TOPOLOGICAL ALGEBRAS, THEIR APPLICATIONS,
AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 67
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

TOPOLOGICAL ALGEBRAS IN WHICH ALL MAXIMAL TWO-SIDED IDEALS ARE CLOSED

MATI ABEL
University of Tartu, 2 Liivi St., 50409 Tartu, Estonia
E-mail: abel@ut.ee

KRZYSZTOF JAROSZ
Southern Illinois University, Edwardsville, IL 62026, U.S.A.
E-mail: kjarosz@siue.edu
http://www.siue.edu/˜kjarosz

Abstract. We characterize unital topological algebras in which all maximal two-sided ideals are closed.

1. Introduction. In a unital Banach algebra all maximal two-sided ideals are obviously closed. The problem of characterizing other classes of topological algebras with the same property has been investigated by a number of authors. For example for a complex unital commutative Fréchet algebra that property is equivalent to being a Q-algebra ([5], Proposition 17); for a complex unital commutative complete locally m-convex algebra for which the set of all non-zero continuous linear multiplicative functionals is compact in the Gelfand topology our property is equivalent to any proper finitely generated ideal being non-dense ([17], Proposition 2). However there are also non-Q-algebras with that property ([9], p. 81). In this note we provide a characterization of that property in terms of compactness of the structure space and properties of finitely generated ideals. We also show that every proper finitely generated two-sided ideal in a complex unital locally m-pseudoconvex Fréchet algebra is contained in a closed maximal two-sided ideal of A if A is topologically almost commutative or $\operatorname{comm} A \neq A$.

Unless otherwise stated the result is valid in both the real and the complex case.

2000 *Mathematics Subject Classification*: Primary 46H05; Secondary 46H20.

Key words and phrases: topological algebras, Fréchet algebras, locally m-pseudoconvex algebras, simplicial algebras, Q-algebras.

Research partially supported by a grant from the National Research Council.

The paper is in final form and no version of it will be published elsewhere.

2. Definitions and notation. By a topological algebra we mean an algebra A that is also a topological vector space such that the multiplication operation is separately continuous.

We call a topological algebra A *locally pseudoconvex* if it has a base of neighborhoods of zero consisting of *balanced* and *pseudoconvex* sets, that is of sets U for which $\mu U \subset U$, whenever $|\mu| \leq 1$, and $U + U \subset \lambda U$ for some $\lambda \geq 2$. In such an algebra the topology can be defined by a family $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms with $k_\lambda \in (0, 1]$, where a seminorm p is called k-homogeneous if $p(\mu a) = |\mu|^k p(a)$ for any $a \in A$ and any scalar μ ([13], p. 4). If A is also metrizable then one can select a countable family $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms. A locally pseudoconvex algebra A is *locally m-pseudoconvex* if every seminorm p_λ in the family $\{p_\lambda : \lambda \in \Lambda\}$ is submultiplicative, is *locally convex*, if $k_\lambda = 1$ for each $\lambda \in \Lambda$, and *k-normed* if the topology of A can be define by a single k-homogeneous norm. A metrizable and complete algebra is called *a Fréchet algebra*.

For a unital ring A, $M(A)$ denotes the set of all maximal two-sided ideals in A. The space $M(A)$ is equipped with the hk-topology: $S \subset M(A)$ is closed if $S = H(K(S))$, where $K(S)$ is the intersection of all ideals in S and $H(I) = \{M \in M(A) : I \subset M\}$ for any two-sided ideal I of A. If A is also equipped with a compatible topology we consider a subset $m(A)$ of $M(A)$ consisting of closed ideals, we use small letters h and k to indicate operations H and K restricted to $m(A)$.

A unital topological algebra A is called *simplicial* (with respect to two-sided ideals) or *normal* in the sense of Michael ([11], p. 68) if every closed two-sided ideal of A is contained in a closed maximal two-sided ideal of A. For an algebra A let $\mathrm{rad}A$ denote the topological radical ([3], p. 27) of A, $\langle a_1, \ldots, a_n \rangle$ the two-sided ideal of A generated by $a_1, \ldots, a_n \in A$, and $\mathrm{comm}A$ the *commutator ideal* of A, that is, the closure of the two-sided ideal of A generated by the set $\{ab - ba : a, b \in A\}$. A is an *topologically almost commutative algebra* if $A/\mathrm{rad}A$ is commutative.

