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ON THE STOKES AND NAVIER-STOKES FLOWS IN A PERTURBED HALF-SPACE

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Abstract. We give the L_p-L_q estimate for the Stokes semigroup in a perturbed half-space and some global in time existence theorems for small solutions to the Navier-Stokes equation.

1. Background. The non-stationary Stokes system is given by the equations:

$$\text{(S)} \qquad \begin{cases} \vec{u}_t - \Delta\vec{u} + \nabla\pi = \vec{f}, \ \ \operatorname{div}\vec{u} = 0, \ \ \text{in } (0,T)\times\Omega, \\ \quad \vec{u}|_{\partial\Omega} = 0, \ \ \vec{u}|_{t=0} = \vec{a} \end{cases}$$

with unknown velocity $\vec{u} = {}^T(u_1, \dots, u_n)$ and pressure π (scalar function) in some domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), whose boundary is denoted by $\partial\Omega$ and assumed to be a $C^{1,1}$ hypersurface at least. Here and hereafter, ${}^TM$ means the transposed M and n-vectors of functions are denoted by letters with arrow. If we define the spaces $J_p(\Omega)$ and $G_p(\Omega)$ by the relations:

$$\begin{aligned} J_p(\Omega) &= \text{the closure of } \{\vec{u} \in C_0^\infty(\Omega)^n \mid \operatorname{div}\vec{u} = 0 \text{ in } \Omega\} \text{ in } L_p(\Omega)^n, \\ G_p(\Omega) &= \{\nabla\pi \in L_p(\Omega)^n \mid \pi \in L_{p,\mathrm{loc}}(\overline{\Omega})\}, \end{aligned}$$

we know the unique decomposition (so called Helmholtz decomposition)

$$\text{(HD)} \qquad L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega)$$

with a linear continuous projection $P : L_p(\Omega)^n \to J_p(\Omega)$ for many types of domains (cf. Fujiwara and Morimoto [18], Farwig and Sohr [16], [17], Galdi [19], Miyakawa [32], Simader and Sohr [37] and references therein). Then, we can define the Stokes operator

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A by

$$A = P(-\Delta) \tag{SO}$$

with definition domain:

$$D_p(A) = \{\vec{u} \in J_p(\Omega) \cap W_p^2(\Omega)^n \mid \vec{u}|_{\partial\Omega} = 0\}. \tag{SD}$$

Having the Stokes operator A in hand, the non-stationary Stokes equation (S) can be formulated as an ordinary differential equation in the Banach space $J_p(\Omega)$:

$$\vec{u}\,'(t) + A\vec{u}(t) = P\vec{f}(t), \;\; \vec{u}(0) = \vec{a}. \tag{O}$$

Hence, the question is whether A generates an analytic semigroup. Through the Laplace transform, this question is related to the resolvent estimate:

$$|\lambda| \|(\lambda + A)^{-1}\vec{f}\|_{L_p(\Omega)} + \|(\lambda + A)^{-1}\vec{f}\|_{W_p^2(\Omega)} \leq C_{\epsilon,p}\|\vec{f}\|_{L_p(\Omega)} \tag{R}$$

for $\lambda \in \Sigma_\epsilon = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| \leq \pi - \epsilon\}$ with some $\epsilon \in (0, \pi/2)$, where $1 < p < \infty$. In fact, once obtaining (R), we have the representation formula:

$$T(t)\vec{f} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)^{-1}\vec{f}\,d\lambda, \quad \vec{f} \in J_p(\Omega) \tag{Rp}$$

where $\Gamma = \{\lambda = e^{i\theta}s \mid s \geq \epsilon\} \cup \{\lambda = e^{-i\theta}s \mid s \geq \epsilon\} \cup \{\lambda = \epsilon e^{is} \mid -\theta \leq s \leq \theta\}$ with some $\theta \in (\pi/2, \pi)$ and $\epsilon > 0$, which combined with (R) implies not only the generation of the analytic semigroup $\{T(t)\}_{t\geq 0}$ by A but also the semigroup estimates:

$$\begin{aligned} \|T(t)\vec{a}\|_{L_p(\Omega)} &\leq C_p e^{\epsilon t}\|\vec{a}\|_{L_p(\Omega)}, \\ \|T(t)\vec{a}\|_{W_p^2(\Omega)} &\leq C_p e^{\epsilon t} t^{-1}\|\vec{a}\|_{L_p(\Omega)} \end{aligned} \tag{SE}$$

for any $t > 0$ (cf. Pazy [33]).

Concerning the references for (R), when $\Omega = \mathbb{R}^n$, since the Helmholtz projection commutes with the Laplacian, the resolvent estimate (R) is reduced to that for the Laplacian. The case of the half-space $\Omega = \mathbb{R}^n_+$ was settled by McCracken [30], where

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R} \mid x_n > 0\},$$

and the case of bounded domains by Giga [20] and Solonnikov [38]. The case of exterior domains was treated by Borchers and Sohr [7], Farwig and Sohr [16], Borchers and Varnhorn [9] and Varnhorn [42]. When Ω is a perturbed half-space which is a domain such that $\Omega \cap B^R = \mathbb{R}^n_+ \cap B^R$ for some $R > 0$ where $B^R = \{x \in \mathbb{R}^n \mid |x| > R\}$, (R) was proved by Farwig and Sohr [16]. The case of cones in $\mathbb{R}^3$ was settled by Deuring [15]. The case of aperture domains was settled by Farwig and Sohr [17]. The case of infinite layers like $\mathbb{R}^{n-1} \times (-1, 1)$ was settled by Wiegner [5] and Abe and Shibata [1] and [2]. The case of the asymptotically flat layer was settled by Abels [4].

