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FRÉCHET ALGEBRAS OF POWER SERIES

H. GARTH DALES
Department of Pure Mathematics, University of Leeds
Leeds, LS2 9JT, UK
E-mail: garth@maths.leeds.ac.uk

SHITAL R. PATEL
Department of Mathematics and Statistics, Indian Institute of Technology
Kanpur, 208 016, Uttar Pradesh, India
E-mail: srpatel.math@gmail.com

CHARLES J. READ
Department of Pure Mathematics, University of Leeds
Leeds, LS2 9JT, UK
E-mail: read@maths.leeds.ac.uk

Abstract. We consider Fréchet algebras which are subalgebras of the algebra $\mathfrak{F} = \mathbb{C}\,[[X]]$ of formal power series in one variable and of $\mathfrak{F}_n = \mathbb{C}\,[[X_1, \ldots, X_n]]$ of formal power series in n variables, where $n \in \mathbb{N}$. In each case, these algebras are taken with the topology of coordinatewise convergence.

We begin with some basic definitions about Fréchet algebras, (F)-algebras, and other topological algebras, and recall some of their properties; we discuss Michael's problem from 1952 on the continuity of characters on these algebras and some results on uniqueness of topology.

A 'test algebra' $\mathcal{U}$ for Michael's problem for commutative Fréchet algebras has been described by Clayton and by Dixon and Esterle. We prove that there is an embedding of $\mathcal{U}$ into $\mathfrak{F}$, and so there is a Fréchet algebra of power series which is a test case for Michael's problem.

We also discuss homomorphisms from Fréchet algebras into $\mathfrak{F}$. We prove that such a homomorphism is either continuous or a surjection, so answering a question of Dales and McClure from 1977. As corollaries, we note that a subalgebra A of $\mathfrak{F}$ containing $\mathbb{C}[X]$ that is a Banach algebra is already a Banach algebra of power series, in the sense that the embedding of A into $\mathfrak{F}$ is automatically continuous, and that each (F)-algebra of power series has a unique (F)-algebra

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topology. We also prove that it is not true that results analogous to the above hold when we replace $\mathfrak{F}$ by $\mathfrak{F}_2$.

1. Algebraic definitions. All the algebras that will arise in this paper will have ground field the complex field, $\mathbb{C}$; for a background in the algebra that we shall use, see [4, 6, 27], for example.

Let A be an algebra over $\mathbb{C}$. As in [6], the product map on A is denoted by

$$m_A : (a, b) \mapsto a \cdot b = ab, \quad A \times A \to A;$$

the set $(A, \cdot)$ is the *multiplicative semigroup* of A.

A *character* on A is a non-zero homomorphism from A onto $\mathbb{C}$; the collection of all characters on A is the *character space* of A, denoted by Φ_A.

Let A be a unital algebra, with identity e_A. Then $a \in A$ is *invertible* if there exists $b \in A$ with $ab = ba = e_A$, and then we write $b = a^{-1}$ for the *inverse* of a; the collection of invertible elements in A is denoted by $\operatorname{Inv} A$, so that $\operatorname{Inv} A$ is a subsemigroup of $(A, \cdot)$. Clearly we have $(ab)^{-1} = b^{-1}a^{-1}$ $(a, b \in \operatorname{Inv} A)$.

We recall that an ideal P in a commutative algebra A is a *prime ideal* if $P \neq A$ and if either $a \in P$ or $b \in P$ whenever $a, b \in A$ and $ab \in P$. Thus P is a prime ideal if and only if the quotient algebra A/P is an integral domain. For example, every maximal modular ideal in A is a prime ideal.

Let A and B be algebras, and let $\theta : A \to B$ be a homomorphism. Then θ is an *embedding* if it is injective, and in this case we often regard A as a subalgebra of B; we say that A *embeds* in B if there is such an embedding. An embedding $\theta : A \to B$ is an *isomorphism* if it is also a surjection; A is *isomorphic* to B, written $A \cong B$, if there is such an isomorphism.

In this paper, we shall consider in particular subalgebras of the algebras of formal power series in one and several variables over $\mathbb{C}$; these latter algebras of formal power series are denoted by

$$\mathfrak{F} = \mathbb{C}[[X]] \quad \text{and} \quad \mathfrak{F}_n = \mathbb{C}[[X_1, \dots, X_n]],$$

respectively, where $n \in \mathbb{N}$. A description of these algebras is given in [6, §1.6]; we recall some notation and some of their basic properties.

Formally $\mathfrak{F}$ consists of sequences $\alpha = (\alpha_k) = (\alpha_k : k \in \mathbb{Z}^+)$, where $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, with coordinatewise addition and scalar multiplication and algebra multiplication determined by the rule that $\delta_k \star \delta_\ell = \delta_{k+\ell}$ for $k, \ell \in \mathbb{Z}^+$, where $\delta_k = (\delta_{j,k} : j \in \mathbb{Z}^+)$, the characteristic function of $\{k\}$. Less formally, $\mathfrak{F}$ consists of the formal sums

$$\sum_{k=0}^{\infty} \alpha_k X^k,$$

with the obvious product. Thus $(\mathfrak{F}, \star)$ is a commutative algebra with an identity denoted by 1; in fact, we shall usually denote the product of two elements of $\mathfrak{F}$ by juxtaposition. We regard the algebra $\mathbb{C}[X]$ of polynomials in one variable as a unital subalgebra of $\mathfrak{F}$ in the obvious way.

Throughout, we shall write

$$\pi_k : \alpha \mapsto \alpha_k, \quad \mathfrak{F} \to \mathbb{C},$$

for the coordinate projections, defined for each $k \in \mathbb{Z}^+$. In particular, π_0 is the unique character on $\mathfrak{F}$. For $f \in \mathfrak{F}$ with $f \neq 0$, the *order* of f is $\mathbf{o}(f) = \min\{k : \pi_k(f) \neq 0\}$; we set $\mathbf{o}(0) = \infty$, and follow usual conventions on the ordering of $\mathbb{Z}^+ \cup \{\infty\}$.

For $k \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$, set

$$M_k = \left\{f = \sum_{k=0}^{\infty} \alpha_k X^k \in \mathfrak{F} : \alpha_0 = \alpha_1 = \cdots = \alpha_{k-1} = 0\right\} = \{f \in \mathfrak{F} : \mathbf{o}(f) \geq k\}$$

(and take $M_0 = \mathfrak{F}$). Then, for each $k \in \mathbb{Z}^+$, the set M_k is an ideal in $\mathfrak{F}$, $M_{k+1} \subset M_k$ with $\dim(M_k/M_{k+1}) = 1$, and every non-zero ideal of $\mathfrak{F}$ has the form M_k for some $k \in \mathbb{Z}^+$. Further, $M = M_1$ is the unique maximal ideal of $\mathfrak{F}$, and

$$M_k = M^{[k]} = M^k = X^k \mathfrak{F} \quad (k \in \mathbb{Z}^+),$$

in the notation of [6]. Clearly $M_k M_\ell = M_{k+\ell}$ $(k, \ell \in \mathbb{Z}^+)$, and so there are precisely two prime ideals in $\mathfrak{F}$, namely the maximal ideal M and $\{0\}$. Further,

$$\operatorname{Inv} \mathfrak{F} = \{f \in \mathfrak{F} : \pi_0(f) \neq 0\}.$$

For $f \in \operatorname{Inv} \mathfrak{F}$ and $k \in \mathbb{N}$, there exists $g \in \operatorname{Inv} \mathfrak{F}$ with $g^k = f$. Indeed, suppose that $f = 1 + \sum_{j=1}^{\infty} \alpha_j X^j$, and we seek g of the form $1 + \sum_{j=1}^{\infty} \beta_j X^j$. Then we take β_1 with $k\beta_1 = \alpha_1$, and then note that, for $j \geq 2$, the formula for β_j is $k\beta_j = \alpha_j + \gamma$, where γ depends on only $\beta_1, \dots, \beta_{j-1}$. It follows that each $f \in \mathfrak{F}$ with $\mathbf{o}(f) = k \in \mathbb{N}$ has the form $(Xg)^k$ for some $g \in \operatorname{Inv} \mathfrak{F}$.

For example, $\exp X \in \mathfrak{F}$ is the series $\sum_{k=0}^{\infty} X^k/k!$.

Let $f \in M$ and $g \in \mathfrak{F}$. Then we can define the 'composition series' $g \circ f \in \mathfrak{F}$ by 'substitution' in the obvious way; for example, we can define $\exp f \in \mathfrak{F}$.

Suppose that $f = \sum_{k=0}^{\infty} \alpha_k X^k \in \mathfrak{F}$ is such that $\sum_{k=0}^{\infty} |\alpha_k| R^k < \infty$ for each $R > 0$. Then we can regard f as an entire function defined on $\mathbb{C}$; in this case, $\exp f$ satisfies the same condition and is also an entire function, and hence an element of $\mathfrak{F}$.

Now take $n \in \mathbb{N}$. Let $r = (r_1, \dots, r_n) \in (\mathbb{Z}^+)^n$, and set

$$|r| = r_1 + \cdots + r_n.$$

A *monomial* is the characteristic function of an element, say r, of $(\mathbb{Z}^+)^n$, and the *degree* of the monomial is $|r|$. For $j = 1, \dots, n$, we write X_j for the monomial corresponding to the element $(\delta_{j,1}, \dots, \delta_{j,n}) \in (\mathbb{Z}^+)^n$. For $k \in \mathbb{Z}^+$, a *homogeneous polynomial of degree* k is a linear combination (necessarily finite) of monomials of degree k. An element of

$$\mathfrak{F}_n = \mathbb{C}[[X_1, \dots, X_n]]$$

is defined to be a sequence $(f_k : k \in \mathbb{Z}^+)$, where each f_k is a homogeneous polynomial of degree k (and f_0 is a multiple of the identity 1). The product of two homogeneous polynomials of degree k and ℓ, respectively, is a homogeneous polynomial of degree $k+\ell$, and in this way we define a product on $\mathfrak{F}_n$ making it into a commutative algebra with identity 1. A generic element of $\mathfrak{F}_n$ is denoted by

$$\sum \{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\} = \sum \left\{\alpha_{(r_1,\dots,r_n)} X_1^{r_1} \cdots X_n^{r_n} : (r_1, \dots, r_n) \in (\mathbb{Z}^+)^n\right\}.$$

The algebra $\mathbb{C}[X_1,\dots,X_n]$ of polynomials in n variables consists of the finite sums of monomials in $\mathfrak{F}_n$, and is identified with a subalgebra of $\mathfrak{F}_n$.

Throughout, we shall write

$$\pi_r : \alpha \mapsto \alpha_r, \quad \mathfrak{F}_n \to \mathbb{C},$$

for the coordinate projections, defined for each $r \in (\mathbb{Z}^+)^n$. In particular, π_0 is the unique character on $\mathfrak{F}_n$ (where $0 = (0,\dots,0)$).

Let $f = (f_k : k \in \mathbb{Z}^+) \in \mathfrak{F}_n$ with $f \neq 0$, where f_k is a homogeneous polynomial of degree k. Then the *order* of f is

$$\mathbf{o}(f) = \min\{k : f_k \neq 0\},$$

and the term f_k is the *initial form* of f [27, p. 130]. Take $f, g \in \mathfrak{F}_n$ with $f, g \neq 0$, and suppose that f_k and g_ℓ are the initial forms of f and g, respectively. Then $fg \neq 0$, so that $\mathfrak{F}_n$ is an integral domain; we have $\mathbf{o}(fg) = \mathbf{o}(f) + \mathbf{o}(g)$ and $f_k g_\ell$ is the initial form of fg. We set $\mathbf{o}(0) = \infty$.

For $k \in \mathbb{Z}^+ \cup \{\infty\}$, set

$$M_k := \{f \in \mathfrak{F}_n : \mathbf{o}(f) \geq k\}.$$

Then, for each $k \in \mathbb{Z}^+ \cup \{\infty\}$, the set M_k is an ideal in $\mathfrak{F}_n$. Also, we see that

$$M_k M_\ell = M_{k+\ell} \quad (k, \ell \in \mathbb{Z}^+)$$

and that, for each $k \in \mathbb{Z}^+$, we have $\dim(M_k/M_{k+1}) = \binom{k+n-1}{k} < \infty$, so that each M_k is an ideal of finite codimension in $\mathfrak{F}_n$, and M_k is generated by the monomials of degree k. Further, M_1, sometimes written as $\mathfrak{M}_n$ (with $\mathfrak{M} = \mathfrak{M}_1$) to show the dependence on n, is the unique maximal ideal in $\mathfrak{F}_n$, and, for each $k \in \mathbb{N}$, we have $M_1^k = M_k$, so that

$$\operatorname{Inv} \mathfrak{F}_n = \{f \in \mathfrak{F}_n : \pi_0(f) \neq 0\} \quad \text{and} \quad M_1^k = \sum\{X^r \mathfrak{F}_n : |r| = k\}.$$

Clearly, $\bigcap\{M_k : k \in \mathbb{N}\} = \{0\}$.

Each ideal in $\mathfrak{F}_n$ is finitely-generated, and so $\mathfrak{F}_n$ is noetherian [27, VII, Corollary p. 139 and Theorem 4′]. However, when $n \geq 2$, there are certainly ideals in $\mathfrak{F}_n$ which are not of finite codimension. For example, this is the case for the ideal $P = X_2\mathfrak{F}_2$ in $\mathfrak{F}_2$. Indeed, it is clear that P is a prime ideal in $\mathfrak{F}_2$ and that $\mathfrak{F}_2/P \cong \mathfrak{F}$.

The topology of coordinatewise convergence, called τ_c, is a metrizable topology on $\mathfrak{F}_n$ (see below). In this topology, a sequence $(f_k)_{k\geq 1}$ in $\mathfrak{F}_n$ converges to $f \in \mathfrak{F}_n$ if and only if $\pi_r(f_k) \to \pi_r(f)$ as $k \to \infty$ for each $r \in (\mathbb{Z}^+)^n$. In particular, a series $\sum_{k=1}^\infty f_k$ in $\mathfrak{F}_n$ converges whenever $(f_k)_{k\geq 1}$ is such that, for each $s \in (\mathbb{Z}^+)^n$, we have $\pi_s(f_k) = 0$ for all sufficiently large $k \in \mathbb{N}$. For example, for each $f \in \mathfrak{M}_n$ and each sequence $(\beta_k)_{k\geq 1}$, the series $\sum_{k=1}^\infty \beta_k f^k$ converges in $\mathfrak{F}_n$.

The following result is given in [27, pp. 135,136]; it is also noted there that each homomorphism from $\mathfrak{F}_m$ to $\mathfrak{F}_n$ has the specified form.

For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{1,\dots,n\}$.

LEMMA 1.1. *Let $m, n \in \mathbb{N}$, and take $f_1,\dots,f_m \in \mathfrak{M}_n$. Then the map*

$$\theta : \sum\{\alpha_r X^r : r \in (\mathbb{Z}^+)^m\} \mapsto \sum\{\alpha_r f_1^{r_1}\cdots f_m^{r_m} : r \in (\mathbb{Z}^+)^m\}, \quad \mathfrak{F}_m \to \mathfrak{F}_n, \tag{1.1}$$

is a continuous homomorphism with $\theta(X_i) = f_i$ $(i \in \mathbb{N}_m)$.

Proof. It suffices to note that, for each $s \in (\mathbb{Z}^+)^n$, we have $\pi_s(f_1^{r_1} \cdots f_m^{r_m}) = 0$ for all but finitely many values of $r \in (\mathbb{Z}^+)^m$, and so the sum on the right-hand side of (1.1) converges in $\mathfrak{F}_n$. It is then clear that θ is a homomorphism. ∎

We shall use the following lemma from [27, p. 136].

LEMMA 1.2. *Let $n \in \mathbb{N}$, and let $f^1, \dots, f^n \in \mathfrak{F}_n$ have initial forms $X_1, \dots, X_n$, respectively. Then the substitution map $\theta : g \mapsto g(f^1, \dots, f^n)$, $\mathfrak{F}_n \to \mathfrak{F}_n$, is an automorphism of $\mathfrak{F}_n$ with $\theta(X_i) = f^i$ $(i \in \mathbb{N}_n)$. Thus there is an automorphism ψ of $\mathfrak{F}_n$ such that $\psi(f^i) = X_i$ $(i \in \mathbb{N}_n)$.* ∎

2. Embeddings of $\mathfrak{F}_m$ in $\mathfrak{F}_n$. As a background to our future results, we shall consider when the algebras $\mathfrak{F}_n$ can be embedded into each other. Of course, there is a trivial embedding of $\mathfrak{F}_m$ into $\mathfrak{F}_n$ whenever $n \geq m$. We shall first show that each $\mathfrak{F}_n$ can be embedded in $\mathfrak{F}_2$; this well-known result is essentially in [27], but we give some details for this specific result.

