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ON STABILITY OF AXIALLY SYMMETRIC SOLUTIONS TO NAVIER-STOKES EQUATIONS IN A CYLINDRICAL DOMAIN AND WITH BOUNDARY SLIP CONDITIONS

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Abstract. The existence of global regular axially symmetric solutions to Navier-Stokes equations in a bounded cylinder and for boundary slip conditions is proved. Next, stability of these solutions is shown.

1. Introduction. We consider the motion of a viscous incompressible fluid in a bounded cylindrical domain $\Omega \subset \mathbb{R}^3$ under boundary slip conditions (see [8])

$$\begin{aligned} & v_{,t} + v \cdot \nabla v - \mathrm{div}\mathbb{T}(v,p) = f && \text{in } \ \Omega^T = \Omega \times (0,T), \\ & \mathrm{div} v = 0 && \text{in } \ \Omega^T, \\ & v \cdot \bar{n} = 0 && \text{on } \ S^T = S \times (0,T), \end{aligned} \tag{1.1}$$

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$$\begin{aligned} &\bar{n}\cdot\mathbb{T}(v,p)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\ &v\big|_{t=0} = v(0) && \text{in } \Omega, \end{aligned}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity vector field, $p = p(x,t) \in \mathbb{R}$ the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\bar{n}$ the unit outward vector normal to $S = \partial\Omega$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\bar{\tau}_1, \bar{\tau}_2$ are unit tangent vectors to S and $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p) = \{\nu(v_{i,x_j} + v_{j,x_i}) - p\delta_{ij}\}_{i,j=1,2,3} \equiv \nu\mathbb{D}(v) - pI,$$

where ν is the constant positive viscosity coefficient, $\mathbb{D}(v)$ the dilatation tensor and I the unit matrix.

We introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$.

By Ω we denote a cylinder with the x_3 axis as the axis of symmetry and with boundary $S = S_1 \cup S_2$. By S_1 we denote the part of the boundary which is parallel to the x_3 axis and is located at the distance $r = R$ from it. Moreover, it is between the planes $x_3 = -a$ and $x_3 = a$. S_2 is composed of two parts which are perpendicular to the x_3 axis and intersect it in two points $x_3 = -a$ and $x_3 = a$, respectively. The intersection of S_1 and S_2 are two circles which are denoted by L.

To introduce axially symmetric solutions we define unit vectors $\bar{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\bar{e}_z = (0,0,1)$, which are directed along the curvilinear coordinates r, φ, z, respectively. Let w be any vector. Then $w_r = w\cdot\bar{e}_r$, $w_\varphi = w\cdot\bar{e}_\varphi$, $w_z = w\cdot\bar{e}_z$.

Definition 1. By an axially symmetric solution to problem (1.1) we mean solution such that $v_\varphi = 0$, $v_{r,\varphi} = 0$, $v_{z,\varphi} = 0$, $p_{,\varphi} = 0$, $f_\varphi = 0$, $f_{r,\varphi} = 0$, $f_{z,\varphi} = 0$, $v_\varphi(0) = 0$, $v_{r,\varphi}(0) = 0$, $v_{z,\varphi}(0) = 0$.

The aim of this paper is to prove stability of axially symmetric solutions in the sense of [6]. In fact we want to prove existence of global solution to (1.1) which remains close to the axially symmetric solution. For this purpose we formulate a problem defining the axially symmetric solutions and prove their existence in Section 3. To prove the global existence of axially symmetric solutions we use ideas from [3, 7], which base on the estimate for vorticity. However, in our case we have different boundary conditions and the estimate for vorticity plays a crucial role.

Having proved global existence of axially symmetric solutions we show its stability by utilizing some ideas from [6]. However our proof is different (see Section 4).

To formulate problems determining an axially symmetric solution to (1.1) we introduce $\Omega_0 = \{x \in \mathbb{R}^3 : \varphi = 0,\ 0 < r < R,\ -a < z < a\}$, $S_{0i} = \{x \in S_i : \varphi = 0\}$, $i = 1,2$, $S_0 = S_{01} \cup S_{02}$. Then from [2, 4] we have

$$\begin{aligned} &v_{r,t} - \nu\left(v_{r,rr} + v_{r,zz} + \frac{1}{r}v_{r,r} - \frac{1}{r^2}v_r\right) + v_z\chi + q_{,r} = f_r && \text{in } \Omega_0^T,\\ &v_{z,t} - \nu\left(v_{z,rr} + v_{z,zz} + \frac{1}{r}v_{z,r}\right) - v_z\chi + q_{,z} = f_z && \text{in } \Omega_0^T, \end{aligned} \tag{1.2}$$

$$
\begin{aligned}
&(rv_r)_{,r}+(rv_z)_{,z}=0 && \text{in } \Omega_0^T,\\
&v_r=0,\quad v_{z,r}=0 && \text{on } S_{01}^T,\\
&v_{r,z}=0,\quad v_z=0 && \text{on } S_{02}^T,\\
&v_r\big|_{t=0}=v_r(0),\quad v_z\big|_{t=0}=v_z(0), && \text{in } \Omega_0,
\end{aligned}
$$

where $\chi=v_{r,z}-v_{z,r}$, $q=p-\frac{v^2}{2}$, and boundary conditions $(1.2)_{4,5}$ are calculated in Lemma 2.1 from [8, Ch. 4].

Following [3, 7] we formulate the problem for χ:

$$
\begin{aligned}
&\chi_{,t}-\nu\bigg(\chi_{,rr}+\chi_{,zz}+\frac{1}{r}\chi_{,r}-\frac{1}{r^2}\chi\bigg)+v_r\chi_{,r}+v_z\chi_{,z}\\
&-\frac{v_r}{r}\chi=F_2\equiv\operatorname{rot}f\cdot\bar e_\varphi && \text{in } \Omega_0^T,\\
&\chi=0 && \text{on } S_0^T,\\
&\chi\big|_{t=0}=\chi(0)\equiv v_r(0)_{,z}-v_z(0)_{,r} && \text{in } \Omega_0,
\end{aligned}
\tag{1.3}
$$

where the boundary condition $(1.3)_2$ is derived in Lemma 2.2 from [8, Ch. 4].

Since considerations in [3, 7] imply that the vorticity problem (1.3) plays a crucial role in the proof of global existence we replace problem (1.1) by problem (1.3), where the velocity is given and the following elliptic problem for v:

$$
\begin{aligned}
&v_{r,z}-v_{z,r}=\chi && \text{in } \Omega_0,\\
&v_{r,r}+v_{z,z}+\frac{v_r}{r}=0 && \text{in } \Omega_0,\\
&v_r\big|_{S_{01}}=0,\quad v_z\big|_{S_{02}}=0,
\end{aligned}
\tag{1.4}
$$

where χ is given.

From Lemma 1.1 from [8, Ch. 3] we know that problems (1.3), (1.4) are equivalent to problem (1.1) if p is a solution to the following problem:

$$
\begin{aligned}
&\Delta p=-\nabla v\cdot\nabla v+\operatorname{div}f && \text{in } \Omega_0,\\
&p_{,r}=f_r && \text{on } S_{01},\\
&p_{,z}=f_z && \text{on } S_{02},
\end{aligned}
\tag{1.5}
$$

where $v=v_r\bar e_r+v_z\bar e_z$ and v_r,v_z do not depend on φ.

Equation $(1.4)_2$ implies existence of a function ψ such that

$$
v_r=\frac{\psi_{,z}}{r},\quad v_z=-\frac{\psi_{,r}}{r}.
\tag{1.6}
$$

Moreover, boundary conditions $(1.4)_3$ give that $\psi_{,r}|_{S_{02}}=0$ and $\psi_{,z}|_{S_{01}}=0$. Since ψ is defined up to an arbitrary constant we replace problem (1.4) by

$$
\begin{aligned}
&\psi_{,rr}+\psi_{,zz}-\frac{\psi_{,r}}{r}=r\chi && \text{in } \Omega_0,\\
&\psi\big|_{S_0}=0,
\end{aligned}
\tag{1.7}
$$

and v_r,v_z are determined by (1.6).

Now we look for a stability problem for axially symmetric solutions. Let v_a,p_a,f_a denote the axially symmetric solution to problem (1.1). Then we are looking for solutions of (1.1) in the form

$$v = v_a + v', \quad p = p_a + p', \quad f = f_a + f', \tag{1.8}$$

where the primed quantities are disturbances which satisfy the problem

$$\begin{aligned}
&v'_{,t} + v'\cdot\nabla v' + v'\cdot\nabla v_a + v_a\cdot\nabla v' - \mathrm{div}\mathbb{T}(v',p') = f' && \text{in } \Omega^T,\\
&\mathrm{div} v' = 0 && \text{in } \Omega^T,\\
&\bar n\cdot v' = 0 && \text{on } S^T,\\
&\bar n\cdot\mathbb{T}(v',p')\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\
&v'\big|_{t=0} = v'(0) && \text{in } \Omega.
\end{aligned}\tag{1.9}$$

Now we formulate the main results of this paper.

THEOREM 1 (global existence of axially symmetric solutions). *Assume that* $v(0) \in L_2(\Omega_0)$, $\frac{v_r(0)}{r} \in L_2(\Omega_0)$, $\frac{\chi(0)}{r} \in L_3(\Omega_0)$, $\chi = v_{r,z} - v_{z,r}$,

$$\sup_k \bigg[\Big\|\frac{f_r}{r}\Big\|_{L_2(kT,(k+1)T;L_2(\Omega_0))} + \Big\|\frac{F}{r}\Big\|_{L_3(kT,(k+1)T;L_3(\Omega_0))} + \|f\|_{L_2(kT,(k+1)T;L_2(\Omega))}$$

$$+ \|\mathrm{div} f\|_{L_2(kT,(k+1)T;L_2(\Omega_0))} + \|f\|_{L_2(kT,(k+1)T;W^{1/3}_{3/2}(\partial\Omega_0))}\bigg] < \infty,$$

$$F = f_{r,z} - f_{z,r}, \quad k \in \mathbb{N}.$$

Then there exists a global axially symmetric solution to problem (1.1) such that

$$v \in C([kT,(k+1)T];H^1(\Omega_0)) \cap W^{2,1}_2(\Omega_0\times(kT,(k+1)T)) \quad \textit{for any } k\in\mathbb{N},$$

which satisfies estimates (3.69) and (3.70).

Let us introduce the quantities

$$\begin{aligned}
A_*(T) = \sup_{k\in\mathbb{N}}\Big(&\|\bar f_a\|_{L_2(kT,(k+1)T)} + \Big\|\frac{f_{ar}}{r}\Big\|_{L_2(\Omega\times(kT,(k+1)T))}\\
&+ \Big\|\frac{F_a}{r}\Big\|_{L_3(\Omega\times(kT,(k+1)T)} + \|\mathrm{div} f_a\|_{L_2(kT,(k+1)T;L_3(\Omega))}\\
&+ \|f_a\|_{L_2(kT,(k+1)T;W^{1/3}_{3/2}(\partial\Omega_0))}\Big)\\
&+ \|v_a(0)\|_{L_2(\Omega)} + \Big\|\frac{\chi_a(0)}{r}\Big\|_{L_3(\Omega)} + \|\chi_a(0)\|_{L_2(\Omega)};
\end{aligned}$$

$$\alpha_*(T) = \frac{1}{\nu^2}\sup_{k\in\mathbb{N}}\Big\|\frac{f_a}{r}\Big\|_{L_2(\Omega\times(kT,(k+1)T))} + \frac{1}{\nu}\Big\|\frac{\chi_a(0)}{r}\Big\|^2_{L_2(\Omega)},$$

$$\begin{aligned}
D_*(T) = \sup_{k\in\mathbb{N}} e^{\alpha_*(T)}\int_{kT}^{(k+1)T}\bigg[&\|f(t)-f_a(t)\|^2_{L_2(\Omega)}\\
&+ \Big|\int_\Omega r|v_\varphi(t) - v_{a\varphi}(t)|^2dx\Big|^2\bigg]dt + \|v(0)-v_a(0)\|^2_{L_2(\Omega)};
\end{aligned}$$

$$\beta_*(T) = D_*(T)\varphi(A_*(T)) + \sup_k \|f - f_a\|_{L_2(\Omega\times(kT,(k+1)T))} \\ + \|v(0) - v_a(0)\|_{H^1(\Omega)},$$

where φ is a polynomial with positive coefficients.

