ (which are zero if $|\gamma+\eta|>r+s$). Then the dots in (5) are zero. Therefore $\varphi^{-1}\times\mathrm{id}_{\mathbf{R}^n}$ does not preserve the right hand side of (1) because $\frac{d^{r+s}}{d(x^1)^{r+s}}\varphi^1(0)\ne 0$ (assumption (d)). On the other hand $\varphi^{-1}\times id_{\mathbf{R}^n}$ preserves the left hand side of (1) because it preserves both the section ρ (assumption (a)) and $j^r_0j^s(x^1,0,\dots,0)$ (as $j^r_0\varphi=\mathrm{id}$ because of assumption (b)). This is a contradiction. ■

Corollary 1. *Let F be a natural vector bundle. Then there is no $\mathcal{FM}_{m,n}$-natural operator $A: F\rightsquigarrow (J^rJ^s, J^sJ^r)$ for $r>s$.*

Proof. It follows from Theorem 1 with $\rho=0$. ■

Corollary 2. *For $r>s$ there is no $\mathcal{FM}_{m,n}$-natural transformation $B: J^rJ^s\to J^sJ^r$.*

Proof. Any such $B: J^rJ^s\to J^sJ^r$ can be treated as the corresponding constant $\mathcal{FM}_{m,n}$-natural operator $B: T\rightsquigarrow (J^rJ^s, J^sJ^r)$. ■

Open problem. Clearly, for $r=s$ Theorem 1 does not hold (we have the identity map $J^rJ^rY\to J^rJ^rY$). On the other hand, we do not know whether Theorem 1 is true in the case $r<s$.

3. Natural transformations $J^rJ^s\to J^sJ^r$ depending on a symplectic structure. From Theorem 1 we obtain easily

Proposition 1. *For $r>s$ there is no $\mathcal{FM}_{2\underline{m},n}$-natural operator $A: SYMP\rightsquigarrow(J^rJ^s, J^sJ^r)$ transforming symplectic structures ω on M into natural transformations $A(\omega): J^rJ^sY\to J^sJ^rY$.*

Proof. In Theorem 1 we put $m = 2\underline{m}$, $F = \bigwedge^2 T^*$,

$$D(M) = \mathrm{SYMP}(M) = \text{the space of local symplectic structures on } M,$$

$$\rho = \sum_{i=1}^{\underline{m}} dx^i \wedge dx^{\underline{m}+i} = \text{the standard symplectic structure on } \mathbf{R}^{2\underline{m}},$$

$$\varphi = \Big(x^1 + \frac{1}{r+s}(x^1)^{r+s}, x^2, \dots, x^{\underline{m}}, \frac{x^{\underline{m}+1}}{1+(x^1)^{r+s-1}}, x^{\underline{m}+2}, \dots, x^{2\underline{m}}\Big).$$

We can see that φ and ρ satisfy assumptions (a)—(d) of Theorem 1. ∎

Now we prove

PROPOSITION 2. *For $s>r$ there is no $\mathcal{FM}_{2\underline{m},n}$-natural operator $A : SYMP \rightsquigarrow (J^rJ^s, J^sJ^r)$ transforming symplectic structures ω on M into natural transformations $A(\omega) : J^rJ^sY \to J^sJ^rY$.*

Proof. Suppose that there exists A in question. Let

$$\tilde{A} : J^r_0J^s\mathbf{R}^{2\underline{m},n} \to J^s_0J^r\mathbf{R}^{2\underline{m},n}$$

be the restriction of $A(\omega^o)$ to the fiber over $0 \in \mathbf{R}^{2\underline{m}}$, where $\omega^o = \sum_{i=1}^{\underline{m}} dx^i \wedge dx^{\underline{m}+i}$ is the standard symplectic structure on $\mathbf{R}^{2\underline{m}}$. Using the invariance of $\tilde{A}$ with respect to the fiber homotheties of $\mathbf{R}^{2\underline{m},n}$ and the homogeneous function theorem we see that $\tilde{A}$ is linear. Next, considering the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-maps