Let A be a commutative, unital algebra, and let $a_1, \dots, a_n$ be distinct elements of A. Then $\{a_1, \dots, a_n\}$ is said to be *algebraically independent* in A if $p(a_1, \dots, a_n) \neq 0$ for each non-zero polynomial $p \in \mathbb{C}[X_1, \dots, X_n]$.

LEMMA 2.1. *There is a sequence $(f_j)_{j\geq 1}$ in $\mathfrak{F}$ such that $\{1, f_1, \dots, f_n\}$ is algebraically independent in $\mathfrak{F}$ for each $n \in \mathbb{N}$.*

Proof. Set $f_0 = 1$ and $f_1 = X$, and then define $(f_j)_{j\geq 2}$ inductively by setting

$$f_{j+1} = \exp f_j \quad (j \in \mathbb{N}).$$

As above, we can regard each f_j as an entire function, and in particular as a function on $\mathbb{R}$. We note that $f_j^m(x)/f_{j+1}(x) \to 0$ as $x \to \infty$ in $\mathbb{R}$ for each $j, m \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Then we *claim* that $\{1, f_1, \dots, f_n\}$ is algebraically independent in $\mathfrak{F}$. Indeed, suppose that $p(1, f_1, \dots, f_n) = 0$, where $p \in \mathbb{C}[X_1, \dots, X_{n+1}]$. Then there exist $\alpha_r \in \mathbb{C}$ such that

$$\sum \{\alpha_r f_1^{r_1} \cdots f_n^{r_n} : r \in (\mathbb{Z}^+)^n\} = 0,$$

where the sum is a finite sum.

Assume towards a contradiction that not all the numbers α_r in this sum are zero. Choose the maximum value of r_n, say s_n, such that $\alpha_r \neq 0$ for some

$$r = (r_1, \dots, r_{n-1}, s_n) \in (\mathbb{Z}^+)^n.$$

Then choose the maximum value of r_{n-1}, say s_{n-1}, such that $\alpha_r \neq 0$ for some

$$r = (r_1, \dots, r_{n-2}, s_{n-1}, s_n) \in (\mathbb{Z}^+)^n.$$

Continue in this way to find a specific $s = (s_1, \dots, s_n) \in (\mathbb{Z}^+)^n$ with $\alpha_s \neq 0$. We see that

$$0 = \sum \{\alpha_r f_1^{r_1}(x) \cdots f_n^{r_n}(x) : r \in (\mathbb{Z}^+)^n\} / f_1^{s_1}(x) \cdots f_n^{s_n}(x) \to \alpha_s \quad \text{as} \quad x \to \infty,$$

a contradiction.

Thus the result holds. ∎

An extension of the following theorem will be given in Theorem 9.1.

THEOREM 2.2. *Let $n \in \mathbb{N}$. Then there is an embedding of $\mathfrak{F}_n$ in $\mathfrak{F}_2$.*

Proof. Set $\mathfrak{F}_n = \mathbb{C}\,[[X_1, \dots, X_n]]$ and $\mathfrak{F}_2 = \mathbb{C}\,[[Y_1, Y_2]]$.

We may suppose that $n \geq 3$. As in Lemma 2.1, there is an algebraically independent set $\{1, f_1, \dots, f_n\}$ in $\mathfrak{F}$. Each element of $\mathfrak{F}_n$ has the form $g = (g_k : k \in \mathbb{Z}^+)$, where g_k is a homogeneous polynomial of degree k for each $k \in \mathbb{Z}^+$. Define

$$\theta : g = (g_k) \mapsto \sum_{k=0}^{\infty} Y_2^k g_k(f_1(Y_1), \dots, f_n(Y_1)), \quad \mathfrak{F}_n \to \mathfrak{F}_2.$$

It is clear that θ is a homomorphism.

Suppose that $\theta(g) = 0$, and take $k \in \mathbb{Z}^+$. Then $g_k(f_1(Y_1), \dots, f_n(Y_1)) = 0$ in $\mathfrak{F}$. However g_k is a polynomial in $\mathbb{C}[X_1, \dots, X_n]$ and $\{1, f_1, \dots, f_n\}$ is algebraically independent, and so $g_k = 0$. Thus $g = 0$, and so θ is an injection, and hence an embedding. ∎

We now seek to show that $\mathfrak{F}_2$ does not embed in $\mathfrak{F}$. This is surely well-known, but we were unable to find a specific reference.

LEMMA 2.3. *Assume that there is an embedding of $\mathfrak{F}_2$ into $\mathfrak{F}$. Then there is an embedding $\overline{\theta} : \mathfrak{F}_2 \to \mathfrak{F}$ and $k \in \mathbb{N}$ such that $X^k \in \overline{\theta}(\mathfrak{F}_2)$.*

Proof. Let $\theta : \mathfrak{F}_2 \to \mathfrak{F}$ be an embedding. Then $\theta(X_1) \in \mathfrak{M} \setminus \{0\}$, and so $\mathbf{o}(\theta(X_1)) = k$ for some $k \in \mathbb{N}$. Hence there exists $f \in \operatorname{Inv} \mathfrak{F}$ with $\theta(X_1) = (Xf)^k$. By Lemma 1.2, there is an automorphism ψ of $\mathfrak{F}$ with $\psi(Xf) = X$. Set $\overline{\theta} = \psi \circ \theta : \mathfrak{F}_2 \to \mathfrak{F}$. Then $\overline{\theta}$ is an embedding, and $\overline{\theta}(X_1) = \psi((Xf)^k) = X^k$. Hence $X^k \in \overline{\theta}(\mathfrak{F}_2)$. ∎

Let A be a unital subalgebra of a unital algebra B. An element $b \in B$ is *integral* over A if there is a monic polynomial $p \in A[X]$ with $p(b) = 0$; the algebra B is *integral over A* if each $b \in B$ is integral over A. Suppose that B is a finitely generated A-module. Then B is integral over A [18, Chapter VIII, Corollary 5.4].

LEMMA 2.4. *Let $\theta : \mathfrak{F}_2 \to \mathfrak{F}$ be an embedding such that $X^k \in \theta(\mathfrak{F}_2)$ for some $k \in \mathbb{N}$. Then $\mathfrak{F}$ is integral over $\theta(\mathfrak{F}_2)$.*

Proof. Set $A = \theta(\mathfrak{F}_2)$. Then it is sufficient to show that $\mathfrak{F}$ is a finitely generated A-module. We shall show that, as an A-module, $\mathfrak{F}$ is generated by $\{1, X, \dots, X^{k-1}\}$.

Let $f \in \mathfrak{F}$, say $f = \sum_{k=0}^{\infty} \alpha_k X^k$. For $j = 0, \dots, k-1$, set $h_j = \sum_{i=0}^{\infty} \alpha_{j+ik} X^{ik}$. Then $h_0, \dots, h_{k-1} \in A$ and $f = h_0 + Xh_1 + \cdots + X^{k-1}h_{k-1}$, and so $\mathfrak{F}$ is generated by $\{1, X, \dots, X^{k-1}\}$. ∎

We shall use the following standard result from [4, Theorem 5.10], for example; it is a precursor of the famous 'going-up' theorem.

LEMMA 2.5. *Let A be a unital subalgebra of an algebra B, and let P be a prime ideal of A. Then there is a prime ideal Q of B with $Q \cap A = P$.* ∎

THEOREM 2.6. *Take $n \geq 2$. Then there is no embedding of $\mathfrak{F}_n$ into $\mathfrak{F}$.*

Proof. Assume towards a contradiction that there is an embedding of $\mathfrak{F}_n$ into $\mathfrak{F}$. Then there is an embedding $\theta : \mathfrak{F}_2 \to \mathfrak{F}$; again set $A = \theta(\mathfrak{F}_2)$. By Lemma 2.3, we may suppose that there exists $k \in \mathbb{N}$ such that $X^k \in A$. By Lemma 2.4, $\mathfrak{F}$ is integral over A. Next set $P = \theta(X_2\mathfrak{F}_2)$, a prime ideal in A. By Lemma 2.5, there is a prime ideal Q of $\mathfrak{F}$

with $Q \cap A = P$. But the only two prime ideals Q of $\mathfrak{F}$ are $\{0\}$ and $\mathfrak{M}$; it is clear that $\{0\} \cap A = \{0\} \subsetneq P$ and that $\mathfrak{M} \cap A = \theta(\mathfrak{M}_2) \supsetneq P$. Thus we have the required contradiction. ∎

A second proof of the above theorem will be given in Theorem 11.8, below.

3. Higher point derivations. We shall be interested in homomorphisms from algebras into $\mathfrak{F}$; these can be defined in terms of certain higher point derivations. For a study of higher point derivations on commutative Banach algebras, see [7, 8, 9].

Definition 3.1. Let A be an algebra, and let τ be a Hausdorff topology on A such that (A, τ) is a topological linear space. Then (A, τ) is a *topological algebra* if the product map m_A is continuous.

Definition 3.2. Let A be an algebra, and let $\varphi \in \Phi_A$. Then a sequence

$$(d_n) = (d_n : n \in \mathbb{Z}^+)$$

of linear functionals on A is a *higher point derivation at* φ if $d_0 = \varphi$ and if

$$d_n(ab) = \sum_{j=0}^{n} d_j(a) d_{n-j}(b) \quad (a, b \in A, n \in \mathbb{N}).$$

A higher point derivation (d_n) is *non-degenerate* if $d_0 \neq 0$ and $d_1 \neq 0$.

Suppose that (A, τ) is a topological algebra. Then a higher point derivation (d_n) on A is *continuous* if each of the linear functionals d_n for $n \in \mathbb{Z}^+$ is continuous with respect to τ, *discontinuous* if at least one of the d_n is discontinuous, and *totally discontinuous* if each of the d_n for $n \in \mathbb{N}$ is discontinuous.

For example, consider $O(\mathbb{D})$, the algebra of all analytic functions on the open unit disc $\mathbb{D}$, and, for $f \in O(\mathbb{D})$, set

$$d_n(f) = \frac{f^{(n)}(0)}{n!} \quad (n \in \mathbb{Z}^+).$$

Then the sequence $(d_n : n \in \mathbb{Z}^+)$ is a non-degenerate, continuous higher point derivation at the evaluation character $\varepsilon_0 : f \mapsto f(0)$ of $O(\mathbb{D})$.

Let A be a unital algebra, and let $\varphi \in \Phi_A$. Suppose that $(d_n : n \in \mathbb{Z}^+)$ is a higher point derivation at φ. Then the map

$$\theta : a \mapsto \sum_{n=0}^{\infty} d_n(a) X^n, \quad A \to \mathfrak{F},$$

is a homomorphism with $\pi_0 \circ \theta = \varphi$. Conversely, if $\theta : A \to \mathfrak{F}$ is a homomorphism, then $(\pi_n \circ \theta : n \in \mathbb{Z}^+)$ is a higher point derivation at the character $\pi_0 \circ \theta$ on A. We shall always identify homomorphisms into $\mathfrak{F}$ with higher point derivations in this way.

Similarly, one can identify homomorphisms from an algebra A into $\mathfrak{F}_n$ (where $n \in \mathbb{N}$) with a suitable sequence $(d_r : r \in (\mathbb{Z}^+)^n)$ of linear functionals on A.

The following easy remark is known.

Proposition 3.3. *Let A be an algebra, and let (d_n) be a non-degenerate higher point derivation at a character of A.*

(i) *The set* $\{d_n : n \in \mathbb{Z}^+\}$ *is linearly independent.*
(ii) *For each* $k \in \mathbb{Z}^+$, *there are* $a_0, \ldots, a_k \in A$ *such that*

$$d_i(a_j) = \delta_{i,j} \quad (i, j = 0, \ldots, k).$$

(iii) *For* $n \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in \ker d_0$, *we have*

$$d_n(a_1 \cdots a_n) = d_1(a_1) \cdots d_n(a_n).$$

Proof. (i) First suppose that $\alpha d_0 + \beta d_1 = 0$. Choose $u \in A$ with $d_0(u) = 1$, so that $d_0(u^2) = 1$ and $d_1(u^2) = 2z$, where $z = d_1(u)$. If $z = 0$, then $\alpha = 0$, and then $\beta = 0$ because $d_1 \neq 0$. If $z \neq 0$, then $\alpha + \beta z = \alpha + 2\beta z = 0$, and so $\alpha = \beta = 0$. Thus $\{d_0, d_1\}$ is linearly independent.

Now choose $v \in A$ with $d_0(v) = 0$ and $d_1(v) = 1$. For $k \in \mathbb{N}$, we have

$$d_0(v^k) = \cdots = d_{k-1}(v^k) = 0$$

and $d_k(v^k) = 1$. It follows easily from this that the set $\{d_n : n \in \mathbb{Z}^+\}$ is linearly independent.

(ii) and (iii) These follow immediately. ■

4. (F)**-algebras and Fréchet algebras.** There is considerable variation of terminology in the literature about these algebras. We shall use the following definitions, copying [6]. An early important source on these algebras is [28]; a fine recent account is that of [14].

A topological linear space E is an (F)*-space* if there is a complete metric defining the topology of E; a locally convex space which is an (F)-space is a *Fréchet space*. The space E is *locally bounded* if there is a bounded neighbourhood of the origin in E.

DEFINITION 4.1. A topological algebra (A, τ) is an (F)*-algebra* if there is a complete metric on A which defines the topology τ.

(These algebras are called 'Fréchet topological algebras' in [14].)

A metric d on a linear space E is *translation-invariant* if

$$d(x + z, y + z) = d(x, y) \quad (x, y, z \in E).$$

In this case $d(x, y) = d(x - y, 0)$ $(x, y \in E)$. Let E be a topological linear space whose topology is specified by a metric. Then its topology is also specified by a translation-invariant metric [26, Theorem 1.24]. We can also suppose that, for each $x \in E$, we have

$$d(\alpha_n x, 0) \to 0 \quad \text{whenever} \quad \alpha_n \to 0 \quad \text{in} \quad \mathbb{C}. \tag{4.1}$$

Thus our (F)-space is the same as an 'F-space' in [26].

Here is an easy remark. Let A be an algebra which is also a complete metrizable space. Suppose that the product $m_A : A \times A \to A$ is separately continuous. Then A is an (F)-algebra with respect to the topology determined by the metric.

Quite a lot of remarks, especially those related to the Baire category theorem, which are normally stated for Banach algebras, are actually true for (F)-algebras. Some particular results hold for *separable* (F)-algebras. For example, if I is a closed ideal in a separable (F)-algebra A, and I^2 has finite codimension in A, then I^2 is automatically closed; see [6].

Note that the Gel'fand–Mazur theorem holds for locally convex (F)-algebras: a locally convex (F)-algebra which is a division algebra is isomorphic to $\mathbb{C}$. It seems to be an open question whether or not every (F)-algebra which is a division algebra is isomorphic to $\mathbb{C}$.

Note that there are topologically simple, commutative locally convex (F)-algebras; of course the existence of topologically simple, commutative Banach algebras is a very famous open problem.

DEFINITION 4.2. Let $A = (A, \tau)$ be an (F)-algebra. Then A is a *Fréchet algebra* if the topology τ can be defined by a sequence $(p_k : k \in \mathbb{N})$ of algebra seminorms.

In this case, we can suppose without loss of generality that the sequence $(p_k : k \in \mathbb{N})$ of algebra seminorms is increasing, in the sense that

$$p_k(a) \le p_{k+1}(a) \quad (a \in A,\ k \in \mathbb{N}).$$

We write $(A, (p_k))$ for the corresponding Fréchet algebra.

Our Fréchet algebras are sometimes called 'complete, metrizable locally m-convex algebras'; Helemskiĭ [17, Chapter V] calls them 'polynormed algebras'. The seminal work is [23]; for a new account that has results on Fréchet algebras, see [2].

For example, define

$$p_k\Big(\sum_{j=0}^{\infty} \alpha_j X^j\Big) = \sum_{j=0}^{k} |\alpha_j| \quad (k \in \mathbb{N})$$

for $\sum \alpha_j X^j \in \mathfrak{F}$. Then $(\mathfrak{F}, (p_k))$ is a Fréchet algebra. The topology so defined on $\mathfrak{F}$ is the *topology of coordinatewise convergence*, τ_c.

