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NONISOTHERMAL SYSTEMS OF SELF-ATTRACTING FERMI–DIRAC PARTICLES

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Abstract. The existence of stationary solutions and blow up of solutions for a system describing the interaction of gravitationally attracting particles that obey the Fermi–Dirac statistics are studied.

1. Introduction.

Parabolic-elliptic systems of the form

$$n_t = \nabla \cdot \left(D_* \left(\nabla p + n \nabla \varphi\right)\right), \tag{1}$$

$$\Delta \varphi = n, \tag{2}$$

appear in the statistical mechanics of self-attracting particles. Here $n = n(x,t) \geq 0$ is the density function defined for $(x,t) \in \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^d$, $\varphi = \varphi(x,t)$ is the Newtonian potential generated by the particles of density n, and the pressure $p \geq 0$ is determined by the density–pressure relation with a sufficiently regular function p

$$p = p(n, \vartheta). \tag{3}$$

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The parameter $\vartheta > 0$ plays the role of the temperature, and $D_* > 0$ is a diffusion coefficient which may depend on n, ϑ, x, t, ... Such systems can be studied either in the *canonical ensemble* (i.e. the *isothermal* setting), when $\vartheta =$ const is fixed, or in the *microcanonical ensemble* with a variable temperature: either $\vartheta = \vartheta(t)$ or $\vartheta = \vartheta(x,t)$. Then, the energy balance is described by a relation like

$$E = c_0 \int_\Omega p\,dx + \frac{1}{2}\int_\Omega n\varphi\,dx = \text{const} \tag{4}$$

(which defines ϑ in an implicit way) in the former case, or by an evolution partial differential equation for ϑ as was in [21, 22, 5] in the latter case. The term $c_0 \int_\Omega p\,dx$ with a fixed $c_0 > 0$ in (4) is interpreted as the kinetic energy of particles while $\frac{1}{2}\int_\Omega n\varphi\,dx$ corresponds to the potential energy.

Thus, (1)–(2) can be viewed as equations for evolution of self-consistent gravitational field. In fact, the starting point of the derivation of such a *mean field* model in astrophysics in [16] is an analysis of kinetic equations whose evolution in time is governed by the Maximum Entropy Production Principle. A short description of this procedure is also given in [6].

The examples of the density–pressure relations (3), studied by P.-H. Chavanis and his collaborators in a series of papers including [10, 11, 12, 13, 14, 15, 16], are

$$p = \vartheta n, \tag{5}$$

which is the Boltzmann relation corresponding to the classical Brownian diffusion of particles. They also considered

$$p = \kappa n^{1+\gamma}, \tag{6}$$

a polytropic equation of state of a gas, and the Fermi–Dirac density–pressure relation

$$p = \frac{\mu}{d}\vartheta^{d/2+1}(I_{d/2} \circ I^{-1}_{d/2-1})(2n(\mu\vartheta^{d/2})^{-1}), \tag{7}$$

which is in a sense intermediate between (5) and (6) with $\gamma = 2/d$, and (4) with $c_0 = d/2$. The symbol I_α in the equation (7) above denotes the Fermi integral of order $\alpha > -1$

$$I_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha\,dy}{1+\lambda e^y} \tag{8}$$

defined for all $\lambda > 0$, and $\mu > 0$ in (7) is a parameter of physical origin, see [6]. In fact, the density and the pressure are defined via a parameter $\lambda > 0$, called the *fugacity*

$$n = \frac{\mu}{2}\vartheta^{d/2} I_{d/2-1}(\lambda), \tag{9}$$

$$p = \frac{\mu}{d}\vartheta^{d/2+1} I_{d/2}(\lambda). \tag{10}$$

Since we will be mainly interested in the Fermi–Dirac model, properties of the Fermi functions collected in Section 5 are of importance.

We consider the system (1)–(2) with the natural no-flux boundary condition on $\partial\Omega$

$$(\nabla p + n\nabla\varphi)\cdot\overline{\nu} = 0 \tag{11}$$

($\overline{\nu}$ is the unit exterior normal vector to $\partial\Omega$), and an initial condition

$$n(x,0) = n_0(x). \tag{12}$$

We impose on the potential φ either the physically acceptable "free" condition

$$\varphi = E_d * n, \tag{13}$$

E_d being the fundamental solution of the Laplacian in $\mathbb{R}^d$, or the homogeneous Dirichlet boundary condition

$$\varphi_{|\partial\Omega} = 0, \tag{14}$$

which is mathematically somewhat simpler. In the case of radially symmetric solutions (14) is equivalent to (13) by adding a constant to the potential φ, cf. also the discussion of this issue in [7, 8, 2, 4].