To obtain L_p–L_q estimates:

$$\|T(t)\vec{a}\|_{L_q(\Omega)} \leq C_{p,q}\, e^{\epsilon t}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\vec{a}\|_{L_p(\Omega)}, \tag{1}$$

$$\|\nabla T(t)\vec{a}\|_{L_q(\Omega)} \leq C_{p,q}\, e^{\epsilon t}\, t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|\vec{a}\|_{L_p(\Omega)}, \tag{2}$$

for $t > 0$ and $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$), we combine (SE) with the Sobolev inequality:

$$\|u\|_{W_q^j(\Omega)} \leq C\|\nabla^m u\|_{L_p(\Omega)}^a \|u\|_{L_p(\Omega)}^{1-a} + \|u\|_{L_p(\Omega)} \tag{3}$$

provided that $0 \leq j < m$, $1 \leq p < \infty$, $m-j-n/p$ is not non-negative integer, $j/m < a \leq 1$ and $1/q = j/n + 1/p - am/n \geq 0$. The estimates (1) and (2) play an important role in the study of Navier-Stokes equation. In fact, by using the Stokes semigroup, we can reduce the Navier-Stokes equation:

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = \nabla\pi + \Delta\vec{u}, \ \ \operatorname{div}\vec{u} = 0 \ \text{ in } (0,T)\times\Omega \\ \qquad \vec{u}|_{\partial\Omega} = 0, \quad \vec{u}|_{t=0} = \vec{a} \end{cases} \tag{NS}$$

to the integral equation:

$$\vec{u}(t) = T(t)\vec{a} - \int_0^t T(t-s)P((\vec{u}(s)\cdot\nabla)\vec{u}(s))\,ds, \tag{I}$$

where we have set

$$(\vec{v}\cdot\nabla)\vec{w} = {}^T\Big(\Big(\sum_{j=1}^n v_j\partial_j\Big)w_1, \ldots, \Big(\sum_{j=1}^n v_j\partial_j\Big)w_n\Big), \ \ \partial_j = \partial/\partial x_j,$$

for the vectors of functions $\vec{v} = {}^T(v_1,\ldots,v_n)$ and $\vec{w} = {}^T(w_1,\ldots,w_n)$. Employing the argument due to Kato [24] and using (1) and (2) we can prove the locally in time existence theorem of (I). More precisely, we see that for any initial data $\vec{a} \in J_n(\Omega)$ there exists a time $t_0 > 0$ such that the integral equation (I) admits a unique solution $\vec{u}(t) \in C^0([0,t_0), J_n(\Omega))$ with $\nabla\vec{u}(t) \in C^0((0,t_0), L_n(\Omega))$ (cf. Giga and Miyakawa [21]).

However, in proving a globally in time existence of solutions to (I) at least with small initial data as well as in the study of time-asymptotic behaviour, we have to show (1) and (2) without $e^{\epsilon t}$. To show this, we need more precise analysis of $(\lambda + A)^{-1}$ near $\lambda = 0$. That $\lambda = 0$ is in the resolvent set was derived in the bounded domain case by Giga [20] and Solonnikov [38], and in the infinite layer case by Abe and Shibata [2], which implies that (1) and (2) hold, replacing $e^{\epsilon t}$ by e^{-ct} with some constant $c > 0$.

When $\Omega = \mathbb{R}^n$, applying the Young inequality to the concrete solution formula, we have (1) and (2) without $e^{\epsilon t}$, namely

$$\|T(t)\vec{a}\|_{L_q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\vec{a}\|_{L_p(\Omega)}, \quad \forall\, t > 0, \tag{4}$$

$$\|\nabla T(t)\vec{a}\|_{L_q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|\vec{a}\|_{L_p(\Omega)}, \quad \forall\, t > 0, \tag{5}$$

for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). When $\Omega = \mathbb{R}^n_+$, applying the Fourier multiplier theorem to the concrete solution formula obtained by Ukai [41] and using the Sobolev inequality:

$$\|\nabla^j u\|_{L_q(\mathbb{R}^n_+)} \leq C\|\nabla^m u\|_{L_p(\mathbb{R}^n_+)}^a \|u\|_{L_p(\mathbb{R}^n_+)}^{1-a} \tag{6}$$

provided that $0 \leq j < m$, $1 \leq p < \infty$, $m-j-n/p$ is not non-negative integer, $j/m < a \leq 1$ and $1/q = j/n + 1/p - am/n \geq 0$, we have (4) and (5) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) (cf. Borchers and Miyakawa [8] and Desch, Hieber and Prüss [14]).