Now fix $n \in \mathbb{N}$, and define

$$p_k\left(\sum \{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\}\right) = \sum \left\{|\alpha_r| : r \in (\mathbb{Z}^+)^n,\ |r| \le k\right\}$$

for $\sum\{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\} \in \mathfrak{F}_n$. Clearly $(\mathfrak{F}_n, \tau_c) = (\mathfrak{F}_n, (p_k))$ is also a Fréchet algebra; the topology τ_c is again that of *coordinatewise convergence*. In this topology, the subalgebra $\mathbb{C}[X_1, \ldots, X_n]$ of polynomials is dense. The space $(\mathfrak{F}_n, \tau_c)$ is not locally bounded.

We note that every ideal in the algebra $\mathfrak{F}_n$ is closed in the topology τ_c. Indeed we note the following pleasant result of Żelazko [29].

PROPOSITION 4.3. *Let A be a commutative Fréchet algebra. Then all ideals in A are closed if and only if A is noetherian.* ∎

5. The continuity of characters. We now consider when characters on a topological algebra are continuous.

DEFINITION 5.1. Let (A, τ) be a topological algebra. The set of continuous characters on A is denoted by Σ_A. The algebra A is *functionally continuous* if every character on A is continuous, so that $\Sigma_A = \Phi_A$.

It is standard fact, proved at the beginning of any course on Banach algebras in a few lines, that all characters on a Banach algebra are continuous. Thus Banach algebras are functionally continuous.

It is a remarkable fact (see [6, §4.10]) that the question whether or not every commutative Fréchet algebra is functionally continuous is open. This question was specifically discussed in the seminal work [23] of Michael, and so it is often called *Michael's problem.* It is likely that the question was already discussed by Mazur in Warsaw before 1939.

It is easy to find non-metrizable, complete LMC algebras that are not functionally continuous. However, we do not know an example of an (F)-algebra, even non-commutative, that is not functionally continuous.

A strong partial result of Arens [3] asserts that each commutative Fréchet algebra A which has a finite subset S that polynomially generates A, in the sense that the subalgebra of elements that are polynomials in the elements of S is dense in A, is functionally continuous; see [6, Corollary 4.10.11]. It follows that Σ_A is dense in Φ_A in the relative topology $\sigma(A^\times, A)$, where $A^\times$ denotes the space of all linear functionals on A. Various other results showing that specific commutative Fréchet algebras are functionally continuous are given in [6, §4.10]. For example, it is shown in [6, Corollary 4.10.12] that each commutative Fréchet algebra for which Σ_A is countable is functionally continuous.

A remarkable result of Dixon and Esterle [11], given as [6, Corollary 4.10.16], shows that, under the assumption that there is a commutative Fréchet algebra which is not functionally continuous, the following result about analytic maps in several complex variables holds true: for each fixed $k \geq 2$ and each sequence $(F_n)_{n\geq 1}$ of analytic maps from $\mathbb{C}^k$ into $\mathbb{C}^k$, the set

$$\left\{(z_n) \in \prod \mathbb{C}^k : F_n(z_{n+1}) = z_n \quad (n \in \mathbb{N})\right\}$$

is non-empty. An example of a sequence $(F_n)_{n\geq 1}$ such that the above set is empty would lead to a proof that each commutative Fréchet algebra is functionally continuous; no such example is known.

Various 'test algebras' for the functional continuity of commutative Fréchet algebras have been given. These are commutative Fréchet algebras A with the property that all commutative Fréchet algebras are functionally continuous provided that this is the case for the specific algebra A. The first such test algebra, called $\mathcal{U}$, is due to Clayton in 1975 [5]. A deep study of Michael's problem and of the test algebra $\mathcal{U}$ is given in [13], where other test algebras are mentioned; we shall describe the algebra $\mathcal{U}$ below.

There are various papers in the literature which claim, explicitly or implicitly, a positive solution to Michael's problem, but none seems to have convinced the community.

Unfortunately we cannot mark our conference with a solution of Michael's problem, much as we would like to in this Polish setting. However we shall make a remark on this question in §9.

6. The separating space. A sequence $(x_n)_{n\geq 1}$ in a topological linear space is a *null sequence* if $x_n \to 0$ as $n \to \infty$.

Let E and F be (F)-spaces, and let $T : E \to F$ be a linear map. Then the *separating space*, $\mathfrak{S}(T)$, of T is defined to be the space of elements $y \in F$ such that there is a null sequence $(x_n)_{n\geq 1}$ in E with $\lim_{n\to\infty} Tx_n = y$. It is easily checked that $\mathfrak{S}(T)$ is a closed linear subspace of F; the closed graph theorem for (F)-spaces (see [2, §2.12] or [26, Theorem 2.15]) asserts that T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Now suppose that A and B are (F)-algebras and that $\theta : A \to B$ is a homomorphism such that $\theta(A)$ is dense in B. Then it is easily checked that $\mathfrak{S}(\theta)$ is a closed ideal in B.

Let B be an (F)-algebra. Then a closed ideal I in B is a *separating ideal* if, for each sequence $(b_n)_{n\geq 1}$ in B, the nest $\left(\overline{b_1 \cdots b_n I} : n \in \mathbb{N}\right)$ of closed right ideals in B stabilizes, in the sense that there exists $n_0 \in \mathbb{N}$ such that

$$\overline{b_1 \cdots b_n I} = \overline{b_1 \cdots b_{n_0} I} \quad (n \geq n_0).$$

The following is a special case of [6, Theorem 5.2.15].

THEOREM 6.1. *Let A be a locally bounded (F)-algebra and B be an (F)-algebra, and let $\theta : B \to A$ be a homomorphism such that $\theta(B)$ is dense in A. Then $\mathfrak{S}(\theta)$ is a separating ideal in A.* ∎

7. Algebras of power series.

The following definition is standard.

DEFINITION 7.1. Let $A = (A, \tau)$ be an (F)-algebra (respectively, a Fréchet algebra, a Banach algebra). Then A is an *(F)-algebra of power series* (respectively, a *Fréchet algebra of power series*, a *Banach algebra of power series*) if $\mathbb{C}[X] \subset A \subset \mathfrak{F}$ and if the embedding of (A, τ) into $(\mathfrak{F}, \tau_c)$ is continuous.

There are many examples of Banach algebras of power series in [6]. An early exposition of Banach algebras of powers series and of their automorphisms and derivations was given by Grabiner in [15]. Fréchet algebras of power series are considered in [1, 13, 16, 21, 22, 24, ?, ?, 25], inter alia.

We also give the obvious generalization of this definition to several variables.

DEFINITION 7.2. Let $n \in \mathbb{N}$, and let $A = (A, \tau)$ be an (F)-algebra (respectively, a Fréchet algebra, a Banach algebra). Then A is an *(F)-algebra* (respectively, a *Fréchet algebra*, a *Banach algebra*) *of power series in n variables* if $\mathbb{C}[X_1, \ldots, X_n] \subset A \subset \mathfrak{F}_n$ and if the embedding of (A, τ) into $(\mathfrak{F}_n, \tau_c)$ is continuous.

We shall discuss the uniqueness of topology for certain topological algebras. Our terminology is the following.

DEFINITION 7.3. Let $A = (A, \tau)$ be an (F)-algebra. Then A has *a unique (F)-algebra topology* if any topology with respect to which A is an (F)-algebra is equal to τ.

Let $A = (A, \tau)$ be a Fréchet algebra. Then A has *a unique Fréchet-algebra topology* if any topology with respect to which A is a Fréchet algebra is equal to τ.

The uniqueness of topology for Banach algebra of power series was first considered in [19] and taken up in [20]. The uniqueness of the Fréchet algebra topology on $\mathfrak{F}$ was first established in [1]. The following theorem is given in [6, Theorem 4.6.1 and Corollary 4.6.2].

THEOREM 7.4. *Let $n \in \mathbb{N}$. Then $(\mathfrak{F}_n, \tau_c)$ is a Fréchet algebra, and $\mathfrak{F}_n$ has a unique (F)-algebra topology. The algebra $(\mathfrak{F}_n, \tau_c)$ is not a Banach algebra with respect to any norm.* ∎

The following is essentially a theorem of Loy [22]; it is proved in [6, Theorem 5.2.20] in the case where $n = 1$ and A is a Banach algebra of power series, but the argument of that proof applies more generally.

THEOREM 7.5. *Let A be a locally bounded Fréchet algebra of power series in n variables, and let B be a functionally continuous Fréchet algebra. Then every homomorphism from B into A is continuous. In particular, A has a unique Fréchet-algebra topology.* ■

This result was generalized by the second author in [24, Theorem 4.1 and Corollary 4.2].

THEOREM 7.6. *Let A be a Fréchet algebra of power series such that $A \subsetneq \mathfrak{F}$, and let B be a Fréchet algebra. Then every homomorphism $\theta : B \to A$ such that $\dim \theta(B) > 1$ is continuous. Further, A has a unique Fréchet algebra topology.* ■

It is necessary to exclude the case where $\dim \theta(B) = 1$ in the above theorem because it may be that there is a discontinuous character φ on B, and this would give a discontinuous homomorphism $b \mapsto \varphi(b)1,\ B \to A$. It is also necessary to exclude the case where $A = \mathfrak{F}$ because it is a theorem of Dales and McClure that there is a discontinuous epimorphism from certain Banach algebras onto $\mathfrak{F}$; see Theorem 11.1, below.

We shall see in Corollary 11.7 that the second part of Theorem 7.6 can be generalized further: each (F)-algebra of power series has a unique (F)-algebra topology. However this leaves open the following queries.

QUERY. Let A be an (F)-algebra of power series, and let B be a functionally continuous (F)-algebra. Is every homomorphism from B into A automatically continuous? Does an (F)-algebra of power series in n variables (where $n \geq 2$) have a unique (F)-algebra topology?

Later, we shall consider the functional continuity of topological algebras of power series in n variables. Here we state an obvious corollary of the theorem of Arens that was mentioned in §5.

THEOREM 7.7. *Let $n \in \mathbb{N}$, and let A be Fréchet algebra of power series in n variables such that $\mathbb{C}[X_1, \ldots, X_n]$ is dense in A. Then A is functionally continuous.* ■

We remark that the algebra $\mathfrak{F}$ has played a key role in automatic continuity theory through the following result that is a special case of a more general theorem of Allan [1]; see also [6, Theorem 5.7.1].

THEOREM 7.8. *There is a norm $\|\cdot\|$ on $\mathfrak{F}$ such that $(\mathfrak{F}, \|\cdot\|)$ is a normed algebra.* ■

The following more general result is due to Haghany [16]; see also [6, Theorem 5.7.7].

THEOREM 7.9. *Let $n \in \mathbb{N}$. Then there is a norm $\|\cdot\|$ on $\mathfrak{F}_n$ such that $(\mathfrak{F}_n, \|\cdot\|)$ is a normed algebra.* ■

All these results, and related results, are given in [6, §5.7].

8. The algebra of absolutely convergent power series

Definition 8.1. A formal power series $\sum \alpha_n X^n$ in $\mathfrak{F}$ is an *absolutely convergent power series* if there exists $\varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} |\alpha_n| \, \varepsilon^n < \infty. \tag{8.1}$$

The collection of all such absolutely convergent power series is clearly a subalgebra of $\mathfrak{F}$ containing $\mathbb{C}[X]$; it is denoted by $\mathbb{C}\{X\}$. The sum of such a series defines an analytic function, say $f \in O(\Delta_\varepsilon)$, where $\Delta_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$, for some $\varepsilon > 0$.

The algebra $\mathbb{C}\{X\}$ is a topological algebra with respect to a certain inductive limit topology; in this topology, we have $f_n \to 0$ if and only if there exists $\varepsilon > 0$ such that each f_n for $n \in \mathbb{N}$ satisfies (8.1) and, further, the corresponding functions in $O(\Delta_\varepsilon)$ converge uniformly on all compact subspaces of Δ_ε. However this inductive limit topology is not metrizable.

We first make an elementary remark on power series. Indeed, consider an element $f = \sum_{n=0}^{\infty} \alpha_n X^n \in \mathbb{C}\{X\}$. Then f has a radius of convergence, denoted by r_f; indeed, by Hadamard's formula, $r_f = 1/\rho$, where

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

We note the triviality that, if $f = \sum_{n=0}^{\infty} \alpha_n X^n$ and $g = \sum_{n=0}^{\infty} \beta_n X^n$ in $\mathbb{C}\{X\}$, where $|\beta_n| \geq |a_n|$, then $r_g \leq r_f$.

Theorem 8.2. *There is no topology τ on $\mathbb{C}\{X\}$ such that $(\mathbb{C}\{X\}, \tau)$ is an (F)-algebra of power series.*

Proof. Assume towards a contradiction that there is a complete metric d that defines the topology τ on $\mathbb{C}\{X\}$; we may suppose that d is translation-invariant and satisfies equation (4.1).

For $n \in \mathbb{N}$, define

$$f_n(z) = (1 - nz)^{-1} = 1 + \sum_{j=1}^{\infty} n^j z^j \quad (z \in \Delta_{1/n}),$$

so that $f_n \in \mathbb{C}\{X\}$ and $r_{f_n} = 1/n$, and then choose $\alpha_n > 0$ such that $d(\alpha_n f_n, 0) < 1/2^n$. Now consider the series

$$\sum_{n=1}^{\infty} \alpha_n f_n,$$

with partial sums $F_n = \sum_{j=1}^{n} \alpha_j f_j$. For $m, n \in \mathbb{N}$ with $m < n$, we have

$$d(F_m, F_n) = d(\alpha_{m+1} f_{m+1} + \cdots + \alpha_n f_n, 0) \leq \sum_{j=m+1}^{n} d(\alpha_j f_j, 0) < 1/2^m,$$

and so the series is a Cauchy series. Since d is a complete metric, the series converges in $(\mathbb{C}\{X\}, \tau)$, say $f = \sum_{n=1}^{\infty} \alpha_n f_n$.

For $k \in \mathbb{Z}^+$, the map $\pi_k : (\mathbb{C}\{X\}, \tau) \to \mathbb{C}$, is continuous, and so

$$\pi_k(f) = \sum_{n=1}^{\infty} \pi_k(\alpha_n f_n) = \sum_{n=1}^{\infty} \alpha_n n^k.$$

In particular, for each $m \in \mathbb{N}$, we have $\pi_k(f) \geq \pi_k(\alpha_m f_m)$, and so

$$r_f \leq r_{\alpha_m f_m} = r_{f_m} = 1/m.$$

This is true for each $m \in \mathbb{N}$, a contradiction of the fact that $r_f > 0$.

The result follows. ∎

9. Formal power series algebras over semigroups. Let S be a semigroup, so that S is a non-empty set with an associative binary operation $(s,t) \mapsto st$, $S \times S \to S$. In the case where S is an abelian semigroup, we shall often write $s + t$ for the image of (s,t).

We shall again write δ_s for the characteristic function of $\{s\}$ for $s \in S$.

We shall consider only countable semigroups S which have a family $\{S_n : n \in \mathbb{N}\}$ of finite subsets satisfying the following conditions, where $I_n = S \setminus S_n$ $(n \in \mathbb{N})$:

$$S_n \subset S_{n+1}, \ \ SI_n \cup I_n S \subset I_n \ \ (n \in \mathbb{N}), \quad \bigcup\{S_n : n \in \mathbb{N}\} = S. \tag{$*$}$$

Note that this implies that, for each $t \in S$, there are only finitely many pairs $(r,s) \in S \times S$ such that $rs = t$. In this case we shall consider $\mathbb{C}^S$, the linear space of all functions from S into $\mathbb{C}$, made into an algebra $(\mathfrak{F}_S, \star)$ by the requirement that $\delta_r \star \delta_s = \delta_{rs}$ for all $r, s \in S$. Thus, for $f, g \in \mathbb{C}^S$ and $t \in S$, we have

$$(f \star g)(t) = \sum\{f(r)g(s) : r, s \in S,\ rs = t\},$$

a finite sum. This algebra is called the *formal power series algebra over* S; it is a Fréchet algebra with respect to the topology τ_c of pointwise convergence on S, which is specified by the increasing sequence $(p_n : n \in \mathbb{N})$ of algebra seminorms, where p_n is given by

$$p_n(f) = \sum\{|f(s)| : s \in S_n\} \quad (f \in \mathbb{C}^S).$$

Clearly, $\mathfrak{F}_S$ is commutative whenever S is abelian. In fact, we shall again denote the product in $\mathfrak{F}_S$ by juxtaposition.