THEOREM 2 (stability). *Assume that there exists the axially symmetric solution determined by Theorem 1. Assume that* $\sup_k \|f - f_a\|_{L_2(kT,(k+1)T;L_2(\Omega))} < \infty$, $k \in \mathbb{N}$ $v(0) - v_a(0) \in H^1(\Omega)$, $|\int_\Omega rv_\varphi(t)dx| < \infty$. *Assume that* T *is so large that*

$$e^{-\frac{\nu'}{2}T} \le \frac{1}{2}, \quad \nu' < \nu \quad \text{and} \quad \alpha_*(T) \le \frac{\nu'}{2}T.$$

Let there exist a number $\sigma > 1$ *sufficiently large and* $D_*(T)$ *sufficiently small such that*

$$c\bigg(\sigma^5 D_*(T)\beta_*^5(T) + \frac{1}{\sigma}\bigg) \le 1,$$

where c *is a constant which depends on the constants from imbedding theorems, estimates of the boundary and on* ν. *Then* $v - v_a \in W_2^{2,1}(\Omega\times(kT,(k+1)T))$, $\nabla(p - p_a) \in L_2(\Omega\times(kT,(k+1)T))$ *for any* $k \in \mathbb{N}$ *and*

$$\sup_k[\|v - v_a\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))} + \|\nabla(p - p_a)\|_{L_2(\Omega\times(kT,(k+1)T))}] \le \sigma\beta_*(T).$$

Stability of axially symmetric solutions, 2d-solutions and helically symmetric solutions is considered in [6]. The authors of [6] assume that the perturbed solution satisfies

$$\int_0^\infty \|\nabla v(t)\|^2_{L_2(\Omega)}dt < \infty. \tag{1.10}$$

In this paper we prove the existence of axially symmetric solutions step by step on intervals $[kT,(k+1)T]$, $k \in \mathbb{N}$, so it does not have to vanish with time. Moreover, in [6] there are considered such boundary conditions that the Poincaré inequality holds, which is not used in this paper because the slip boundary conditions are considered. In [5] stability of solutions on 3d-torus is examined. We have to underline that stability in [5] is also considered step by step so (1.10) does not hold.

2. Notation and auxiliary results. Let u be a scalar. Then $|u|$ is the absolute value of u. Let $u = (u_1, u_2, u_3)$ be a vector. Then $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$. By c we denote the generic constant and $c = c(\sigma)$ is an increasing positive function.

To simplify considerations we introduce the notation

$$\begin{aligned}
&|u|_{p,Q} = \|u\|_{L_p(Q)}, \quad Q \in \{\Omega, S, \Omega^T, S^T\}, \quad \Omega^T = \Omega\times(0,T),\\
&S^T = S\times(0,T), \quad p \in [1,\infty];\\
&\|u\|_{s,Q} = \|u\|_{H^s(Q)}, \quad Q \in \{\Omega, S\}, \quad H^s(Q) = W_2^s(Q), \quad s \in \mathbb{R}_+;\\
&\|u\|_{s,Q^T} = \|u\|_{W_2^{s,s/2}(Q^T)}, \quad Q \in \{\Omega, S\}, \quad s \in \mathbb{R}_+;\\
&\|u\|_{s,p,Q} = \|u\|_{W_p^s(Q)}, \quad \|u\|_{s,p,Q^T} = \|u\|_{W_p^{s,s/2}(Q^T)}, \quad Q \in \{\Omega, S\},\\
&s \in \mathbb{R}_+, \quad p \in [1,\infty],
\end{aligned}$$

$$|u|_{p,q,\Omega^T}=\bigg(\int\limits_0^T\bigg(\int\limits_\Omega|u(x,t)|^pdx\bigg)^{q/p}dt\bigg)^{1/q},\quad p,q\in[1,\infty].$$

Moreover, we assume that

$$\|u\|_{L_p(\Omega_0)}=\bigg(\int\limits_{-a}^a\int\limits_0^R|u(x)|^prdrdz\bigg)^{1/p},$$

$$\|u\|_{L_p(\Omega)}=\bigg(\int\limits_{-a}^a dz\int\limits_0^R rdr\int\limits_0^{2\pi}d\varphi|u(x)|^p\bigg)^{1/p},$$

where r,φ,z are the cylindrical coordinates and similarly we define other spaces based on L_p-spaces.

We use $\bar u=(u_r,u_z)$, $\bar\nabla=(\partial_r,\partial_z)$, $\Omega_0=\{(r,z)\in\mathbb{R}^2:0<r<R,-a<z<z\}$, $\Omega_{0\varepsilon}=\{(r,z)\in\mathbb{R}^2:0<\varepsilon<r<R,-a<z<a\}$. Moreover, $dx_0=rdrdz$.

From [8, Ch. 4] we have the Korn inequality

$$\|v'\|_{1,\Omega}^2\le c_1\bigg(|\mathbb{D}(v')|_{2,\Omega}^2+\bigg|\int\limits_\Omega rv'_\varphi dx\bigg|^2\bigg).\tag{2.1}$$

To estimate the last term we use the following conservation law which holds for solutions of (1.9) (see [8, Ch. 4, Lemma 2.3])

$$\int\limits_\Omega rv'_\varphi dx=\int\limits_\Omega rv'_\varphi(0)dx+\int\limits_0^t dt'\int\limits_\Omega rf'_\varphi dx.\tag{2.2}$$

For axially symmetric solution we also have the conservation law

$$\int\limits_\Omega rv_{a\varphi}dx=\int\limits_\Omega rv_{a\varphi}(0)dx+\int\limits_0^t dt'\int\limits_\Omega rf_{a\varphi}dx.\tag{2.3}$$

3. Global existence of axially symmetric solutions. In this section we follow the results on global existence of axially symmetric solutions of Ladyzhenskaya [3] and Yudovich-Ukhovskij [7]. Since we are interested in proving global existence of solutions which do not decay with time we consider problem (1.1) step by step in time. Let $k\in\mathbb{N}$ and $T>0$ be given. Then instead of (1.1) we consider the following axially symmetric problem:

$$\begin{aligned}
&v_{,t}-\mathrm{div}\mathbb{T}(v,p)=-v\cdot\nabla v+f && \text{in}\ \ \Omega_0\times(kT,(k+1)T),\\
&\mathrm{div}v=0 && \text{in}\ \ \Omega_0\times(kT,(k+1)T),\\
&v\cdot\bar n=0,\quad \bar n\cdot\mathbb{T}(v,p)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, && \text{on}\ \ S_0\times(kT,(k+1)T),\\
&v\big|_{t=0}=v(kT) && \text{in}\ \ \Omega_0.
\end{aligned}\tag{3.1}$$

Lemma 3.1. *Assume that $v(0)\in L_2(\Omega_0)$, $f\in L_2(kT,(k+1)T;L_{6/5}(\Omega_0))$ for any $k\in\mathbb{N}$ and*

$$B_1(T) = \left(\sup_k \int_{kT}^{(k+1)T} |f(t')|_{6/5,\Omega_0}^2 dt' \right)^{1/2} < \infty. \tag{3.2}$$

Assume that $\nu = \nu_1 + \nu_2$, $\nu_i \geq 0$, $i = 1, 2$. *Then*

$$\begin{aligned} |\bar{v}(kT)|_{2,\Omega_0}^2 &\leq \frac{c_1}{\nu} \frac{B_1^2(T)}{1 - e^{-\nu_1 T}} + e^{-\nu_1 kT} |\bar{v}(0)|_{2,\Omega_0}^2 \\ &\leq \frac{c_1}{\nu} \frac{B_1^2(T)}{1 - e^{-\nu_1 T}} + |\bar{v}(0)|_{2,\Omega_0}^2 \equiv A_1^2(T), \\ |\bar{v}(t)|_{2,\Omega_0}^2 &\leq \frac{c_1}{\nu} \int_{kT}^{t} |f(t')|_{6/5,\Omega_0}^2 dt' + |\bar{v}(kT)|_{2,\Omega_0}^2 \leq 2A_1^2(T), \end{aligned} \tag{3.3}$$

$$\int_{kT}^{t} \left(|\bar{\nabla}\bar{v}|_{2,\Omega_0}^2 + \left| \frac{v_r}{r} \right|_{2,\Omega_0}^2 \right) dt' \leq \frac{1}{\nu} \left(\frac{c_1}{\nu} \int_{kT}^{t} |f(t')|_{6/5,\Omega_0}^2 dt' + |\bar{v}(kT)|_{2,\Omega_0}^2 \right) \leq \frac{2}{\nu} A_1^2(T), \tag{3.4}$$

where $t \in [kT, (k+1)T]$.

Proof. From [3] it follows that the axially symmetric solutions to (3.1) satisfy

$$\frac{d}{dt} |\bar{v}|_{2,\Omega_{0\varepsilon}}^2 + \nu \left(|\bar{\nabla}\bar{v}|_{2,\Omega_{0\varepsilon}}^2 + \left| \frac{v_r}{r} \right|_{2,\Omega_{0\varepsilon}}^2 \right) \leq \frac{c_1}{\nu} |f|_{6/5,\Omega_{0\varepsilon}}^2, \tag{3.5}$$

where c_1 is the constant from the imbedding $H^1(\Omega_{0\varepsilon}) \subset L_6(\Omega_{0\varepsilon})$. From (3.5) we have

$$\frac{d}{dt} (|\bar{v}|_{2,\Omega_{0\varepsilon}}^2 e^{\nu_1 t}) + \left(\nu_2 |\bar{\nabla}\bar{v}|_{2,\Omega_{0\varepsilon}}^2 + \nu \left| \frac{v_r}{r} \right|_{2,\Omega_{0\varepsilon}}^2 \right) e^{\nu_1 t} \leq \frac{c_1}{\nu} |f|_{6/5,\Omega_{0\varepsilon}}^2 e^{\nu_1 t}. \tag{3.6}$$

Integrating (3.6) with respect to time from kT to $t \in [kT, (k+1)T]$ we obtain

$$\begin{aligned} &|\bar{v}(t)|_{2,\Omega_{0\varepsilon}}^2 + e^{-\nu_1 t} \int_{kT}^{t} \left(\nu_2 |\bar{\nabla}\bar{v}(t')|_{2,\Omega_{0\varepsilon}}^2 + \nu \left| \frac{v_r(t')}{r} \right|_{2,\Omega_{0\varepsilon}}^2 \right) e^{\nu_1 t'} dt' \\ &\leq \frac{c_1}{\nu} e^{-\nu_1 t} \int_{kT}^{t} |f(t')|_{6/5,\Omega_{0\varepsilon}}^2 e^{\nu_1 t'} dt' + |\bar{v}(kT)|_{2,\Omega_{0\varepsilon}}^2 e^{-\nu_1 (t - kT)}. \end{aligned} \tag{3.7}$$

From (3.7) we have

$$|\bar{v}((k+1)T)|_{2,\Omega_{0\varepsilon}}^2 \leq \frac{c_1}{\nu} \int_{kT}^{(k+1)T} |f(t')|_{6/5,\Omega_{0\varepsilon}}^2 dt' + |\bar{v}(kT)|_{2,\Omega_{0\varepsilon}}^2 e^{-\nu_1' T}. \tag{3.8}$$

In view of (3.2) and the inductive considerations inequality (3.8) implies

$$|\bar{v}(kT)|_{2,\Omega_{0\varepsilon}}^2 \leq \frac{c_1}{\nu} \frac{B_1^2(T)}{1 - e^{-\nu_1 T}} + e^{-\nu_1 kT} |\bar{v}(0)|_{2,\Omega_{0\varepsilon}}. \tag{3.9}$$

Passing with ε to 0 in (3.9) yields (3.3). Simplifying (3.7) gives

$$|\bar{v}(t)|_{2,\Omega_{0\varepsilon}}^2 + \nu_2 e^{-\nu_1 T} \int_{kT}^{t} \left(|\bar{\nabla}\bar{v}|_{2,\Omega_{0\varepsilon}}^2 + \left| \frac{v_r}{r} \right|_{2,\Omega_{0\varepsilon}}^2 \right) dt' \tag{3.10}$$

$$\leq \frac{c_1}{\nu}\int_{kT}^{t}|f(t')|^2_{6/5,\Omega_{0\varepsilon}}dt' + |\bar{v}(kT)|^2_{2,\Omega_{0\varepsilon}},$$

where $t\in[kT,(k+1)T]$. Passing with ε to 0 in inequality (3.10) and inserting $\nu_1=0$, $\nu_2=\nu$ implies (3.4). This concludes the proof.

Lemma 3.2. *Assume that* $\frac{\chi(0)}{r}\in L_2(\Omega_0)$, $\frac{f}{r}\in L_2(kT,(k+1)T;L_2(\Omega_0))$, $k\in\mathbb{N}$ *and*

$$B_2(T)=\left(\sup_k\int_{kT}^{(k+1)T}\left|\frac{f(t)}{r}\right|^2_{2,\Omega_0}dt\right)^{1/2}<\infty. \tag{3.11}$$

Assume that $\nu=\nu_1+\nu_2$, $\nu_i\geq 0$, $i=1,2$. *Then*

$$\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_0}\leq\frac{1}{\nu}\frac{B_2^2(T)}{1-e^{-\nu_1T}}+e^{-\nu_1kT}\left|\frac{\chi(0)}{r}\right|^2_{2,\Omega_0}\leq\frac{1}{\nu}\frac{B_2^2(T)}{1-e^{-\nu_1T}}+\left|\frac{\chi(0)}{r}\right|^2_{2,\Omega_0}\equiv A_2^2(T), \tag{3.12}$$

$$\begin{aligned}&\left|\frac{\chi(t)}{r}\right|^2_{2,\Omega_0}\leq\frac{1}{\nu}\int_{kT}^{t}\left|\frac{f(t')}{r}\right|^2_{2,\Omega_0}dt'+\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_0}\leq 2A_2^2(T),\\ &\int_{kT}^{t}\left|\nabla\frac{\chi(t')}{r}\right|^2_{2,\Omega_0}dt'\leq\frac{1}{\nu}\left(\frac{1}{\nu}\int_{kT}^{t}\left|\frac{f(t')}{r}\right|^2_{2,\Omega_0}dt'+\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_0}\right)\leq\frac{2}{\nu}A_2^2(T),\end{aligned} \tag{3.13}$$

where $t\in[kT,(k+1)T]$.