As a consequence of (11), the total mass of particles

$$M = \int_\Omega n(x,t)\,dx \tag{15}$$

is conserved during the evolution. Moreover, sufficiently regular solutions of the evolution problem with $n_0 \geq 0$ are positive, cf., e.g. [6].

The main mathematical questions concerning these systems are:

- existence, nonexistence and multiplicity of steady states, either for given M, ϑ or for M, E fixed,
- local in time existence of solutions of the evolution problem,
- asymptotics of global in time solutions,
- possibility of finite time blow up of solutions (corresponding either to a gravitational collapse or an explosion).

The model of self-gravitating Brownian particles, which consists of (1)–(2), (5) supplemented by (4) with $c_0 = d/2$, has been considered in [16, 13] for radially symmetric solutions (n, φ), and in [17, 9] without those symmetry assumptions. Studies of the corresponding isothermal problem with $\vartheta \equiv 1$ had been conducted earlier, see e.g. [7, 4, 1]. However, the motivations had been a bit different—stemming from statistical mechanics of interacting charged particles in semiconductors, electrolytes, plasmas (see also [3] for different density–pressure relations) with (2) replaced by $\Delta\varphi = -n$, and afterwards for gravitating particles.

We refer the reader to [13, 17, 9, 2, 8] for the Brownian particles models. The main issues are:

- gravitational collapse is possible for $d \geq 2$ in the isothermal model ([7, 1]), and for $d \geq 3$ in the nonisothermal model ([9]),
- the existence of steady states with prescribed mass and energy in $d \geq 3$ dimensions is controlled by the parameter E/M^2 which should be large enough ([2, 8]).

The Fermi–Dirac model involves nonlinear diffusion, and thus even local in time existence of solutions is much harder to establish than in the Brownian (linear diffusion) case, see [6] where a specific choice of the diffusion coefficient $D_* = D$ in (19) has been considered in the isothermal case. There are also many results on radially symmetric stationary solutions in [10, 11, 12]. In particular:

- gravitational collapse cannot occur in $d \leq 3$ dimensions ([6]),

- structure of the set of steady states with given M and ϑ is different (and less complicated) than in the Brownian case ([11]).

It is worth noting that while "local" results on the existence of steady states (i.e. for a small range of control parameters M, ϑ, E) are quite similar for general $p = p(n, \vartheta)$, the global structure of the set of steady states is rather sensitive to variations of the form of p in (3).

The main outcome of our present study is that for the systems of Fermi–Dirac self-attracting particles, the gravitational collapse is possible for $d \geq 4$ for suitable initial data in the nonisothermal case, in contrast to the case of the isothermal problem in $d \leq 3$ dimensions.

All this shows that the Fermi–Dirac diffusion prevents the overcrowding of particles in lower dimensions, thus is more suitable to describe evolution of real stellar systems than Brownian diffusion models.

In particular, we will study in this paper:

- nonexistence of global in time solutions of (1)–(2) with general density–pressure relations (3), $D_* = 1$, one of the conditions (13), (14), and low energy in Section 2 (thus, *a fortiori*, we obtain nonexistence of steady states for arbitrary D_*),
- various estimates of the pressure, energy and entropy functionals in Section 3,
- existence of steady states with prescribed mass and the energy, E sufficiently large, in Section 4,
- properties of the Fermi functions in Section 5.

We do not consider here the question of the local in time existence of solutions of the evolution problem. This will be the subject of a separate paper.

Notation. In the sequel $|\,.\,|_p$ will denote the $L^p(\Omega)$ norm. The letter C will denote inessential constants which may vary from line to line.

2. Nonexistence of global in time solutions. *Star-shaped* domains $0 \in \Omega \subset \mathbb{R}^d$ are defined by the condition $\beta \geq 0$, where

$$\beta = \inf_{x \in \partial\Omega} x \cdot \overline{\nu}. \tag{16}$$

Similarly, *strictly star-shaped* domains are those with $\beta > 0$. Geometric assumptions on the shape of the domain Ω expressed in terms of β permit us to prove, under some restrictions on the ratio E/M^2, nonexistence of steady states of (1)–(2), (4), (11) with fairly general density–pressure relations (3). In this section, by solutions we mean the classical ones $n \in C^2(\Omega \times (0, \infty)) \cap C^1(\overline{\Omega} \times [0, \infty))$.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 3$, and $c_0 \geq d/(d-2)$, $E/M^2 < ((d-2)c_0 - d)/\left(2d\sigma_d(d-2)(\operatorname{diam}\Omega)^{d-2}\right)$. Then sufficiently regular, positive solutions of the problem (1)–(2), (11), with a general density–pressure relation (3), $D_* = 1$, $n_0 \geq 0$ in (12), and (13), satisfying the energy relation (4) cannot be defined globally in time.*

REMARK. There exist initial data (n_0, ϑ_0) leading to $E/M^2 \ll 0$. It suffices to consider an arbitrary $0 \leq n_0 \not\equiv 0$ in (12), $\vartheta_0 > 0$, and take the density Mn_0 with $M \gg 1$ large enough, cf. Lemma 3.5, (ii). A similar property holds for the energy (4) in the Brownian model.