$$\Big(\tau_1x^1, \dots, \tau_{\underline{m}}x^{\underline{m}}, \frac{1}{\tau_1}x^{\underline{m}+1}, \dots, \frac{1}{\tau_{\underline{m}}}x^{2\underline{m}}, y^1, \tau y^2, \dots, \tau y^n\Big)$$

preserving ω^o one can easily show that

$$\tilde{A}(j^r_0j^s(x^1, 0, \dots, 0)) = j^s_0j^r(ax^1 + b\underline{x}^1 + \dots, 0, \dots, 0),$$

where the dots mean some combination of monomials in $x, \underline{x}$ of degree ≥ 2. Taking into account the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$\Big(x^1 + (x^1)^{r+1}, x^2, \dots, x^{\underline{m}}, \frac{x^{\underline{m}+1}}{1+(r+1)(x^1)^r}, x^{\underline{m}+2}, \dots, x^{2\underline{m}}, y^1, \dots, y^n\Big)^{-1}$$

preserving ω^o and $j^r_0j^sx^1$ we deduce

$$j^s_0j^r(a(x^1)^{r+1} + b(\underline{x}^1)^{r+1} + \dots, 0, \dots, 0) = 0, \tag{6}$$

where the dots denote some expression of monomials of degree $\geq r+2$. Applying $(t\mathrm{id}_{\mathbf{R}^{2\underline{m}}} \times \mathrm{id}_{\mathbf{R}^n})$ to both sides of (6) we get

$$j^s_0j^r(a(x^1)^{r+1} + b(\underline{x}^1)^{r+1}, 0, \dots, 0) = 0\ . \tag{7}$$

But

$$j^s_0j^r(\underline{x}^1)^{r+1} = j^s_0j^r(\underline{x}^1 - x^1 + x^1)^{r+1} = j^s_0j^r\Big(\sum_{k=0}^{r} C^{r+1}_k(\underline{x}^1 - x^1)^k(x^1)^{r+1-k}\Big).$$

From (7) and the assumption $s > r$ we get $a + b = 0$ and $b = 0$. Then we have

$$\tilde{A}(j^r_0j^s(x^1, 0, \dots, 0)) = j^s_0j^r(*, 0, \dots, 0), \tag{8}$$

where $*$ denote some linear combination of monomials in $x, \underline{x}$ of degree ≥ 2. Further, using the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1, \dots, x^{2\underline{m}}, y^1 - x^1, y^2, \dots, y^n)$$

preserving ω^o we get from (8)

$$\tilde{A}(j_0^r j^s(x^1 - \underline{x}^1, 0, \dots, 0)) = j_0^s j^r(-\underline{x}^1 + *, 0, \dots, 0). \tag{9}$$

Then using the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1, \dots, x^{2\underline{m}}, y^1 + (y^1)^{s+1}, y^2, \dots, y^n)$$

we obtain from (9)

$$\tilde{A}(j_0^r j^s((\underline{x}^1 - x^1)^{s+1}, 0, \dots, 0)) = j_0^s j^r((\underline{x}^1)^{s+1} + **, 0, \dots, 0), \tag{10}$$

where $**$ is some linear combination of monomials in $x, \underline{x}$ of degree $\geq s+2$. But $j_0^r j^s((\underline{x}^1 - x^1)^{s+1}) = 0$. So from (10) we have

$$j_0^s j^r((\underline{x}^1)^{s+1}) = 0.$$

This is a contradiction as $j_0^s j^r((\underline{x}^1)^{s+1}) = j_0^{r+s}((\underline{x}^1)^{s+1}) \neq 0$. ■

From Corollary 2 and Proposition 3 we obtain

PROPOSITION 3. *For $r \neq s$ there is no natural transformation $A : J^r J^s \to J^s J^r$.*

Proof. It suffices to prove the case $s > r$. Obviously, such A can be treated as a natural operator $A : \mathrm{SYMP} \rightsquigarrow (J^r J^s, J^s J^r)$ constant with respect to elements from SYMP. By Proposition 3 the proof is complete. ■

It is interesting to point out that the only natural transformation $J^r J^s \to J^r J^s$ is the identity, [1].