Proof. From [3] we have

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^2_{2,\Omega_{0\varepsilon}}+\nu\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega_{0\varepsilon}}\leq\frac{1}{\nu}\left|\frac{f}{r}\right|^2_{2,\Omega_{0\varepsilon}}, \tag{3.14}$$

where $|\nabla\frac{\chi}{r}|=|\bar{\nabla}\frac{\chi}{r}|$. From (3.14) we obtain

$$\frac{d}{dt}\left(\left|\frac{\chi}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t}\right)+\nu_2\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t}\leq\frac{1}{\nu}\left|\frac{f}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t}. \tag{3.15}$$

Integrating (3.15) with respect to time from kT to $t\in[kT,(k+1)T]$ implies

$$\begin{aligned}&\left|\frac{\chi(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}}+\nu_2e^{-\nu_1t}\int_{kT}^{t}\left|\nabla\frac{\chi(t')}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t'}dt'\\ &\leq\frac{1}{\nu}e^{-\nu_1t}\int_{kT}^{t}\left|\frac{f(t')}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t'}dt'+\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{-\nu_1(t-kT)}.\end{aligned} \tag{3.16}$$

Inserting $t=(k+1)T$ in (3.16) we get

$$\left|\frac{\chi((k+1)T)}{r}\right|^2_{2,\Omega_{0\varepsilon}}\leq\frac{1}{\nu}\int_{kT}^{(k+1)T}\left|\frac{f(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}}dt+\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{-\nu_1T}. \tag{3.17}$$

In view of (3.11) and the inductive considerations inequality (3.17) yields

$$\left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}} \le \frac{1}{\nu}\frac{B_2^2(T)}{1-e^{-\nu_1 T}} + e^{-\nu_1 kT}\left|\frac{\chi(0)}{r}\right|^2_{2,\Omega_{0\varepsilon}}. \tag{3.18}$$

Passing with ε to 0 in (3.18) gives (3.12). Simplifying (3.16) implies

$$\left|\frac{\chi(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu_2 e^{-\nu_1 T}\int_{kT}^{t}\left|\nabla\frac{\chi(t')}{r}\right|^2_{2,\Omega_{0\varepsilon}}dt' \le \frac{1}{\nu}\int_{kT}^{t}\left|\frac{f(t')}{r}\right|^2_{2,\Omega_{0\varepsilon}}dt' + \left|\frac{\chi(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}, \tag{3.19}$$

where $t\in[kT,(k+1)T]$. Passing with ε to 0 in inequality (3.19) and putting $\nu_1 = 0$, $\nu_2=\nu$ yields (3.13). This concludes the proof.

LEMMA 3.3. *Assume that* $\frac{\chi(0)}{r}\in L_s(\Omega_0)$, $\frac{F}{r}\in L_s(kT,(k+1)T;L_s(\Omega_0))$, $s>1$, $k\in\mathbb{N}$ *and*

$$B_3(s,T) = \left(\sup_k \int_{kT}^{(k+1)T}\left|\frac{F(t)}{r}\right|^s_{s,\Omega_0}dt\right)^{1/s} < \infty. \tag{3.20}$$

Assume that $\nu=\nu_1+\nu_2$, $\nu_i\ge 0$, $i=1,2$. *Then*

$$\begin{aligned}\left|\frac{\chi(kT)}{r}\right|^s_{s,\Omega_0} &\le c_2\frac{B_3^s(s,T)}{1-e^{-\nu_1 T}} + e^{-\nu_1 kT}\left|\frac{\chi(0)}{r}\right|^s_{s,\Omega_0}\\ &\le c_2\frac{B_3^s(s,T)}{1-e^{-\nu_1 T}} + \left|\frac{\chi(0)}{r}\right|^s_{s,\Omega_0} \equiv A_3^s(s,T),\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left|\frac{\chi(t)}{r}\right|^s_{s,\Omega_0} + \nu\int_{kT}^{t}\int_{\Omega_0}\left|\nabla\left|\frac{\chi(t')}{r}\right|^{s/2}\right|^2 dx_0dt'\\ &\le c_2\int_{kT}^{t}\left|\frac{F(t')}{r}\right|^s_{s,\Omega_0}dt' + \left|\frac{\chi(kT)}{r}\right|^s_{s,\Omega_0} \le 2A_3^s(s,T),\end{aligned} \tag{3.22}$$

where $t\in[kT,(k+1)T]$.

Proof. From [7] we have the inequality

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \frac{4(s-1)}{s}\nu\int_{\Omega_{0\varepsilon}}\left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2dx_0 \le s\int_{\Omega_{0\varepsilon}}\left|\frac{F}{r}\right|\left|\frac{\chi}{r}\right|^{s-1}dx_0. \tag{3.23}$$

Let $s\in[2,\infty)$. Then (3.23) implies

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + 2\nu_1\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + 2\nu_2\int_{\Omega_{0\varepsilon}}\left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2dx_0 \le s\int_{\Omega_{0\varepsilon}}\left|\frac{F}{r}\right|\left|\frac{\chi}{r}\right|^{s-1}dx_0. \tag{3.24}$$

We shall perform estimates in (3.24) in two different ways. First we estimate the r.h.s. of (3.24) by

$$s\left|\frac{F}{r}\right|_{s,\Omega_{0\varepsilon}}\left|\frac{\chi}{r}\right|^{s-1}_{s,\Omega_{0\varepsilon}} \le (s-1)\varepsilon^{\frac{s}{s-1}}\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \frac{1}{\varepsilon^s}\left|\frac{F}{r}\right|^s_{s,\Omega_{0\varepsilon}} \equiv I_1.$$

Assuming that $(s-1)\varepsilon^{\frac{s}{s-1}} = \nu_1$ we obtain

$$I_1 = \nu_1 \left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \left(\frac{s-1}{\nu_1}\right)^{s-1} \left|\frac{F}{r}\right|^s_{s,\Omega_{0\varepsilon}}.$$

Then (3.24) takes the form

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \nu_1 \left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \nu_2 \int\limits_{\Omega_{0\varepsilon}} \left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2 dx_0 \le \left(\frac{s-1}{\nu}\right)^{s-1} \left|\frac{F}{r}\right|^s_{s,\Omega_{0\varepsilon}}. \tag{3.25$'$}$$

In the second case we use the imbedding

$$\int\limits_{\Omega_{0\varepsilon}} \left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2 dx_0 \ge \frac{1}{c_1}\left|\frac{\chi}{r}\right|^s_{3s,\Omega_{0\varepsilon}},$$

where c_1 is the constant from the imbedding $H^1(\Omega_{0\varepsilon}) \subset L_6(\Omega_{0\varepsilon})$. Then the r.h.s. of (3.24) is bounded by

$$s\left|\frac{F}{r}\right|_{\frac{3s}{2s+1},\Omega_{0\varepsilon}} \left|\frac{\chi}{r}\right|^{s-1}_{3s,\Omega_{0\varepsilon}} \le (s-1)\varepsilon^{\frac{s}{s-1}} \left|\frac{\chi}{r}\right|^s_{3s,\Omega_{0\varepsilon}} + \frac{1}{\varepsilon^s}\left|\frac{F}{r}\right|^s_{\frac{3s}{2s+1},\Omega_{0\varepsilon}} \equiv I_2.$$

Imposing that $(s-1)\varepsilon^{\frac{s}{s-1}} = \frac{\nu_2}{c_1}$ we have that

$$I_2 = \frac{\nu_2}{c_1}\left|\frac{\chi}{r}\right|^s_{3s,\Omega_{0\varepsilon}} + \left(\frac{(s-1)c_1}{\nu_2}\right)^{s-1} \left|\frac{F}{r}\right|^s_{\frac{3s}{2s+1},\Omega_{0\varepsilon}}.$$

Then (3.24) takes the form

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + 2\nu \left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \nu_2 \int\limits_{\Omega_{0\varepsilon}} \left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2 dx_0 \le \left(\frac{(s-1)c_1}{\nu_2}\right)^{s-1} \left|\frac{F}{r}\right|^s_{\frac{3s}{2s+1},\Omega_{0\varepsilon}}. \tag{3.25$''$}$$

We shall restrict our considerations to the first case. Then (3.25′) implies

$$\frac{d}{dt}\left(\left|\frac{\chi}{r}\right|^s_{s,\Omega_{0\varepsilon}} e^{\nu_1 t}\right) + \nu_2 \int\limits_{\Omega_{0\varepsilon}} \left|\nabla\left|\frac{\chi}{r}\right|^{s/2}\right|^2 dx' e^{\nu_1 t} \le c_2 \left|\frac{F}{r}\right|^s_{s,\Omega_{0\varepsilon}} e^{\nu_1 t}, \tag{3.26}$$

where $c_2 = (\frac{s-1}{\nu})^{s-1}$. Integrating (3.26) with respect to time from kT to $t \in [kT,(k+1)T]$ gives

$$\begin{aligned} &\left|\frac{\chi(t)}{r}\right|^s_{s,\Omega_{0\varepsilon}} + \nu_2 e^{-\nu_1 t} \int\limits_{kT}^{t} \int\limits_{\Omega_{0\varepsilon}} \left|\nabla\left|\frac{\chi(t')}{r}\right|^{s/2}\right|^2 dx' e^{\nu_1 t'} dt' \\ &\quad \le c_2 e^{-\nu_1 t} \int\limits_{kT}^{t} \left|\frac{F(t')}{r}\right|^s_{s,\Omega_{0\varepsilon}} e^{\nu_1 t'} dt' + \left|\frac{\chi(kT)}{r}\right|^s_{s,\Omega_{0\varepsilon}} e^{-\nu_1(t-kT)}. \end{aligned} \tag{3.27}$$

From (3.27) we obtain

$$\left|\frac{\chi((k+1)T)}{r}\right|^s_{s,\Omega_{0\varepsilon}} \le c_2 \int\limits_{kT}^{(k+1)T} \left|\frac{F(t')}{r}\right|^s_{s,\Omega_{0\varepsilon}} dt' + \left|\frac{\chi(kT)}{r}\right|^s_{s,\Omega_{0\varepsilon}} e^{-\nu_1 T}. \tag{3.28}$$

In view of (3.20) and inductive considerations we have

$$\left|\frac{\chi(kT)}{r}\right|_{s,\Omega_{0\varepsilon}}^{s} \le c_2\frac{B_3^s(s,T)}{1-e^{-\nu_1 T}} + e^{-\nu_1 kT}\left|\frac{\chi(0)}{r}\right|_{s,\Omega_{0\varepsilon}}^{s}. \tag{3.29}$$

Passing with ε to 0 yields (3.21). Simplifying (3.27) yields

$$\begin{aligned}&\left|\frac{\chi(t)}{r}\right|_{s,\Omega_{0\varepsilon}}^{s} + \nu_2 e^{-\nu_1 T}\int_{kT}^{t}\int_{\Omega_{0\varepsilon}}\left|\nabla\left|\frac{\chi(t')}{r}\right|^{s/2}\right|^2 dx_0dt'\\ &\le c_2\int_{kT}^{t}\left|\frac{F(t')}{r}\right|_{s,\Omega_{0\varepsilon}}^{s}dt' + \left|\frac{\chi(kT)}{r}\right|_{s,\Omega_{0\varepsilon}}^{s},\end{aligned} \tag{3.30}$$

where $t\in[kT,(k+1)T]$. Passing with ε to 0 and inserting $\nu_1=0$, $\nu_2=\nu$, inequality (3.30) implies (3.22). This concludes the proof.

Finally we shall obtain estimates for $\frac{v_r}{r}$.