Proof. Let us multiply (1) by $|x|^2$ and integrate over Ω, which leads to

$$\begin{aligned}\frac{d}{dt}\int_\Omega n|x|^2\,dx &= -2\int_\Omega \nabla p\cdot x\,dx - 2\int_\Omega n\nabla\varphi\cdot x\,dx\\ &= 2d\int_\Omega p\,dx - 2\int_{\partial\Omega} p\,x\cdot\overline{\nu}\,d\sigma - \frac{2}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^d}(x-y)\cdot x\,dx\,dy.\end{aligned}$$

After symmetrization of the double integral above, we arrive at

$$\frac{d}{dt}\int_\Omega n|x|^2\,dx \le 2d\int_\Omega p\,dx - \frac{1}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy$$

since $(x-y)\cdot x + (y-x)\cdot y = |x-y|^2$. Then, from (4) we conclude that

$$\begin{aligned}\frac{d}{dt}\int_\Omega n|x|^2\,dx &\le \frac{2d}{c_0}E + \frac{1}{\sigma_d}\left(\frac{d}{(d-2)c_0} - 1\right)\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy\\ &\le \frac{1}{c_0}\left(2dE + \frac{1}{\sigma_d}\frac{d-(d-2)c_0}{d-2}M^2(\operatorname{diam}\Omega)^{2-d}\right).\end{aligned}$$

Under the assumptions of Theorem 2.1 we obtain

$$\frac{d}{dt}\int_\Omega n|x|^2\,dx \le -\varepsilon < 0$$

for some $\varepsilon > 0$. This leads to a contradiction with the existence of a positive solution n for $t \ge T_{\max} = \int_\Omega n_0|x|^2\,dx/\varepsilon$. ∎

A counterpart of Theorem 2.1 for the Dirichlet condition (14) is

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \ge 3$, $c_0 \ge d/(d-2)$ and $E/M^2 < c_0\beta/(2d|\partial\Omega|)$. Then sufficiently regular, positive solutions of the problem (1)–(2), (11), with a general density–pressure relation (3), $D_* = 1$, $n_0 \ge 0$ in (12), and (14), satisfying the energy relation (4) cannot be defined globally in time.*

Proof. Proceeding as in the proof of Theorem 2.1, let us multiply (1) by $|x|^2$ and integrate over Ω. Taking into account the no-flux boundary condition (11), and the Pohozaev–Rellich identity for $\varphi \in C^2(\Omega)\cap C^1(\overline{\Omega})$, $\varphi_{|\partial\Omega} = 0$

$$\int_\Omega \Delta\varphi\,\nabla\varphi\cdot x\,dx = \frac{1}{2}\int_{\partial\Omega} x\cdot\overline{\nu}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma + \frac{d-2}{2}\int_\Omega |\nabla\varphi|^2\,dx,$$

we get a differential inequality for the second moment of the distribution $n(x,t)$. Namely, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega n|x|^2\,dx &= -2\int_\Omega \nabla p\cdot x\,dx - 2\int_\Omega n\nabla\varphi\cdot x\,dx\\ &= 2d\int_\Omega p\,dx - 2\int_{\partial\Omega} p\,x\cdot\overline{\nu}\,d\sigma - \int_{\partial\Omega} x\cdot\overline{\nu}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma - (d-2)\int_\Omega |\nabla\varphi|^2\,dx\\ &\le \frac{2d}{c_0}\left(E + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\,dx\right) - \beta\int_{\partial\Omega}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma - (d-2)\int_\Omega |\nabla\varphi|^2\,dx\\ &= \frac{2d}{c_0}E + \left(\frac{d}{c_0} + 2 - d\right)\int_\Omega |\nabla\varphi|^2\,dx - \beta\frac{M^2}{|\partial\Omega|} < 0.\end{aligned}$$

The remainder of the reasoning is as in the proof of Theorem 2.1. ∎

COROLLARY 2.3. *There is no steady state of the Fermi–Dirac system (1)–(2), (7), (11) with the energy (4) and* $c_0 = d/2$, *arbitrary* $D_* > 0$, *in star-shaped domain* $\Omega \subset \mathbb{R}^d$ *with either* $d \geq 4$, $E < 0$ *or* $d > 4$, $E \leq 0$, *and one of the boundary conditions (13), (14).*