3. Topological algebras in which all maximal two-sided ideals are closed. The next theorem characterizes topological algebras with closed maximal ideals.

THEOREM 1. *Let A be a unital topological algebra. Then $M(A) = m(A)$ if and only if*

(i) *every proper finitely generated two-sided ideal of A is contained in a closed maximal two-sided ideal of A, and*

(ii) *$m(A)$ is compact in the hk-topology.*

To prove the theorem we will need the compactness of $M(A)$, that fact is well known but typically it is presented only for certain classes of algebras or topological algebras (e.g. [8], p. 301, [12], p. 84, or [16]). Below we show that the standard arguments work for an arbitrary unital ring.

PROPOSITION 1. *Let A be a ring with a unit e. Then the space $M(A)$ is compact in the hk-topology.*

Proof. Assume $\mathcal{F} = \{F_j : j \in J\}$ is a nonempty family of hk-closed subsets of $M(A)$ with $\bigcap_{j \in J} F_j = \varnothing$. We have

$$\begin{aligned}\varnothing &= \bigcap_{j\in J} F_j = \bigcap_{j\in J} HK(F_j) = \bigcap_{j\in J}\{M \in M(A) : K(F_j) \subset M\}\\ &= \{M \in M(A) : K(F_j) \subset M \text{ for all } j\} = \Big\{M \in M(A) : \bigcup_{j\in J} K(F_j) \subset M\Big\}\end{aligned}$$

so the two-sided ideal generated by $\bigcup_{j\in J} K(F_j)$ is equal to the entire ring A. Hence there are elements $a_1, \dots, a_n$ (not necessarily all different) of $\bigcup_{j\in J} K(F_j)$ and $b_1, \dots, b_n$, $b'_1, \dots, b'_n$ in A with $\sum_{k=1}^n b'_k a_k b_k = e$; let $F_1, \dots, F_n \in \mathcal{F}$ be such that $a_k \in K(F_k)$ for $k = 1, 2, \dots, n$. It follows that the two-sided ideal generated by $\bigcup_{k=1}^n K(F_k)$ is equal to the entire ring A and consequently we can reverse the last sequence of equalities:

$$\begin{aligned}\varnothing &= \Big\{M \in M(A) : \bigcup_{k=1}^{n} K(F_k) \subset M\Big\} = \{M \in M(A) : K(F_k) \subset M \text{ for } k = 1, \dots, n\}\\ &= \bigcap_{k=1}^{n}\{M \in M(A) : K(F_k) \subset M\} = \bigcap_{k=1}^{n} HK(F_k) = \bigcap_{k=1}^{n} F_k,\end{aligned}$$

which shows that $M(A)$ is compact. ■

Proof of Theorem 1. Assume $M(A) = m(A)$. In a unital algebra any proper two-sided ideal is contained in a maximal ideal. Hence, if $M(A) = m(A)$ any proper two-sided ideal is contained in a closed maximal two-sided ideal. Since $M(A)$ is compact in the hk-topology so is $m(A)$.

Assume now A is a unital topological algebra which satisfies the conditions of the Theorem and let $M_0 \in M(A)$. Put

$$Z(a) \stackrel{df}{=} \{M \in m(A) : a \in M\}, \qquad \text{for } a \in M_0.$$

Since for any $a_1, \dots, a_n \in M_0$ we have $I \stackrel{df}{=} \langle a_1, \dots, a_n\rangle \subset M_0 \neq A$, by the first condition there is an ideal $M \in m(A)$ with $\{a_1, \dots, a_n\} \subset I \subset M$. Hence $M \in \bigcap_{k=1}^n Z(a_k)$, so $\{Z(a) : a \in M_0\}$ is a collection of hk-closed subsets of $m(A)$ having the finite intersection property. By (ii) there is a closed maximal two-sided ideal $M_1 \in \bigcap_{a\in M_0} Z(a)$. The ideal M_1 contains all elements of M_0 so $M_0 \subset M_1$, however M_0 is maximal so $M_0 = M_1 \in m(A)$. ■

COROLLARY 1. *Let A be a unital simplicial (with respect to two-sided ideals) algebra. If*

(a) *no proper finitely generated two-sided ideal of A is dense in A, and*
(b) *$m(A)$ is compact in the hk-topology,*

then $M(A) = m(A)$.