When Ω is an exterior domain, (4) holds for $1 \le p \le q \le \infty$ ($p \ne \infty$, $q \ne 1$) but (5) holds only for $1 \le p \le q \le n$ ($q \ne 1$). This result was first proved by Iwashita [23] for $1 < p \le q < \infty$ in (4) and $1 < p \le q \le n$ in (5) when $n \ge 3$. The refinement of his result was done by the following authors: Chen [10] ($n = 3$, $q = \infty$), Shibata [35] ($n = 3$, $q = \infty$), Borchers and Varnhorn [9] ($n = 2$, (4) for $p = q$), Dan and Shibata [11], [12] ($n = 2$), Dan, Kobayashi and Shibata [13] ($n = 2, 3$), and Maremonti and Solonnikov [31] ($n \ge 2$). Especially, that Iwashita's restriction: $q \le n$ in (5) is unavoidable was shown by Maremonti and Solonnikov [31].

When Ω is an aperture domain, Abels [3] proved (4) for $1 < p \le q < \infty$ and (5) for $1 < p \le q < n$ when $n \ge 3$; and Hishida [22] proved (4) for $1 \le p \le q \le \infty$ and (5) for $1 \le p \le q \le n$ ($q \ne 1$) and $1 \le p < n < q < \infty$ when $n \ge 3$.

Moreover, (I) was solved globally in time for small initial data in $J_n(\Omega)$ by using (4) and (5) in the following papers: Kato [24] in the whole space case; Ukai [41] and Borchers and Miyakawa [8] in the half-space case; Iwashita [23] and Dan and Shibata [11] in the exterior domain cases; Abe and Shibata [2] in the infinite layer case; Hishida [22] in the aperture domain case.

In this paper, we report on the results about (4) and (5) and the related topics in the perturbed half-space cases. And also, we report on some results on the Navier-Stokes flow in the perturbed half-space case. The detailed proofs of the results stated below are found in the papers due to Kubo and Shibata [28] and [29].

2. Notation. Before stating our main theorem precisely, we outline our notation used throughout the paper. If X is a subset in the complex number field $\mathbb{C}$ or functional space, then X^n denotes the n-th product:

$$X^n = \{(x_1, \ldots, x_n) \mid x_j \in X,\ j = 1, \ldots, n\}.$$

If X be a subset of $\mathbb{C}$, then $X \setminus (-\infty, 0]$ is defined by

$$X \setminus (-\infty, 0] = X \setminus \{x + i0 \in \mathbb{C} \mid -\infty < x \le 0\}.$$

Given an n-vector of functions $\vec{v} = {}^T(v_1, \ldots, v_n)$ and point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we set

$$\vec{v}\,' = {}^T(v_1, \ldots, v_{n-1}), \quad x' = (x_1, \ldots, x_{n-1}).$$

The dot $\cdot$ stands for the usual inner products both of $\mathbb{R}^n$ and of $\mathbb{R}^{n-1}$. Given $R > 0$, we set

$$B_R = \{x \in \mathbb{R}^n \mid |x| < R\},\ \ B^R = \{x \in \mathbb{R}^n \mid |x| > R\},\ \ B_R^+ = B_R \cap \mathbb{R}^n_+.$$

For the differentiation, we use the symbols:

$$\begin{aligned} &\partial_x^\alpha u = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, \\ &\partial_{x'}^{\alpha'} u = \partial_1^{\alpha_1} \cdots \partial_{n-1}^{\alpha_{n-1}} u \text{ for } \alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}, \\ &\partial_x^\alpha \vec{u} = {}^T(\partial_x^\alpha u_1, \ldots, \partial_x^\alpha u_n),\ \ \partial_{x'}^{\alpha'} \vec{u} = {}^T(\partial_{x'}^{\alpha'} u_1, \ldots, \partial_{x'}^{\alpha'} u_n) \end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of all natural numbers, and moreover we set

$$\nabla^j u = (\partial_x^\alpha u \mid |\alpha| = j),\ \ \nabla u = \nabla^1 u,\ \ \nabla^j \vec{u} = {}^T(\nabla^j u_1, \ldots, \nabla^j u_n).$$

Sobolev spaces of vector-valued functions are used as well as of scalar functions. Thus, given a domain D in $\mathbb{R}^n$, $\|\cdot\|_{L_p(D)}$ denotes the usual L_p norm on D and we set

$$\|u\|_{W_p^m(D)} = \sum_{|\alpha|\le m} \|\partial_x^\alpha u\|_{L_p(D)}, \quad \|\vec{u}\|_{L_p(D)} = \sum_{j=1}^{n} \|u_j\|_{L_p(D)}, \quad \|\vec{u}\|_{W_p^m(D)} = \sum_{j=1}^{m} \|u_j\|_{W_p^m(D)}.$$