Proof. Evidently, steady states do not depend on $D_* > 0$, cf. (38), so that $\frac{d}{dt}\int_\Omega n|x|^2\,dx \equiv 0$ for each steady state. Thus, the proofs of Theorems 2.1, 2.2 apply in this situation. Moreover, even if $E/M^2 < (d-4)/\big(4(d-2)\sigma_d(\operatorname{diam}\Omega)^{d-2}\big)$, then there is no steady state satisfying (13), and similarly if $E/M^2 < \beta/(4|\partial\Omega|)$—with (14). ∎

REMARK. Using the comparison results discussed in [1] (see also [9, Sec. 3]) one can show that

$$\lim_{t\nearrow T_{\max}} \int_\Omega n(x,t)|x|^2\,dx = 0$$

implies by the Shannon inequality (e.g. [1, Lemma 1])

$$\lim_{t\nearrow T_{\max}} \int_\Omega n(x,t)\log n(x,t)\,dx = \infty,$$

and thus for each $\delta > 0$

$$\lim_{t\nearrow T_{\max}} \int_\Omega n^{1+\delta}(x,t)\,dx = \infty,$$

so by the energy relation (4) one infers that

$$\lim_{t\nearrow T_{\max}} \int_\Omega n(x,t)\varphi(x,t)\,dx = -\infty.$$

This gives an insight into the problem: How do the blowing up solutions behave near the maximal existence time (which can be, of course, strictly less than $T_{\max}$)? The question is also related to necessary regularity conditions on initial data guaranteeing the local existence of smooth solutions of the evolution system.

3. A priori estimates of energy and entropy

3.1. *Pressure.* The Fermi–Dirac model results, in a certain sense, as an interpolation between mean field models involving the Brownian diffusion and diffusion in gases.

In the *classical limit* $\lambda \nearrow \infty$ (i.e. $n/\vartheta^{d/2} \searrow 0$, e.g. for fixed $\vartheta > 0$ and the small density $n \to 0$) the relation

$$\frac{p}{n} = \frac{2}{d}\vartheta\frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)} \sim \vartheta \tag{17}$$

holds by (57). This limit corresponds to the linear Brownian diffusion as in [13, 9, 17].

The completely degenerate case (*white dwarf* in astrophysics), $\lambda \searrow 0$ (i.e. $n/\vartheta^{d/2} \nearrow \infty$, e.g. for $\vartheta > 0$ fixed, $n \to \infty$), by (56) corresponds to the relation

$$\frac{p}{n^{1+2/d}} \sim \frac{2}{d+2}\left(\frac{d}{\mu}\right)^{2/d} = \kappa = \text{const}, \tag{18}$$

i.e., to a polytropic equation of state of a gas. This defines an evolution equation with a nonlinear diffusion as, e.g., in the porous media equation.

In order to eliminate the variable λ we introduce auxiliary functions D, V, F and H defined by

$$D(z) = -\frac{z}{I_{d/2-1}^{-1}(2z(\mu\vartheta^{d/2})^{-1})(I'_{d/2-1}\circ I_{d/2-1}^{-1})(2z(\mu\vartheta^{d/2})^{-1})}, \tag{19}$$

$$V(z) = zD(z), \tag{20}$$

$$F'(z) = (D(z))^2, \quad F(0) = 0, \tag{21}$$

$$h(z) = \int_1^z \frac{F'(y)}{V(y)}\, dy = \int_1^z \frac{D(y)}{y}\, dy, \quad H(z) = \int_1^z h(y)\, dy. \tag{22}$$

We gather some properties of these functions of z, depending also on ϑ, in the next lemma, see [6, Lemma 2.1].

Lemma 3.1. *The function V belongs to $\mathcal{C}^2([0,\infty))$, is nonnegative, and can be extended as the odd function to an element (still denoted by V) of $\mathcal{C}^2(\mathbb{R})$.*

Similarly, the function F belongs to $\mathcal{C}^3([0,\infty))$, is nonnegative, increasing and convex. It can be extended to a convex increasing function in $\mathcal{C}^3([-\delta,\infty))$ for some $\delta > 0$. The function H is nonnegative and convex on $[0,\infty)$.

Moreover, the asymptotic relations following from (56)

$$V(z) \sim Kz^{1+2/d}, \quad V'(z) \sim \frac{d+2}{d}Kz^{2/d}, \tag{23}$$

$$F'(z) \sim K^2 z^{4/d}, \tag{24}$$

$$p(z) \sim \vartheta\frac{d}{d+2}Kz^{1+2/d}, \quad p'(z) \sim \vartheta K z^{2/d}, \tag{25}$$

$$h(z) \sim \frac{d}{2}Kz^{2/d}, \quad H(z) \sim \frac{d^2}{2(d+2)}Kz^{1+2/d} \tag{26}$$

hold as $z \to \infty$ for some positive constant K. In fact, $K = \frac{d+2}{d\vartheta}\kappa$ is related to the constant κ in (18).