For example, consider the case where $S = \mathbb{Z}^+$ or $S = (\mathbb{Z}^+)^n$, where $n \in \mathbb{N}$. Then $\mathfrak{F}_S$ is equal to $\mathfrak{F}$ or $\mathfrak{F}_n$, respectively, algebras which we have already discussed.

Now let $S = (\mathbb{Z}^+)^{\omega}$, the abelian semigroup of all $\mathbb{Z}^+$-valued sequences, with coordinatewise addition (so that S does not satisfy $(*)$), and the subsemigroup $S = (\mathbb{Z}^+)^{<\omega}$ consisting of all sequences in $(\mathbb{Z}^+)^{\omega}$ that are eventually 0; this latter semigroup is countable and does satisfy $(*)$, where we take the subsets S_n to satisfy $(*)$ to consist of the sequences $s = (s_k) \in (\mathbb{Z}^+)^{<\omega}$ such that $s_k = 0$ $(k > n)$ and $s_1 + \cdots + s_n \leq n$. A generic element s of $(\mathbb{Z}^+)^{<\omega}$ which is not equal to the zero sequence $(0, 0, \ldots,)$ can be written uniquely as

$$s = (s_1, \ldots, s_n, 0, 0, \ldots)$$

with $n \in \mathbb{N}$ defined by the requirement that $s_n \in \mathbb{N}$; when we specify a non-zero element of $(\mathbb{Z}^+)^{<\omega}$, we shall suppose that it has this form. The corresponding formal power series algebra over $(\mathbb{Z}^+)^{<\omega}$ is denoted by $\mathfrak{F}_\infty$. (In [13] and elsewhere, this algebra is denoted by $\mathbb{C}_{\mathbb{N}}[[X]]$.) Thus a generic element of $\mathfrak{F}_\infty$ again has the form

$$\sum\left\{\alpha_r X^r : r \in (\mathbb{Z}^+)^n\right\} = \sum\{\alpha_{(r_1,\dots,r_n)} X_1^{r_1}\cdots X_n^{r_n} : (r_1,\dots,r_n) \in (\mathbb{Z}^+)^n\},$$

but now there is no restriction on the value of $n \in \mathbb{N}$. Further, the seminorms p_n such that $(p_n : n \in \mathbb{N})$ defines the Fréchet-algebra topology τ_c on $\mathfrak{F}_\infty$ are given by

$$p_n\left(\sum \alpha_r X^r\right) = \sum\left\{|\alpha_r| : r \in (\mathbb{Z}^+)^n,\, |r| \le n\right\} \quad (n \in \mathbb{Z}^+),$$

as in [13, p. 545]. We may regard each algebra $\mathfrak{F}_n$ as a subalgebra of $\mathfrak{F}_\infty$ in an obvious way, and then $\bigcup\{\mathfrak{F}_n : n \in \mathbb{N}\}$ is a dense subalgebra of $(\mathfrak{F}_\infty, \tau_c)$.

This algebra $\mathfrak{F}_\infty$ is not noetherian. For example, consider the ideal I generated by the elements $X_1, X_2, \dots$ in $\mathfrak{F}_\infty$. Then the element

$$\sum\left\{\frac{1}{j}X_j : j \in \mathbb{N}\right\}$$

belongs to $\overline{I}$, but not to I, and so I is not closed in $\mathfrak{F}_\infty$. (In [13], Esterle remarks that principal ideals in $\mathfrak{F}_\infty$ are closed, but that he does not know whether or not all finitely-generated ideals in $\mathfrak{F}_\infty$ are closed.)

Essentially as before, a *monomial* is the characteristic function of an element, say r, of $(\mathbb{Z}^+)^{<\omega}$, and the *degree* of the monomial is $|r|$. For $k \in \mathbb{Z}^+$, a *homogeneous element of degree* k is an 'infinite linear combination' of monomials of degree k; the set of these elements is the linear subspace $\mathfrak{F}_\infty^{(k)}$ of $\mathfrak{F}_\infty$, and the component of an element $f \in \mathfrak{F}_\infty$ in $\mathfrak{F}_\infty^{(k)}$ is denoted by $f^{(k)}$, so that $f = \sum_{k=0}^\infty f^{(k)}$ in $(\mathfrak{F}_\infty, \tau_c)$. Clearly we have

$$\mathfrak{F}_\infty^{(k)} \cdot \mathfrak{F}_\infty^{(\ell)} \subset \mathfrak{F}_\infty^{(k+\ell)} \quad (k, \ell \in \mathbb{Z}^+),$$

and so

$$\mathfrak{F}_\infty = \bigcup\{\mathfrak{F}_\infty^{(k)} : k \in \mathbb{Z}^+\}$$

is a graded algebra. This algebra is an integral domain.

There is another way of writing elements of $\mathfrak{F}_\infty$; for this, each monomial $X_1^{r_1}\cdots X_n^{r_n}$ is written uniquely as

$$X_{t_1}X_{t_2}\cdots X_{t_m}, \quad \text{where} \quad t_1 \le t_2 \le \cdots \le t_m \quad \text{and} \quad m = |r|\,. \tag{9.1}$$

We note that there is a unique character on $\mathfrak{F}_\infty$, namely the evaluation character

$$\varepsilon_0 : f \mapsto f(0,0,\dots), \quad \mathfrak{F}_\infty \to \mathbb{C}.$$

Indeed, let φ be a character on $\mathfrak{F}_\infty$. Then $\varphi \mid \mathfrak{F}_n$ is a character on $\mathfrak{F}_n$ for each $n \in \mathbb{N}$, and so $\varphi(X^r) = 0$ for each monomial X^r. It follows that the only continuous character on $\mathfrak{F}_\infty$ is ε_0. By an earlier remark, this implies that $\mathfrak{F}_\infty$ is functionally continuous, and so the only character on $\mathfrak{F}_\infty$ is ε_0. Alternatively, let $f \in \mathfrak{F}_\infty$ be such that $f(0,0,\dots) \ne 0$. Then the argument of [27, Theorem 2] shows directly that $f \in \operatorname{Inv}\mathfrak{F}_\infty$, and it follows that the unique character is ε_0; this remark shows that $\ker\varepsilon_0$ is the unique maximal ideal in $\mathfrak{F}_\infty$, as noted in [13].

We now note that there is an embedding of $\mathfrak{F}_\infty$ into $\mathfrak{F}_2$, so extending Theorem 2.2.

For $r = (r_1,\dots,r_n,0,0,\dots) \in (\mathbb{Z}^+)^{<\omega}$, set

$$w(r) = r_1 + 2r_2 + \cdots + nr_n$$

for the *weighted order* of r. Thus $w(r+s) = w(r) + w(s)\ \ (r, s \in (\mathbb{Z}^+)^{<\omega})$. We note that, for each $k \in \mathbb{Z}^+$, there are only finitely many elements r of the semigroup $(\mathbb{Z}^+)^{<\omega}$ with $w(r) = k$, and so each element of $\mathfrak{F}_\infty$ can be written as

$$f = \sum_{k=0}^{\infty} \left\{ \sum \alpha_r X^r : r \in (\mathbb{Z}^+)^n \text{ with } w(r) = k \right\},$$

where the inner sum is finite.

THEOREM 9.1. *There is an embedding of $\mathfrak{F}_\infty$ in $\mathfrak{F}_2$.*

Proof. As before we write $\mathfrak{F}_2 = \mathbb{C}[[Y_1, Y_2]]$. Let $(f_j)_{j=1}^{\infty}$ in $\mathfrak{F}$ be the sequence in $\mathfrak{F}$ specified in Lemma 2.1 such that $\{1, f_1, \dots, f_n\}$ is algebraically independent for each $n \in \mathbb{N}$.

Take $f \in \mathfrak{F}_\infty$, as above, and set

$$\theta(f) = \sum_{k=0}^{\infty} Y_2^k \left\{ \sum \alpha_r f_1^{r_1} \cdots f_n^{r_n} : r \in (\mathbb{Z}^+)^n \text{ with } w(r) = k \right\}.$$

Then it is clear that $\theta : \mathfrak{F}_\infty \to \mathfrak{F}_2$ is a continuous homomorphism (using the fact that $w(r+s) = w(r) + w(s)\ \ (r, s \in (\mathbb{Z}^+)^n)$). Suppose that $\theta(f) = 0$. Then, for each $k \in \mathbb{Z}^+$, we have

$$\left\{ \sum \alpha_r f_1^{r_1} \cdots f_n^{r_n} : r \in (\mathbb{Z}^+)^n \text{ with } w(r) = k \right\} = 0.$$

Since this sum is finite and since $\{1, f_1, \dots, f_n\}$ is algebraically independent in $\mathfrak{F}$, it follows that $\alpha_r = 0$ for each $r \in (\mathbb{Z}^+)^n$ with $w(r) = k$, and so $f = 0$. Thus θ is an embedding. ∎

DEFINITION 9.2. For $m \in \mathbb{N}$, set

$$\mathcal{U}_m = \left\{ f = \sum \{\alpha_r X^r : r \in (\mathbb{Z}^+)^{<\omega}\} \in \mathfrak{F}_\infty : q_m(f) := \sum |\alpha_r|\, m^{|r|} < \infty \right\},$$

and then set

$$\mathcal{U} = \bigcap \{\mathcal{U}_m : m \in \mathbb{N}\}.$$

It is clear that each $\mathcal{U}_m$ is a unital subalgebra of $\mathfrak{F}_\infty$ and that $(\mathcal{U}_m, q_m)$ is a Banach algebra continuously embedded in $\mathfrak{F}_\infty$. Thus $\mathcal{U}$ is a unital subalgebra of $\mathfrak{F}_\infty$, and $(\mathcal{U}, (q_m))$ is a unital, commutative Fréchet algebra continuously embedded in $\mathfrak{F}_\infty$. The algebra $\mathcal{U}$ contains each monomial X^r.

The algebra $\mathcal{U}$ was first introduced in this context by Clayton [5]; it is studied further in [11, 13].

It is noted in [11] that the map

$$\varphi \mapsto (\varphi(X_i) : i \in \mathbb{N}), \quad \Phi_{\mathcal{U}} \to \ell^\infty,$$

is a continuous bijection. It can be said that $\mathcal{U}$ is the *algebra of all entire functions on ℓ^∞*.

Extended versions of the following theorem are given in [5, 11, 13]; in [11, Proposition 2.1], there is a non-commutative version of the theorem. We write $\mathcal{M}$ for the closed maximal ideal $\{f \in \mathcal{U} : f(0, 0, \dots) = 0\}$ and

$$\mathcal{I} = \bigcup \{X_1\mathcal{U} + \cdots + X_n\mathcal{U} : n \in \mathbb{N}\},$$

a prime ideal in $\mathcal{U}$.

Theorem 9.3. *The following statements are equivalent:*

(a) *all characters on the commutative Fréchet algebra* $(\mathcal{U},(q_m))$ *are continuous;*
(b) *there is a non-zero character on the quotient algebra* $\mathcal{M}/\mathcal{I}$*;*
(c) *every commutative Fréchet algebra is functionally continuous.* ∎

There is a study of the quotient algebra $\mathcal{M}/\mathcal{I}$ in [13].

In distinction from the uniqueness of topology results that we stated for each algebra $\mathfrak{F}_n$ in Theorem 7.4, we have the following result from [25].

Theorem 9.4. *The algebra* $(\mathfrak{F}_\infty,\tau_c)$ *is a Fréchet algebra, but it does not have a unique Fréchet algebra topology.* ∎

We shall also require in a future proof the non-commutative version of $\mathfrak{F}_\infty$.

We now take S to be the free semigroup in countably many (non-commuting) elements $X_1,X_2,\dots$. Thus, S consists of finite sequences $i=(i_1,\dots,i_m)$ in $\mathbb{N}^m$ for some $m\in\mathbb{N}$, and the product is given by concatenation, so that

$$(i_1,\dots,i_m)+(j_1,\dots,j_n)=(i_1,\dots,i_m,j_1,\dots,j_n);$$

we shall write $X^{\otimes i}=X_{i_1}\otimes X_{i_2}\otimes\cdots\otimes X_{i_n}$ for a generic element of S. This semigroup S is countable and also satisfies condition $(*)$, above, and so we can consider $\mathfrak{F}_S$, the formal power series algebra over S; as in [25], we shall set

$$\mathfrak{B}=\mathfrak{F}_S=\mathbb{C}_{nc}[[X_1,X_2,\dots]]$$

for the corresponding algebra. In the case where $i=(i_1,\dots,i_n)$ in $\mathbb{N}^n$, we obtain a 'non-commutative monomial' of *rank* n, and, almost as before, the space of 'infinite linear combinations' of monomials of rank n forms a linear subspace $\mathfrak{B}^{(n)}$ of $\mathfrak{B}$, the natural decomposition making $\mathfrak{B}$ into a graded algebra. We can write each $b\in\mathfrak{B}$ uniquely as $b=\sum_{n=1}^\infty b^{(n)}$, essentially as before.

We shall also require the 'averaging map' on $\mathfrak{B}$. For $n\in\mathbb{N}$, let $\mathfrak{S}_n$ be the symmetric group on n symbols, and define $\widetilde{\sigma}$ on $\mathfrak{B}^{(n)}$ by

$$\widetilde{\sigma}(X_{i_1}\otimes X_{i_2}\otimes\cdots\otimes X_{i_n})=\frac{1}{n!}\sum\left\{X_{i_{\sigma(1)}}\otimes X_{i_{\sigma(2)}}\otimes\cdots\otimes X_{i_{\sigma(n)}}:\sigma\in\mathfrak{S}_n\right\}.$$

We then extend $\widetilde{\sigma}$ to a continuous linear map on $\mathfrak{B}$ to obtain the *symmetrizing map* $\widetilde{\sigma}$ (*cf*. [6, p. 27]). The elements $b\in\mathfrak{B}$ with $\widetilde{\sigma}(b)=b$ are the *symmetric* elements of $\mathfrak{B}$.

For $n\in\mathbb{N}$, there is a continuous linear embedding $\varepsilon_n:\mathfrak{F}_\infty^{(n)}\to\mathfrak{B}^{(n)}$ defined by the requirement that

$$\varepsilon_n(X_{i_1}\cdots X_{i_n})=\widetilde{\sigma}(X_{i_1}\otimes X_{i_2}\otimes\cdots\otimes X_{i_n});$$

the map ε_n is well-defined because

$$\widetilde{\sigma}(X_{i_1}\otimes X_{i_2}\otimes\cdots\otimes X_{i_n})=\widetilde{\sigma}(X_{j_1}\otimes X_{j_2}\otimes\cdots\otimes X_{j_n})$$

whenever $X_{i_1}\cdots X_{i_n}=X_{j_1}\cdots X_{j_n}$, the latter happening exactly when $\{i_1,\dots,i_n\}$ is a permutation of $\{j_1,\dots,j_n\}$. From these maps, we obtain a continuous linear embedding $\varepsilon:\mathfrak{F}_\infty\to\mathfrak{B}$. Clearly, the symmetrizing map $\widetilde{\sigma}$ is a projection from $\mathfrak{B}$ onto the subspace $\mathfrak{B}_{\mathrm{sym}}$ of $\mathfrak{B}$ consisting of the symmetric elements. There is a product in $\mathfrak{B}_{\mathrm{sym}}$, denoted by $\vee$, so that

$$u\vee v=\widetilde{\sigma}(u\otimes v)\quad(u,v\in\mathfrak{B}_{\mathrm{sym}});$$

now $(\mathfrak{B}_{\mathrm{sym}},\vee)$ is a commutative, unital algebra.

Proposition 9.5. *Let $m, n \in \mathbb{N}$. Then $\widetilde{\sigma}(\varepsilon_m(f) \otimes \varepsilon_n(g)) = \varepsilon_{m+n}(fg)$ for all $f \in \mathfrak{F}_\infty^{(m)}$ and $g \in \mathfrak{F}_\infty^{(n)}$.*

Proof. This is clear in the special case where $f = X_{i_1} \cdots X_{i_m}$ and $g = X_{j_1} \cdots X_{j_n}$. The general case follows because ε_m, ε_n and ε_{m+n} are continuous linear maps. ∎

It follows that $(\mathfrak{B}_{\mathrm{sym}}, \vee)$ is naturally identified with $\varepsilon(\mathfrak{F}_\infty)$ as an algebra.

We shall require the concept of 'tensor products by rows', taken from [25].