$L_p(D)$ denotes the usual L_p space on D and $C_0^\infty(D)$ the set of all functions in $C^\infty(\mathbb{R}^n)$ whose support is compact and contained in D. Moreover, we set

$$\begin{aligned}
&L_{p,R}(D) = \{u \in L_p(D) \mid u(x) = 0 \text{ for } x \notin B_R\},\\
&W^m_{p,\mathrm{loc}}(\overline{D}) = \{u \in L_{p,\mathrm{loc}}(\overline{D}) \mid \partial_x^\alpha u \in L_{p,\mathrm{loc}}(\overline{D}), |\alpha| \le m\},\\
&\hat{W}_p^1(D) = \{u \in L_{p,\mathrm{loc}}(\overline{D}) \mid \partial_j u \in L_p(D),\ \ j = 1, \dots, n\},\\
&W_p^m(D) = \{u \in L_p(D) \mid \partial_x^\alpha u \in L_p(D), |\alpha| \le m\}.
\end{aligned}$$

By $C_{A,B,\cdots}$ we denote the constants depending on the quantities A, B, $\ldots$. For two Banach spaces X and Y, $\mathscr{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y. $\mathscr{A}(U_\epsilon, X)$ denotes the set of all X-valued holomorphic functions defined on $U_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$. $BC(I; X)$ and $C^k(I; X)$ denote the set of all X-valued bounded continuous functions and C^k functions defined on I, respectively.

3. Main results about Stokes flow in the perturbed half-space. In this section, we will state our main results concerning the Stokes system (S) in the half-space and the perturbed half-space, which is defined as follows.

Definition 1. (1) The half-space $\mathbb{R}^n_+$ is defined by

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

(2) Let Ω be a domain in $\mathbb{R}^n$. We call Ω a perturbed half-space if there exists a number $R > 0$ such that

$$\Omega \cap B^R = \mathbb{R}^n_+ \cap B^R. \tag{7}$$

As we already stated in section 1, when Ω is a perturbed half-space, Farwig and Sohr [16] proved the Helmholtz decomposition (HD) and the resolvent estimate (R) on Ω. Therefore, we know that the Stokes operator (SO) with domain (SD) generates the analytic semigroup $\{T(t)\}_{t\ge 0}$ on $J_p(\Omega)$. Then, we have the following theorem.

Theorem 1. *Let Ω be a perturbed half-space in $\mathbb{R}^n$ $(n \ge 2)$ whose boundary $\partial\Omega$ is a $C^{1,1}$ hypersurface. Then, the Stokes semigroup $\{T(t)\}_{t\ge 0}$ satisfies the following two estimates:*

$$\|T(t)\vec{a}\|_{L_q(\Omega)} \le C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\vec{a}\|_{L_p(\Omega)}, \tag{8}$$

$$\|\nabla T(t)\vec{a}\|_{L_q(\Omega)} \le C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|\vec{a}\|_{L_p(\Omega)}, \tag{9}$$

for any $\vec{a} \in J_p(\Omega)$, $t > 0$ and $1 \le p \le q \le \infty$ $(p \ne \infty,\ q \ne 1)$.

The main step in our proof of Theorem 1 is to show the following local energy decay estimate.

Theorem 2. *Let Ω be a perturbed half-space in $\mathbb{R}^n$ $(n \ge 2)$ whose boundary $\partial\Omega$ is a $C^{1,1}$ hypersurface. Let $1 < p < \infty$ and R be a number such that (7) holds. Then, the Stokes* *semigroup* $\{T(t)\}_{t\geq 0}$ *satisfies the following estimate:*

$$\|\partial_t^j T(t)P\vec{a}\|_{W_p^2(\Omega\cap B_R)} \leq C_{p,R} t^{-\frac{n+1}{2}-j} \|\vec{a}\|_{L_p(\Omega)} \tag{10}$$

for any $t \geq 1$, $j \in \mathbb{N}_0$ *and* $\vec{a} \in L_{p,R}(\Omega)$.

If we consider the Stokes system in the half-space:

$$\begin{aligned} &\vec{v}_t - \Delta\vec{v} + \nabla\pi = 0, \ \ \operatorname{div}\vec{v} = 0 \ \text{ in } (0,\infty)\times\mathbb{R}^n_+, \\ &\vec{v}|_{x_n=0} = 0, \quad \vec{v}|_{t=0} = \vec{b}, \end{aligned} \tag{11}$$

then we know by Ukai [41] and Borchers and Miyakawa [8] that the solution $\vec{v}$ of (11) satisfies the L_p–L_q estimate:

$$\|\vec{v}(t)\|_{L_q(\mathbb{R}^n_+)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\vec{b}\|_{L_p(\mathbb{R}^n_+)}, \tag{12}$$

$$\|\nabla\vec{v}(t)\|_{L_q(\mathbb{R}^n_+)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|\vec{b}\|_{L_p(\mathbb{R}^n_+)} \tag{13}$$

for any $t > 0$ and $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). Since

$$\|\vec{v}(t)\|_{L_p(B_R^+)} \leq C_R \|\nabla\vec{v}(t)\|_{L_p(B_R^+)}$$

as follows from the boundary condition: $\vec{v}|_{x_n=0} = 0$, using (13) and Theorem 2, we have

$$\|T(t)P\vec{a}\|_{W_p^2(\Omega\cap B_R)} \leq C_{p,R} t^{-\frac{n}{2p}-\frac{1}{2}} \|\vec{a}\|_{L_p(\Omega)} \tag{14}$$

for any $\vec{a} \in L_p(\Omega)$ and $t \geq 1$. Combining (12), (13) and (14) by the cut-off technique and following the argument due to Hishida [22, the proof of Theorem 2.1], we can show Theorem 1.