Recalling (7) we see that

$$\frac{\partial p}{\partial n} = p'(n) = \vartheta D(n), \tag{27}$$

so that the relation $\nabla p = \frac{\partial p}{\partial n}\nabla n = \vartheta D(n)\nabla n$ holds.

Lemma 3.2. *The pressure function p in the Fermi–Dirac model (7) is*

- *increasing and strictly convex with respect to the density n*

$$p''(n) = \frac{\partial^2 p}{\partial n^2} > 0 \quad \textit{for} \quad n \geq 0, \tag{28}$$

- $$p \geq \max\{\vartheta n, \kappa n^{1+2/d}\}, \tag{29}$$

- *strictly monotone with respect to the temperature ϑ*

$$\frac{\partial p}{\partial \vartheta} > 0 \quad \textit{for all} \quad n > 0,\ \vartheta > 0. \tag{30}$$

Proof. By (27), the desired property (28) of p as a function of n follows from Lemma 5.2 with $\alpha = d/2$, $\beta = d/2 - 1$.

The first part of (29) is a consequence of (17) and the strict monotonicity of the quotient

$$\frac{p}{\vartheta n} = \frac{2}{d}\frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)} = \frac{2}{d+2}\frac{I'_{d/2+1}(\lambda)}{I'_{d/2}(\lambda)} > 1,$$

which follows from Lemma 5.2 with $\alpha = d/2+1$ and $\beta = d/2$.

The second part of the inequality (29) follows from (18) and the strict monotonicity of

$$\frac{p}{\kappa n^{1+2/d}} = \frac{d+2}{2}\left(\frac{2}{d}\right)^{1+2/d}\frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)^{1+2/d}} > 1,$$

which is implied by Lemma 5.4 with $\alpha = d/2$ and $\beta = d/2-1$.

To prove (30) let us represent the considered derivative as

$$\frac{\partial p}{\partial\vartheta} = \left(\frac{d}{2}+1\right)\frac{p}{\vartheta} - \frac{d}{2}nD = \left(\frac{d}{2}+1\right)\frac{1}{d}I_{d/2}(\lambda) + \frac{1}{2}\frac{d}{2}\frac{I^2_{d/2-1}(\lambda)}{\lambda I'_{d/2-1}(\lambda)}.$$

Then, to conclude we can apply Lemma 5.3 with $\alpha = d/2$ and $\beta = 1$. Indeed, by (54)

$$\operatorname{sgn}\left\{\frac{\partial p}{\partial\vartheta}\right\} = \operatorname{sgn}\left\{-\lambda I'_{d/2-1}(\lambda) + \frac{2}{d}\lambda^2\frac{(I'_{d/2}(\lambda))^2}{\lambda I'_{d/2-1}(\lambda)}\right\} > 0. \blacksquare$$

3.2. *Energy.* The energy relation (4) for the Fermi–Dirac model leads to interesting *a priori* estimates in low dimensions.

LEMMA 3.3. *If $d \le 3$, then the total energy (4) controls the thermal energy $\frac{d}{2}\int_\Omega p\,dx$ and the absolute value of the potential energy $\frac{1}{2}\left|\int_\Omega n\varphi\,dx\right|$ from above. More precisely, if $d = 3$, then for each $0 < c_1 < 3/2$ there exists $c_2 = c_2(c_1, \Omega)$ such that*

$$E \ge c_1\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - c_2M^{7/3}. \tag{31}$$

Similarly, if $d \le 2$ then

$$E \ge c_1\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - c_2M^2 \tag{32}$$

for each $0 < c_1 < d/2$ and suitable $c_2 > 0$. If $d = 4$ such an estimate is meaningful for small values of mass M only, namely for $0 < M < M_0$,

$$E \ge (2 - c_3M^{1/2})\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - c_4 \tag{33}$$

for some c_3, $c_4 > 0$.

Proof. Clearly, the integral $\int_\Omega n\varphi\,dx$ is bounded from above. In fact, for either $d \ge 3$ or $d \ge 1$ and (14), we have $\int_\Omega n\varphi\,dx \le 0$.

For $d = 3$, using the Hölder inequality and the Sobolev imbedding theorem we obtain for each $\varepsilon > 0$

$$\left|\int_\Omega n\varphi\,dx\right| \le |n|_{6/5}|\varphi|_6 \le C|n|^2_{6/5} \le C|n|^{5/6}_{5/3}|n|^{7/6}_1 \le \varepsilon|n|^{5/3}_{5/3} + C_\varepsilon M^{7/3}$$

with a suitable $C_\varepsilon > 0$ independent of ϑ. Since p satisfies (29), the claimed estimate follows.