Proof. Let I be a proper finitely generated two-sided ideal in A. Since $\mathrm{cl}_A(I)$ is a closed two-sided ideal in a simplicial algebra A, by (a) there is $M \in m(A)$ with $\mathrm{cl}_A(I) \subset M$. Hence A has the property (i) of Theorem 1 and consequently $M(A) = m(A)$. ■

EXAMPLE 1. We construct a commutative unital normed algebra A such that both $M(A)$ and $m(A)$ are compact but different. Let A be the space of all C^∞ (real or complex valued) functions f defined on the real line $\mathbb{R}$ such that both restrictions $f_{|(-\infty,0]}$ and $f_{|[1,\infty)}$ are polynomials. We equip A with a submultiplicative norm:

$$\|f\| = \sup\{|f(t)| : 0 \le t \le 1\}.$$

Notice that $\|\cdot\|$ is indeed a norm rather then just a seminorm since if $f = 0$ on the unit segment, then all the derivatives of f at the points 0 and 1 are equal to zero and consequently $f = 0$ on the entire line.

Let I be a maximal proper ideal of A. Assume there is $f \in I$ which does not vanish on $[0,1]$. If p_n is a sequence of polynomials convergent uniformly to $1/f$ on $[0,1]$ then the sequence fp_n converges in A to the unit of that algebra, so I is not closed. On the other hand if all of the functions from I vanish somewhere on $[0,1]$ then the sup norm closure of I in the Banach algebra $C[0,1]$ is a proper ideal so $I \subset \{f : f(x_I) = 0\}$ for some x_I in $[0,1]$. Hence $m(A)$ can be identified with the unit segment.

We show that the hk-topology on $[0,1]$, denoted by τ_{hk}, coincides with the usual topology τ_{std} of that segment. Clearly any hk-closed set is closed in the standard topology so $\tau_{hk} \subset \tau_{std}$. On the other hand, since for any $0 \le a < b \le 1$ there is a function $f_0 \in A$ such that $f_0(t) \ne 0$ exactly when $a < t < b$, we have $hk([0,1] \setminus (a,b)) = [0,1] \setminus (a,b)$ so all of the segments (a,b) are hk-open; as such segments form a basis of τ_{std} we get $\tau_{std} \subset \tau_{hk}$.

Hence $m(A) = [0,1]$ while $M(A)$ is much bigger containing as a proper subset all of $\mathbb{R}$.

4. Properties of locally m-pseudoconvex Fréchet algebras. The next theorem shows that the first condition considered in the previous section is valid for a large class of topological algebras.

THEOREM 2. *Let A be a complex unital locally m-pseudoconvex Fréchet algebra. If* $\operatorname{comm}A \ne A$ *or A is topologically almost commutative then every proper finitely generated two-sided ideal in A is contained in a closed maximal two-sided ideal of A.*

The idea of proof comes from [6]; we first will need the following lemma ([8], p. 233).

LEMMA 1. *Let A and B be topological algebras with unit elements e_A and e_B, respectively and let h be a homomorphism from A onto a dense subset of B with $h(e_A) = e_B$. Assume $a_1, \dots, a_n, c_1, \dots, c_n \in A$ and $d_1, \dots, d_n \in B$ are such that*

$$\sum_{v=1}^{n} c_v a_v = e_A \ \textit{and} \ \sum_{v=1}^{n} d_v h(a_v) = e_B.$$

Then for any neighborhood O of zero in B there exist $b_1, \dots, b_n \in A$ such that

$$\sum_{v=1}^{n} b_v a_v = e_A \ \textit{and} \ h(b_v) \in d_v + O, \ \textit{for} \ v = 1, \dots, n.$$

Proof of Theorem 2. Assume A is a complex locally m-pseudoconvex Fréchet algebra with a unit element e_A and the topology given by a family $\{p_n : n \in \mathbb{N}\}$ of k_n-homogeneous submultiplicative seminorms, with $k_n \in (0,1]$, for $n \in \mathbb{N}$. We may also assume ([7], Proposition 4.6.1), that

$$p_n(a)^{\frac{1}{k_n}} \leqslant p_{n+1}(a)^{\frac{1}{k_{n+1}}}, \quad \text{for } n \in \mathbb{N},\, a \in A,\, k_{n+1} \leqslant k_n. \tag{1}$$

