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ON (CO)HOMOLOGY OF TRIANGULAR BANACH ALGEBRAS

MOHAMMAD SAL MOSLEHIAN

Ferdowsi University of Mashhad and
Institute for Studies in Theoretical Physics and Mathematics (IPM), Iran
E-mail: msalm@math.um.ac.ir
Home: http://www.um.ac.ir/˜moslehian/

Abstract. Suppose that A and B are unital Banach algebras with units 1_A and 1_B, respectively, M is a unital Banach A,B-module, $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ is the triangular Banach algebra, X is a unital $\mathcal{T}$-bimodule, $X_{AA} = 1_A X 1_A$, $X_{BB} = 1_B X 1_B$, $X_{AB} = 1_A X 1_B$ and $X_{BA} = 1_B X 1_A$. Applying two nice long exact sequences related to A, B, $\mathcal{T}$, X, X_{AA}, X_{BB}, X_{AB} and X_{BA} we establish some results on (co)homology of triangular Banach algebras.

1. Introduction. Topological homology arose from the problems concerning extensions by H. Kamowitz who introduced the Banach version of Hochschild cohomology groups in 1962 [11], derivations by R. V. Kadison and J. R. Ringrose [9, 10] and amenability by B. E. Johnson [8] and has been extensively developed by A. Ya. Helemskii and his school. In addition, this area includes a lot of problems concerning automorphisms, fixed point theorems, perturbations, invariant means, topology of spectrum, … [6].

This article deals with the cohomology and homology of triangular Banach algebras, i.e. algebras of the form $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ in which A and B are unital Banach algebras and M is a unital Banach A,B-module. These algebras were introduced by Forrest and Marcoux [1], motivated by work of Gilfeather and Smith in [4]. Forrest and Marcoux also studied and directly computed some cohomology groups of triangular Banach algebras (see [2] and [3]). In this paper, after some preliminaries, we present two long exact sequences and apply them to give some significant isomorphisms and vanishing theorems.

2. Preliminaries. We begin with some observations concerning cohomology and homology of Banach algebras. Some sources of references are [6] and [7].

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Let **Lin** denote the category of linear spaces and linear operators. A sequence $\cdots \leftarrow X_n \xleftarrow{d_n} X_{n+1} \leftarrow \cdots$, $\mathcal{X} = \{X, d\}$ (resp. $\cdots \to X^n \xrightarrow{\delta^n} X^{n+1} \to \cdots$, $\mathcal{X} = \{X, \delta\}$) in a subcategory of **Lin** is said to be a (chain) complex (resp. (cochain) complex) if $d_{n-1} \circ d_n = 0$ (resp. $\delta^n \circ \delta^{n-1} = 0$).

Suppose that A is a Banach algebra and X is a Banach A-bimodule.

For $n = 0, 1, 2, \ldots$, let $C^n(A, X)$ be the Banach space of all bounded n-linear mappings from $A \times \cdots \times A$ into X together with multilinear operator norm

$$\|f\| = \sup\{\|f(a_1, \ldots, a_n)\|;\ a_i \in A, \|a_i\| \leq 1, 1 \leq i \leq n\},$$

and $C^0(A, X) = X$. The elements of $C^n(A, X)$ are called n-dimensional cochains. Consider the sequence

$$0 \to C^0(A, X) \xrightarrow{\delta^0} C^1(A, X) \xrightarrow{\delta^1} \cdots \qquad (\widetilde{C}(A, X)),$$

where $\delta^0 x(a) = ax - xa$ and for $n = 0, 1, 2, \ldots$

$$\begin{aligned} \delta^n f(a_1, \ldots, a_{n+1}) = & \ a_1 f(a_2, \ldots, a_{n+1}) \\ & + \sum_{k=1}^{n} (-1)^k f(a_1, \ldots, a_{k-1}, a_k a_{k+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1} \end{aligned}$$

where $x \in X$, $a, a_1, \ldots, a_{n+1} \in A$, $f \in C^n(A, X)$.