Lemma 3.4. *Let the assumptions of Lemma 3.1 and Lemma 3.3 for $s=3$ hold. Let*

$$\sup_{k\in\mathbb{N}}\bigg(\|\mathrm{div} f\|_{L_2(kT,(k+1)T;L_2(\Omega_0))} + \|f\|_{L_2(kT,(k+1)T);W_{3/2}^{1/3}(\partial\Omega_0))} + \left\|\frac{f_r}{r}\right\|_{L_2(kT,(k+1)T;L_2(\Omega_0))}\bigg) < \infty.$$

Let

$$\begin{aligned}B_4^2(T) &= A_1^2(T)A_3^2(3,T) + R^4A_3^4(3,T)\\ &+ \sup_k\int_{kT}^{(k+1)T}\left(|\mathrm{div} f(t)|_{3,\Omega_0}^2 + \|f(t)\|_{1/3,3/2,\partial\Omega_0}^2 + \left|\frac{f_r(t)}{r}\right|_{2,\Omega_0}^2\right)dt.\end{aligned} \tag{3.31}$$

Then

$$\begin{aligned}\left|\frac{v_r(kT)}{r}\right|_{2,\Omega_0}^2 &\le \frac{cB_4^2(T)}{1-e^{-\nu_1 T}} + e^{-\nu_1 kT}\left|\frac{v_r(0)}{r}\right|_{2,\Omega_0}\\ &\le \frac{cB_4^2(T)}{1-e^{-\nu_1 T}} + \left|\frac{v_r(0)}{r}\right|_{2,\Omega_0}^2 \equiv A_4^2(T),\end{aligned} \tag{3.32}$$

and

$$\left|\frac{v_r(t)}{r}\right|_{2,\Omega_0}^2 + \nu\int_{kT}^{t}\left|\bar\nabla\frac{v_r(t')}{r}\right|_{2,\Omega_0}^2dt' \le cB_4^2(T) + \left|\frac{v_r(kT)}{r}\right|_{2,\Omega_0}^2 \le 2A_4^2(T). \tag{3.33}$$

Proof. From $(1.2)_1$ and boundary conditions we obtain the problem

$$\begin{aligned}&v_{r,t} - \nu\left[\left(r\left(\frac{v_r}{r}\right)_{,r}\right)_{,r} + 2\left(\frac{v_r}{r}\right)_{,r} + v_{r,zz}\right] + v_z\chi\\ &= -q_{,r} + f_r && \text{in } \Omega_{0\varepsilon}\times(kT,(k+1)T),\\ &v_r\big|_{r=R,\varepsilon} = 0,\\ &v_{r,z}\big|_{z=\pm a} = 0,\\ &v_r\big|_{t=kT} = v_r(kT) && \text{in } \Omega_{0\varepsilon},\end{aligned} \tag{3.34}$$

where $q = p - \frac{v^2}{2}$. Multiplying $(3.34)_1$ by $\frac{v_r}{r^2}$, integrating the result over $\Omega_{0\varepsilon}$ with the measure $rdrdz$ and using the boundary conditions we get

$$\begin{aligned}(3.35)\qquad &\frac{1}{2}\frac{d}{dt}\left|\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu\left(\left|\left(\frac{v_r}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\left(\frac{v_r}{r}\right)_{,z}\right|^2_{2,\Omega_{0\varepsilon}}\right)\\ &+ \nu\int_{\Omega_{0\varepsilon}} 2\left(\frac{v_r}{r}\right)_{,r}\frac{v_r}{r}drdz = -\int_{\Omega_{0\varepsilon}} v_z\chi\frac{v_r}{r}drdz\\ &-\int_{\Omega_{0\varepsilon}} q_{,r}\frac{v_r}{r}drdz + \int_{\Omega_{0\varepsilon}}\frac{f_r}{r}\frac{v_r}{r}dx_0.\end{aligned}$$

In view of boundary conditions the last term on the l.h.s. of (3.35) vanishes. The first term on the r.h.s. of (3.35) is estimated by

$$\varepsilon_1\left|\frac{v_r}{r}\right|^2_{6,\Omega_{0\varepsilon}} + c(1/\varepsilon_1)\left|\frac{\chi}{r}\right|^2_{3,\Omega_{0\varepsilon}}|v_z|^2_{2,\Omega_{0\varepsilon}}.$$

We express the second term on the r.h.s. of (3.35) in the form

$$-\int_{\Omega_{0\varepsilon}} p_{,r}\frac{v_r}{r}drdz + \int_{\Omega_{0\varepsilon}} vv_{,r}\frac{v_r}{r}drdz \equiv I_1 + I_2.$$

First we examine

$$I_1 = -\int_{\Omega_{0\varepsilon}}(p-\mathring{p})_{,r}\frac{v_r}{r}drdz = \int_{\Omega_{0\varepsilon}}(p-\mathring{p})\left(\frac{v_r}{r}\right)_{,r}drdz = \int_{\Omega_{0\varepsilon}}\frac{p-\mathring{p}}{r}\left(\frac{v_r}{r}\right)_{,r}dx_0,$$

where $\mathring{p} = p|_{r=0}$. Continuing, we have

$$|I_1| \le \varepsilon_2\left|\left(\frac{v_r}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + c(1/\varepsilon_2)\left|\frac{p-\mathring{p}}{r}\right|^2_{2,\Omega_{0\varepsilon}}.$$

Next we examine

$$I_2 = \frac{1}{2}\int_{\Omega_{0\varepsilon}}(v^2)_{,r}\frac{v_r}{r}drdz = \frac{1}{2}\int_{\Omega_{0\varepsilon}}(v^2-\mathring{v}^2)_{,r}\frac{v_r}{r}drdz = -\frac{1}{2}\int_{\Omega_{0\varepsilon}}(v^2-\mathring{v}^2)\left(\frac{v_r}{r}\right)_{,r}drdz,$$

where $\mathring{v} = v|_{r=0}$. By the Hölder and Young inequalities we have

$$|I_2| \le \varepsilon_3\left|\left(\frac{v_r}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + c(1/\varepsilon_3)\left|\frac{v^2-\mathring{v}^2}{r}\right|^2_{2,\Omega_{0\varepsilon}},$$

where the second integral equals

$$\int_{\Omega_{0\varepsilon}}\left|\frac{(v-\mathring{v})\cdot(v+\mathring{v})}{r}\right|^2dx_0 \le \left|\frac{v-\mathring{v}}{r}\right|^2_{2\lambda_1,\Omega_{0\varepsilon}}|v+\mathring{v}|^2_{2\lambda_2,\Omega_{0\varepsilon}},$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Finally, we estimate the last term on the r.h.s. of (3.35) by

$$\varepsilon_4\left|\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}} + c(1/\varepsilon_4)\left|\frac{f_r}{r}\right|^2_{2,\Omega_{0\varepsilon}}.$$

Utilizing the above estimates in (3.35), assuming that $\varepsilon_1 - \varepsilon_4$ are sufficiently small and exploiting the Poincaré inequality we obtain

$$\begin{aligned}(3.36)\quad &\frac{d}{dt}\left|\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu\left(\left|\left(\frac{v_r}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\left(\frac{v_r}{r}\right)_{,z}\right|^2_{2,\Omega_{0\varepsilon}}\right)\\ &\le c\left(\left|\frac{\chi}{r}\right|^2_{3,\Omega_{0\varepsilon}}|v_z|^2_{2,\Omega_{0\varepsilon}} + \left|\frac{p-\mathring{p}}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\frac{v-\mathring{v}}{r}\right|^2_{2\lambda_1,\Omega_{0\varepsilon}}|v+\mathring{v}|^2_{2\lambda_2,\Omega_{0\varepsilon}} + \left|\frac{f_r}{r}\right|^2_{2,\Omega_{0\varepsilon}}\right),\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. In view of the Poincaré inequality we can express (3.36) in the form

$$\begin{aligned}(3.37)\quad &\frac{d}{dt}\left|\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu_1\left|\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu_2\left(\left|\left(\frac{v_r}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\left(\frac{v_r}{r}\right)_{,z}\right|^2_{2,\Omega_{0\varepsilon}}\right)\\ &\le c\left(\left|\frac{\chi}{r}\right|^2_{3,\Omega_{0\varepsilon}}|v_z|^2_{2,\Omega_{0\varepsilon}} + \left|\frac{p-\mathring{p}}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\frac{v-\mathring{v}}{r}\right|^2_{2\lambda_1,\Omega_{0\varepsilon}}|v+\mathring{v}|^2_{2\lambda_2,\Omega_{0\varepsilon}} + \left|\frac{f_r}{r}\right|^2_{2,\Omega_{0\varepsilon}}\right),\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Multiplying (3.37) by $e^{\nu_1 t}$ and integrating with respect to time from kT to $t \in (kT,(k+1)T]$ yields

$$\begin{aligned}(3.38)\quad &\left|\frac{v_r(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}} + \nu_2 e^{-\nu_1 t}\int_{kT}^{t}\left(\left|\left(\frac{v_r(t')}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}} + \left|\left(\frac{v_r(t')}{r}\right)_{,z}\right|^2_{2,\Omega_{0\varepsilon}}\right)e^{\nu_1 t'}dt'\\ &\le ce^{-\nu_1 t}\int_{kT}^{t}\left(\left|\frac{\chi}{r}\right|^2_{3,\Omega_{0\varepsilon}}|v_z|^2_{2,\Omega_{0\varepsilon}} + \left|\frac{p-\mathring{p}}{r}\right|^2_{2,\Omega_{0\varepsilon}}\right.\\ &\left.+\left|\frac{v-\mathring{v}}{r}\right|^2_{2\lambda_1,\Omega_{0\varepsilon}}|v+\mathring{v}|^2_{2\lambda_1,\Omega_{0\varepsilon}} + \left|\frac{f_r}{r}\right|^2_{2,\Omega_{0\varepsilon}}\right)e^{\nu_1 t'}dt'\\ &+\left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{-\nu_1(t-kT)},\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Now we shall estimate the first term on the r.h.s. of (3.38). In view of $(3.4)_1$ and (3.22) we have that

$$\begin{aligned}&ce^{-\nu_1 t}\int_{kT}^{t}\left|\frac{\chi}{r}\right|^2_{3,\Omega_{0\varepsilon}}|v_z|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1 t'}dt' \le ce^{-\nu_1 t}\int_{kT}^{t}\left|\frac{\chi}{r}\right|^2_{3,\Omega_0}|v_z|^2_{2,\Omega_0}e^{\nu_1 t'}dt'\\ &\le cA_1^2(T)A_3^2(3,T)e^{-\nu_1 t}\int_{kT}^{t}e^{\nu_1 t'}dt' \le cA_1^2(T)A_3^2(3,T).\end{aligned}$$

From (3.22) we have that

$$(3.39)\qquad |\chi(t)|_{s,\Omega_0} \le R\left|\frac{\chi(t)}{r}\right|_{s,\Omega_0} \le 2RA_3(s,T).$$

Then considering the problem

$$
\begin{aligned}
&\operatorname{rot} v = \alpha = \chi e_\varphi && \text{in } \Omega_{0\varepsilon}, \\
&\operatorname{div} v = 0 && \text{in } \Omega_{0\varepsilon}, \\
&v \cdot \bar{n} = 0 && \text{on } S_{0\varepsilon},
\end{aligned}
\tag{3.40}
$$

we obtain the estimate

$$
\|v\|_{1,s,\Omega_{0\varepsilon}} \le c|\chi|_{s,\Omega_{0\varepsilon}} \le cRA_3(s,T). \tag{3.41}
$$

Let us consider the third expression under the time integral on the r.h.s. of (3.38). Let $\lambda_1 = \frac{3}{2}$, $\lambda_2 = 3$. Then it is bounded by

$$
\left|\frac{v - \overset{\circ}{v}}{r}\right|^2_{3,\Omega_0} |v + \overset{\circ}{v}|^2_{6,\Omega_0} \equiv I_1.
$$

By the Hardy inequality the first factor in I_1 is estimated by

$$
c|v_{,r}|^2_{3,\Omega_0} \equiv I_2,
$$

so in view of (3.41) we have

$$
I_2 \le cR^2 A_3^2(3,T).
$$

In view of the imbedding

$$
|v|_{r=0}|_{6,\Omega_0} \le c\|v\|_{1,3,\Omega_0},
$$

the second factor in I_1 is bounded by

$$
c\|v\|^2_{1,3,\Omega_0} \le cR^2 A_3^2(3,T).
$$

Hence,

$$
I_1 \le cR^4 A_3^4(3,T).
$$

Finally, the term with pressure under the time integral on the r.h.s. (3.38) is estimated by

$$
c\left|\frac{p - \overset{\circ}{p}}{r}\right|_{3,\Omega_{0\varepsilon}} \le c\left|\frac{p - \overset{\circ}{p}}{r}\right|_{3,\Omega_0} \le c|p_{,r}|_{3,\Omega_0} \equiv I_3,
$$

where in the second inequality the Hardy inequality was applied.