Put

$$B \stackrel{df}{=} \begin{cases} A/\mathrm{comm}A & \text{if } \mathrm{comm}A \neq A \\ A/\mathrm{rad}A & \text{if } A \text{ is topologically almost commutative,} \end{cases}$$

let $\kappa : A \to B$ be the canonical homomorphism, and let

$$q_n(b) \stackrel{df}{=} \inf \{p_n(a) : \kappa(a) = b, a \in A\}, \quad n \in \mathbb{N}, b \in B.$$

Since comm$A$ and rad$A$ are closed two-sided ideals, B is a commutative complex locally m-pseudoconvex Fréchet algebra with a unit element $e_B = \kappa(e_A)$ and the topology given by the family $\{q_n : n \in \mathbb{N}\}$ of k_n-homogeneous submultiplicative seminorms. Furthermore by (1)

$$q_n(b)^{\frac{1}{k_n}} \leqslant q_{n+1}(b)^{\frac{1}{k_{n+1}}}, \quad \text{for } n \in \mathbb{N}, b \in B, k_{n+1} \leqslant k_n. \tag{2}$$

Next for $n \in \mathbb{N}$ let π_n be the canonical homomorphism of B onto $B_n \stackrel{df}{=} B/\ker q_n$ and let r_n be the quotient k_n-homogeneous norm on B_n. Let $\widetilde{B}_n$ be the completion of B_n, let $\tilde{r}_n$ be the extension of r_n to a k_n-homogeneous norm on $\widetilde{B}_n$, and denote by $\mu_n : B \to \widetilde{B}_n$ the composition of π_n with the embedding into $\widetilde{B}_n$.

For $m \leqslant n$ we have $B_n \subset B_m$ and by (2)

$$r_m(\pi_m(b))^{\frac{1}{k_m}} = q_m(b)^{\frac{1}{k_m}} \leqslant q_n(b)^{\frac{1}{k_n}} = r_n(\pi_n(b))^{\frac{1}{k_n}}, \quad b \in B, \ m \leqslant n. \tag{3}$$

For $n, m \in \mathbb{N}$ with $m \leqslant n$ let

$$f_{m,n} : B_n \to B_m, \quad \text{be defined by } f_{m,n}(\pi_n(b)) = \pi_m(b).$$

By (3) the homomorphism f_{mn} is a uniformly continuous map from B_n onto B_m ([7], Theorem 4.3.11) so it can be continuously extended to a homomorphism $\tilde{f}_{m,n}$ from the commutative k_n-Banach algebra $\widetilde{B}_n$ onto a dense subalgebra of k_m-Banach algebra $\widetilde{B}_m$ ([10], Proposition 5, p. 129). We have

$$\tilde{f}_{l,n} = \tilde{f}_{l,m} \circ \tilde{f}_{m,n}, \text{ and } \tilde{r}_m(\tilde{f}_{m,n}(\tilde{b}_n))^{\frac{1}{k_m}} \leqslant \tilde{r}_n(\tilde{b}_n)^{\frac{1}{k_n}}, \quad \text{for } l \leqslant m \leqslant n, \ \tilde{b}_n \in \widetilde{B}_n. \tag{4}$$

To finish the proof assume I is a finitely generated ideal in A and let $a_1, \ldots, a_s \in A$ be such that $I = \langle a_1, \ldots, a_s \rangle$. Suppose that

$$\langle \mu_n(\kappa(a_1)), \ldots, \mu_n(\kappa(a_s)) \rangle = \tilde{B}_n, \quad \text{for all } n \in \mathbb{N},$$

and let $\tilde{b}_1^n, \ldots, \tilde{b}_s^n \in \tilde{B}_n$ be such that

$$\sum_{v=1}^{s} \tilde{b}_v^n \mu_n(\kappa(a_v)) = \mu_n(\kappa(e_A)) = e_{\widetilde{B}_n}.$$

Put $\tilde{d}_v^{n-1} = \tilde{f}_{n-1,n}(\tilde{b}_v^n)$. We have

$$\sum_{v=1}^{s} \tilde{d}_v^{n-1} \mu_{n-1}(\kappa(a_v)) = \mu_{n-1}(\kappa(e_A)) = e_{\widetilde{B}_{n-1}}.$$