In order to prove Theorem 2, we need some precise information about solutions to the resolvent problem in $\mathbb{R}^n_+$:

$$\begin{aligned} &(\lambda - \Delta)\vec{w} + \nabla\theta = \vec{f}, \ \ \operatorname{div}\vec{w} = 0 \ \ \text{ in } \mathbb{R}^n_+, \\ &\vec{w}|_{x_n=0} = 0, \end{aligned} \tag{15}$$

which is stated in the following two theorems.

THEOREM 3. *Let* $R(\lambda)$ *and* $\Pi(\lambda)$ *denote the solution operators of* (15) *which are defined by*

$$\vec{w} = R(\lambda)\vec{f} = {}^T(R_1(\lambda)\vec{f}, \ldots, R_n(\lambda)\vec{f}) \quad \textit{and} \quad \theta = \Pi(\lambda)\vec{f}$$

for $\lambda \in \mathbb{C}\setminus(-\infty,0]$. *Let* $R > 0$, $1 < p < \infty$ *and set*

$$\mathscr{B}^j_{p,R} = \mathscr{L}(L_{p,R}(\mathbb{R}^n_+)^n, W_p^j(B_R^+))$$

for $j = 1, 2$. *Then there exist operators* $G_j^k(\lambda) \in \mathscr{A}(U_{1/16}, \mathscr{B}^2_{p,R})$, $k = 1,2,3$, $j = 1,\ldots,n$, *and* $G_\pi^k(\lambda) \in \mathscr{B}^1_{p,R}$, $k = 1,2,3$ *such that*

$$\begin{aligned} R_j(\lambda)\vec{f} &= \lambda^{\frac{n-1}{2}} G_j^1(\lambda)\vec{f} + (\lambda^{\frac{n}{2}}\log\lambda) G_j^2(\lambda)\vec{f} + G_j^3(\lambda), \\ \Pi(\lambda)\vec{f} &= \lambda^{\frac{n-1}{2}} G_\pi^1(\lambda)\vec{f} + (\lambda^{\frac{n}{2}}\log\lambda) G_\pi^2(\lambda)\vec{f} + G_\pi^3(\lambda), \end{aligned} \tag{16}$$

in B_R^+ *when* $n \geq 2$ *and* n *is even; and*

$$R_j(\lambda)\vec{f} = \lambda^{\frac{n}{2}} G_j^1(\lambda)\vec{f} + (\lambda^{\frac{n-1}{2}} \log \lambda) G_j^2(\lambda)\vec{f} + G_j^3(\lambda),$$
$$\Pi(\lambda)\vec{f} = \lambda^{\frac{n}{2}} G_\pi^1(\lambda)\vec{f} + (\lambda^{\frac{n-1}{2}} \log \lambda) G_\pi^2(\lambda)\vec{f} + G_\pi^3(\lambda), \tag{17}$$

in B_R^+ *when* $n \geq 3$ *and* n *is odd, provided that* $\lambda \in U_{1/16} \setminus (-\infty, 0]$ *and* $\vec{f} \in L_{p,R}(\Omega)$.

THEOREM 4. *Let* $1 < p < \infty$, $0 < \epsilon < \pi/2$, *and let* $R(\lambda)$ *and* $\Pi(\lambda)$ *be the operators given in Theorem 3 for* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. *Let* Σ_ϵ *be the set in* $\mathbb{C}$ *defined by*

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}. \tag{18}$$

Then, there exist operators $R(0) \in \mathscr{L}(L_{p,R}(\mathbb{R}_+^n)^n, W_{p,\mathrm{loc}}^2(\mathbb{R}_+^n)^n)$ *and* $\Pi(0) \in \mathscr{L}(L_{p,R}(\mathbb{R}_+^n)^n, W_{p,\mathrm{loc}}^1(\mathbb{R}_+^n))$ *which satisfy the following three conditions:*

(i) *Given* $\vec{f} \in L_{p,R}(\mathbb{R}_+^n)$, $\vec{v} = R(0)\vec{f}$ *and* $\theta = \Pi(0)\vec{f}$ *satisfy the equation:*

$$-\Delta\vec{v} + \nabla\theta = \vec{f}, \quad \operatorname{div}\vec{v} = 0 \quad \text{in } \mathbb{R}_+^n, \quad \vec{v}|_{x_n=0} = 0. \tag{19}$$

(ii)

$$\|R(\lambda)\vec{f} - R(0)\vec{f}\|_{W_p^1(B_R^+)} + \|\Pi(\lambda)\vec{f} - \Pi(0)\vec{f}\|_{L_p(B_R^+)} \leq C_{p,R,\epsilon}|\lambda|^{\frac{1}{4}}\|\vec{f}\|_{L_p(\mathbb{R}^n)}$$

for any $\vec{f} \in L_{p,R}(\mathbb{R}_+^n)$ *and* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \leq 1/16$, *where* $C_{p,R,\epsilon}$ *is a constant independent of* $\vec{f}$ *and* λ.