In the case $d \le 2$ a similar argument applies to $\left|\int_\Omega n\varphi\,dx\right| \le |n|_q|\varphi|_{q'}$ with q close to 1, so that with arbitrarily large q'.

For $d = 4$ the inequality

$$\left|\int_\Omega n\varphi\,dx\right| \le C|n|_{3/2}^{3/2} M^{1/2}$$

holds and (33) follows immediately. ■

COROLLARY 3.4. *If either $d \le 3$ or $d = 4$ and $M > 0$ is small enough, then the relation (4) gives a uniform apriori estimate from above for $\int_\Omega p(x,t)\,dx$, $\left|\int_\Omega n(x,t)\varphi(x,t)\,dx\right|$ and $\vartheta(t)$ in the evolution problem.*

Proof. If $d = 3$, Lemma 3.3 and (29) imply $E \ge c_1\vartheta M - c_2 M^{7/3}$. This leads to $\vartheta(t) \le \vartheta_1 < \infty$ for all $t \ge 0$, whenever solutions $n(x,t)$ with fixed M, E exist. ■

Simple properties of the energy function $(n,\vartheta) \mapsto E$ are gathered in

LEMMA 3.5. *Define for a bounded domain $\Omega \subset \mathbb{R}^d$ and $M > 0$ the convex subsets of $L^{1+2/d}(\Omega)$*

$$\mathcal{M}_d = \{n \in L^{1+2/d}(\Omega) : n \ge 0\}, \quad \mathcal{M}_{d,M} = \left\{n \in \mathcal{M}_d : \int_\Omega n\,dx = M\right\}.$$

The following properties of the energy function E hold for all $\vartheta > 0$:

(i) if $d \ge 1$ then $\sup_{\mathcal{M}_{d,1}} E = \infty$,
(ii) if $d > 2$ then $\inf_{\mathcal{M}_d} E = -\infty$,
(iii) if $d = 2$, then $\inf_{\mathcal{M}_d} E/M^2 > -\infty$,
(iv) if $d = 4$, then for each sufficiently small $M < M_0$, $\inf_{\mathcal{M}_{4,M}} E > -\infty$, but $\inf_{\mathcal{M}_{4,M}} E = -\infty$ for all $M > M_0$,
(v) if $d \ge 5$ then for each $M > 0$, $\inf_{\mathcal{M}_{d,M}} E = -\infty$,
(vi) given $d > 2$, $n \in \mathcal{M}_{d,1}$ and $\vartheta > 0$, if $M > 0$ is sufficiently small, then $E > 0$ for (Mn, ϑ),
(vii) for each fixed n with $\int_\Omega n\,dx = M > 0$, $\lim_{\vartheta\to\infty} E(n,\vartheta) = \infty$.

Proof. (i) follows from the calculation of the function E for $n = \mathbb{1}_\omega/|\omega|$ with $\omega \subset\subset \Omega$, $0 < |\omega| \ll 1$. Indeed, $E \sim |\omega|^{-2/d} - \text{const}$ for such n.

(ii) Fix $\vartheta > 0$ and take a smooth function $n \in \mathcal{M}_{d,1}$ with compact support, and consider the densities Mn, $M \gg 1$. For these functions $E \sim cM^{1+2/d} - CM^2$ holds.

(iii) Observe that, by Lemma 3.3, for $d = 2$ $\ell_1 \equiv \inf_{\mathcal{M}_{d,1}} E > -\infty$ holds. Then the scaling of the both terms in E leads to the desired estimate.

(vii) This follows from the strict monotonicity of p with respect to ϑ in (30) and (29). ■

REMARK. Given E and the instantaneous value of $n = n(x,t)$, the property (vii) in Lemma 3.5 above permits us to define the temperature $\vartheta = \vartheta(t)$ (and thus $p = p(n,\vartheta)$) in a unique way.

3.3. *Entropy.* For particular density–pressure relations (3) there may exist important functionals which play the role of a Lyapunov function for the dynamical system associated with (1)–(2), (11)–(12), (4). For instance, in the isothermal Brownian case

$\int_\Omega(\vartheta n\log n+\frac12 n\varphi)\,dx$ decreases in time. Similarly, in the nonisothermal Brownian case, the classical Boltzmann (neg)entropy $\int_\Omega n(\log n-c_0\log\vartheta)\,dx$ is a decreasing function of t, cf. e.g. [13, 9, 5].