To estimate I_3 we consider the problem for pressure

$$
\begin{aligned}
&\Delta p = -\nabla v \cdot \nabla v + \operatorname{div} f && \text{in } \Omega_0, \\
&p_{,r} = f_r && \text{for } r = R, \\
&p_{,z} = f_z && \text{for } z = \mp a.
\end{aligned}
\tag{3.42}
$$

For solutions of problem (3.42) we have

$$
\|\nabla p\|_{1,\sigma,\Omega_0} \le c(|\nabla v|^2_{2\sigma,\Omega_0} + |\operatorname{div} f|_{\sigma,\Omega_0} + \|f\|_{1-1/\sigma,\sigma,\partial\Omega_0}). \tag{3.43}
$$

Hence

$$
|p_{,r}|_{3,\Omega_0} \le \|\nabla p\|_{1,3/2,\Omega_0} \le c(|\nabla v|^2_{3,\Omega_0} + |\operatorname{div} f|_{3,\Omega_0} + \|f\|_{1/3,3/2,\partial\Omega_0}). \tag{3.44}
$$

Then (3.41) yields

$$
|p_{,r}|_{3,\Omega_0} \le c(R^2 A_3^2(3,T) + |\operatorname{div} f|_{3,\Omega_0} + \|f\|_{1/3,3/2,\partial\Omega_0}). \tag{3.45}
$$

Summarizing, (3.38) takes the form

$$\begin{aligned}(3.46)\quad &\left|\frac{v_r(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}}+\nu_2e^{-\nu_1t}\int\limits_{kT}^{t}\left(\left|\left(\frac{v_r(t')}{r}\right)_{,r}\right|^2_{2,\Omega_{0\varepsilon}}+\left|\left(\frac{v_r(t')}{r}\right)_{,z}\right|^2_{2,\Omega_{0\varepsilon}}\right)e^{\nu_1t'}dt'\\ &\le c(A_1^2(T)A_3^2(3,T)+R^4A_3^4(3,T))\\ &+c\int\limits_{kT}^{t}\left(|\mathrm{div}f(t')|^2_{3,\Omega_0}+\|f(t')\|_{1/3,3/2,\partial\Omega_0}+\left|\frac{f_r(t')}{r}\right|^2_{2,\Omega_0}\right)dt'\\ &+\left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_0}e^{-\nu_1(t-kT)}.\end{aligned}$$

In view of (3.31) inequality (3.46) takes the form

$$\begin{aligned}(3.47)\quad &\left|\frac{v_r(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}}+\nu_2e^{-\nu_1t}\int\limits_{kT}^{t}\left|\bar\nabla\frac{v_r}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{\nu_1t'}dt'\\ &\qquad\le cB_4^2(T)+\left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{-\nu_1(t-kT)},\end{aligned}$$

where $t\in[kT,(k+1)T]$. Inserting $t=(k+1)T$ into (3.47) we obtain

$$(3.48)\qquad \left|\frac{v_r((k+1)T)}{r}\right|^2_{2,\Omega_{0\varepsilon}}\le cB_4^2(T)+\left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}e^{-\nu_1T}.$$

Hence by the inductive considerations we obtain

$$(3.49)\qquad \left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}}\le\frac{cB_4^2(T)}{1-e^{-\nu_1T}}+e^{-\nu_1kT}\left|\frac{v_r(0)}{r}\right|^2_{2,\Omega_{0\varepsilon}}.$$

Passing with ε to 0 yields (3.32).

Simplifying (3.47) yields

$$(3.50)\qquad \left|\frac{v_r(t)}{r}\right|^2_{2,\Omega_{0\varepsilon}}+\nu_2e^{-\nu_1T}\int\limits_{kT}^{t}\left|\bar\nabla\frac{v_r(t')}{r}\right|^2_{2,\Omega_{0\varepsilon}}dt'\le cB_4^2(T)+\left|\frac{v_r(kT)}{r}\right|^2_{2,\Omega_{0\varepsilon}},$$

where $t\in[kT,(k+1)T]$. Passing with ε to 0 and inserting $\nu_1=0$, $\nu_2=\nu$, we obtain from (3.50) inequality (3.33). This concludes the proof.

Finally we need

LEMMA 3.5. *Assume that $A_5(T)$ is defined by the r.h.s. of (3.60), $B_6(T)$ by (3.66), $A_2(T)$ by (3.12), $A_4(T)$ by (3.32). Then axially symmetric solutions to problem (1.1) are such that $v\in W_2^{2,1}(\Omega_0\times(kT,(k+1)T))\cap C([kT,(k+1)T];H^1(\Omega_0))$ for any $k\in\mathbb{N}$ and satisfy the inequalities*

$$(3.51)\qquad \begin{aligned}&\|\bar v\|_{2,\Omega_0\times(kT,(k+1)T)}\le A_6(T),\\ &\|\bar v\|_{C([kT,(k+1)T];H^1(\Omega_0))}\le A_7(T),\end{aligned}$$

where

$$A_6(T) = c[A_4^2(T) + A_5^2(T) + B_6(T) + RA_2(T)],$$
$$A_7(T) = cRA_3(2,T),$$

where $A_5(T)$ is defined in (3.60) and $B_6(T)$ by (3.66).

Proof. To have the full norm of $H^1(\Omega_{0\varepsilon})$ in the second term on the l.h.s. of (3.5) we use the inequalitties

$$|v_r|_{2,\Omega_{0\varepsilon}}^2 \le R^2 \left|\frac{v_r}{r}\right|_{2,\Omega_{0\varepsilon}}^2$$

and

$$|v_z|_{2,\Omega_{0\varepsilon}}^2 \le 4a^2 |v_{z,z}|_{2,\Omega_{0\varepsilon}}^2,$$

where the second inequality is the Poincaré inequality. Then (3.5) assumes the form

$$\frac{d}{dt}|\bar v|_{2,\Omega_{0\varepsilon}}^2 + \frac{\nu}{2} c_3 \|\bar v\|_{1,\Omega_{0\varepsilon}}^2 \le \frac{c_1}{\nu} |\bar f|_{6/5,\Omega_{0\varepsilon}}^2, \tag{3.52}$$

where $c_3 = \min\{1, \frac{1}{R^2}, \frac{1}{4a^2}\}$. After the same considerations as in Lemma 3.1 we obtain

$$\begin{aligned} &|\bar v(kT)|_{2,\Omega_0}^2 \le A_1^2(T), \\ &|\bar v(t)|_{2,\Omega_0}^2 + \frac{\nu}{2} c_3 \int_{kT}^{t} \|\bar v(t')\|_{1,\Omega_0}^2 dt' \le \frac{2}{\nu} A_1^2(T), \end{aligned} \tag{3.53}$$

for $t \in [kT, (k+1)T]$, $k \in \mathbb{N}$.

From [3] we have

$$\frac{1}{2}\frac{d}{dt}|\chi|_{2,\Omega_{0\varepsilon}}^2 + \nu\left(|\nabla\chi|_{2,\Omega_{0\varepsilon}}^2 + \left|\frac{\chi}{r}\right|_{2,\Omega_{0\varepsilon}}^2\right) \le c(\|v_r\|_{1,\Omega_{0\varepsilon}}^2 + |\bar f|_{2,\Omega_{0\varepsilon}}^2). \tag{3.54}$$

Using that $\chi|_{\partial\Omega_{0\varepsilon}} = 0$ we can express (3.54) in the form

$$\frac{d}{dt}|\chi|_{2,\Omega_{0\varepsilon}}^2 + \nu\left(\|\chi\|_{1,\Omega_{0\varepsilon}}^2 + \left|\frac{\chi}{r}\right|_{2,\Omega_{0\varepsilon}}^2\right) \le c(\|v_r\|_{1,\Omega_{0\varepsilon}}^2 + |\bar f|_{2,\Omega_{0\varepsilon}}^2). \tag{3.55}$$

Utilizing that $\nu = \nu_1 + \nu_2$ and multiplying (3.55) by $e^{\nu_1 t}$ we obtain

$$\frac{d}{dt}(|\chi|_{2,\Omega_{0\varepsilon}}^2 e^{\nu_1 t}) + \nu_2\left(\|\chi\|_{1,\Omega_{0\varepsilon}}^2 + \left|\frac{\chi}{r}\right|_{2,\Omega_{0\varepsilon}}^2\right)e^{\nu_1 t} \le c(\|v_r\|_{1,\Omega_{0\varepsilon}}^2 + |\bar f|_{2,\Omega_{0\varepsilon}}^2)e^{\nu_1 t}. \tag{3.56}$$

Integrating (3.56) with respect to time from kT to $t \in (kT, (k+1)T]$ and neglecting the second term on the l.h.s. we obtain

$$|\chi(t)|_{2,\Omega_{0\varepsilon}}^2 \le ce^{-\nu_1 t}\int_{kT}^{t}(\|v_r(t')\|_{1,\Omega_{0\varepsilon}}^2 + |\bar f(t')|_{2,\Omega_{0\varepsilon}}^2)e^{\nu_1 t'}dt' + |\chi(kT)|_{2,\Omega_{0\varepsilon}}^2 e^{-\nu_1(t-kT)}. \tag{3.57}$$

Inserting $t = (k+1)T$ and utilizing (3.53) we get

$$|\chi((k+1)T)|_{2,\Omega_{0\varepsilon}}^2 \le cB_5^2(T) + |\chi(kT)|_{2,\Omega_{0\varepsilon}}^2 e^{-\nu_1 T}, \tag{3.58}$$

where

$$B_5^2(T) = A_1^2(T) + \sup_{k\in\mathbb{N}} \int_{kT}^{(k+1)T} |\bar f(t')|_{2,\Omega_0}^2 dt'. \tag{3.59}$$

Inductive considerations and passing with ε to 0 implies

$$(3.60)\quad |\chi(kT)|^2_{2,\Omega_0} \le \frac{cB_5^2(T)}{1-e^{-\nu_1 T}} + |\chi(0)|^2_{2,\Omega_0} e^{-\nu_1 kT} \le \frac{cB_5^2(T)}{1-e^{-\nu_1 T}} + |\chi(0)|^2_{2,\Omega_0} \equiv A_5^2(T).$$

Integrating (3.55) with respect to time from kT to $t \in (kT,(k+1)T]$, passing with ε to 0, utilizing $(3.53)_2$ and (3.60) we have

$$(3.61)\qquad |\chi(t)|^2_{2,\Omega_0} + \nu \int_{kT}^{t} \left(\|\chi(t')\|^2_{1,\Omega_0} + \left|\frac{\chi(t')}{r}\right|^2_{2,\Omega_0} \right) dt' \le cA_5^2(T).$$

Let us consider the problem

$$(3.62)\qquad \begin{aligned} & v_{r,z} - v_{z,r} = \chi && \text{in } \Omega_{0\varepsilon}, \\ & v_{r,z} + v_{z,z} = -\frac{v_r}{r} && \text{in } \Omega_{0\varepsilon}, \\ & v_r\big|_{r=R,\varepsilon} = 0, \quad v_z\big|_{z=\mp a} = 0. \end{aligned}$$

In view of (3.33) and (3.61) we have

$$(3.63)\qquad \begin{aligned} & \|\bar v\|_{L_\infty(kT,t;H^1(\Omega_{0\varepsilon}))} + \|\bar v\|_{L_2(kT,t;H^2(\Omega_{0\varepsilon}))} \\ & \le c(A_4(T) + A_5(T)), \quad t \in (kT,(k+1)T], \quad k \in \mathbb{N}. \end{aligned}$$

Passing with ε to 0 yields

$$(3.64)\qquad \begin{aligned} & \|\bar v\|_{L_\infty(kT,t;H^1(\Omega_0))} + \|\bar v\|_{L_2(kT,t;H^2(\Omega_0))} \\ & \le c(A_4(T) + A_5(T)), \quad t \in (kT,(k+1)T], \quad k \in \mathbb{N}. \end{aligned}$$

Let us consider the problem

$$(3.65)\qquad \begin{aligned} & \bar v_{,t} - \mathrm{div}\mathbb{T}(\bar v, p) = -\bar v \cdot \nabla \bar v + f && \text{in } \Omega_0 \times (kT,(k+1)T), \\ & \mathrm{div}\bar v = 0 && \text{in } \Omega_0 \times (kT,(k+1)T), \\ & \bar v \cdot \bar n = 0, \quad \bar n \cdot \mathbb{T}(\bar v, p) \cdot \bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_0 \times (kT,(k+1)T), \\ & \bar v\big|_{t=kT} = \bar v(kT) && \text{in } \Omega_0. \end{aligned}$$

In view of (3.64) we have

$$\begin{aligned} & |\bar v \cdot \nabla \bar v|_{2,\Omega_0\times(kT,(k+1)T)} \le \|\bar v\|_{L_\infty(kT,(k+1)T;L_4(\Omega_0))} \cdot \\ & \cdot \|\nabla \bar v\|_{L_2(kT,(k+1)T;L_4(\Omega_0))} \le c(A_4^2(T) + A_5^2(T)). \end{aligned}$$

Assuming that

$$(3.66)\qquad B_6(T) = \sup_{k\in\mathbb{N}} |\bar f|_{2,\Omega_0\times(kT,(k+1)T)}$$

we obtain for solutions of problem (3.65) the estimate

$$(3.67)\qquad \|\bar v\|_{2,\Omega_0\times(kT,(k+1)T)} \le c(A_4^2(T) + A_5^2(T) + B_6(T) + \|\bar v(kT)\|_{1,\Omega_0}).$$

Finally, we have to show that $\|\bar v(kT)\|_{1,\Omega_0}$ can be estimated by a constant independent of k. For this purpose we use problem (3.40). Since (3.12) holds we have

$$(3.68)\qquad \|\bar v(kT)\|_{1,\Omega_0} \le cR\left|\frac{\chi(kT)}{r}\right|_{2,\Omega_0} \le cRA_2(T).$$

From (3.67) and (3.68) estimate $(3.51)_1$ follows. Finally to show $(3.51)_2$ we recall that

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|_{s,\Omega_{0\varepsilon}} \le \left|\frac{F}{r}\right|_{s,\Omega_{0\varepsilon}},$$

so $|\frac{\chi}{r}|_{s,\Omega_0} \in C(\mathbb{R}_+)$. Hence by (3.40) we obtain that $\bar{v} \in C(\mathbb{R}_+; W_s^1(\Omega_0))$, so $(3.51)_2$ holds. This concludes the proof.

Finally, we derive an explicit form of estimate (3.5).