4. Non-identical natural transformations $J^r J^r Y \to J^r J^r Y$ depending on a symplectic structure Clearly, for $r = s$ we have the trivial $\mathcal{FM}_{2\underline{m},n}$-natural operator $A^o : \mathrm{SYMP} \rightsquigarrow (J^r J^r, J^r J^r)$ such that $A^o(\omega) = \mathrm{id}_{J^r J^r Y}$ for any $\mathcal{FM}_{2\underline{m},n}$-object $Y \to M$ and any symplectic form ω on M. On the other hand, in the case $r = s$ we formulate the following hypothesis.

HYPOTHESIS. There is no non-trivial $\mathcal{FM}_{2\underline{m},n}$-natural operator $A : \mathrm{SYMP} \rightsquigarrow (J^r J^r, J^r J^r)$ transforming symplectic structures ω on M into natural transformations $A(\omega) : J^r J^r Y \to J^r J^r Y$.

It seems that the verification of this hypothesis will be technically complicated. Bellow we prove only

PROPOSITION 4. *The hypothesis is true for $r = s = 1$.*

Proof. Suppose that there exists A in question. Let

$$\tilde{A} : J_0^1 J^1 \mathbf{R}^{2\underline{m},n} \to J_0^1 J^1 \mathbf{R}^{2\underline{m},n}$$

be the restriction of $A(\omega^o)$ to the fiber over $0 \in \mathbf{R}^{2\underline{m}}$, where $\omega^o = \sum_{i=1}^{\underline{m}} dx^i \wedge dx^{\underline{m}+i}$ is the standard symplectic structure on $\mathbf{R}^{2\underline{m}}$. Quite similarly to the proof of Proposition 3,

$\tilde{A}$ is linear and we can write

$$\tilde{A}(j^1_0 j^1(x^1, 0, \dots, 0)) = j^1_0 j^1(ax^1 + b\underline{x}^1 + *, 0, \dots, 0),$$

where * is some linear combination of $x^i \underline{x}^j$ for $i, j = 1, \dots, 2\underline{m}$. Using the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-maps $(tid_{\underline{m}} \times \frac{1}{t} id_{\underline{m}} \times id_{\mathbf{R}^n})$ preserving ω^o we deduce that $* = 0$. Analogously to the proof of Proposition 3 we deduce

$$j^1_0 j^1(b(\underline{x}^1)^2) = 0,$$

which reads $b = 0$. Further, using the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1, \dots, x^{2\underline{m}}, y^1 - x^1, y^2, \dots, y^n)$$

preserving ω^o we get

$$\tilde{A}(j^1 j^1(x^1 - \underline{x}^1, 0, \dots, 0)) = j^1_0 j^1(ax^1 - \underline{x}^1).$$

Taking into account the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1, \dots, x^{2\underline{m}}, y^1 + (y^1)^2, y^2, \dots, y^n)$$

preserving ω^o we obtain

$$\tilde{A}(j^1_0 j^1((x^1 - \underline{x}^1)^2, 0, \dots, 0)) = j^1_0 j^1((ax^1 - \underline{x}^1)^2, 0, \dots, 0).$$

Then as $j^1_0 j^1(\underline{x}^1 - x^1)^2 = 0$ and $j^1_0 j^1(x^1)^2 = 0$, we have

$$0 = \tilde{A}(j^1_0 j^1(0, \dots, 0)) = \tilde{A}(j((x^1 - \underline{x}^1)^2, 0, \dots, 0))$$

and

$$j^1_0 j^1(ax^1 - \underline{x}^1)^2 = -2(a-1) j^1_0 j^1(x^1(\underline{x}^1 - x^1)).$$