Moreover, for the polytropic case $W_\gamma=\int_\Omega(\frac{\kappa}{\gamma}n^{1+\gamma}+\frac12 n\varphi)\,dx$ decreases in t (the first term in this entropy is known as Rényi entropy by mathematicians, and Tsallis entropy by physicists). Indeed,

$$\frac{dW_\gamma}{dt}=-\int_\Omega nD_*\left|\kappa\frac{1+\gamma}{\gamma}\nabla(n^\gamma)+\nabla\varphi\right|^2dx\le 0. \tag{34}$$

Finally, in the Fermi–Dirac case $W_{\rm iso}=\int_\Omega\left(\vartheta H(n)+\frac12 n\varphi\right)dx$ is an entropy for the isothermal model, see [6], and

$$W=\int_\Omega\left(H(n)+\frac12\frac{p}{\vartheta}\right)dx \tag{35}$$

is an entropy in nonisothermal case, see [16]. Here $H=H(n,\vartheta)$ is the convex function defined in (22). Indeed, the entropy W satisfies, for all sufficiently regular solutions of the evolution problem the relation

$$\frac{dW}{dt}=-\int_\Omega\vartheta nD_*\left|\nabla\left(h(n)+\frac{\varphi}{\vartheta}\right)\right|^2dx\le 0. \tag{36}$$

The proof of the above entropy production formula follows from multiplication of (1) by $h(n)+\varphi/\vartheta$ and integration by parts. We use also the relations

$$\frac{d}{dt}E=\frac{d}{2}\int_\Omega p_t\,dx+\int_\Omega n_t\varphi\,dx=0,$$
$$p_t=\left(\frac{d}{2}+1\right)p\frac{d}{dt}\left(\log\vartheta\right)+n\vartheta\frac{d}{dt}h(n),$$

and the properties (22) of the functions h, H.

The asymptotic properties of the Fermi integrals in Lemma 5.1 permit us to show

LEMMA 3.6. *For $d>2$ the entropy functional W defined in (35) is of order*

$$W\sim -C\vartheta\int_\Omega n^{1-2/d}\,dx$$

with some constant $C>0$. In particular, this implies that if $W(0)$ is strictly negative, then a lower bound for the temperature $\vartheta(t)\ge\vartheta_1>0$ holds.

Proof. Indeed, using the notation in (59), we get

$$W=\int_\Omega\left(-n\log\lambda-\frac{d+2}{2}\frac{p}{\vartheta}\right)dx$$
$$=\frac{\mu}{2}\vartheta^{d/2}\int_\Omega(-\log\lambda)^{d/2-1}R_{d/2-1}(\lambda)\,dx\ge -C\vartheta\int_\Omega n^{1-2/d}\,dx$$

with a constant $C>\pi^2/3$ by the properties of R_α proved in Lemma 5.1 for large values of n.

Note that this entropy functional is not convex since the second derivative with respect to n of the entropy density (i.e. the integrand in (35)) equals $\frac1n D-\frac{d}{2}D'=\frac1n\frac{\partial^2 p}{\partial n\,\partial\vartheta}$, and this expression changes the sign: for $0<n\ll 1$ it is positive, while for $n\gg 1$ is negative. ∎

4. The steady state problem with fixed mass and energy. For general density–pressure relations (3) the steady states (N, Φ) of (1)–(2) are determined from the equation

$$h(N) + \frac{\Phi}{\vartheta} = c, \tag{37}$$

where the constant c is chosen from the mass constraint $\int_\Omega N\,dx = M$, and the function $h = h(n, \vartheta)$ is defined in a similar way as in the Fermi–Dirac case (22)

$$\frac{\partial h}{\partial n} = \frac{1}{\vartheta n}\frac{\partial p}{\partial n}.$$

Indeed, multiplying the stationary version $\nabla\cdot(D_*(\nabla P + N\nabla\Phi)) = 0$ of (1) by $\vartheta h(N) + \Phi$ and integrating over Ω, we obtain

$$\int_\Omega \vartheta^2 N D_* \left|\nabla\left(h(N) + \frac{\Phi}{\vartheta}\right)\right|^2 dx = 0, \tag{38}$$

which implies (37).

The Brownian case with $h(n) = \log n$ has been considered in [2] and [8]. The polytropic case with $h(n) = \frac{\kappa}{\vartheta}\frac{1+\gamma}{\gamma}n^\gamma$ was studied using monotonicity methods in e.g. [18], and then, in the radially symmetric case, in [14]. Variational methods are also available in certain cases when $\gamma \geq 1 - 2/d$.

Whenever an entropy exists for (1)–(2) with a general relation (3), there is an alternative way to derive the equation (37) for steady states. Namely, the production of entropy formula leads to (37) as in (34), (36), and in the Brownian case studied in [2, 8].