First, for each $n \in \mathbb{Z}^+$, let $P_n : \mathfrak{B} \to \mathfrak{B}$ be the linear map such that $P_n(1) = 0$ and

$$P_n(X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m}) = \begin{cases} 0 & \text{when } i_1 \neq n, \\ X_{i_2} \otimes \cdots \otimes X_{i_m} & \text{when } i_1 = n. \end{cases}$$

Now let $\lambda_1, \lambda_2 : \mathfrak{B}^{(1)} \to \mathbb{C}$ be two linear functionals. We define the *tensor product by rows*, $\lambda_1 \otimes \lambda_2 : \mathfrak{B}^{(2)} \to \mathbb{C}$, by

$$(\lambda_1 \otimes \lambda_2)(b) = \lambda_1\Big(\sum_{j=1}^{\infty} \lambda_2(P_j b) X_j\Big) \quad (b \in \mathfrak{B}^{(2)}).$$

Finally, let $n \in \mathbb{N}$, and let $\lambda_1, \ldots, \lambda_n : \mathfrak{B}^{(1)} \to \mathbb{C}$ be n linear functionals. Then we define the *tensor product by rows*, $\lambda_1 \otimes \cdots \otimes \lambda_n : \mathfrak{B}^{(n)} \to \mathbb{C}$, inductively by

$$(\lambda_1 \otimes \cdots \otimes \lambda_n)(b) = \lambda_1\Big(\sum_{j=1}^{\infty} (\lambda_2 \otimes \cdots \otimes \lambda_n)(P_j b) X_j\Big) \quad (b \in \mathfrak{B}^{(n)}).$$

The first lemma that we shall use is the following; it is essentially obvious.

Lemma 9.6. *Let S be a semigroup satisfying $(*)$, and suppose that there are linear functionals $\lambda_s : \mathfrak{B}^{(1)} \to \mathbb{C}$ for each $s \in S$. Let $s \in S$ and $n \in \mathbb{N}$, and set*

$$\lambda = \sum \{\lambda_{r_1} \otimes \cdots \otimes \lambda_{r_n} : r_1, \ldots, r_n \in S,\ r_1 + \cdots + r_n = s\}.$$

Then $\lambda = \lambda \circ \widetilde{\sigma}$. ∎

The second lemma that we shall use is the following, taken from [25, Lemma 1.10]. In this lemma, the tensor product of *no* linear functionals is deemed to be the identity map, regarded as a linear functional on $\mathbb{C} = \mathfrak{B}^{(0)}$.

Lemma 9.7. *Let $m, n \in \mathbb{N}$, and let $\lambda_1, \ldots, \lambda_{m+n}$ be linear functionals on $\mathfrak{B}^{(1)}$. Then*

$$(\lambda_1 \otimes \cdots \otimes \lambda_{m+n})(a \otimes b) = (\lambda_1 \otimes \cdots \otimes \lambda_m)(a)(\lambda_{m+1} \otimes \cdots \otimes \lambda_{m+n})(b)$$

for each $a \in \mathfrak{B}^{(m)}$ and $b \in \mathfrak{B}^{(n)}$. ∎

10. Semigroup algebras. Now let S be an arbitrary semigroup. Then the Banach space $\ell^1(S)$ consists of all sums

$$f = \sum \{\alpha_s \delta_s : s \in S\},$$

where $\alpha_s \in \mathbb{C}$ $(s \in S)$, such that $\sum\{|\alpha_s| : s \in S\} < \infty$. Of course, this space is a Banach space for the norm $\|\cdot\|_1$, specified by

$$\|f\|_1 = \sum \{|\alpha_s| : s \in S\} \quad \Big(f = \sum_{s \in S} \alpha_s \delta_s \in \ell^1(S)\Big),$$

and it is a Banach algebra with respect to a unique product $\star$ again specified by the condition that $\delta_s \star \delta_t = \delta_{st}$ for all $s, t \in S$. This algebra is called the *semigroup algebra over* S. There have been many recent studies of this Banach algebra; for example, see [10], where more details are given.

For example, consider the semigroup $S = (\mathbb{Z}^+)^{<\omega}$, described above. For $k \in \mathbb{Z}^+$, we set

$$S^{(k)} = \Big\{(r_j) \in S : |r| = \sum_{j=1}^{\infty} r_j = k\Big\}.$$

Then $S = \bigcup\{S^{(k)} : k \in \mathbb{Z}^+\}$, and $S^{(k)} \cdot S^{(\ell)} \subset S^{(k+\ell)}$ for $k, \ell \in \mathbb{Z}^+$, so that S is graded in a natural way. Further,

$$\ell^1(S) = \Big(\bigoplus_{k=0}^{\infty} \ell^1(S^{(k)})\Big)_1,$$

is a graded algebra; here $(\,\cdot\,)_1$ denotes an ℓ_1-sum. We shall often write $A = \ell^1(S)$ for this semigroup algebra, and then $A^{(k)} = \ell^1(S^{(k)})$ and $A = \sum\{A^{(k)} : k \in \mathbb{Z}^+\}$ is a graded algebra. There is a natural embedding of A into $\mathfrak{F}_\infty$, and this embedding takes each $A^{(k)}$ into $\mathfrak{F}_\infty^{(k)}$, so that A is a graded subalgebra of $\mathfrak{F}_\infty$.

Again, a generic element of A can also be written as

$$f = \sum \beta_{(t_1,\dots,t_m)} X_{t_1} X_{t_2} \cdots X_{t_m}, \tag{10.1}$$

where $t_1 \le t_2 \le \cdots \le t_m$, as in equation (9.1), and $\sum \left|\beta_{(t_1,\dots,t_m)}\right| = \|f\|_1$.

Let $\mathcal{U}$ be the test algebra which was described above for Michael's problem. Then clearly there is a continuous embedding of $\mathcal{U}$ into $\ell^1(S)$.

Set $E = \ell^1(\mathbb{Z}^+)$, so that E is a Banach space, and recall that, for each $n \in \mathbb{N}$, the Banach space $\ell^1((\mathbb{Z}^+)^n)$ can be identified as a Banach space with the n-fold projective tensor product

$$E_n := \widehat{\bigotimes}^n E = E \,\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi\, E.$$

As in [6, Example 2.2.46(ii)], we form the *projective tensor algebra* of E; this is

$$\widehat{\bigotimes} E = \{u = (u_n) : u_n \in E_n \ \ (n \in \mathbb{N})\},$$

with product denoted by $\otimes$, so that

$$(u_p) \otimes (v_q) = \Big(\sum_{p+q=r} u_p \otimes v_q : r \in \mathbb{Z}^+\Big);$$

we obtain a non-commutative, unital algebra.

We again have the concept of a *symmetric element* and a *symmetrizing map* $\widetilde{\sigma}$, as in [6]. The subspace of $\widehat{\bigotimes} E$ consisting of the symmetric elements is denoted by $\widehat{\bigvee} E$; it is the range of the map $\widetilde{\sigma}$, and is itself an algebra with respect to the product $\vee$, where

$$(u_p) \vee (v_q) = \Big(\sum_{p+q=r} \widetilde{\sigma}(u_p \otimes v_q) : r \in \mathbb{Z}^+\Big);$$

we obtain a commutative, unital algebra $(\widehat{\bigvee} E, \vee)$, called the *projective symmetric algebra* of E.

For $n \in \mathbb{N}$, define

$$p_n(u) = \sum_{i=0}^{n} \|u_i\|_1 \quad \Big(u = (u_i) \in \widehat{\bigvee} E\Big).$$

Then each p_n is an algebra seminorm on $\widehat{\bigvee} E$, and $(\widehat{\bigvee} E, (p_n)_{n\geq 1}, \vee)$ is a commutative, unital Fréchet algebra which is naturally identified with a subalgebra of $(\mathfrak{B}_{\mathrm{sym}}, \vee)$.

We now set

$$B = \Big\{u = (u_n) \in \widehat{\bigvee} E : \|u\|_1 := \sum_{n=0}^{\infty} \|u_n\|_1 < \infty\Big\}.$$

As in [6, Example 2.2.46(ii)], $(B, \|\cdot\|, \vee)$ is a commutative, unital Banach algebra; it is a subalgebra of the projective symmetric algebra $(\widehat{\bigvee} E, \vee)$.

Again set $A = \ell^1(S)$, where $S = (\mathbb{Z}^+)^{<\omega}$. The restriction of the map ε to A is an isometric unital isomorphism of A onto the above algebra B.

It was shown in [6, §5.5] how to construct continuous higher point derivations of infinite order on the above algebra $A = \ell^1(S)$, and hence how to construct continuous homomorphisms from $(A, \|\cdot\|_1)$ into $(\mathfrak{F}, \tau_c)$. However, it is not clear to us how to modify this argument to obtain a continuous *embedding* of A into $\mathfrak{F}$; such an embedding will be exhibited in the following theorem.

THEOREM 10.1. (i) *There is a continuous embedding* θ *of* $\ell^1((\mathbb{Z}^+)^{<\omega})$ *into* $(\mathfrak{F}, \tau_c)$ *such that* $\theta(X_1) = X$, *and so the Banach algebra* $\ell^1((\mathbb{Z}^+)^{<\omega})$ *is* (*isometrically isomorphic to*) *a Banach algebra of power series.*

(ii) *The Fréchet algebra* $\mathcal{U}$ *is* (*isometrically isomorphic to*) *a Fréchet algebra of power series.*

Proof. Set $S = (\mathbb{Z}^+)^{<\omega}$ and $A = \ell^1(S)$, as above. We shall construct a continuous, unital homomorphism $\theta : (\mathfrak{F}_\infty, \tau_c) \to (\mathfrak{F}, \tau_c)$ such that $\theta \mid A : (A, \|\cdot\|_1) \to (\mathfrak{F}, \tau_c)$ is a continuous embedding with $\theta(X_1) = X$, and so $\theta(\mathcal{U}) \supset \mathbb{C}[X]$. In this case, $\theta(A)$ is a Banach algebra of power series, with respect to the norm transfered from A, and so A is isometrically isomorphic to a Banach algebra of power series. Since the embedding of $\mathcal{U}$ into A is continuous, $\theta(\mathcal{U})$ is a Fréchet algebra of power series. Thus the result will be established.

Our first remark is the following. Let $(g_i : i \in \mathbb{N})$ be a sequence in $\mathfrak{F}$ with $g_1 = X$ such that $\mathbf{o}(g_i) \geq i \ (i \in \mathbb{N})$. Then there is a unique continuous, unital homomorphism $\theta : (\mathfrak{F}_\infty, \tau_c) \to (\mathfrak{F}, \tau_c)$ with $\theta(X_i) = g_i \ (i \in \mathbb{N})$. Since $\theta(X_1) = X$, we have $\theta(\mathcal{U}) \supset \mathbb{C}[X]$, and so all the required conditions are satisfied save perhaps for the fact that $\theta \mid A$ is an injection. (We note for future reference in Theorem 12.3 that the element

$$\theta\Big(\sum_{i=2}^{\infty} X_i/i^2\Big)$$

belongs to $X^2\mathfrak{F}$.)

Our main *claim* is that we can choose the sequence $(g_i : i \in \mathbb{N})$ so that the corresponding map θ is indeed an injection.

The first step in our construction is to specify a function

$$\gamma : \mathbb{N} \to \mathbb{N}$$

with the following properties: we have $\gamma_i \leq i \ (i \in \mathbb{N})$ and, for each $n \in \mathbb{N}$ and each $r = (r_1, \dots, r_n) \in \mathbb{N}^n$, there exists $k \in \mathbb{N}$ such that

$$(\gamma(k+1), \dots, \gamma(k+n)) = (r_1, \dots, r_n). \tag{10.2}$$

Such a function is easily constructed by listing all the elements in the countable set

$$\bigcup \{(r_1, \dots, r_n) \in \mathbb{N}^n : n \in \mathbb{N}\}$$

in one sequence and by regarding the elements in this listing as successive parts of a function in $\mathbb{N}^{\mathbb{N}}$. Note that, in this case, there are infinitely many values of $k \in \mathbb{N}$ such that equation (10.2) holds for each specified value of r.

For each $i \in \mathbb{N}$ with $i \geq 2$, we define

$$E_i = \{j \in \mathbb{N} \setminus \{1\} : \gamma(j) = i\},$$

and we take $E_1 = \{1\}$ so that $\{E_i : i \in \mathbb{N}\}$ is a partition of $\mathbb{N}$, and each E_i save for E_1 is infinite. Further, $\min E_i \geq i \ (i \in \mathbb{N})$.

We now take a 'rapidly increasing sequence' $(c_i : i \in \mathbb{N}) \in \mathbb{N}^{\mathbb{N}}$ with $c_1 = 1$.

In fact, we shall write $(c_j : j \in \mathbb{N})$ as $(a_1, b_1, a_2, b_2, \dots)$, where

$$1 = a_1 < b_1 < a_2 < b_2 < \cdots .$$

The growth conditions that we shall impose are:

$$a_{i+1} > i a_i \quad (i \in \mathbb{N}) \tag{10.3}$$

and

$$b_i > i \cdot (i(1+a_i))! \cdot i^{i(1+a_i)} \cdot b_{i-1}^{i(1+a_i)} \quad (i \geq 2). \tag{10.4}$$

Clearly, we can choose the sequence $(c_i : i \in \mathbb{N})$ to satisfy these constraints.

For each $i \in \mathbb{N}$, we define

$$g_i = \sum \{b_j X^{a_j} : j \in E_i\} \in M \subset \mathfrak{F}, \tag{10.5}$$

Note that, since $a_i \geq i$ and $\min E_i \geq i$ for each $i \in \mathbb{N}$, we have $\mathbf{o}(g_i) \geq i \ (i \in \mathbb{N})$, as required in the above remarks.

Our claim will follow easily from the following lemma. We continue to denote the semigroup $(\mathbb{Z}^+)^{<\omega}$ by S.

Lemma 10.2. *Let $m \in \mathbb{N}$. Let $(r_1, \dots, r_m, 0, 0, \dots) \in S$ be such that $r_1 \leq r_2 \leq \cdots \leq r_m$, and let $k \in \mathbb{N}$ be such that $k > m$ and $(\gamma(k+1), \dots, \gamma(k+m)) = (r_1, \dots, r_m)$. Set $P = \sum_{i=1}^m a_{k+i}$ and $Q = \prod_{i=1}^m b_{k+i}$.*

(i) *We have*

$$\pi_P(g_{r_1} \cdots g_{r_m}) \geq Q.$$

(ii) *Provided that the sequence $(c_j : j \in \mathbb{N})$ satisfies equations* (10.3) *and* (10.4), *we have*

$$\pi_P(g_{s_1} \cdots g_{s_n}) \leq Q/k.$$

for each $(s_1, \dots, s_n, 0, 0, \dots) \in S$ with $\{s_1, \dots, s_n\} \neq \{r_1, \dots, r_m\}$.

We now prove that the fact that θ is injective follows from Lemma 10.2.

As in equation (10.1), each element $f \in A$ can be written in the form

$$f = \sum \beta_{(t_1,\dots,t_m)} X_{t_1} \cdots X_{t_m},$$

where $t_1 \le t_2 \le \cdots \le t_m$. Take such an element with $f \neq 0$; we shall show that $\theta(f) \neq 0$. We may suppose for convenience that $\|f\|_1 = 1$. Choose a specific element $t = (t_1, \dots, t_m, 0, 0, \dots) \in S$ for which $\beta_t \neq 0$. Then there exists $k \in \mathbb{N}$ with $k > 1/|\beta_t|$ and such that $(\gamma(k+1), \dots, \gamma(k+m)) = (t_1, \dots, t_m)$. Define P and Q with respect to the elements $t \in S$ and $k \in \mathbb{N}$ as in Lemma 10.2. By clauses (i) and (ii) of that lemma, we have

$$|\pi_P(\beta_t g_{t_1} \cdots g_{t_m})| \ge Q\,|\beta_t|\,.$$

and

$$\begin{aligned} |\pi_P(\theta(f) - \beta_t g_{t_1} \cdots g_{t_m})| &= \Big|\pi_P\Big(\sum_{s\in S} \beta_s g_{s_1} \cdots g_{s_n} : \{s_1, \dots, s_n\} \neq \{t_1, \dots, t_m\}\Big)\Big| \\ &\le \sup\{\pi_P(g_{s_1} \cdots g_{s_n}) : \{s_1, \dots, s_n\} \neq \{t_1, \dots, t_m\}\} \\ &\le Q/k, \end{aligned}$$

where we recall that $\sum_{s\in S} |\beta_s| = 1$. It follows that

$$|\pi_P(\theta(f))| \ge Q \cdot (|\beta_t| - 1/k) > 0,$$

and so $\theta(f) \neq 0$ in $\mathfrak{F}$, as required to complete the proof of Theorem 10.1.