It is straightforward to verify that the above sequence is a complex. $\widetilde{C}(A, X)$ is called the standard cohomology complex or Hochschild-Kamowitz complex for A and X. The nth cohomology group of $\widetilde{C}(A, X)$ is said to be the n-dimensional (ordinary or Hochschild) cohomology group of A with coefficients in X and denoted by $H^n(A, X)$. The spaces $Ker\delta^n$ and $Im\delta^{n-1}$ are denoted by $Z^n(A, X)$ and $B^n(A, X)$, and their elements are called n-dimensional cocycles and n-dimensional coboundaries, respectively. Hence $H^n(A, X) = Z^n(A, X)/B^n(A, X)$. Note that $H^n(A, X)$, generally speaking, is a complete seminormed space.

Assume that $C_0(A, X) = X$ and for $n = 1, 2, \ldots$

$$C_n(A, X) = \underbrace{A \hat{\otimes} \cdots \hat{\otimes} A}_{n} \hat{\otimes} X$$

in which $\hat{\otimes}$ denotes the projective tensor product of Banach spaces. The elements of $C_n(A, X)$ are called n-dimensional chains. Consider the complex

$$0 \leftarrow C_0(A, X) \xleftarrow{d_0} C_1(A, X) \xleftarrow{d_1} \cdots \qquad (\widehat{C}(A, X)),$$

where

$$\begin{aligned} d_n(a_1 \otimes \ldots \otimes a_{n+1} \otimes x) = & \ a_2 \otimes \ldots \otimes a_{n+1} \otimes a_1 x \\ & + \sum_{k=1}^{n} (-1)^k a_1 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1} \otimes x \\ & + (-1)^{n+1} a_1 \otimes \ldots \otimes a_n \otimes x a_{n+1}. \end{aligned}$$

The n-th homology group of $\widehat{C}(A, X)$ is called the (ordinary) homology group of A with coefficients in X. It is denoted by $H_n(A, X)$ which is a complete seminormed space.

The dual X^* of the Banach A-bimodule X is again a Banach A-bimodule with respect to the following actions:

$$(af)(x) = f(xa), \quad (fa)(x) = f(ax); \quad f \in X^*, a \in A, x \in X,$$

In particular, A^* is a Banach bimodule over A.

A complex $\mathcal{X} = \{X, d\}$ in a category of Banach modules is called admissible if it splits as a complex of Banach spaces and continuous linear operators, i.e. the kernels of all its morphisms are topologically complemented.

An additive functor F is said to be exact if for every admissible complex $\mathcal{X} = \{X, d\}$ the complex $\underline{F}(\mathcal{X}) = \{F(X), F(d)\}$ is exact in the category **Lin**. Notice that $\underline{F}$ is a functor.

A unital left Banach module P over a unital Banach algebra A is said to be projective if the functor ${}_Ah(P, ?)$ is exact. Recall that for left Banach A-modules X and Y, ${}_Ah(?, ?)$ takes left A-modules X and Y to ${}_Ah(X, Y) = \{f : X \to Y; f$ is a bounded left A-module map$\}$. Indeed ${}_Ah(?, ?)$ is a bifunctor contravariant in the first variable and covariant in the second.

A left Banach A-module (resp. right Banach A-module, Banach A, B-bimodule) X is called projective if X as a left Banach unital A_+-module (resp. left Banach unital A_+^{op}-module, left Banach unital $A_+\hat{\otimes}B_+^{op}$-module) is projective, where $A_+ = A \oplus \mathbf{C}$ denotes the unitization of the Banach algebra $A.A^{op}$, the so-called opposite to A, is the space A equipped with the multiplication $a \circ b = ba$.

A complex $0 \leftarrow X_0 \overset{d_0}{\leftarrow} X_1 \leftarrow \cdots$ $(\mathcal{X})$ is called a resolution of the A-module X if the complex $0 \leftarrow X \overset{\varepsilon}{\leftarrow} X_0 \overset{d_0}{\leftarrow} X_1 \leftarrow \cdots$ is admissible. By a projective resolution we mean one in which the X_i's are projective.