(iii)

$$|[R(0)\vec{f}](x)| \leq C_{p,R}|x|^{-(n-1)}\|\vec{f}\|_{L_p(\mathbb{R}_+^n)},$$
$$|\nabla[R(0)\vec{f}](x)| \leq C_{p,R}|x|^{-(n-1)}\|\vec{f}\|_{L_p(\mathbb{R}_+^n)},$$
$$|[\Pi(0)\vec{f}](x)| \leq C_{p,R}|x|^{-(n-1)}\|\vec{f}\|_{L_p(\mathbb{R}_+^n)},$$

for any $\vec{f} \in L_{p,R}(\mathbb{R}_+^n)$ *and* $x \in \mathbb{R}_+^n$ *with* $|x| \geq 2\sqrt{2}R$, *where* $C_{p,R}$ *is a constant independent of* $\vec{f}$ *and* x.

Constructing a parametrix of the resolvent problem in a perturbed half-space, we can derive from Theorem 3 and Theorem 4 that the resolvent operator $(\lambda + A)^{-1}$ has the expansion formula of the same type near $\lambda = 0$ in the space $\mathscr{L}(L_{p,R}(\Omega)^n, W_p^2(\Omega \cap B_R)^n)$ as in the half-space case, which is applied to (Rp) implies Theorem 2. The detailed proof of Theorems 1 and 2 is given in Kubo and Shibata [29] and that of Theorems 3 and 4 in Kubo-Shibata [28]. The fundamental idea of the proofs of Theorems 1 and 2 by using Theorems 3 and 4 goes back to a paper due to Shibata [34].

4. The Navier-Stokes flow in a perturbed half-space. Following the arguments due to Kato [24], Kozono [25], Hishida [22] and Wiegner [43] and using Theorem 1 we can show the following theorem.

THEOREM 5. *Let* $n \geq 2$. *There exists a constant* $\delta = \delta(\Omega, n) > 0$ *with the following property: if* $\vec{a} \in J^n(\Omega)$ *satsifies* $\|\vec{a}\|_{L_n(\Omega)} \leq \delta$, *then the integral equation*

$$\vec{u}(t) = T(t)\vec{a} - \int_0^t T(t-s)P((\vec{u}(s)\cdot\nabla)\vec{u}(s))ds$$

admits a unique strong solution $\vec{u}(t)\in BC([0,\infty);J_n(\Omega))$ with $\nabla\vec{u}(t)\in C^0((0,\infty);L_n(\Omega))$. Moreover as $t\to\infty$,

$$\|\vec{u}(t)\|_{L_r(\Omega)} = o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad \textit{for} \quad n\le r\le\infty,$$
$$\|\nabla\vec{u}(t)\|_{L_r(\Omega)} = o(t^{-1+\frac{n}{2r}}) \qquad \textit{for} \quad n\le r<\infty.$$

THEOREM 6. *Let $n\ge 2$. There exists a constant $\eta=\eta(\Omega,n)\in(0,\delta]$ with the following property: if $\vec{a}\in L_1(\Omega)\cap J_n(\Omega)$ satisfies $\|\vec{a}\|_{L_n(\Omega)}\le\eta$, then the solution $\vec{u}(t)$ obtained in Theorem 5 satisfies the estimates:*

$$\|\vec{u}(t)\|_{L_r(\Omega)} = O(t^{-(n-\frac{n}{r})/2}) \qquad \textit{for } 1<r\le\infty,$$
$$\|\nabla\vec{u}(t)\|_{L_r(\Omega)} = O(t^{-(n-\frac{n}{r})/2-\frac{1}{2}}) \qquad \textit{for } 1<r<\infty,$$

as $t\to\infty$.

5. On the periodic solution of the Navier-Stokes equation in the perturbed half-space. We can show that if the incompressible fluid in the perturbed half-space is governed by the periodic external force, the Navier-Stokes equations have a periodic strong solution with the same period as the external force. Let Ω be a domain in $\mathbb{R}^n (n\ge 3)$. Let us consider the following Navier-Stokes equations in Ω:

$$\text{(NSP)} \qquad \begin{cases} \dfrac{\partial\vec{u}}{\partial t} - \Delta\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla\pi = \vec{f}, & x\in\Omega,\ t\in\mathbb{R}, \\ \operatorname{div}\vec{u} = 0, & x\in\Omega,\ t\in\mathbb{R}, \\ \vec{u}|_{\partial\Omega} = 0. \end{cases}$$

Applying the projection operator P_r to both sides of the first equation of (NSP), we have

$$\text{(E)} \qquad \frac{d\vec{u}}{dt} + A_r\vec{u} + P_r(\vec{u}\cdot\nabla\vec{u}) = P_r\vec{f}$$

on $J_r(\Omega)$ for $t\in\mathbb{R}$. The above (E) can be further transformed to the following integral equation:

$$\text{(I-E)} \qquad \vec{u}(t) = \int_{-\infty}^{t} T(t-s)P_r\vec{f}(s)ds - \int_{-\infty}^{t} T(t-s)P_r((\vec{u}\cdot\nabla)\vec{u}(s))ds.$$

Concerning the external force $\vec{f}$, we impose the following assumption:

ASSUMPTION 1. Let the exponents r and q be such as $2<r<n$, $\frac{n}{2}<q<n$. When $n\ge 4$, we assume that

$$\text{(20)} \qquad \vec{f}\in BC(\mathbb{R};\ L_p(\Omega)\cap L_\ell(\Omega))$$

for $1<p,\ell<\infty$ with $1/r+2/n<1/p, 1/q<1/\ell<1/q+1/n$.