Remark 3.6. First we have to estimate expressions from the r.h.s. of (3.5). Let T be so large that $e^{-\nu_1 T/2} \le \frac{1}{2}$. From Lemma 3.1 we have

$$A_1(T) \le c(\sup_k |f|_{6/5,2,\Omega_0\times(kT,(k+1)T)} + |\bar{v}(0)|_{2,\Omega_0}).$$

Lemma 3.2 implies

$$A_2(T) \le c\biggl(\sup_k \left|\frac{f}{r}\right|_{2,\Omega_0\times(kT,(k+1)T)} + \left|\frac{\chi(0)}{r}\right|_{2,\Omega_0}\biggr).$$

Lemma 3.3 yields

$$A_3(3,T) \le c\biggl(\sup_k \left|\frac{F}{r}\right|_{3,\Omega_0\times(kT,(k+1)T)} + \left|\frac{\chi(0)}{r}\right|_{3,\Omega_0}\biggr).$$

Lemma 3.4 shows that

$$\begin{aligned} A_4(T) \le c\biggl[&A_1(T)A_3(3,T) + R^2 A_3^2(3,T) \\ &+ \sup_k \biggl(|\operatorname{div} f|_{3,2,\Omega_0\times(kT,(k+1)T)} + \|f\|_{L_2(kT,(k+1)T;W_{3/2}^{1/3}(\partial\Omega_0))} \\ &+ \left|\frac{f_r}{r}\right|_{2,\Omega_0\times(kT,(k+1)T)}\biggr)\biggr] + \left|\frac{v_r(0)}{r}\right|_{2,\Omega_0}. \end{aligned}$$

From (3.59) and (3.60) we have

$$A_5(T) \le c(\sup_k |f|_{2,\Omega_0\times(kT,(k+1)T)} + |v(0)|_{2,\Omega_0} + |\chi(0)|_{2,\Omega_0}).$$

Finally,

$$B_6(T) = \sup_{k\in\mathbb{N}} |\bar{f}|_{2,\Omega_0\times(kT,(k+1)T)}.$$

In view of the above estimates, (3.51) takes the form

$$\begin{aligned} \|\bar{v}\|_{2,\Omega_0\times(kT,(k+1)T)} \le c \sup_{k\in\mathbb{N}} \biggl(&|f|^4_{2,\Omega_0\times(kT,(k+1)T)} \\ &+ \left|\frac{f_r}{r}\right|^2_{2,\Omega_0\times(kT,(k+1)T)} + \left|\frac{f_r}{r}\right|_{2,\Omega_0\times(kT,(k+1)T)} \\ &+ \left|\frac{F}{r}\right|^4_{3,\Omega_0\times(kT,(k+1)T)} + |\operatorname{div} f|^2_{3,2,\Omega_0\times(kT,(k+1)T)} \\ &+ \|f\|^2_{L_2(kT,(k+1)T;W_{3/2}^{1/3}(\partial\Omega_0))}\biggr) \end{aligned} \tag{3.69}$$

$$+c\left(|\bar{v}(0)|_{2,\Omega_0}^4+|\bar{v}(0)|_{2,\Omega_0}^2+\left|\frac{\chi(0)}{r}\right|_{3,\Omega_0}^4+\left|\frac{\chi(0)}{r}\right|_{2,\Omega_0}+|\chi(0)|_{2,\Omega_0}^2\right),$$

and

$$\|v\|_{C([kT,(k+1)T];H^1(\Omega_0))}\le c\left(\sup_k\left|\frac{F}{r}\right|_{2,\Omega_0\times(kT,(k+1)T)}+\left|\frac{\chi(0)}{r}\right|_{2,\Omega_0}\right). \tag{3.70}$$

From Lemmas 3.1–3.5, Remark 3.6 and the result of global existence of regular axially symmetric solutions proved in [3] we obtain Theorem 1.

4. Stability problem. First we obtain an L_2-estimate for v'.

LEMMA 4.1. *Assume that* $v_a\in L_\infty(kT,(k+1)T;W_3^1(\Omega_0))$, $\frac{f_a}{r}\in L_2(kT(k+1)T;L_2(\Omega))$, $k\in\mathbb{N}$, $\frac{\chi(0)}{r}\in L_2(\Omega_0)$. *Let* $\nu_1'<\nu'=\frac{\nu}{c_1}$ *and* c_1 *is from (4.3). Let* T *be so large that*

$$\alpha(T)\equiv\frac{1}{\nu^2}\sup_k\left|\frac{f_a}{r}\right|_{2,\Omega\times(kT,(k+1)T)}^2+\frac{1}{\nu}\left|\frac{\chi(kT)}{r}\right|_{2,\Omega}^2\le\frac{\nu_1'}{2}T.$$

Then

$$|v'((k+1)T)|_{2,\Omega}^2\le D_1^2(T)+e^{-(\nu_1'/2)T}|v'(kT)|_{2,\Omega}^2, \tag{4.1}$$

where

$$D_1^2(T)=\sup_{k\in\mathbb{N}}c_3e^{\alpha(T)}\int_{kT}^{(k+1)T}\left(|f'(t)|_{2,\Omega}^2+\left|\int_\Omega rv_\varphi'(t)dx\right|^2\right)dt. \tag{4.1'}$$

Proof. Multiplying (1.9) by v', integrating over Ω and using the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}|v'|_{2,\Omega}^2+\nu|\mathbb{D}(v')|_{2,\Omega}^2+\int_\Omega v'\cdot\nabla v_a\cdot v'dx=\int_\Omega f'\cdot v'dx. \tag{4.2}$$

Utilizing the Korn inequality (see [8, Ch. 4])

$$\|v'(t)\|_{1,\Omega}^2\le c_1\left(|\mathbb{D}(v')|_{2,\Omega}^2+\left|\int_\Omega rv_\varphi'(t)dx\right|^2\right) \tag{4.3}$$

in (4.2) yields

$$\frac{1}{2}\frac{d}{dt}|v'|_{2,\Omega}^2+\nu'\|v'\|_{1,\Omega}^2+\int_\Omega v'\cdot\nabla v_a\cdot v'dx\le\int_\Omega f'\cdot v'dx+\left|\int_\Omega rv_\varphi'(t)dx\right|^2, \tag{4.4}$$

where $\nu'=\nu/c_1$. Applying the Hölder and Young inequalities in (4.4) implies

$$\begin{aligned}\frac{d}{dt}|v'|_{2,\Omega}^2+\nu_1'|v'|_{2,\Omega}^2+\nu_2'\|v'\|_{1,\Omega}^2&\le c_2|\nabla v_a|_{3,\Omega}^2|v'|_{2,\Omega}^2\\&\quad+c|f'|_{6/5,\Omega}^2+\left|\int_\Omega rv_\varphi'(t)dx\right|^2.\end{aligned} \tag{4.5}$$

Continuing, we have

$$\frac{d}{dt}\left(|v'|_{2,\Omega}^2e^{\nu_1't-c_2\int_{kT}^t|\nabla v_a|_{3,\Omega}^2dt'}\right)+\nu_2'\|v'\|_{1,\Omega}^2e^{\nu_1't-c_2\int_{kT}^t|\nabla v_a|_{3,\Omega}^2dt'} \tag{4.6}$$

$$\leq c_3\bigg(|f'|^2_{6/5,\Omega}+\bigg|\int\limits_\Omega rv'_\varphi(t)dx\bigg|^2\bigg)e^{\nu'_1t-c_2\int_{kT}^t|\nabla v_a|^2_{3,\Omega}dt'}.$$

Integrating (4.6) with respect to t from kT to $t\in[kT,(k+1)T]$ we obtain

$$\tag{4.7} \begin{aligned} &|v'(t)|^2_{2,\Omega}e^{\nu'_1t-c_2\int_{kT}^t|\nabla v_a(t')|^2_{3,\Omega}dt'}+\nu'_2\int\limits_{kT}^t\|v'(t')\|^2_{1,\Omega}e^{\nu'_1t'-c_2\int_{kT}^{t'}|\nabla v_a(t'')|^2_{3,\Omega}dt''}dt'\\ &\leq c_3\int\limits_{kT}^t\bigg(|f'(t')|^2_{6/5,\Omega}+\bigg|\int\limits_\Omega rv'_\varphi(t')dx\bigg|^2\bigg)e^{\nu'_1t'-c_2\int_{kT}^{t'}|\nabla v_a(t'')|^2_{3,\Omega}dt''}dt'\\ &+|v'(kT)|^2_{2,\Omega}e^{\nu'_1kT}. \end{aligned}$$

Simplifying (4.7) yields

$$\tag{4.8} \begin{aligned} &|v'(t)|^2_{2,\Omega}+\nu'_2e^{-\nu'_1t}\int\limits_{kT}^t\|v'(t')\|^2_{1,\Omega}e^{\nu'_1t'}dt'\\ &\leq c_3e^{c_2\int_{kT}^t|\nabla v_a|^2_{3,\Omega}dt'}\int\limits_{kT}^t\bigg(|f'(t')|^2_{6/5,\Omega}+\bigg|\int\limits_\Omega rv'_\varphi(t')dx\bigg|^2\bigg)dt'\\ &+e^{-\nu'_1(t-kT)+c_2\int_{kT}^t|\nabla v_a(t')|^2_{3,\Omega}dt'}|v'(kT)|^2_{2,\Omega}, \end{aligned}$$

where $t\in[kT,(k+1)T]$. Inserting $t=(k+1)T$ and cancelling the second term on the l.h.s. of (4.8) implies

$$\tag{4.9} \begin{aligned} |v'((k+1)T)|^2_{2,\Omega}&\leq c_3e^{c_2\int_{kT}^{(k+1)T}|\nabla v_a(t')|^2_{3,\Omega}dt'}\int\limits_{kT}^{(k+1)T}\bigg(f'(t)|^2_{2,\Omega}+\bigg|\int\limits_\Omega rv_\varphi(t)dx\bigg|^2\bigg)dt\\ &+e^{-\nu'_1T+c_2\int_{kT}^{(k+1)T}|\nabla v_a(t)|^2_{3,\Omega}dt}|v'(kT)|^2_{2,\Omega}. \end{aligned}$$

In view of (3.8) we have

$$\int\limits_{kT}^{(k+1)T}|\nabla v_a(t)|^2_{3,\Omega}dt\leq\frac{1}{\nu^2}\bigg|\frac{f_a}{r}\bigg|^2_{2,\Omega\times(kT,(k+1)T)}+\frac{1}{\nu}\bigg|\frac{\chi(kT)}{r}\bigg|^2_{2,\Omega}\leq\alpha(T).$$

Utilizing the inequality and assumptions of the lemma we obtain (4.1). This ends the proof.

By the inductive considerations we obtain from (4.1) the estimate

$$\tag{4.10} |v'(kT)|^2_{2,\Omega}\leq\frac{D_1^2(T)}{1-e^{-(\nu'_1/2)T}}+e^{-\frac{\nu'_1}{2}kT}|v'(0)|^2_{2,\Omega}\leq\frac{D_1^2(T)}{1-e^{-(\nu'_1/2)T}}+|v'(0)|^2_{2,\Omega}\equiv G_1(T).$$

Moreover, from (4.8) we have

$$\tag{4.11} |v'(t)|^2_{2,\Omega}+\nu'_2e^{-\nu'_1t}\int\limits_{kT}^t\|v'(t')\|^2_{1,\Omega}e^{\nu'_1t'}dt'\leq D_1^2(T)+e^{-(\nu'_1/2)T}|v'(kT)|^2_{2,\Omega},$$

where $t \in [kT, (k+1)T]$. From (4.11) we obtain the estimates

$$
\begin{aligned}
&|v'(t)|^2_{2,\Omega} \le D_1^2(T) + |v'(kT)|^2_{2,\Omega},\\
&\int_{kT}^{t} \|v'(t')\|^2_{1,\Omega} dt' \le e^{\nu_1' T}(D_1^2(T) + |v'(kT)|^2_{2,\Omega}),
\end{aligned}
\tag{4.12}
$$

where $t \in [kT, (k+1)T]$.

Now we prove the existence of local solutions to problem (1.9).

LEMMA 4.2. *Let the assumptions of Theorem 1 be satisfied. Let $v'(0) \in L_2(\Omega)$, $f' \in L_2(\Omega \times (kT, (k+1)T)$ for any $k \in \mathbb{N}$. Let $D_1(T)$ (see (4.1′)), $G_1(T)$ (see (4.10)), $D_2(T)$ (see (4.14)) be sufficiently small. Let*

$$\beta(k) = D_2(T)(A_6^6(T) + A_6^4(T)) + (|f'|_{2,\Omega\times(kT,(k+1)T)} + \|v'(kT)\|_{1,\Omega})$$

and let there exist a number $\sigma > 1$ such that

$$cD_2\sigma^5\beta^5 + \frac{c}{\sigma} \le 1.$$

Then there exists a solution to problem (1.9) such that $v' \in W_2^{2,1}(\Omega \times (kT, (k+1)T))$ and

$$\|v'\|_{2,\Omega\times(kT,(k+1)T)} + |\nabla p'|_{2,\Omega\times(kT,(k+1)T)} \le A(k),$$

where $A(k)$ defined by inequality (4.19) is such that $A(k) \le \sigma\beta(k)$.