Every left A-module X admits sufficiently many projective resolutions, especially it admits the normalized bar-resolution $\mathcal{B}(X)$ as follows:

Consider free modules $B_n(X) = A_+\hat{\otimes}(\underbrace{A\hat{\otimes}\cdots\hat{\otimes}A}_{n}\hat{\otimes}X)$ and $B_0(X) = A_+\hat{\otimes}X$, and operators $\pi : A_+\hat{\otimes}X \to X$ and $d_n : B_{n+1}(X) \to B_n(X)$, well-defined by

$$\begin{aligned}
\pi(a \otimes x) &= ax, \\
d_n(a \otimes a_1 \otimes \ldots \otimes a_{n+1} \otimes x) &= aa_1 \otimes a_2 \otimes \ldots \otimes a_{n+1} \otimes x \\
&\quad + \sum_{k=1}^{n} (-1)^k a \otimes a_1 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1} \otimes x \\
&\quad + (-1)^{n+1} a \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}x.
\end{aligned}$$

Then the sequence $0 \leftarrow X \overset{\pi}{\leftarrow} B_0(X) \overset{d_0}{\leftarrow} B_1(X) \overset{d_1}{\leftarrow} \cdots$ is a projective resolution of X. We denote the complex $0 \leftarrow B_0(X) \overset{d_0}{\leftarrow} B_1(X) \overset{d_1}{\leftarrow} \cdots$ by $\mathcal{B}(X)$. In fact $\mathcal{B}$ induces a functor.

Let F be an additive functor. Then the functor $F_n = H_n \circ \underline{F} \circ \mathcal{B}$ is called the n-th projective derived functor of F. F_n is independent of the choice of resulotion. The projective derived cofunctors could be defined in a similar way.

For a left Banach A-module Y, let $Ext_A^n(?, Y)$ denote the n-th projective derived cofunctor of ${}_Ah(?, Y)$. Given a right Banach A-module X, denote the n-th projective derived functor of $X\overset{\wedge}{\otimes}_A ?$ by $Tor_n^A(X,?)$. Recall that for a right Banach A-module X and a left Banach A-module Y the projective tensor product of modules X and Y is defined to be the quotient space $X\hat{\otimes}_A Y = (X\hat{\otimes}Y)/L$ where L denotes the closed linear span of all elements of the form $xa\otimes y - x\otimes ay$ $(x\in X, a\in A, y\in Y)$. In fact $?\hat{\otimes}_A ?$ is a bifunctor covariant in both variables.

3. Main results. Suppose that A and B are unital Banach algebras with units 1_A and 1_B, and Banach space M is a unital Banach A,B-module. Then

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix};\ a\in A, m\in M, b\in B \right\}$$

with the usual 2×2 matrix addition and formal multiplication equipped with the norm $\| \left[\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right] \| = \|a\| + \|m\| + \|b\|$ is a Banach algebra which is called a triangular Banach algebra [1].

Let X is a unital Banach $\mathcal{T}$-bimodule, $X_{AA} = 1_A X 1_A, X_{BB} = 1_B X 1_B, X_{AB} = 1_A X 1_B$ and $X_{BA} = 1_B X 1_A$.

Applying homological techniques we can establish the following long exact sequences (see [5]):