When $n=3$, we assume that

$$\text{(21)} \qquad P_p\vec{f}(s) = A_p^s\vec{g}(s)\ (s\in\mathbb{R}) \text{ with some } \vec{g}\in BC(\mathbb{R};D(A_p^s))$$

for $1<p<\min(r,q)$ and $\delta>0$ satisfying $3/2p+\delta\ge\max(1+3/2r, 1/2+3/2q)$ and that

$$\vec{f}\in BC(\mathbb{R};L_\ell(\Omega))$$

for $1/q<1/\ell<1/q+1/3$.

Using the method due to Kozono and Nakao [26], we can show the following two theorems in the perturbed half-space case.

THEOREM 7. *Let Ω and $\vec{f}$ satisfy Assumption 1. Suppose that $\vec{f}(t) = \vec{f}(t+\omega)$ for all $t \in \mathbb{R}$ with some $\omega > 0$. Then there is a constant $\eta = \eta(n, r, q, p, \ell, \delta) > 0$ such that if*

$$\sup_{s\in\mathbb{R}} \|P_p\vec{f}(s)\|_{L_p(\Omega)} + \sup_{s\in\mathbb{R}} \|P_\ell\vec{f}(s)\|_{L_\ell(\Omega)} \leq \eta \text{ when } n \geq 4,$$

$$\sup_{s\in\mathbb{R}} \|\vec{g}(s)\|_{L_p(\Omega)} + \sup_{s\in\mathbb{R}} \|P_\ell\vec{f}(s)\|_{L_\ell(\Omega)} \leq \eta \text{ when } n = 3,$$

we have a periodic solution $\vec{u}$ of (I-E) *with the same period ω as $\vec{f}$ in the class $BC(\mathbb{R}; J_r(\Omega))$ with $\nabla\vec{u} \in BC(\mathbb{R}; L_q(\Omega))$.*

Such a solution $\vec{u}$ is unique within this class provided $\sup_{s\in\mathbb{R}} \|\vec{u}(s)\|_{L_r(\Omega)} + \sup_{s\in\mathbb{R}} \|\nabla\vec{u}(s)\|_{L_q(\Omega)}$ *is sufficiently small.*

THEOREM 8. *In addition to the hypotheses of Theorem 7, let us assume furthermore that $\vec{f}$ is a Hölder continuous function on $\mathbb{R}$ with values in $L_n(\Omega)$. Then the periodic solution $\vec{u}$ given by Theorem 7 has the following additional properties:*

(i) $\vec{u} \in BC(\mathbb{R}; J_n(\Omega)) \cap C^1(\mathbb{R}; J_n(\Omega))$;
(ii) $\vec{u}(t) \in D(A_n)$ *for all* $t \in \mathbb{R}$ *and* $A_n\vec{u} \in C^0(\mathbb{R}; J_n(\Omega))$;
(iii) $\vec{u}$ *satisfies* (E) *in* $J_n(\Omega)$ *for all* $t \in \mathbb{R}$.

References

[1] T. Abe and Y. Shibata, *On a resolvent estimate of the Stokes equation on an infinite layer*, J. Math. Soc. Japan 55 (2003), 469–497.

[2] T. Abe and Y. Shibata, *On a resolvent estimate of the Stokes equation on an infinite layer. II. $\lambda = 0$ case*, J. Math. Fluid Mech. 5 (2003), 245–274.

[3] H. Abels, *L_q-L_r estimates for the non-stationary Stokes equations in an aperture domain*, Z. Anal. Anwendungen 21 (2002), 159–178.

[4] H. Abels, *Stokes equations in asymptotically flat domains and the motion of a free surface*, Doctor Thesis, Technische Univ. Darmstadt, Shaker Verlag, Aachen, 2003.

[5] H. Abels and M. Wiegner, *Resolvent estimates for the Stokes operator on an infinite layer*, Diff. Int. Eqns. 18 (2005), 1081–1110.

[6] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[7] W. Borchers and H. Sohr, *On the semigroup of the Stokes operator for exterior domains*, Math. Z. 196 (1987), 415–425.

[8] W. Borchers and T. Miyakawa, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. 282 (1988), 139–155.

[9] W. Borchers and W. Varnhorn, *On the boundedness of the Stokes semigroup in two dimensional exterior domains*, Math. Z. 213 (1993), 275–299.

[10] Z. M. Chen, *Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains*, Pacific J. Math. 159 (1993), 227–240.