It remains to prove the two clauses of Lemma 10.2. Let k, P, and Q be as in that lemma. We recall that, for each $(r_1, \dots, r_m, 0, 0, \dots) \in S$, there does indeed exist $k \in \mathbb{N}$ such that $k > m$ and $(\gamma(k+1), \dots, \gamma(k+m)) = (r_1, \dots, r_m)$.

(i) For each $j \in \mathbb{N}_m$, we have $\gamma(k+j) = r_j$, so that $k + j \in E_{r_j}$, and hence equation (10.5) shows that $\pi_{a_{k+j}}(g_{r_j}) = b_{k+j}$. It follows that

$$\begin{aligned} \pi_P(g_{r_1} \cdots g_{r_m}) &= \sum\{\pi_{p_1}(g_{r_1}) \cdots \pi_{p_m}(g_{r_m}) : p_1, \dots, p_m \in \mathbb{Z}^+,\ p_1 + \cdots + p_m = P\} \\ &\ge \pi_{a_{k+1}}(g_{r_1}) \cdots \pi_{a_{k+m}}(g_{r_m}) \\ &= b_{k+1} \cdots b_{k+m} = Q. \end{aligned}$$

This establishes clause (i).

(ii) The proof of this clause is more complicated.

We first define the (reverse) lexicographic ordering on $S = (\mathbb{Z}^+)^{<\omega}$. Indeed, let

$$s = (s_1, \dots, s_m, 0, 0, \dots), \quad t = (t_1, \dots, t_n, 0, 0, \dots) \in S,$$

and set $s > t$ if $s_j > t_j$, where $j = \max\{i \in \mathbb{N} : s_i \neq t_i\}$. (Such a maximum exists.) Further, set $s \ge t$ if $s > t$ or $s = t$. Then it is clear that $(S, \le)$ is a well-ordered set. (In fact, $(S, \le)$ is a well-ordered semigroup, in the terminology of [6, Definition 1.2.11].)

We define $\alpha : S \to \mathbb{Z}^+$ and $\beta : S \to \mathbb{N}$ by

$$\alpha(t) = \sum t_i a_i, \quad \beta(t) = \prod b_i^{t_i} \quad (t = (t_1, \dots, t_n, 0, 0, \dots) \in S).$$

(Of course, this sum and product are finite.)

For each $R \in \mathbb{Z}^+$, we define

$$\mathcal{N}_R = \{t \in S : \alpha(t) = R\}.$$

Thus each set $\mathcal{N}_R$ is finite and $\{\mathcal{N}_R : R \in \mathbb{N}\}$ is a partition of S. Further, for each $R, M \in \mathbb{Z}^+$, we define

$$\mathcal{N}_R^{(M)} = \mathcal{N}_R \cap S^{(M)} = \{r \in \mathcal{N}_R : |r| = M\}.$$

We shall be particularly interested in the case where $R = P$, in the notation of our lemma.

Let $u = (u_i)$ be the element of S such that

$$u_{k+1} = \cdots = u_{k+m} = 1, \quad u_i = 0 \quad (i \notin \{k+1, \ldots, k+m\}),$$

so that $u \in \mathcal{N}_P^{(m)}$. Our subsidiary *claim* is that u is the maximum element of $(\mathcal{N}_P, \leq)$.

Indeed, assume towards a contradiction that $v \in \mathcal{N}_P$ with $v > u$, and define

$$j = \max\{i \in \mathbb{N} : v_i \neq u_i\}.$$

Suppose that $j > k + m$, so that $v_j \geq 1$. Then

$$\alpha(v) \geq a_j \geq a_{k+m+1} > (k+m)a_{k+m}$$

by (10.3), and so $\alpha(v) > a_{k+1} + \cdots + a_{k+m} = P$, a contradiction of the fact that $v \in \mathcal{N}_P$.

Suppose that $k < j \leq k + m$. Then $v_j \geq 2$, and now

$$0 = \alpha(v) - \alpha(u) = \sum_{i=1}^{j} (v_i - u_i)a_i \geq a_j - \sum_{i=k+1}^{j-1} a_i > 0$$

by (10.3), again a contradiction.

Finally, suppose that $j \leq k$. Then $v_j \geq 1$ and $v_{k+1} = \cdots = v_{k+m} = 1$, and so

$$\alpha(v) \geq v_j + P > P,$$

again a contradiction of the fact that $v \in \mathcal{N}_P$.

Thus, for each possible choice of j, we have a contradiction, and so our subsidiary claim is proved.

Next, for each $n \in \mathbb{N}$, define $\eta_n : (\mathbb{Z}^+)^n \to S$ by

$$\eta_n(s_1, \ldots, s_n) = (\eta_n(s_1, \ldots, s_n)(i) : i \in \mathbb{N}),$$

where

$$\eta_n(s_1, \ldots, s_n)(i) = \begin{cases} 1 & \text{when } i \in \{s_1, \ldots, s_n\}, \\ 0 & \text{when } i \notin \{s_1, \ldots, s_n\}. \end{cases}$$

The map η_n is not injective; indeed, we have $\eta_n(s_1, \ldots, s_n) = \eta_n(t_1, \ldots, t_n)$ if and only if $\{s_1, \ldots, s_n\} = \{t_1, \ldots, t_n\}$, and so the inverse image of each element of the range of η_n has cardinality at most $n!$.

Now take $(s_1, \ldots, s_n, 0, 0, \ldots) \in S$ with $\{s_1, \ldots, s_n\} \neq \{r_1, \ldots, r_m\}$ and $s_1 \leq \cdots \leq s_n$. We have

$$\pi_P(g_{s_1} \cdots g_{s_n}) = \sum b_{p_1} \cdots b_{p_n},$$

where the sum is taken over all elements $p_1, \ldots, p_n \in \mathbb{N}$ such that $a_{p_1} + \cdots + a_{p_n} = P$ and $p_i \in E_{s_i}$ $(i \in \mathbb{N}_n)$. The above sum involves only sequences $(p_1, \ldots, p_n)$ such that $\eta_n(p_1, \ldots, p_n) \in \mathcal{N}_P^{(n)}$. (For example, we could take $n = m$ and

$$(p_1, \ldots, p_n) = (k+1, \ldots, k+m)).$$

Since $\{s_1, \dots, s_n\} \neq \{r_1, \dots, r_m\}$, we have $\eta_n(p_1, \dots, p_n) \neq u$. (This last constraint is only applicable in the special case where $n = m$.) Thus we have the estimate

$$0 \leq \pi_P(g_{s_1} \cdots g_{s_n}) \leq n! \cdot \sum\{\beta(v) : v \in \mathcal{N}_P^{(n)},\ v \neq u\ \}. \tag{10.6}$$

Take $v \in \mathcal{N}_P^{(n)}$ with $v \neq u$, and set $j = \max\{i \in \mathbb{N} : v_i \neq u_i\}$. Since $v < u$, we have $v_j < u_j$, so that $j \in \{k+1, \dots, k+m\}$, $v_j = 0$, $v_{j+1} = \cdots = v_{k+m} = 1$, and $v_i = 0\ (i \geq k+m+1)$. This shows that

$$|\{v \in \mathcal{N}_P^{(n)},\ v \neq u\}| \leq (j-1)^n. \tag{10.7}$$

We have

$$\sum_{i=k+1}^{k+m} a_i = P = \alpha(v) = \sum_{i=j+1}^{k+m} a_i + \sum_{i=1}^{j-1} v_i a_i. \tag{10.8}$$

However, $a_i \geq 1\ (i \in \mathbb{N})$ and

$$n = |v| = k + m - j + \sum_{i=1}^{j-1} v_i,$$

so that

$$\sum_{i=1}^{j-1} v_i \geq n - m,$$

and hence it follows from equation (10.8) that

$$\sum_{i=k+1}^{j} a_i = \sum_{i=1}^{j-1} v_i a_i \geq n - m.$$

Thus we have

$$n \leq m + \sum_{i=k+1}^{j} a_i \leq m(1 + a_j).$$

Since $m \leq k < j$, it follows that

$$n \leq j(1 + a_j). \tag{10.9}$$

We also have $v_j = 0$, $u_j = 1$, and $\sum v_i = n$, and so

$$\frac{\beta(v)}{Q} = \frac{\beta(v)}{b_{k+1} \cdots b_{k+m}} = \frac{\beta(v)}{\beta(u)} = b_j^{-1} \cdot \prod_{i=1}^{j-1} b_i^{v_j - u_j} \leq b_j^{-1} \cdot b_{j-1}^n,$$

whence

$$\beta(v) \leq Q \cdot b_j^{-1} \cdot b_{j-1}^n. \tag{10.10}$$

It follows from equations (10.6), (10.7), (10.9), and (10.10) that

$$0 \leq \pi_P(g_{s_1} \cdots g_{s_n}) \leq (j(1 + a_j))! \cdot (j-1)^{j(1+a_j)} \cdot Q \cdot b_j^{-1} \cdot b_{j-1}^{j(1+a_j)}.$$

From equation (10.4), we have

$$0 \leq \pi_P(g_{s_1} \cdots g_{s_n}) \leq Q/j.$$

Since $j > k$, we have $\pi_P(g_{s_1} \cdots g_{s_n}) \leq Q/k$, and thus we have established clause (ii) of Lemma 10.2.

This completes the proof of Theorem 10.1. ∎

COROLLARY 10.3. *There is a Fréchet algebra of power series which is a test case for the functional continuity of the class of commutative Fréchet algebras.* ∎

Since $\mathfrak{F}_2$ is a subalgebra of $\mathfrak{F}_\infty$, it follows from Theorem 2.6 that there is no embedding of $\mathfrak{F}_\infty$ into $\mathfrak{F}$.

The above proof shows that the semigroup algebra $\ell^1(S)$, where $S = (\mathbb{Z}^+)^{<\omega}$ is the free semigroup on countably many generators is a Banach algebra of power series. We shall now show the somewhat surprising fact that the 'much bigger' semigroup algebra $\ell^1(S_\mathfrak{c})$, where $S_\mathfrak{c}$ denotes the free semigroup on $\mathfrak{c}$ generators, is also Banach algebra of power series. Of course, $\mathfrak{c}$ is the largest cardinal for which such a statement could be true. The proof depends on the following lemma that is surely well known.

LEMMA 10.4. *There is a family $\{E_\alpha : \alpha < \mathfrak{c}\}$ of subsets of $\mathbb{N}$ such that the set*

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$$

is an infinite subset of $\mathbb{N}$ for each $n \in \mathbb{N}$ and each $\alpha_1, \ldots, \alpha_n < \mathfrak{c}$, where each set F_α is equal to either E_α or to its complement $\mathbb{N} \setminus E_\alpha$.

Proof. Let $D = \{0,1\}^\mathfrak{c}$ be the Cantor cube of size $\mathfrak{c}$, so that D is a compact, Hausdorff space with respect to the product topology. It is a special case of the famous Hewitt–Marczewski–Pondiczery theorem (see [12, 2.3.15]) that D is separable; let C be a countable, dense subset of D. Since D has no isolated points, it is clear that $U \cap C$ is infinite for each non-empty, open subset U of D.

A generic element of D has the form $\varepsilon = (\varepsilon_\alpha : \alpha < \mathfrak{c})$, where each ε_α is 0 or 1. For each $\alpha < \mathfrak{c}$, set $D_\alpha = \{\varepsilon \in D : \varepsilon_\alpha = 0\}$, so that the complement of D_α in D is the set $D'_\alpha = \{\varepsilon \in D : \varepsilon_\alpha = 1\}$. A family of basic open sets for D consists of the finite intersections U of sets of the form D_α or D'_α, and $U \cap C$ is infinite for each such set U.

Set $E_\alpha = D_\alpha \cap C$ for $\alpha < \mathfrak{c}$, and identify C bijectively with $\mathbb{N}$. It is clear that the family $\{E_\alpha : \alpha < \mathfrak{c}\}$ has the required property. ∎

THEOREM 10.5. *There is a continuous embedding of the semigroup algebra $\ell^1(S_\mathfrak{c})$ into $\mathfrak{F}$ such that the range contains $\mathbb{C}[X]$, and so $\ell^1(S_\mathfrak{c})$ is a Banach algebra of power series.*

Proof. In fact, there is a continuous embedding (of norm 1) of $\ell^1(S_\mathfrak{c})$ into $\ell^1(S)$, where $S = (\mathbb{Z}^+)^{<\omega}$, such that the range contains the specific element X_1. Given this, it will follow immediately from Theorem 10.1(i) that the required continuous embedding will exist.

Choose a sequence (r_i) for which $r_{i+1} > r_i^2$ $(i \in \mathbb{N})$, and then use Lemma 10.4 to choose a family $\{E_\alpha : \alpha < \mathfrak{c}\}$ of subsets of $R := \{r_i : i \in \mathbb{N}\}$ such that $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$ is an infinite subset of R for each $n \in \mathbb{N}$ and each $\alpha_1, \ldots, \alpha_n < \mathfrak{c}$, where each set F_α is equal to either E_α or to its complement $R \setminus E_\alpha$.

For each $K, M \in \mathbb{N}$ with $K \le M$, the integers of the form $\sum_{i=K}^M n_i r_i$, with $n_i \in \mathbb{Z}^+$ and $n_i < r_K$ for $i = K, \ldots, M$, are all distinct. Indeed, the minimum distance between any two distinct integers of this form is r_K. Suppose that $n_i \in \mathbb{Z}^+$ and $n_i < r_K$ for $i = K, \ldots, M$ and that two sums $\sum_{i=K}^M m_i r_i$ and $\sum_{i=1}^M n_i r_i$ are equal, where $m_i, n_i \in \mathbb{Z}^+$ for $i \in \mathbb{N}$ and $K, M \in \mathbb{N}$ with $K \le M$, then either $m_i = n_i$ $(i \in \mathbb{N})$, or the sum $\sum_{i=1}^{K-1} m_i \ge r_K/r_{K-1} \ge \sqrt{r_K}$.

We now define the map $\theta : \ell^1(S_{\mathfrak{c}}) \to \ell^1(S)$ to be the unique continuous homomorphism such that, for each $\alpha < \mathfrak{c}$, we have

$$\theta(X_\alpha) = \sum \left\{ \frac{1}{2^i} X_i : i \in E_\alpha \right\}.$$

It is obvious that such a map θ exists and that θ is a homomorphism with $\|\theta\| = 1$.

We *claim* that θ is also injective. To see this, assume towards a contradiction that $\theta(f) = 0$ for some $f \in \ell^1(S_{\mathfrak{c}})$, where f has a coefficient equal to 1 at the monomial $\prod_{i=1}^N X_{\alpha_i}^{n_i}$ (where the α_i are distinct ordinals, with each $\alpha_i < \mathfrak{c}$). Write $d = \sum_{i=1}^N n_i$ for the total degree of this latter monomial, and choose an element $g \in \ell^1(S_{\mathfrak{c}})$ of finite support such that $\|f - g\|_1 < 1/2d!$, say the support of g is $\{\beta_i : i \in \mathbb{N}_M\}$ for some $M \geq N$. Take $i \in \mathbb{N}_M$. By Lemma 10.4, the set $E_{\beta_i} \setminus \bigcup_{j \neq i} E_{\beta_j}$ is infinite, and so we may choose $s_i \in E_{\beta_i} \setminus \bigcup_{j \neq i} E_{\beta_j}$. Set $R = \sum_{i=1}^N s_i n_i$. Then the coefficient of the monomial $Q := \prod_{i=1}^N X_{s_i}^{n_i}$ in $\theta(g)$ is exactly 2^{-R}. However, the coefficient of Q in $\theta(X_{\beta_1} X_{\beta_2} \cdots X_{\beta_k})$ is zero unless we have $k = d$ and we can rearrange the β_j in such a way that $s_1 \in E_{\beta_j}$ for $j = 1, \dots, n_1$, $s_2 \in E_{\beta_j}$ for $j = n_1 + 1, \dots, n_1 + n_2$, and so on. In this latter case, the coefficient we obtain is $2^{-R} \cdot p$, where p is the number of such rearrangements divided by a combinatorial factor, which is 1 if the β_j are themselves distinct, but will be greater than 1 if there are some repetitions in the sequence β_j. Of course, p cannot exceed $d\,!$, and so the coefficient of Q in $\theta(f - g)$ is at most $2^{-R} \cdot d\,! \cdot \|f - g\|_1 \leq 2^{-R-1}$. Thus $\theta(f)$ has a coefficient of at least 2^{-R-1} in Q, so that $\theta(f) \neq 0$, contrary to hypothesis.