Hence by Lemma 1, with $A = \tilde{B}_n$, $B = \tilde{B}_{n-1}$, $O = \{x \in \tilde{B}_{n-1} : \tilde{r}_{n-1}(x) < 2^{-n}\}$, and $h = \tilde{f}_{n-1,n}$ there are $\tilde{d}_1^n, \ldots, \tilde{d}_s^n \in \tilde{B}_n$ such that

$$\sum_{v=1}^{s} \tilde{d}_v^n \mu_n(\kappa(a_v)) = e_{\widetilde{B}_n} \tag{5}$$

and

$$\tilde{r}_{n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1}) < 2^{-n}, \quad n \in \mathbb{N},\ v = 1, \ldots, s.$$

By (4) for $m \leqslant n-1$ and $v = 1, \ldots, s$

$$\begin{aligned}\tilde{r}_m(\tilde{f}_{m,n}(\tilde{d}_v^n) - \tilde{f}_{m,n-1}(\tilde{d}_v^{n-1})) &= \tilde{r}_m(\tilde{f}_{m,n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n)) - \tilde{f}_{m,n-1}(\tilde{d}_v^{n-1}))\\ &= \tilde{r}_m(\tilde{f}_{m,n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1}))\\ &\leqslant (\tilde{r}_{n-1}(\tilde{f}_{n-1,n}(\tilde{d}_v^n) - \tilde{d}_v^{n-1}))^{\frac{k_m}{k_{n-1}}} \leqslant \left(\frac{1}{2^n}\right)^{\frac{k_m}{k_{n-1}}}\end{aligned}$$

therefore, since $k_l \leqslant k_n$ for $l \geqslant n$, for any $n \leqslant p < q$ we get

$$\tilde{r}_n(\tilde{f}_{n,q}(\tilde{d}_v^q) - \tilde{f}_{n,p}(\tilde{d}_v^p)) \leqslant \sum_{t=p+1}^{q} \tilde{r}_n(\tilde{f}_{n,t}(\tilde{d}_v^t) - \tilde{f}_{n,t-1}(\tilde{d}_v^{t-1})) < \sum_{t=p+1}^{q} \left(\frac{1}{2^t}\right)^{\frac{k_n}{k_{t-1}}} \leqslant \sum_{t=p+1}^{q} \left(\frac{1}{2}\right)^t.$$

Hence, as $\sum_{t=0}^{\infty}(\frac{1}{2})^t$ is convergent, $(\tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}))_{l\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{B}_n$, for any $n \in \mathbb{N}$ and $v = 1, \ldots, s$; as $\tilde{B}_n$ are complete we may put

$$\lim_{l\to\infty} \tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}) \stackrel{df}{=} \tilde{e}_v^n \in \tilde{B}_n.$$

By (4) $(\tilde{e}_v^n)_{n\in\mathbb{N}} \in \varprojlim \left\{\tilde{B}_n, \tilde{f}_{m,n}, \mathbb{N}\right\}$ for each $v = 1, \ldots, s$. Since

$$B \ni b \mapsto (\mu_n(b)) \in \varprojlim\{\tilde{B}_n, \tilde{f}_{m,n}, \mathbb{N}\}$$

is a surjective topological isomorphism ([2], pp. 18–22, or [7], Theorem 4.5.3) there are elements $e_1, \ldots, e_s \in A$ such that

$$\mu_n(\kappa(e_v)) = \tilde{e}_v^n = \lim_{l\to\infty} \tilde{f}_{n,n+l}(\tilde{d}_v^{n+l}), \quad \text{for all } v \text{ and } n. \tag{6}$$

Therefore by (5) and (6) for each $n \in \mathbb{N}$ we get

$$\begin{aligned}p_n\Big(\sum_{v=1}^{s} e_v a_v - e_A\Big) &= \tilde{r}_n\Big(\mu_n\Big(\kappa\Big(\sum_{v=1}^{s} e_v a_v - e_A\Big)\Big)\Big)\\ &= \tilde{r}_n\Big(\sum_{v=1}^{s} \mu_n(\kappa(e_v))\mu_n(\kappa(a_v)) - \mu_n(\kappa(e_A))\Big)\\ &= \tilde{r}_n\Big(\sum_{v=1}^{s} \lim_{l\to\infty} \tilde{f}_{n,n+l}(\tilde{d}_v^{n+l})\mu_n(\kappa(a_v)) - \mu_n(\kappa(e_A))\Big)\\ &= \lim_{l\to\infty} \tilde{r}_n\Big(\tilde{f}_{n,n+l}\Big(\sum_{v=1}^{s} \tilde{d}_v^{n+l}\mu_{n+l}(\kappa(a_v)) - \mu_{n+l}(\kappa(e_A))\Big)\Big) = 0,\end{aligned}$$