Proof. To prove the existence of solutions to problem (1.9) we apply the following method of successive approximations:

$$
\begin{aligned}
&v'_{m+1,t} + v'_m \cdot \nabla v'_{m+1} + v'_{m+1} \cdot \nabla v_a + v_a \cdot \nabla v'_{m+1} - \operatorname{div}\mathbb{T}(v'_{m+1}, p'_{m+1}) = f',\\
&\operatorname{div} v'_{m+1} = 0,\\
&\bar{n} \cdot v'_{m+1} = 0,\\
&\bar{n} \cdot \mathbb{T}(v'_{m+1}, p'_{m+1}) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2,\\
&v'_{m+1}\big|_{t=kT} = v'(kT).
\end{aligned}
\tag{4.13}
$$

By (4.11) the weak solution to problem (4.13) satisfies the inequality

$$|v'_{m+1}(t)|^2_{2,\Omega} + \nu \int_{kT}^{t} \|v'_{m+1}(t')\|^2_{1,\Omega} dt' \le (1 + e^{\nu_1' T})[D_1^2(T) + |v'(kT)|^2_{2,\Omega}] \equiv D_2^2(T). \tag{4.14}$$

Let c_4 be the constant in the estimate of the Stokes system for (4.13). Then for solutions of (4.13) we have

$$
\begin{aligned}
&\|v'_{m+1}\|_{2,\Omega\times(kT,(k+1)T)} + |\nabla p'_{m+1}|_{2,\Omega\times(kT,(k+1)T)}\\
&\le c_4(|v'_m \cdot \nabla v'_{m+1}|_{2,\Omega\times(kT,(k+1)T)}\\
&+ |v'_{m+1} \cdot \nabla v_a|_{2,\Omega\times(kT,(k+1)T)} + |v_a \cdot \nabla v'_{m+1}|_{2,\Omega\times(kT,(k+1)T)}\\
&+ |f'|_{2,\Omega\times(kT,(k+1)T)} + \|v'_{m+1}(kT)\|_{1,\Omega}).
\end{aligned}
\tag{4.15}
$$

Now, we estimate the particular terms on the r.h.s. of (4.15). We estimate the first and the third terms by

$$\begin{aligned}&|v'_m+v_a|_{6,\Omega\times(kT,(k+1)T)}|\nabla v'_{m+1}|_{3,\Omega\times(kT,(k+1)T)}\\&\le(\varepsilon^{1/6}\|\nabla v'_{m+1}\|_{1,\Omega\times(kT,(k+1)T)}+c\varepsilon^{-5/6}|\nabla v'_{m+1}|_{2,\Omega\times(kT,(k+1)T)})\cdot\\&\cdot|v'_m+v_a|_{6,\Omega\times(kT,(k+1)T)}\\&\le\varepsilon_1^{1/6}\|\nabla v'_{m+1}\|_{1,\Omega\times(kT,(k+1)T)}+c(1/\varepsilon_1)|v'_m+v_a|^6_{6,\Omega\times(kT,(k+1)T)}D_2(T)\\&\le\varepsilon_1^{1/6}\|\nabla v'_{m+1}\|_{1,\Omega\times(kT,(k+1)T)}+c(1/\varepsilon_1)D_2(T)\cdot\\&\cdot(\|v'_m\|^6_{2,\Omega\times(kT,(k+1)T)}+\|v_a\|^6_{2,\Omega\times(kT,(k+1)T)}).\end{aligned}$$

By the Hölder inequality the second term on the r.h.s. of (4.15) is bounded by

$$\begin{aligned}&|v'_{m+1}|_{5,\Omega\times(kT,(k+1)T)}|\nabla v_a|_{\frac{10}{3},\Omega\times(kT,(k+1)T)}\le[\varepsilon_2^{1/4}\|v'_{m+1}\|_{2,\Omega\times(kT,(k+1)T)}\\&\quad+c\varepsilon_2^{-3/4}|v'_{m+1}|_{10/3,\Omega\times(kT,(k+1)T)}]A_6(T)\\&\le\varepsilon_3^{1/4}\|v'_{m+1}\|_{2,\Omega\times(kT,(k+1)T)}+c\varepsilon_2^{-3/4}c(T)D_2(T)A_6^4(T).\end{aligned}$$

Utilizing the above estimates in (4.15) yields

$$\begin{aligned}&\|v'_{m+1}\|_{2,\Omega\times(kT,(k+1)T)}+|\nabla p'_{m+1}|_{2,\Omega\times(kT,(k+1)T)}\\&\le cD_2(T)(\|v'_m\|^6_{2,\Omega\times(kT,(k+1)T)}\\&+A_6^6(T)+A_6^4(T))+c_4(|f'|_{2,\Omega\times(kT,(k+1)T)}+\|v'(kT)\|_{1,\Omega}),\end{aligned}\tag{4.16}$$

where by (4.14) and (4.10) we have

$$D_2(T)=(1+e^{\nu'_1T})\left[\frac{2-e^{-(\nu'_1/2)T}}{1-e^{-(\nu'_1/2)T}}D_1(T)+|v'(0)|_{2,\Omega}\right].\tag{4.17}$$

To show that the constructed sequence $\{v'_m,p'_m\}$ is uniformly bounded we assume that

$$\|v'_m\|_{2,\Omega\times(kT,(k+1)T)}+|\nabla p'_m|_{2,\Omega\times(kT,(k+1)T)}\le A(k).\tag{4.18}$$

Next for $D_2(T)$ and $A(k)$ sufficiently small we get

$$\begin{aligned}&cD_2(T)A^6(k)+cD_2(T)(A_6^6(T)+A_6^4(T))+c_4(|f'|_{2,\Omega\times(kT,(k+1)T)}+\|v'(kT)\|_{1,\Omega})\\&\le A(k).\end{aligned}\tag{4.19}$$

Then (4.19) implies

$$\|v'_{m+1}\|_{2,\Omega\times(kT,(k+1)T)}+|\nabla p'_{m+1}|_{2,\Omega\times(kT,(k+1)T)}\le A(k).\tag{4.20}$$

To show (4.18) for all $m\in\mathbb{N}$ we assume that $v'_0=0$ and v'_1 is a solution to the problem

$$\begin{aligned}&v'_{1,t}+v'_1\cdot\nabla v_a+v_a\cdot\nabla v'_1-\mathrm{div}\mathbb{T}(v'_1,p'_1)=f',\\&\mathrm{div}v'_1=0,\\&\bar n\cdot v'_1=0,\quad\bar n\cdot\mathbb{T}(v'_1,p'_1)\cdot\bar\tau_\alpha=0,\quad\alpha=1,2,\\&v'_1\big|_{t=kT}=v'(kT).\end{aligned}\tag{4.21}$$

For solutions of problem (4.21) we have

$$\begin{aligned}&\|v'_1\|_{2,\Omega\times(kT,(k+1)T)}\le c_4[|v'_1\cdot\nabla v_a|_{2,\Omega\times(kT,(k+1)T)}\\&+|v_a\cdot\nabla v'_1|_{2,\Omega\times(kT,(k+1)T)}+|f'|_{2,\Omega\times(kT,(k+1)T)}+\|v'(kT)\|_{1,\Omega}].\end{aligned}\tag{4.22}$$

Repeating the considerations leading to (4.16) we obtain

$$\begin{aligned}\|v_1'\|_{2,\Omega\times(kT,(k+1)T)} + |\nabla p_1'|_{2,\Omega\times(kT,(k+1)T)} \le cD_2(T)(A_6^6(T)+A_6^4(T))\\ + c_4(|f'|_{2,\Omega\times(kT,(k+1)T)} + \|v'(kT)\|_{1,\Omega}).\end{aligned} \tag{4.23}$$

Assuming that

$$cD_2(T)(A_6^6(T)+A_6^4(T)) + c_4(|f'|_{2,\Omega\times(kT,(k+1)T)} + \|v'(kT)\|_{1,\Omega}) \le A(k), \tag{4.24}$$

which is also justified by (4.19) we see by the inductive argument that (4.18) holds for any $m\in\mathbb{N}$.

Finally we must show convergence. Let $V'_{m+1} = v'_{m+1} - v'_m$, $P'_{m+1} = p'_{m+1} - p'_m$. Then (4.19) implies

$$\begin{aligned}&V'_{m+1,t} + v'_m\cdot\nabla V'_{m+1} + V'_m\cdot\nabla v'_m + V'_{m+1}\cdot\nabla v_a + v_a\cdot\nabla V'_{m+1}\\ &\quad - \operatorname{div}\mathbb{T}(V'_{m+1},P'_{m+1}) = 0,\\ &\operatorname{div}V'_{m+1} = 0,\\ &\bar n\cdot V'_{m+1} = 0,\quad \bar n\cdot\mathbb{T}(V'_{m+1},P'_{m+1})\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2,\\ &V'_{m+1}\big|_{t=kT} = 0.\end{aligned} \tag{4.25}$$

To show convergence we divide the interval $[kT,(k+1)T]$ into n subintervals with the length $\Delta t = \frac{T}{n}$, which will be chosen sufficiently small. Having proved the existence in the interval $[kT,kT+\Delta t]$ we obtain instead of the last equation in (4.25) the condition $V'_{m+1}|_{t=kT+\Delta t} = 0$. Then considerations from the interval $[kT,kT+\Delta t]$ can be repeated in intervals $[kT+(s-1)\Delta t, kT+s\Delta t]$, $s\le n$. Hence we shall restrict our considerations to the interval $[kT,kT+\Delta t]$ only.

Multiplying $(4.25)_1$ by V'_{m+1} and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}|V'_{m+1}|_{2,\Omega}^2 + \int\limits_\Omega V'_m\cdot\nabla v'_m\cdot V'_{m+1}dx + \int\limits_\Omega V'_{m+1}\cdot\nabla v_a\cdot V'_{m+1}dx + \nu\|V'_{m+1}\|_{1,\Omega}^2 = 0.$$

By the Hölder and Young inequalities we have

$$\frac{d}{dt}|V'_{m+1}|_{2,\Omega}^2 + \nu\|V'_{m+1}\|_{1,\Omega}^2 \le c|\nabla v_a|_{3,\Omega}^2|V'_{m+1}|_{2,\Omega}^2 + c|\nabla v'_m|_{2,\Omega}^2|V'_m|_{3,\Omega}^2.$$

Integrating with respect to time implies

$$\begin{aligned}|V'_{m+1}(t)|_{2,\Omega}^2 &+ \nu\int\limits_{kT}^{kT+\Delta t}\|V'_{m+1}(t')\|_{1,\Omega}^2dt'\\ &\le c\exp\bigg(\int\limits_{kT}^{kT+\Delta t}|\nabla v_a(t')|_{3,\Omega}^2dt'\bigg)\int\limits_{kT}^{kT+\Delta t}|\nabla v'_m|_{2,\Omega}^2\|V'_m\|_{1,\Omega}^2dt'\\ &\le ce^{cA_6^2}\sup_{t'}\|v'_m(t')\|_{1,\Omega}^2\sup_{t'}\|V'_m(t')\|_{1,\Omega}^2\Delta t\\ &\le ce^{cA_6^2}A^2(k)\|V'_m\|_{2,\Omega\times(kT,kT+\Delta t)}^2\Delta t.\end{aligned} \tag{4.26}$$

From (4.25) we have also

$$\|V'_{m+1}\|_{2,\Omega\times(kT,kT+\Delta t)} \le c(|v'_m\cdot\nabla V'_{m+1}|_{2,\Omega\times(kT,kT+\Delta t)} \tag{4.27}$$

$$+ |V'_m \cdot \nabla v'_m|_{2,\Omega\times(kT,kT+\Delta t)} + |V'_{m+1} \cdot \nabla v_a|_{2,\Omega\times(kT,kT+\Delta t)}$$
$$+ |v_a \cdot \nabla V'_{m+1}|_{2,\Omega\times(kT,kT+\Delta t)}).$$

Now we estimate the particular terms on the r.h.s. of (4.27). We bound the first term by

$$\int_{kT}^{kT+\Delta t} dt|v'_m \cdot \nabla V'_{m+1}|^2_{2,\Omega} \le \int_{kT}^{kT+\Delta t} dt|v'_m|^2_{4,\Omega}|\nabla V'_{m+1}|^2_{4,\Omega}$$
$$\le \sup_t |v'_m|^2_{4,\Omega}\bigg(\varepsilon_1 \int_{kT}^{kT+\Delta t} |V'_{m+1,xx}|^2_{2,\Omega}dt + c(1/\varepsilon_1) \int_{kT}^{kT+\Delta t} |V'_{m+1}|^2_{2,\Omega}dt\bigg)$$
$$\le cA^2(\varepsilon_1|V'_{m+1,xx}|^2_{2,\Omega\times(kT,kT+\Delta t)} + c(1/\varepsilon_1)|V'_{m+1}|^2_{2,\Omega\times(kT,kT+\Delta t)}),$$

the second by

$$\int_{kT}^{kT+\Delta t} dt|V'_m|^2_{6,\Omega}|\nabla v'_m|^2_{3,\Omega} \le \sup_t \|V'_m\|^2_{1,\Omega} \int_{kT}^{kT+\Delta t} dt|\nabla v'_m|^2_{3,\Omega}$$
$$\le \sup_t \|V'_m\|^2_{1,\Omega}(\Delta t)^{1/2}\bigg(\int_{kT}^{kT+\Delta t} dt|\nabla v'_m|^4_{3,\Omega}\bigg)^{1/2}$$
$$\le cA^2(\Delta t)^{1/2}\|V'_m\|^2_{2,\Omega\times(kT,kT+\Delta t)},$$

the third by

$$\int_{kT}^{kT+\Delta t} dt|\nabla v_a|^2_{2,\Omega}|V'_{m+1}|^2_{\infty,\Omega} \le \sup_t |\nabla v_a|^2_{2,\Omega}(\varepsilon_2|V'_{m+1,xx}|^2_{2,\Omega\times(kT,kT+\Delta t)}$$
$$+ c(1/\varepsilon_2)|V'_{m+1}|^2_{2,\Omega\times(kT,kT+\Delta t)})$$
$$\le cA_6^2(\varepsilon_2|V'_{m+1,xx}|^2_{2,\Omega\times(kT,kT+\Delta t)} + c(1/\varepsilon_2)|V'_{m+1}|^2_{2,\Omega\times(kT,kT+\Delta t)}).$$