Therefore θ is injective, as required. ∎

11. Homomorphisms into $\mathfrak{F}$. At one stage, it was conjectured that every homomorphism from a Banach algebra into $\mathfrak{F}$ would be automatically continuous. This was proved to be false by a construction of Dales and McClure [8]; for an improved version of this construction, see [6, Theorem 5.5.19].

THEOREM 11.1. *There is a commutative, unital Banach algebra A which has a totally discontinuous higher point derivation at a character of A, and such that this higher point derivation defines a discontinuous epimorphism from A onto $\mathfrak{F}$.* ∎

It is noted in [8] that the algebra A of the above theorem can be taken to be a uniform algebra or a regular Banach function algebra.

The authors of [8] also asked (somewhat casually) if *every* discontinuous homomorphism from a Banach algebra into $\mathfrak{F}$ had to be an epimorphism. This question was discussed in [24]. We shall now prove that this is indeed the case; in fact, we establish a stronger form of this conjecture.

THEOREM 11.2. *Let A be an (F)-algebra, and let $(d_n : n \in \mathbb{Z}^+)$ be a non-degenerate, discontinuous higher point derivation on A. Then the map*

$$\theta : a \mapsto \sum_{n=0}^{\infty} d_n(a) X^n, \quad A \to \mathfrak{F},$$

is an epimorphism.

Proof. The topology of A is given by a complete, translation-invariant metric, say ρ.

We first note that, if d_0 is discontinuous, then so is d_1. Indeed, take $(a_n)_{n\geq 1}$ to be a null sequence in A with $d_0(a_n) = 1$ $(n \in \mathbb{N})$, and choose $b \in A$ with $d_0(b) = 0$ and $d_1(b) = 1$. Then $a_n b \to 0$ in A and $d_1(a_n b) = 1$ $(n \in \mathbb{N})$, and so d_1 is discontinuous.

We define k to be the minimum value of $n \in \mathbb{N}$ such that d_n is discontinuous; such a value of k exists.

By Proposition 3.3, there are $b_0, \dots, b_k \in A$ such that

$$d_i(b_j) = \delta_{i,j} \quad (i, j = 0, \dots, k);$$

we fix these elements $b_0, \dots, b_k$.

We first *claim* that there is a null sequence $(a_n)_{n\geq 1}$ in A such that, for each $n \in \mathbb{N}$, we have

$$d_j(a_n) = 0 \quad (j = 0, \dots, k-1) \quad \text{and} \quad d_k(a_n) = 1. \tag{11.1}$$

Indeed, if d_0 is discontinuous, so that $k = 1$, the above sequence $(a_n b)_{n\geq 1}$ satisfies the requirement. Now suppose that d_0 is continuous. Then there is a null sequence $(c_n)_{n\geq 1}$ in A with $d_k(c_n) = 1$ $(n \in \mathbb{N})$. Set

$$a_n = c_n - \sum_{i=0}^{k-1} d_i(c_n) b_i \quad (n \in \mathbb{N}).$$

Since $d_0, \dots, d_{k-1}$ are continuous, $(a_n)_{n\geq 1}$ is also a null sequence. Also, equation (11.1) holds. This gives the claim.

Now consider a fixed sequence $(\alpha_n : n \in \mathbb{Z}^+)$; we shall seek an element $c \in A$ such that

$$\theta(c) = \sum_{n=0}^{\infty} \alpha_n X^n.$$

The element c will be $\lim_{i\to\infty} c_i$, where the sequence $(c_i : i \geq k-1)$ is defined inductively as follows. First, we set

$$c_{k-1} = \sum_{j=0}^{k-1} \alpha_j b_j.$$

Next, fix $i \geq k$, and assume inductively that $c_{k-1}, \dots, c_{i-1}$ have been specified. Then we set

$$c_i = c_{i-1} + \beta_i b_1^{i-k} a_{m_i},$$

where $\beta_i = \alpha_i - d_i(c_{i-1})$ and $m_i \in \mathbb{N}$ is chosen so that, for each $\ell = k, \dots, i$, we have

$$\rho(\beta_i b_1^{i-\ell} a_{m_i}, 0) = \rho\Big(\sum_{j=\ell}^{i-1} \beta_j b_1^{j-\ell} a_{m_j}, \sum_{j=\ell}^{i} \beta_j b_1^{j-\ell} a_{m_j}\Big) \leq \frac{1}{2^i}. \tag{11.2}$$

The latter condition can be satisfied because $a_n \to 0$ as $n \to \infty$, since the product in A is continuous, and since the metric is translation-invariant. This completes the inductive definition of the sequence $(c_i : i \geq k-1)$.

We note that

$$d_j(c_i) = d_j(c_{i-1}) \quad (j = k, \dots, i)$$

and that the choice of the elements c_i is such that

$$d_j(c_i) = \alpha_j \quad (j = 0, \ldots, i).$$

Thus the limit $\lim_{i\to\infty} \theta(c_i)$ exists, and is equal to $\sum_{n=0}^{\infty} \alpha_n X^n$. On the other hand, it is clear from equation (11.2) that the series

$$\beta_k a_{m_k} + \beta_{k+1} b_1 a_{m_{k+1}} + \beta_{k+2} b_1^2 a_{m_{k+2}} + \cdots$$

converges in A, and so $\lim_{i\to\infty} c_i$ exists in A, say $\lim_{i\to\infty} c_i = c$. We now *claim* that $\theta(c) = \lim_{i\to\infty} \theta(c_i)$, which will complete the proof.

To establish this claim, it suffices to show that, for each $n \in \mathbb{N}$ and each $\ell \geq k+n-1$, the difference

$$\theta(c) - \theta(c_\ell)$$

belongs to M^n, where $M = X\mathfrak{F}$ is the maximal ideal of $\mathfrak{F}$. However,

$$\begin{aligned} c - c_\ell &= \beta_{\ell+1} b_1^{\ell+1-k} a_{m_{\ell+1}} + \beta_{\ell+2} b_1^{\ell+2-k} a_{m_{\ell+2}} + \beta_{\ell+3} b_1^{\ell+3-k} a_{m_{\ell+2}} + \cdots \\ &= b_1^{\ell+1-k} \left(\beta_{\ell+1} a_{m_{\ell+1}} + \beta_{\ell+2} b_1 a_{m_{\ell+2}} + \beta_{\ell+3} b_1 a_{m_{\ell+3}} + \cdots\right), \end{aligned}$$

and the inner sum converges by equation (11.2). Thus $c - c_\ell \in b_1^{\ell+1-k} A \subset b_1^n A$. This implies that $\theta(c) - \theta(c_\ell) \in \theta(b_1)^n \mathfrak{F} \subset M^n$, as required for the claim.

This concludes the proof of the theorem. ∎

The first corollary shows that the time-honoured definition of a Banach algebra of power series contains a redundant clause.

Corollary 11.3. *Let A be a subalgebra of $\mathfrak{F}$ containing $\mathbb{C}[X]$ such that $(A, \|\cdot\|)$ is a Banach algebra with respect to some norm. Then $(A, \|\cdot\|)$ is a Banach algebra of power series.*

Proof. We must show that the embedding of $(A, \|\cdot\|)$ into $(\mathfrak{F}, \tau_c)$ is continuous.

Assume that the embedding is discontinuous. Then, by the theorem, $A = \mathfrak{F}$. By Theorem 7.4, $\mathfrak{F}$ has a unique (F)-algebra topology, and so $(\mathfrak{F}, \|\cdot\|)$ is a Banach algebra, a contradiction of Theorem 7.4. ∎

Essentially the same argument shows the following.

Corollary 11.4. *Let A be a subalgebra of $\mathfrak{F}$ containing $\mathbb{C}[X]$ such that (A, τ) is an (F)-algebra (respectively, a Fréchet algebra) with respect to some topology τ. Then (A, τ) is an (F)-algebra (respectively, a Fréchet algebra) of power series.* ∎

We do not know whether or not a Fréchet algebra of power series is functionally continuous. However we can state the following (rather trivial) immediate consequence of Corollary 11.4.

Corollary 11.5. *Let A be a subalgebra of $\mathfrak{F}$ containing $\mathbb{C}[X]$ such that (A, τ) is an (F)-algebra with respect to some topology τ. Then the character $\pi_0 : A \to \mathbb{C}$ is continuous.* ∎

Corollary 11.6. *There is no topology τ on $\mathbb{C}\{X\}$ such that $(\mathbb{C}\{X\}, \tau)$ is an (F)-algebra.*

Proof. Assume towards a contradiction that there is such a topology. Then, by Corollary 11.4, $(\mathbb{C}\{X\}, \tau)$ is an (F)-algebra of power series. But this is a contradiction of Theorem 8.2. ∎

The next corollary generalizes [6, Theorem 4.6.1] and [24, Corollary 4.2].

COROLLARY 11.7. *Let (A, τ) be an (F)-algebra of power series. Then A has a unique (F)-algebra topology.*

Proof. Let (A, σ) be an (F)-algebra for a topology σ. By Corollary 11.4, (A, σ) is an (F)-algebra of power series. Let $(a_n)_{n\geq 1}$ be a sequence in A such that $a_n \to 0$ in (A, τ) and $a_n \to a$ in (A, σ). For each $k \in \mathbb{N}$, the functional π_k is continuous on both (A, τ) and (A, σ), and so $\pi_k(a) = 0$, whence $a = 0$. By the closed graph theorem for (F)-spaces, the embedding $\iota : (A, \tau) \to (A, \sigma)$ is a linear homeomorphism, and so $\sigma = \tau$. ∎

We now note that the above results lead to a different proof of Theorem 2.6, which we restate in the form below.

THEOREM 11.8. *There is no embedding of $\mathfrak{F}_2$ into $\mathfrak{F}$.*

Proof. Assume towards a contradiction that $\theta : \mathfrak{F}_2 \to \mathfrak{F}$ is an embedding. Then θ is not a surjection, for this would imply that $\mathfrak{F}_2 \cong \mathfrak{F}$, and this is impossible, for example because $\mathfrak{F}_2$ has many prime ideals, but $\mathfrak{F}$ has only two prime ideals. Thus, by Theorem 11.2, the embedding $\theta : \mathfrak{F}_2 \to \mathfrak{F}$ is continuous, and so we may regard $A := \theta(\mathfrak{F}_2)$ as a Fréchet subalgebra of $\mathfrak{F}$.

By [24, Theorem 3.3], the topology of A is given by a countable family of norms (not just seminorms), say by the sequence $(\|\cdot\|_n)_{n\geq 1}$. By Theorem 7.6, A has a unique Fréchet algebra topology, and so the topology given by the sequence $(\|\cdot\|_n)_{n\geq 1}$ on A is equivalent to the usual topology τ_c, given by the sequence $(p_n)_{n\geq 1}$ of seminorms. In particular, there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\|f\|_1 \leq C p_n(f) \quad (f \in \mathfrak{F}_2).$$

But now $\left\|X^{n+1}\right\|_1 \leq C p_n(X^{n+1}) = 0$, a contradiction of the fact that $\|\cdot\|_1$ is a norm on $\mathfrak{F}_2$.

Thus there is no such embedding $\theta : \mathfrak{F}_2 \to \mathfrak{F}$. ∎

12. Homomorphisms into $\mathfrak{F}_n$. Throughout this section, we fix $n \geq 2$ in $\mathbb{N}$. Our first query is to seek an analogous result to Corollaries 11.3 and 11.4 when $\mathfrak{F}$ is replaced by $\mathfrak{F}_n$. Indeed, the natural guess is that the following holds.

'Let A be a subalgebra of $\mathfrak{F}_n$ containing $\mathbb{C}[X_1, \ldots, X_n]$ such that A is an (F)-algebra with respect to some topology τ. Then A is an (F)-algebra of power series in n variables.'

In fact, this is not true, as we shall show soon. However, we can prove a considerably weaker positive result. A major hurdle that arises when we replace $\mathfrak{F}$ by $\mathfrak{F}_n$ is that non-zero (necessarily closed) ideals in $\mathfrak{F}_n$ are not necessarily of finite codimension. The version of the above that we shall prove is the following. We recall that the separating space $\mathfrak{S}(\theta)$ of a homomorphism θ was defined in §6.

THEOREM 12.1. *Let $n \in \mathbb{N}$, let A be an (F)-algebra, and let $\theta : A \to \mathfrak{F}_n$ be a homomorphism such that $\theta(A)$ is dense in $(\mathfrak{F}_n, \tau_c)$. Assume that $\mathfrak{S}(\theta)$ has finite codimension in $\mathfrak{F}_n$. Then θ is a surjection.*

We note that, in the case where $n = 1$, every non-zero ideal in $\mathfrak{F}$ has finite codimension in $\mathfrak{F}$, and so the above result subsumes Theorem 11.2.

We shall first give a lemma; we maintain the notation of the theorem. The space of all linear functionals on $\mathfrak{F}_n$ is denoted by $\mathfrak{F}_n^*$, with the duality specified by the pairing $\langle \cdot, \cdot \rangle$, and the annihilator in $\mathfrak{F}_n^*$ of a subspace E of $\mathfrak{F}_n$ is denoted by $E^\perp$.

LEMMA 12.2. *Let I be a proper ideal of finite codimension in $\mathfrak{F}_n$, and let $f \in \mathfrak{F}_n$. Then there exists $a \in A$ such that $\theta(a) \in f + I$.*

Suppose, further, that $f \in \mathfrak{S}(\theta)$. Then there is a null sequence $(a_k)_{k\geq 1}$ in A such that $\theta(a_k) \in f + I$ $(k \in \mathbb{N})$ and $\theta(a_k) \to f$ in $(\mathfrak{F}_n, \tau_c)$ as $k \to \infty$.

Proof. Let $\pi : \mathfrak{F}_n \to \mathfrak{F}_n/I$ be the quotient map. Since $\theta(A)$ is dense in $\mathfrak{F}_n$, it is clear that $(\pi \circ \theta)(A)$ is a dense linear subspace of the finite-dimensional space $\mathfrak{F}_n/I$, and so necessarily $(\pi \circ \theta)(A) = \mathfrak{F}_n/I$. This gives the first part of the lemma.

The space $I^\perp$ is finite-dimensional, with basis $\{\lambda_1, \dots, \lambda_m\}$, say, and there exist $f_1, \dots, f_m \in \mathfrak{F}_n$ such that $\langle f_i, \lambda_j \rangle = \delta_{i,j}$ $(i, j = 1, \dots, m)$. By the first clause, there exist $x_1, \dots, x_m \in A$ with $\theta(x_i) \in f_i + I$ $(i = 1, \dots, m)$.

Now take a null sequence $(b_k)_{k\geq 1}$ in A such that $\theta(b_k) \to f$, and define

$$a_k = b_k + \sum_{i=1}^{m} \langle f - \theta(b_k), \lambda_i \rangle x_i \quad (k \in \mathbb{N}).$$

Then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$ and $\lim_{k\to\infty} \theta(a_k) = \lim_{k\to\infty} \theta(b_k) = f$. Take $k \in \mathbb{N}$. Then

$$\langle \theta(a_k), \lambda_j \rangle = \langle \theta(b_k), \lambda_j \rangle + \sum_{i=1}^{m} \langle f - \theta(b_k), \lambda_i \rangle \delta_{i,j} = \langle f, \lambda_j \rangle \quad (j = 1, \dots, m),$$

and so $\theta(a_k) \in f + I$. ∎

We shall now give our proof of Theorem 12.1. In the proof we shall write M for $\mathfrak{M}_n$, the maximal ideal of $\mathfrak{F}_n$. Also, we take $f_1, \dots, f_p \in \mathfrak{S}$ to be the generators of the ideal $\mathfrak{S} := \mathfrak{S}(\theta)$, so that

$$\mathfrak{S} = f_1 \mathfrak{F}_n + \cdots + f_p \mathfrak{F}_n.$$

As before, the topology of A is given by a complete, translation-invariant metric, say ρ; for $\eta > 0$, we set $A_{[\eta]} = \{a \in A : \rho(a, 0) < \eta\}$.

Proof of Theorem 12.1. Let $f \in \mathfrak{F}_n$ be fixed; we are seeking an element $a \in A$ with $\theta(a) = f$.