[11] W. Dan and Y. Shibata, *On the L_q-L_r estimates of the Stokes semigroup in a two dimensional exterior domain*, J. Math. Soc. Japan 51 (1999), 181–207.

[12] W. Dan and Y. Shibata, *Remark on the L_q-L_∞ estimate of the Stokes semigroup in a 2-dimensional exterior domain*, Pacific J. Math. 189 (1999), 223–240.

[13] W. Dan, T. Kobayashi and Y. Shibata, *On the local energy decay approach to some fluid flow in exterior domain*, in: Recent Topics on Mathematical Theory of Viscous Incompressible Fluid, Lecture Notes Numer. Appl. Math. 16, Kinokuniya, Tokyo, 1998, 1–51.

[14] W. Desch, M. Hieber and J. Prüss, *L^p theory of the Stokes equation in a half-space*, J. Evol. Equations 1 (2001), 115–142.

[15] P. Deuring, *The Stokes System in an Infinite Cone*, Math. Research 78, Akademie-Verlag, Berlin, 1994.

[16] R. Farwig and H. Sohr, *Generalized resolvent estimates for the Stokes operator in bounded and unbounded domains*, J. Math. Soc. Japan 46 (1994), 607–643.

[17] R. Farwig and H. Sohr, *Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q-spaces*, Analysis 16 (1996), 1–26.

[18] D. Fujiwara and H. Morimoto *An L_r-theory of the Helmholtz decomposition of vector fields*, J. Fac. Sc. Univ. Tokyo, Sect. Math. 24 (1977), 685–700.

[19] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes equations, Vol. I: Linear Steady Problems, Vol. II: Nonlinear Steady Problems*, Springer Tracts in Nat. Ph. 38, 39, Springer-Verlag, New York, 1994, 2nd ed., 1998.

[20] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L_r spaces*, Math. Z. 178 (1981), 297–329.

[21] Y. Giga and T. Miyakawa, *Solutions in L_r to the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. 89 (1985), 267–281.

[22] T. Hishida, *The nonstationary Stokes and Navier-Stokes equations in aperture domains*, in: Elliptic and Parabolic Problems (Rolduc/Gaeta, 2001), World Sci., River Edge, NJ, 2002, 126–134.

[23] H. Iwashita, *L_q-L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_p spaces*, Math. Ann. 285 (1989), 265–288.

[24] T. Kato, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z. 189 (1984), 471–480.

[25] H. Kozono, *L^1-solutions of the Navier-Stokes equations in exterior domains*, Math. Ann. 312 (1998), 319–340.

[26] H. Kozono and M. Nakao, *Periodic solutions of the Navier-Stokes equations in unbounded domains*, Tohoku Math. J. 48 (1996), 33–50.

[27] T. Kubo, *On the Stokes and Navier-Stokes flows through an aperture*, preprint, 2004.

[28] T. Kubo and Y. Shibata, *On some properties of solutions to the Stokes equation in the half-space*, Adv. Diff. Eqns. 10 (2005), 695–720.

[29] T. Kubo and Y. Shibata, *On the Stokes and Navier-Stokes flows in a perturbed half-space*, preprint, 2004.

[30] M. McCracken, *The resolvent problem for the Stokes equation on halfspace in L_p*, SIAM J. Math. Anal. 12 (1981), 201–228.

[31] P. Maremonti and V. A. Solonnikov, *On nonstationary Stokes problem in exterior domains*, Ann. Sc. Norm. Sup. Pisa 24 (1997), 395–449.

[32] T. Miyakawa, *The Helmholtz decomposition of vector fields in some unbounded domains*, Math. J. Toyama Univ. 17 (1994), 115–149.

[33] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[34] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. 7 (1983), 1–68.

[35] Y. Shibata, *On an exterior initial boundary value problem for Navier-Stokes equations*, Quart. Appl. Math. 57 (1999), 117–155.

[36] Y. Shibata and S. Shimizu, *A decay property of the Fourier transform and its application to the Stokes problem*, J. Math. Fluid Mech. 3 (2001), 213–230.

[37] C. G. Simader and H. Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains*, in: Math. Probl. Relating to the Navier-Stokes Equations, Ser. Adv. Math. Appl. Sci. 11, World Scientific, River Edge, NJ, 1992, 1–35.

[38] V. A. Solonnikov, *Estimates for solutions of nonstationary Navier-Stokes equations*, Zap. Nauch. Sem. Len. Otdel. Mat. Inst. Steklov (LOMI) 38 (1973), 153–234; English transl.: J. Soviet Math. 8 (1977), 467–528.

[39] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[40] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd edition, Johann Ambrosius Barth, Heidelberg, 1995.

[41] S. Ukai, *A solution formula for the Stokes equation in $\mathbb{R}^n_+$*, Comm. Pure Appl. Math. 40, 6111–621.

[42] W. Varnhorn, *The Stokes Equations*, Math. Research 76, Akademie-Verl., Berlin, 1994.

[43] M. Wiegner, *Decay estimates for strong solutions of the Navier-Stokes equations in exterior domains*, Ann. Univ. Ferrara Sez. VII. Sc. Mat. 46 (2000), 61–79.