Finally the last term on the r.h.s. of (4.27) is estimated by

$$\int_{kT}^{kT+\Delta t} dt|v_a|^2_{4,\Omega}|\nabla V'_{m+1}|^2_{4,\Omega} \le \sup_t |v_a|^2_{4,\Omega} \int_{kT}^{kT+\Delta t} dt|\nabla V'_{m+1}|^2_{4,\Omega}$$
$$\le cA_6^2(\varepsilon_3|V'_{m+1,xx}|^2_{2,\Omega\times(kT,kT+\Delta t)} + c(1/\varepsilon_3)|V'_{m+1}|^2_{2,\Omega\times(kT,kT+\Delta t)}).$$

Utilizing the above estimates in (4.27), assuming that $\varepsilon_1 - \varepsilon_3$ are sufficiently small and using (4.26) we obtain

$$\|V'_{m+1}\|_{2,\Omega\times(kT,kT+\Delta t)} \le \varphi(A_6, A)(\Delta t)^{1/2}\|V'_m\|_{2,\Omega\times(kT,kT+\Delta t)} \tag{4.28}$$

where φ is an increasing positive function. Hence we have convergence. This concludes the proof.

Therefore, we have proved local existence of solutions to problem (1.9) in the interval $[kT,(k+1)T]$, where T is finite and fixed. To prove global existence we have to show that $\|v'(kT)\|_{1,\Omega}$ can be estimated by a quantity independent of k. Then by (4.19) $A(k)$ can be chosen independently of k too. Hence we need

LEMMA 4.3. *Assume that $G_1(T)$ and $G_2(T) = A(0) + \|v(0)\|_{1,\Omega}$ are sufficiently small. Assume that*

$$v_a \in L_2(kT,(k+1)T;H^2(\Omega)) \cap L_4(kT,(k+1)T;H^1(\Omega)), \quad k \in \mathbb{N}.$$

Then

$$\|v'(kT)\|_{1,\Omega} \le \|v'(0)\|_{1,\Omega}. \tag{4.29}$$

Proof. Multiplying (1.9) by $\mathrm{div}\mathbb{D}(v')$ and integrating over Ω implies

$$\begin{aligned}
&\int_\Omega v'_t \cdot \mathrm{div}\mathbb{D}(v')dx - \int_\Omega |\mathrm{div}\mathbb{D}(v')|^2 dx + \int_\Omega v' \cdot \nabla v' \cdot \mathrm{div}\mathbb{D}(v')dx \\
&+ \int_\Omega v' \cdot \nabla v_a \cdot \mathrm{div}\mathbb{D}(v')dx + \int_\Omega v_a \cdot \nabla v' \cdot \mathrm{div}\mathbb{D}(v')dx \\
&= \int_\Omega f' \cdot \mathrm{div}\mathbb{D}(v')dx.
\end{aligned} \tag{4.30}$$

The first term in (4.30) equals

$$\begin{aligned}
&\int_\Omega \mathrm{div}(v'_t \cdot \mathbb{D}(v'))dx - \int_\Omega \nabla v'_t \cdot \mathbb{D}(v')dx = \int_S v'_t \cdot \bar{n} \cdot \mathbb{D}(v')dS \\
&- \int_\Omega \nabla v'_t \cdot \mathbb{D}(v')dx = \int_S (v'_{nt}\bar{n} + v'_{\tau_\alpha t}\bar{\tau}_\alpha) \cdot \bar{n} \cdot \mathbb{D}(v')dS \\
&- \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbb{D}(v')|^2 dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbb{D}(v')|^2 dx,
\end{aligned}$$

where the boundary conditions were used.

To continue considerations we have to examine the following elliptic problems:

$$\begin{aligned}
&\mathrm{div}\mathbb{D}(v) = f, \\
&v \cdot \bar{n}\big|_S = 0, \\
&\bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha\big|_S = 0, \quad \alpha = 1,2,
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
&\mathrm{div}\mathbb{T}(v,p) = f, \\
&\mathrm{div}v = 0, \\
&v \cdot \bar{n}\big|_S = 0, \\
&\bar{n} \cdot \mathbb{T}(v,p) \cdot \bar{\tau}_\alpha\big|_S = 0, \quad \alpha = 1,2.
\end{aligned} \tag{4.32}$$

Since the function $\eta = b \times x$, where b is any constant vector, belongs to the kernels of operators (4.31) and (4.32), we see that $\int_\Omega r v_\varphi dx\, \eta$ belongs also to the kernels. Hence we have the following estimates for solutions of problems (4.31) and (4.32):

$$\|v\|_{2,\Omega} \le c(|f|_{2,\Omega} + |v|_{2,\Omega}), \tag{4.33}$$

and

$$\|v\|_{2,\Omega} + |p_x|_{2,\Omega} \le c(|f|_{2,\Omega} + |v|_{2,\Omega}). \tag{4.34}$$

By the Hölder and interpolation inequalities (see [1, Ch. 3, Sec. 15]) we estimate the third term on the l.h.s. of (4.30) by

$$
\begin{aligned}
&c|v'|_{6,\Omega}|v'_{,x}|_{3,\Omega}|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le c|v'|_{6,\Omega}(|v'_{,x}|_{2,\Omega}^{1/2}\|v'\|_{2,\Omega}^{1/2}+|v'_{,x}|_{2,\Omega})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le c\|v'\|_{1,\Omega}^{3/2}(|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^{1/2}+|v'|_{2,\Omega}^{1/2})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}+c\|v'\|_{1,\Omega}^{2}|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le \varepsilon|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^2+c(1/\varepsilon)(\|v'\|_{1,\Omega}^6+\|v'\|_{1,\Omega}^4),
\end{aligned}
$$

the fourth term by

$$
\begin{aligned}
&c|v'|_{\infty,\Omega}|\nabla v_a|_{2,\Omega}|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\le c|\nabla v_a|_{2,\Omega}(|v'|_{6,\Omega}^{1/2}\|v'\|_{2,\Omega}^{1/2}+|v'|_{2,\Omega})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le c|\nabla v_a|_{2,\Omega}\|v'\|_{1,\Omega}^{1/2}(|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^{1/2}+\|v'\|_{1,\Omega}^{1/2})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le \varepsilon|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^2+c(1/\varepsilon)(|\nabla v_a|_{2,\Omega}^4\|v'\|_{1,\Omega}^2+|\nabla v_a|_{2,\Omega}^2\|v'\|_{1,\Omega}^2),
\end{aligned}
$$

where the interpolation inequality is taken from [1, Ch. 3, Sec. 15], and finally, the fifth term by

$$
\begin{aligned}
&|v_a|_{6,\Omega}|v'_{,x}|_{3,\Omega}|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\le c\|v_a\|_{1,\Omega}(|v'_{,x}|_{2,\Omega}^{1/2}\|v'_{,x}\|_{1,\Omega}^{1/2}+|v'_{,x}|_{2,\Omega})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le c\|v_a\|_{1,\Omega}\|v'\|_{1,\Omega}^{1/2}(|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^{1/2}+\|v'\|_{1,\Omega}^{1/2})|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}\\
&\le \varepsilon|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^2+c(1/\varepsilon)(\|v_a\|_{1,\Omega}^4+\|v_a\|_{1,\Omega}^2)\|v'\|_{1,\Omega}^2.
\end{aligned}
$$

In view of the above considerations (4.30) implies

$$
\begin{aligned}
\frac{d}{dt}|\mathbb{D}(v')|_{2,\Omega}^2+\nu|\mathrm{div}\mathbb{D}(v')|_{2,\Omega}^2&\le c(1+\|v'\|_{1,\Omega}^2)\|v'\|_{1,\Omega}^4\\
&+c(\|v_a\|_{1,\Omega}^2+\|v_a\|_{1,\Omega}^4)\|v'\|_{1,\Omega}^2+c|f'|_{2,\Omega}^2.
\end{aligned}\tag{4.35}
$$

From (4.5) and (4.35) we have

$$
\begin{aligned}
&\frac{d}{dt}(|v'|_{2,\Omega}^2+|\mathbb{D}(v')|_{2,\Omega}^2)+\nu(|v'|_{2,\Omega}^2+|\mathbb{D}(v')|_{2,\Omega}^2)\\
&\le c(\|v_a\|_{1,\Omega}^2+\|v_a\|_{1,\Omega}^4+|\nabla v_a|_{3,\Omega}^2)(|v'|_{2,\Omega}^2+|\mathbb{D}(v')|_{2,\Omega}^2)\\
&+c(1+\|v'\|_{1,\Omega}^2)\|v'\|_{1,\Omega}^4+c|f'|_{2,\Omega}^2\\
&+c\bigg|\int_\Omega rv'_\varphi(0)dx\bigg|^2+c\bigg|\int_0^t dt'\int_\Omega rf'_\varphi dx\bigg|^2.
\end{aligned}\tag{4.36}
$$

Let us introduce the notation

$$
\begin{aligned}
X(t)&=|v'(t)|_{2,\Omega}^2+|\mathbb{D}(v'(t))|_{2,\Omega}^2,\\
\beta(t)&=\|v_a(t)\|_{1,\Omega}^2+\|v_a(t)\|_{1,\Omega}^4+|\nabla v_a(t)|_{3,\Omega}^2.
\end{aligned}
$$

Then (4.36) implies

$$
\begin{aligned}
&\frac{d}{dt}(X(t)e^{\nu t-c\int_0^t\beta(t')dt'})\le c\bigg[(1+\|v'\|_{1,\Omega}^2)\|v'\|_{1,\Omega}^4\\
&+|f'|_{2,\Omega}^2+\bigg|\int_\Omega rv'_\varphi(0)dx\bigg|^2+\bigg|\int_0^t dt'\int_\Omega rf'_\varphi(t')dx\bigg|^2\bigg]e^{\nu t-c\int_0^t\beta(t')dt'}.
\end{aligned}\tag{4.37}
$$

Integrating (4.37) with respect to time from kT to $t \in (kT,(k+1)T]$ yields

$$X(t) \le e^{-\nu t + c\int_{kT}^t \beta(t')dt'} \int_{kT}^t [(1+\|v'(t')\|_{1,\Omega}^2)\|v'(t')\|_{1,\Omega}^4 + |f'(t')|_{2,\Omega}^2] e^{\nu t'} dt' + ce^{c\int_{kT}^t \beta(t')dt'} \bigg| \int_\Omega r v'_\varphi(kT)dx \bigg|^2 + e^{-\nu t + c\int_{kT}^t \beta(t')dt'} c \int_{kT}^t dt' \bigg| \int_{kT}^{t'} dt'' \int_\Omega r f'_\varphi(t'')dx \bigg|^2 e^{\nu t'} + e^{-\nu(t-kT) + c\int_{kT}^t \beta(t')dt'} X(kT). \tag{4.38}$$

Let us introduce the quantities

$$B(T) = \sup_k \int_{kT}^{(k+1)T} \beta(t)dt,$$

$$H(T) = \sup_k \bigg[\int_{kT}^{(k+1)T} |f'(t)|_{2,\Omega}^2 dt + \bigg| \int_\Omega r v'_\varphi(kT)dx \bigg|^2 + \int_{kT}^{(k+1)T} dt \bigg| \int_{kT}^t dt' \int_\Omega r f'_\varphi(t')dx \bigg|^2 \bigg]. \tag{4.39}$$

Inserting $t = (k+1)T$ into (4.38) and using that

$$\sup_{t \in [kT,(k+1)T]} \|v'(t)\|_{1,\Omega} \le A(k) + \|v'(kT)\|_{1,\Omega} \equiv G_2(k,T)$$

we obtain from (4.38) the inequality

$$X((k+1)T) \le e^{cB(T)}(G_2^4(k,T) + H(T)) + e^{-\nu T + cB(T)} X(kT). \tag{4.40}$$

Assuming that $X(kT)$, $G_2(k,T)$, $H(T)$ are sufficiently small and T is sufficiently large we are able to show that

$$X((k+1)T) \le X(kT). \tag{4.41}$$

Starting from $k = 0$ we can show step by step by applying Lemmas 4.1 and 4.2 that $G_2(k,T)$ can be chosen independent of k and then

$$X(kT) \le X(0) \quad \text{for any } k \in \mathbb{N}.$$

Hence (4.29) holds. This concludes the proof.

Lemmas 4.1–4.3 imply Theorem 2.

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