Since M^2 is a proper ideal of finite codimension in $\mathfrak{F}_n$, it follows from Lemma 12.2 that there exist $x_1, \dots, x_m \in A$ such that $\theta(x_i) \in X_i + M^2$ $(i = 1, \dots, m)$. Set

$$N = \max\{\rho(x_1, 0), \dots, \rho(x_m, 0)\}.$$

For $k \in \mathbb{Z}^+$, take L_k to be the number of monomials (in n variables) of degree k, and choose $\varepsilon_k > 0$ such that

$$L_k \cdot N^{2k} \cdot \varepsilon_k < \frac{1}{(k+1)^2}. \tag{12.1}$$

We *claim* that there is a sequence $(a_k)_{k\geq 1}$ in A such that, for each $k \in \mathbb{N}$, we have

$$\theta(a_k) - f \in M^k \mathfrak{S} = \Big(\sum_{|r|=k} X^r \mathfrak{F}_n \Big) \mathfrak{S} \tag{12.2}$$

and

$$a_{k+1} - a_k = \sum_{|r|=k} x^r \, b_{k,r}, \tag{12.3}$$

where $b_{k,r} \in A_{[\varepsilon_k]}$ for $r \in (\mathbb{Z}^+)^n$ with $|r| = k$. (Here we write $x^r = x_1^{r_1} \cdots x_n^{r_n} \in A$ when $r = (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n$.)

Since $M\,\mathfrak{G} = f_1 M + \cdots + f_p M$ is an ideal of finite codimension in $\mathfrak{F}_n$, it follows from Lemma 12.2 that there exists $a_1 \in A$ with $\theta(a_1) - f \in M\,\mathfrak{G}$.

We can write

$$\theta(a_1) - f = \sum_{j=1}^{p} \sum_{i=1}^{n} X_i f_j v_{i,j},$$

where $v_{i,j} \in \mathfrak{F}_n$ for $i = 1, \ldots, n$ and $j = 1, \ldots, p$. It follows from Lemma 12.2 that, for each $i = 1, \ldots, n$ and $j = 1, \ldots, p$, there exists $b_{i,j} \in A$ such that $\rho(b_{i,j}, 0) < \varepsilon_1/p$ and $\theta(b_{i,j}) \in f_j v_{i,j} + M^2\,\mathfrak{G}$.

Now define

$$a_2 = a_1 + \sum_{j=1}^{p} \sum_{i=1}^{n} b_{i,j} x_i = a_1 + \sum_{i=1}^{n} c_i x_i,$$

say, where $c_1, \ldots, c_n \in A_{[1]}$. Thus we have (12.3) in the case where $k = 1$. Also

$$\begin{aligned}
\theta(a_2) - f &= \theta(a_1) - f - \sum_{j=1}^{p} \sum_{i=1}^{n} \theta(b_{i,j})\theta(x_i) \\
&= \sum_{j=1}^{p} \sum_{i=1}^{n} (X_i f_j v_{i,j} - \theta(b_{i,j})\theta(x_i)) \\
&= \sum_{j=1}^{p} \sum_{i=1}^{n} (f_j(X_i - \theta(x_i))v_{i,j} + \theta(x_i)(f_j v_{i,j} - \theta(b_{i,j}))) \in M^2\,\mathfrak{G}
\end{aligned}$$

because $f_j \in \mathfrak{G}$, $X_i - \theta(x_i) \in M^2$, and $f_j v_{i,j} - \theta(b_{i,j}) \in M^2\,\mathfrak{G}$. Thus we have (12.2) in the case where $k = 2$.

Assume inductively that we have chosen $a_k \in A$ such that (12.2) holds, say

$$\theta(a_k) - f = \sum_{j=1}^{p} \sum_{|r|=k} X^r f_j v_{r,j},$$

where $v_{r,j} \in \mathfrak{F}_n$ for $|r| = k$ and $j = 1, \ldots, p$. Then, for each r and j, there exists $b_{r,j} \in A$ such that $\rho(b_{r,j}, 0) < \varepsilon_k/p$ and $\theta(b_{r,j}) \in f_j v_{r,j} + M^{k+1}\,\mathfrak{G}$. Now define

$$a_{k+1} = a_k + \sum_{j=1}^{p} \sum_{|r|=k} b_{r,j} x^r = a_k + \sum_{|r|=k} c_r x^r,$$

say, where $c_1, \ldots, c_n \in A_{[\varepsilon_k]}$. Then we have (12.3) for k.

Essentially the same calculation as above gives (12.2) for $k + 1$: we use the facts that $X^r - \theta(x^r) \in M^{k+1}$ and $f_j v_{r,j} - \theta(b_{r,j}) \in M^{k+1}\,\mathfrak{G}$ when $|r| = k$.

This completes the inductive step in the proof of the claim. By induction we obtain the required sequence $(a_k)_{k\geq 1}$ in A.

It follows from equations (12.1) and (12.3) that the sequence $(a_k)_{k\geq 1}$ converges in A, say $a = \lim_{k\to\infty} a_k$. We shall prove that $\theta(a) = f$; for this, it is sufficient to show that

$$\theta(a) - f \in M^R \quad \text{for each} \quad R \in \mathbb{N}. \tag{12.4}$$

Fix $R \in \mathbb{N}$, and take $k \geq R$. From (12.3), we can write $a_{k+1} - a_k$ as

$$\sum_{|s|=k} x^s d_{k,s},$$

where

$$\rho(d_{k,s}, 0) \leq N^{k-R} \cdot \sum_r \rho(b_{k,r}, 0) \leq L_k \cdot N^{2k-R} \cdot \varepsilon_k < \frac{1}{(k+1)^2}$$

for each $s \in (\mathbb{Z}^+)^n$. It follows that $d_s := \sum_{k=R}^{\infty} d_{k,s}$ exists in A for each $s \in (\mathbb{Z}^+)^n$, and that

$$a - a_R = \sum_{|s|=R} x^s d_s.$$

Thus

$$\theta(a) - \theta(a_R) = \sum_{|s|=R} \theta(x^s d_s) \in M^R.$$

But also $\theta(a_R) - f \in M^R$, and so (12.4) follows.

This completes the proof of Theorem 12.1. ■

We shall now show that the obvious analogue for $\mathfrak{F}_2$ of Corollary 11.3 is false.

THEOREM 12.3. *There exists a Banach algebra $(A, \|\cdot\|)$ such that $\mathbb{C}[X_1, X_2] \subset A \subset \mathfrak{F}_2$, but such that the embedding $(A, \|\cdot\|) \to (\mathfrak{F}_2, \tau_c)$ is not continuous.*

Proof. We set $S = (\mathbb{Z}^+)^{<\omega}$ and $A = \ell^1(S)$, as in Theorem 10.1. In fact, it is convenient to write $\mathfrak{F}_2$ as $\mathbb{C}[[X, Y]]$ and to reserve X_i for elements of A, as before. We regard $\mathfrak{F} = \mathbb{C}[[X]]$ as a subalgebra of $\mathbb{C}[[X, Y]]$; the obvious quotient map from $\mathbb{C}[[X, Y]]$ obtained by setting $Y = 0$ is denoted by

$$\pi : \sum_{i,j=0}^{\infty} \alpha_{i,j} X^i Y^j \mapsto \sum_{i=0}^{\infty} \alpha_{i,0} X^i, \quad \mathfrak{F}_2 \to \mathfrak{F}.$$

By Theorem 10.1, there is a continuous, unital embedding $\theta : A \to \mathfrak{F}$ such that $\theta(A) \supset \mathbb{C}[X]$. We set $f_i = \theta(X_i)$ $(i \in \mathbb{N})$. As in Theorem 10.1, we may suppose that $f_1 = X \in \mathfrak{F}$.

As before, we denote by $A^{(1)}$ the closed linear subspace of A spanned by the elements X_i for $i \in \mathbb{N}$, so that

$$A^{(1)} = \left\{ \sum_{i=1}^{\infty} \alpha_i X_i : \sum_{i=1}^{\infty} |\alpha_i| < \infty \right\},$$

and $A^{(1)}$ is isometrically isomorphic to ℓ^1. Choose a non-zero linear functional λ on $A^{(1)}$ such that $\lambda(X_i) = 0$ $(i \in \mathbb{N})$ and

$$\lambda\left(\sum_{i=2}^{\infty} \frac{1}{i^2} X_i \right) = 1,$$

so that λ is discontinuous, and then define a linear map

$$\psi : u \mapsto \theta(u) + \lambda(u)Y, \quad A^{(1)} \to \mathfrak{F}_2.$$

Our main *claim* is that ψ can be extended to a homomorphism $\Psi : A \to \mathfrak{F}_2$ such that $\pi \circ \Psi = \theta$. To establish this claim, we shall prove the following slightly more general theorem, in which we maintain the above notation. Further, we again write $\mathfrak{M}_2$ for the unique maximal ideal of $\mathfrak{F}_2$.

THEOREM 12.4. *Let $\beta : A^{(1)} \to \mathfrak{M}_2$ be a linear map such that $\pi \circ \beta : A^{(1)} \to \mathfrak{F}$ is continuous. Then there is a unital homomorphism $\overline{\beta} : A \to \mathfrak{F}_2$, extending β, such that $\pi \circ \overline{\beta} : A \to \mathfrak{F}$ is continuous.*

Proof. For each $i, j \in \mathbb{Z}^+$, there is a linear functional $\beta_{(i,j)} : A^{(1)} \to \mathbb{C}$ such that

$$\beta(f) = \sum\{\beta_{(i,j)}(f)X^iY^j : i, j \in \mathbb{Z}^+\} \quad (f \in A^{(1)}). \tag{12.5}$$

Note that $\beta_{(0,0)} = 0$ because the range of β on $A^{(1)}$ is contained in $\mathfrak{M}_2$. We extend each linear functional $\beta_{(i,j)}$ to a linear functional $\beta_{(i,j)} : \mathfrak{F}_\infty^{(1)} \to \mathbb{C}$.

Next, we define a linear functional $\beta^{(n)}_{(i,j)}$ on $\mathfrak{F}_\infty^{(n)}$ for each $n \in \mathbb{N}$ by the following formula:

$$\beta^{(n)}_{(i,j)}(f) = \sum\{(\beta_{(i^{(1)},j^{(1)})} \otimes \cdots \otimes \beta_{(i^{(n)},j^{(n)})})(\varepsilon_n(f))\} \quad (f \in \mathfrak{F}_\infty^{(n)}), \tag{12.6}$$

where the sum is taken over all n-tuples $((i^{(1)}, j^{(1)}), \dots, (i^{(n)}, j^{(n)})) \in ((\mathbb{Z}^+)^2)^n$ such that $(i^{(1)}, j^{(1)}) + \cdots + (i^{(n)}, j^{(n)}) = (i, j)$.

We now *claim* that the map $\overline{\beta} : A \to \mathfrak{F}_2$, defined for $f \in \mathfrak{F}_\infty$ by the formula

$$\overline{\beta}(f) = \sum_{k=0}^{\infty} \left\{ \left(\sum \left\{ \beta^{(n)}_{(i,j)}(f^{(n)}) : n \in \mathbb{N}_{i+j} \right\} \right) X^iY^j : i, j \in \mathbb{Z}^+,\, i + j = k \right\}, \tag{12.7}$$

where we set $\overline{\beta}^{(0)}(f) = f(0,0)1$, is a unital homomorphism $\overline{\beta} : A \to \mathfrak{F}_2$ satisfying the stated conditions.

First, we shall show that the map $\overline{\beta}$ is a homomorphism. The map $\overline{\beta}$ satisfies the equation

$$\overline{\beta}(f) = \sum_{n=1}^{\infty} \overline{\beta}(f^{(n)}) \quad (f \in A). \tag{12.8}$$

Thus, to prove that $\overline{\beta}(fg) = \overline{\beta}(f)\overline{\beta}(g)$ for all $f, g \in A$, it suffices to do this in the special case where $f = f^{(r)}$ and $g = g^{(n-r)}$ for some $n \in \mathbb{N}$ and $r \in \{0, \dots, n\}$. The result in this case is immediate if $r = 0$ or $r = n$, and so we may suppose that $n \geq 2$ and that $0 < r < n$. Further inspection shows that it is sufficient to show that

$$\overline{\beta}^{(n)}_{(i,j)}(fg) = \sum\{\overline{\beta}^{(r)}_{(i_1,j_1)}(f)\overline{\beta}^{(n-r)}_{(i_2,j_2)}(g) : (i_1, j_1) + (i_2, j_2) = (i, j)\}$$

whenever $i + j \geq n$.

By the definition in (12.6), we must verify that

$$\sum\{(\beta_{(i^{(1)},j^{(1)})} \otimes \cdots \otimes \beta_{(i^{(n)},j^{(n)})})(\varepsilon_n(fg))\}$$

is equal to the product

$$\sum\{(\beta_{(i^{(1)},j^{(1)})}\otimes\cdots\otimes\beta_{(i^{(r)},j^{(r)})})(\varepsilon_r(f))\}\sum\{(\beta_{(i^{(1)},j^{(1)})}\otimes\cdots\otimes\beta_{(i^{(n-r)},j^{(n-r)})})(\varepsilon_{n-r}(g))\},$$

where the sums are taken over all n-tuples $((i^{(1)},j^{(1)}),\dots,(i^{(n)},j^{(n)}))\in((\mathbb{Z}^+)^2)^n$ such that $(i^{(1)},j^{(1)})+\cdots+(i^{(n)},j^{(n)})=(i,j)$. However, this follows from Proposition 9.5, Lemma 9.6, and Lemma 9.7, taking $a=\varepsilon_r(f)$ and $b=\varepsilon_{n-r}(g)$ in Proposition 9.5.

Thus $\overline{\beta}$ is a homomorphism. Clearly $\overline{\beta}$ is unital.

We next show that $\overline{\beta}$ extends β. Suppose that $f\in A^{(1)}$. Then equation (12.7) becomes

$$\overline{\beta}(f)=\sum_{k=0}^{\infty}\{\beta^{(1)}_{(i,j)}(f)X^iY^j : i+j=k\};$$

by (12.5), the right-hand side is just $\beta(f)$, as required.

Finally, we claim that $\pi\circ\overline{\beta}:A\to\mathfrak{F}$ is continuous. Evidently $\pi\circ\overline{\beta}$ maps $A^{(r)}$ into $\mathfrak{M}^r$ for each $r\in\mathbb{Z}^+$, and so it is enough to show that $(\pi\circ\overline{\beta})\mid A^{(r)}$ is continuous for each $r\in\mathbb{Z}^+$. From equation (12.7), we see that

$$(\pi\circ\overline{\beta})(f)=\sum_{i=r}^{\infty}\beta^{(r)}_{(i,0)}(f)X^i\quad(f\in A^{(r)}),$$

and so it is enough to show that each $\beta^{(r)}_{(i,0)}$ is continuous for $i\geq r$ and $r\in\mathbb{Z}^+$. But the fact that $\pi\circ\beta$ is continuous implies that the linear functionals $\beta_{(i,0)}$ are all continuous. Further, the 'tensor product by rows' agrees with the usual tensor product when the linear functionals are continuous, and so $\beta^{(r)}_{(i,0)}$, being a finite sum of r-fold tensor products of continuous linear functionals of the form $\beta_{(j,0)}$, is indeed continuous for each $i\geq r$ and $r\in\mathbb{Z}^+$, and so $\pi\circ\overline{\beta}$ is continuous. ∎

We can now complete the proof of Theorem 12.3.

The above theorem shows that ψ can be extended to a homomorphism $\Psi:A\to\mathfrak{F}_2$ such that $\pi\circ\Psi=\theta$. The map Ψ is an embedding because θ is injective, and it is manifest that Ψ is discontinuous. It is clear that $\Psi(A)$ contains $\mathfrak{F}$ and the element $\Psi(\sum_{i=2}^{\infty}X_i/i^2)$, which, by a remark in the proof of Theorem 10.1, has the form X^2f+Y for some $f\in\mathfrak{F}$. By Lemma 1.2, there is a continuous, unital automorphism χ of $\mathfrak{F}_2$ such that $\chi(X)=X$ and $\chi(X^2f+Y)=Y$, and so $\chi\circ\Psi:A\to\mathfrak{F}_2$ is a discontinuous embedding whose range contains $\mathbb{C}[X,Y]$, as required for the proof of the theorem (where we identify A with its image in $\mathfrak{F}_2$ under $\chi\circ\Psi$). ∎

Corollary 12.5. *There is a discontinuous embedding of the semigroup algebra $\ell^1(S_{\mathfrak{c}})$ into $\mathfrak{F}_2$ such that the range contains $\mathbb{C}[X_1,X_2]$.*

Proof. This follows easily from Theorem 10.5 and the above proof. ∎

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