hence $I = A$. So for every proper finitely generated ideal I of A there is $n_0 \in \mathbb{N}$ such that $\mu_{n_0}(\kappa(I)) \neq \tilde{B}_{n_0}$. Because $\mu_{n_0}(\kappa(I))$ is an ideal in a complex unital commutative k_{n_0}-Banach algebra $\tilde{B}_{n_0}$ there is a nontrivial continuous multiplicative linear functional φ_0 on $\tilde{B}_{n_0}$ such that $\mu_{n_0}(\kappa(I)) \subset \ker\varphi_0$ ([14], Proposition 4.3, or [15], Theorem 4.1). Let $\phi = \varphi_0 \circ \mu_{n_0} \circ \kappa$. Then ϕ is a nontrivial continuous linear multiplicative functional on A, $\ker\phi$ is a closed maximal two-sided ideal in A and $I \subset \ker\phi$. ∎

COROLLARY 2. *Assume A is a complex unital locally m-pseudoconvex Fréchet algebra that is topologically almost commutative, or such that* $\mathrm{comm}A \neq A$. *Then $M(A) = m(A)$ if and only if $m(A)$ is compact in the hk-topology.*

5. Simplicial algebras with compact topological strong structure space. In this section we discuss the second condition listed in Theorem 1. The following proposition generalizes Corollary 3.9 of ([1]); notice that every commutative Q-algebra satisfies the condition (7) below.

PROPOSITION 2. *Let A be a unital simplicial algebra. If*

$$\mathrm{cl}_A\Big(\bigcup_{M\in m(A)} M\Big) \subset \bigcup_{M\in M(A)} M, \tag{7}$$

then $m(A)$ is compact in the hk-topology.

Proof. Suppose $m(A)$ is not compact and let $(F_\gamma)_{\gamma\in\Gamma}$ be a family of hk-closed subsets of $m(A)$ with the finite intersection property and such that $\bigcap_{\gamma\in\Gamma} F_\gamma = \varnothing$. Let J be the two-sided ideal in A generated by $\{k(F_\gamma) : \gamma \in \Gamma\}$. Since A is simplicial, if $\mathrm{cl}_A(J) \neq A$ then $\mathrm{cl}_A(J) \subset M$ for some $M \in m(A)$. As $k(F_\gamma) \subset \mathrm{cl}_A(J) \subset M$ for every $\gamma \in \Gamma$, then $M \in h(k(F_\gamma)) = F_\gamma$ for each $\gamma \in \Gamma$, which contradicts our assumption. Hence $\mathrm{cl}(J) = A$.

Fix $a \in A$ and a neighborhood of zero O in A. Let O' be a neighborhood of zero in A such that $O'a \subset O$. Since A is unital $(e_A + O') \cap J \neq \varnothing$ and we can find $o \in O'$, $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $a_{\gamma_k} \in k(F_{\gamma_k})$ such that

$$e_A + o = \sum_{k=1}^{n} a_{\gamma_k} \in \sum_{k=1}^{n} k(F_{\gamma_k}).$$

Let $M_o \in \bigcap_{k=1}^{n} F_{\gamma_k}$. Since $M_o \in F_{\gamma_k} = h(k(F_{\gamma_k}))$ so $k(F_{\gamma_k}) \subset M_o$, and we get

$$e_A + o \in M_o \subset \bigcup_{M\in m(A)} M.$$

Hence

$$(e_A + O) \bigcap \Big(\bigcup_{M\in m(A)} M\Big) \neq \varnothing, \quad \text{for any neighborhood } O \text{ of zero in } A.$$

Consequently

$$e_A \in \mathrm{cl}_A\Big(\bigcup_{M\in m(A)} M\Big) \subset \bigcup_{M\in M(A)} M,$$

which is impossible. ∎

COROLLARY 3. *Assume A is a complex unital locally m-pseudoconvex Fréchet algebra that is topologically almost commutative, or such that* $\mathrm{comm}A \neq A$. *If A satisfies the condition* (7) *then $M(A) = m(A)$.*

PROPOSITION 3. *A commutative unital simplicial topological algebra A is a Q-algebra if and only if it satisfies* (7) *and $M(A) = m(A)$.*

Proof. If A is a commutative Q-algebra then $M(A) = m(A)$ and (7) holds because $A \backslash InvA$ is equal to the union of all ideals from $M(A)$. If A is a commutative unital topological algebra and $M(A) = m(A)$ then the union of all ideals from $m(A)$ and the union of all ideals from $M(A)$ coincides; by (7) the last union is closed so A is a Q-algebra. ∎

It is clear that every Q-algebra is simplicial. The next result characterizes those simplicial algebras which are Q-algebras.