This yields $a = 1$, i.e.

$$\tilde{A}(j^1_0 j^1(x^1, 0, \dots, 0)) = j^1_0 j^1(x^1, 0, \dots, 0). \tag{11}$$

Considering the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1 + x^{\underline{m}+1}, x^2, \dots, x^{2\underline{m}}, y^1, \dots, y^n)^{-1}$$

preserving ω^o and $j^1_0 j^1 x^1$ we deduce from (11)

$$\tilde{A}(j^1_0 j^1(x^{\underline{m}+1}, 0, \dots, 0)) = j^1_0 j^1(x^{\underline{m}+1}, 0, \dots, 0). \tag{12}$$

Next, applying the invariance of $\tilde{A}$ with respect to permutations of first $\underline{m}$ base coordinates and respective second $\underline{m}$ base coordinates (preserving ω^o) we deduce from (11) and (12)

$$\tilde{A}(j^1_0 j^1(x^i, 0, \dots, 0)) = j^1_0 j^1(x^i, 0, \dots, 0) \tag{13}$$

for $i = 1, \dots, 2\underline{m}$. Then the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$-map

$$(x^1, \dots, x^{2\underline{m}}, y^1 + x^j y^1, y^2, \dots, y^n)$$

preserving ω^o yields

$$\tilde{A}(j^1_0 j^1(\underline{x}^j x^i, 0, \dots, 0)) = j^1_0 j^1(\underline{x}^j x^i, 0, \dots, 0) \tag{14}$$

for $i, j = 1, \dots, 2\underline{m}$. Taking into account the invariance of $\tilde{A}$ with respect to the $\mathcal{FM}_{2\underline{m},n}$ maps

$$(x^1, \dots, x^{2\underline{m}}, y^1 - x^\beta, y^2, \dots, y^n)$$

preserving ω^o, where β are $2\underline{m}$-tuples of non-negative integers, from the clear equality $\tilde{A}(j^1_0 j^1(0,\dots,0)) = j^1_0 j^1(0,\dots,0)$ we get

$$\tilde{A}(j^1_0 j^1(1,0,\dots,0)) = j^1_0 j^1(1,0,\dots,0) \tag{15}$$

and

$$\tilde{A}(j^1_0 j^1(\underline{x}^i,0,\dots,0)) = j^1_0 j(\underline{x}^i,0,\dots,0) \tag{16}$$

for $i = 1,,\dots,2\underline{m}$. Finally, using the linearity of $\tilde{A}$ and the invariance of $\tilde{A}$ with respect to permutations of fiber coordinates from (14),(15) and (16) we obtain $\tilde{A} = \mathrm{id}$ (these elements generate the vector space $J^1_0 J^1 \mathbf{R}^{2\underline{m},n}$). Then A is trivial because of the Darboux theorem, which is contradiction. ■

Denote by $\mathrm{ex}_\Lambda : J^1J^1 \to J^1J^1$ the involution depending on a classical linear connection Λ on the base of Y, which was constructed by M. Modugno [9]. Clearly, ex_Λ is a non-identic natural equivalence. From Proposition 4 it follows directly the following result, which was also proved in [8].

Proposition 5. *Symplectic structures do not induce canonically classical linear connections.*

Remark 1. Let $\Gamma : Y \to J^1Y$ be a connection on a fibered manifold $Y \to M$. Using the exchange transformation $\mathrm{ex}_\Lambda : J^1J^1 \to J^1J^1$, one can construct a connection $\mathcal{J}^1(\Gamma,\Lambda)$ on $J^1Y \to M$ by means of a classical linear connection Λ on M by

$$\mathcal{J}^1(\Gamma,\Lambda) := (\mathrm{ex}_\Lambda)_Y \circ J^1\Gamma,$$

see [9]. By propositions 1, 2 and 4, it is impossible to construct in a similar way a connection on $J^rY \to M$ from a connection Γ by means of a symplectic form on M.

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