$$\begin{aligned}
&0 \overset{\pi^{-1}}{\to} H^0(\mathcal{T},X) \overset{\phi^0}{\to} H^0(A,X_{AA})\oplus H^0(B,X_{BB}) \overset{\delta^0}{\to} Ext^0_{A\hat{\otimes}B^{op}}(M,X_{AB})\\
&\overset{\pi^0}{\to} H^1(\mathcal{T},X) \overset{\phi^1}{\to} H^1(A,X_{AA})\oplus H^1(B,X_{BB}) \overset{\delta^1}{\to} Ext^1_{A\hat{\otimes}B^{op}}(M,X_{AB})\\
&\overset{\pi^1}{\to} H^2(\mathcal{T},X) \overset{\phi^2}{\to} H^2(A,X_{AA})\oplus H^2(B,X_{BB}) \overset{\delta^2}{\to} Ext^2_{A\hat{\otimes}B^{op}}(M,X_{AB}),\\
\to \cdots &\overset{\rho_2}{\to} Tor_2^{A\hat{\otimes}B^{op}}(M,X_{BA}) \overset{d_1}{\to} H_2(A,X_{AA})\oplus H_2(B,X_{BB}) \overset{\psi_2}{\to} H_2(\mathcal{T},X)\\
&\overset{\rho_1}{\to} Tor_1^{A\hat{\otimes}B^{op}}(M,X_{BA}) \overset{d_1}{\to} H_1(A,X_{AA})\oplus H_1(B,X_{BB}) \overset{\psi_1}{\to} H_1(\mathcal{T},X)\\
&\overset{\rho_0}{\to} Tor_0^{A\hat{\otimes}B^{op}}(M,X_{BA}) \overset{d_0}{\to} H_0(A,X_{AA})\oplus H_0(B,X_{BB}) \overset{\psi_0}{\to} H_0(\mathcal{T},X) \overset{\rho_{-1}}{\to} 0,
\end{aligned}$$

Using these nice sequences we shall obtain some significant results:

THEOREM 1 ([12, Proposition 2.11]). *Let* $X_{AB}=0$ *and* $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$. *Then* $H^n(\mathcal{T},X) \simeq H^n(A,X_{AA})\oplus H^n(B,X_{BB})$ *for all* $n\geq 0$.

Proof. If $X_{AB}=0$, then $Ext^{n-1}_{A\hat{\otimes}B^{op}}(M,X_{AB}) = Ext^{n}_{A\hat{\otimes}B^{op}}(M,X_{AB}) = 0$.

Hence $0 \overset{\pi^{n-1}}{\to} H^n(\mathcal{T},X) \overset{\phi^n}{\to} H^n(A,X_{AA})\oplus H^n(B,X_{BB}) \overset{\delta^n}{\to} 0$.

COROLLARY 1 ([2, Corollary 3.5]). $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ *is weakly amenable iff so are* A *and* B.

Proof. $X = \mathcal{T}^*$ is a Banach $\mathcal{T}$-bimodule for which clearly $X_{AA} = A^*, X_{BB} = B^*, X_{AB} = 0$ and $X_{BA} = M^*$. Then the previous theorem, with $n=1$, implies that

$$H^1(\mathcal{T},\mathcal{T}^*) = H^1(A,A^*)\oplus H^1(B,B^*).$$

Hence $\mathcal{T}$ is weakly amenable iff so are A and B.

THEOREM 2. *Let* $X_{BA}=0$ *and* $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$. *Then* $H_n(\mathcal{T},X)\simeq H_n(A,X_{AA})\oplus H_n(B,X_{BB})$ *for all* $n\geq 0$.

Proof. If $X_{BA} = 0$, then $Tor_{n-1}^{A\hat{\otimes}B^{op}}(M, X_{AB}) = Tor_n^{A\hat{\otimes}B^{op}}(M, X_{AB}) = 0$. Hence $0 \xrightarrow{d_n} H_n(A, X_{AA}) \oplus H_n(B, X_{BB}) \xrightarrow{\psi_n} H_n(\mathcal{T}, X) \xrightarrow{\rho_{n-1}} 0$.

COROLLARY 2. *$H_n(\mathcal{T}, \mathcal{T}) \simeq H_n(A, A) \oplus H_n(B, B)$ if $\mathcal{T} = [\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}]$.*

Proof. For $X = \mathcal{T}$ we have $X_{AA} = A, X_{BB} = B, X_{AB} = M$ and $X_{BA} = 0$.

COROLLARY 3. *If $\mathcal{T} = [\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}]$, then $H_n(\mathcal{T}, M) = 0$. In particular, with $\mathcal{T}_m = \left[\begin{smallmatrix} A & \mathcal{T}_{m-1} \\ 0 & B \end{smallmatrix}\right]$ and $\mathcal{T}_0 = \mathcal{T}$, we conclude that $H_n(\mathcal{T}, \mathcal{T}_m) = 0$.*

Proof. Note that $X_{AA} = A, X_{BB} = B, X_{AB} = M$ and $X_{BA} = 0$, if $X = M$.