THEOREM 3. *Let A be a commutative unital simplicial algebra. Then A is a Q-algebra if and only if A satisfies* (7) *and condition (a) of Corollary 1.*

Proof. If A is a commutative Q-algebra then $M(A) = m(A)$ and $\bigcup_{M \in M(A)} M = A \backslash \mathrm{Inv}A$ is closed in A.

Assume A is a commutative unital simplicial algebra and I is a proper finitely generated ideal in A. If A satisfies the condition (a) of Corollary 1 then there is an ideal $M \in m(A)$ such that $I \subset M$. By Theorem 1 and Proposition 2 we have $M(A) = m(A)$. Hence, by (7) $\mathrm{Inv}A = A \backslash \bigcup_{M \in M(A)} M$ is open in A and consequently A is a Q-algebra. ∎

COROLLARY 4. *Let A be a commutative unital complex locally m-pseudoconvex Fréchet algebra. Then A is a Q-algebra if and only if A satisfies* (7).

Proof. By Theorem 4.2 of [4] algebra A is simplicial and by Theorem 2 it satisfies condition (a) of Corollary 1. Hence the result follows from Theorem 3. ∎

References

[1] Mart Abel, *Structure of Gelfand-Mazur algebras*, Dissertationes Math. Univ. Tartuensis 31, Tartu Univ. Press, Tartu, 2003.

[2] Mati Abel, *Projective limits of topological algebras*, Tartu. Ülik. Toimetised 836 (1989), 3–27 (in Russian).

[3] Mati Abel, *Descriptions of the topological radical in topological algebras*, in: General topological algebras (Tartu, 1999), Math. Stud. (Tartu) 1, Est. Math. Soc., Tartu, 2001, 25–31.

[4] Mati Abel, *Inductive limits of Gelfand-Mazur algebras*, Int. J. Pure Appl. Math. 16 (2004), 363–378.

[5] M. Akkar et C. Nacir, *Continuité automatique dans les limites inductives localement comvexes et Q-algèbres de Fréchet*, Ann. Sci. Math. Québec 19 (1995), 85–96.

[6] R. F. Arens, *Dense inverse limit rings*, Michigan Math. J. 5 (1958), 169–182.

[7] V. K. Balachandran, *Topological Algebras*, North-Holland Math. Stud. 185, Elsevier, Amsterdam, 2000.

[8] E. Beckenstein, L. Narici and Ch. Suffel, *Topological Algebras.* North-Holland Math. Studies 24, North-Holland, Amsterdam-New York-Oxford, 1977.

[9] R. Choukri, *Sur certaines questions concernant les Q-algèbres*, Extracta Math. 16 (2001), 79–82.

[10] J. Horváth, *Topological Vector Spaces and Distributions*, Vol. I, Addison-Wesley, Reading, Mass., 1966.

[11] E. A. Michael, *Locally multiplicatively-convex topological algebras.* Mem. Amer. Math. Soc., 1952.

[12] C. E. Rickart, *General Theory of Banach Algebras*, D. van Nostrand, Princeton, 1960.

[13] L. Waelbroeck, *Topological Vector Spaces and Algebras*, Lecture Notes in Math. 230, Springer-Verlag, Berlin, 1973.

[14] W. Żelazko, *Metrical generalizations of Banach algebras*, Dissertationes Math. 47 (1965).

[15] W. Żelazko, *Selected Topics in Topological Algebras*, Lect. Notes Ser. 31, Aarhus Univ., 1971.

[16] W. Żelazko, *Banach Algebras*, Elsevier, Amsterdam and PWN–Polish Scientific Publishers, Warsaw, 1973.

[17] W. Żelazko, *On maximal ideals in commutative m-convex algebras*, Studia Math. 57 (1976), 291–298.