THEOREM 3. *Denote by $\tau(D)$ the set of all bounded traces over the Banach algebra D, i.e. $\tau(D) = \{f \in D^*; f(d_1 d_2) = f(d_2 d_1)$, for all $d_1, d_2 \in D\}$. Then $\tau(\mathcal{T}) \simeq \tau(A) \oplus \tau(B)$ if $\mathcal{T} = [\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}]$.*

Proof.

$$0 \to H^0(\mathcal{T}, \mathcal{T}^*) \xrightarrow{\phi^0} H^0(A, A^*) \oplus H^0(B, B^*) \xrightarrow{\delta^0}$$

$Ext^0_{A\hat{\otimes}B^{op}}(M, \mathcal{T}^*_{AB}) = 0$, since $\mathcal{T}^*_{AB} = 0$. Note that then $H^0(D, D^*) = \tau(D)$.

REMARK. Thanks to Niels Jakob Laustsen for his comment on the fact that there is a direct proof for Theorem 3.6:

The equality

$$\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

implies that $f([\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}]) = 0$ for every $f \in \tau(\mathcal{T})$. Then $f_A(a) = f([\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}])$ and $f_B(b) = f([\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}])$ give two bounded traces over A and B respectively. Conversely, if we have two bounded traces f_1 and f_2 on A and B, resp., then $f([\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}] = f_1(a) + f_2(b)$ defines a bounded trace over $\mathcal{T}$.

THEOREM 4 ([3, Corollary 4.2]). *Let A be a unital Banach algebra with $H^n(A, A) = 0$ for all $n > 1$, and M be a left Banach A-module, then $H^n(\mathcal{T}, \mathcal{T}) \simeq H^{n-1}(A, B(M))$ in which $\mathcal{T} = [\begin{smallmatrix} A & M \\ 0 & \mathbf{C} \end{smallmatrix}]$.*

Proof. Put $X = \mathcal{T}$. By $Ext^n_A(X, Y) \simeq H^n(A, B(X, Y))$ and $H^n(\mathbf{C}, \mathbf{C}) = 0$, the exact sequence

$$\cdots \xrightarrow{\phi^{n-1}} H^{n-1}(A, A) \oplus H^{n-1}(\mathbf{C}, \mathbf{C}) \xrightarrow{\delta^{n-1}} Ext^{n-1}_{A\otimes\mathbf{C}^{op}}(M, M)$$

$$\xrightarrow{\pi^{n-1}} H^n(\mathcal{T}, \mathcal{T}) \xrightarrow{\phi^n} H^n(A, A) \oplus H^n(\mathbf{C}, \mathbf{C}) \xrightarrow{\delta^n} \cdots$$

gives rise to

$$\cdots \to 0 \to H^{n-1}(A, M) \to H^n(\mathcal{T}, \mathcal{T}) \to 0 \to \cdots.$$

Hence $H^n(\mathcal{T}, \mathcal{T}) \simeq H^{n-1}(A, B(M))$.

EXAMPLE. Suppose that A is a hyperfinite von Neumann algebra acting on a Hilbert space H. $M = H$ is a left A-module via $a.\xi = a(\xi), a \in A, \xi \in H$. It follows from [13, Corollary 3.4.6] $H^n(A, A) = 0$ for all n. So $H^n([\begin{smallmatrix} A & H \\ 0 & \mathbf{C} \end{smallmatrix}], [\begin{smallmatrix} A & H \\ 0 & \mathbf{C} \end{smallmatrix}]) = H^{n-1}(A, B(H))$. In particular,

$$H^2\left(\begin{bmatrix} A & H \\ 0 & \mathbf{C} \end{bmatrix}, \begin{bmatrix} A & H \\ 0 & \mathbf{C} \end{bmatrix}\right) = H^1(A, B(H)) = 0$$

by [13, Theorem 2.4.3]. (See [3, Example 4.2].)

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