In the problem of finding steady states for fixed E and M, (37) is supplemented by (4) and (15), so that the temperature ϑ is to be determined from these constraints.

In the Fermi–Dirac case the stationary solutions (N, Φ) of the problem (1)–(2), (11), (14) (or (1)–(2), (11), (13)) with an arbitrary diffusion coefficient D_* and a constant ϑ satisfying (4), are characterized by the identity $\nabla(\vartheta\log\Lambda - \Phi) = 0$ equivalent to (37). Here, the relation between Λ and N is as for λ and n in (9). Taking into account the relations (7), (9) and the Poisson equation (2), we arrive at the relation

$$N = \Delta\Phi = \frac{\mu}{2}\vartheta^{d/2} I_{d/2-1}(e^{\Phi/\vartheta - c}), \tag{39}$$

for the stationary potential Φ, the density N and a constant temperature ϑ, called the Poisson–Fermi–Dirac equation. The constant c is such that the mass constraint

$$\int_\Omega N\,dx = \int_\Omega \Delta\Phi\,dx = M \tag{40}$$

is satisfied. The equation (39) can be simplified a bit by introducing the new potential $\Psi = -\Phi/\vartheta$ leading to

$$\Delta\Psi + \frac{\mu}{2}\vartheta^{d/2-1} I_{d/2-1}(e^{-\Psi - c}) = 0 \tag{41}$$

with the constant c satisfying the mass constraint

$$\frac{\mu}{2}\vartheta^{d/2-1}\int_\Omega I_{d/2-1}(e^{-\Psi-c})\,dx = \frac{M}{\vartheta}. \tag{42}$$

We are looking for solutions satisfying a boundary condition and also the energy constraint (4) which is now for $P = P(N, \vartheta)$:

$$E = \frac{d}{2}\int_\Omega P\,dx - \frac{\vartheta}{2}\int_\Omega N\Psi\,dx \tag{43}$$
$$= \frac{\mu}{2}\vartheta^{d/2}\int_\Omega (I_{d/2}(e^{-\Psi-c}) - \frac{1}{2}\Psi I_{d/2-1}(e^{-\Psi-c}))\,dx.$$

First, we recall results on steady states with a prescribed temperature $\vartheta > 0$ from [6] (based on constructions in [23]), and [19, 20].

PROPOSITION 4.1. *For $d = 1, 2, 3$, given $M > 0$ there exists at least one solution Φ of the Poisson–Fermi–Dirac equation (39) satisfying the Dirichlet condition (14) and (40). For $d = 4$ such a solution exists for all sufficiently small $M > 0$.*

Minimizing the entropy (either (35) or W_{iso}) at a fixed temperature $\vartheta > 0$ and under the mass constraint $M > 0$, one obtains existence results similar to those in Proposition 4.1.

PROPOSITION 4.2. *For $d = 1, 2, 3$, given $M > 0$ there exists at least one solution Φ of the Poisson–Fermi–Dirac equation (39) satisfying the free condition (13) and (40).*

For $d = 4$ such a solution exists for all sufficiently small $M > 0$. ■

The question of the existence of multiple solutions of the equation (39), and their stability as solutions of the evolution problem (1)–(12), is rather delicate. There are some numerical results in the case of radially symmetric solutions in the ball of $\mathbb{R}^3$ in [10, 11, 12].

REMARK. For another approach to the problem of the existence of solutions of (41), (42) with Ψ satisfying the Dirichlet condition (14) for small $M > 0$ and each $d \geq 3$, we refer the reader to [19]. These results are proved in the spirit of fixed point theorems based on the compactness properties of the operator $N \mapsto \Psi$ in [19], and using contraction arguments in [20]. For the free condition (13) and $d \geq 3$, the proofs of analogous results are practically the same, as well as in the two-dimensional case for either of the boundary conditions (13), (14). Also, by an application of the Pohozaev identity, it is shown in [19] that for $d \geq 5$, the equation (41) with the Dirichlet boundary condition in star-shaped domains has no solution for sufficiently large M, say $M \geq M(\vartheta)$ with $M(\vartheta)/\vartheta \gg 1$.

Now we are ready to study stationary solutions satisfying the energy and mass constraints (i.e. in the *microcanonical ensemble*). The details of the fixed point arguments can be found in [19, 20]. However, it should be noted that in the aforementioned papers the dependence on ϑ was not explicitly stated (as irrelevant in the *canonical ensemble*), while it is of crucial importance herein.

To simplify the notation, define the parameter $\varrho = \frac{\mu}{2}\vartheta^{d/2}$, and the function $g(z) = I_{d/2-1}(e^{-z})$, $d \geq 2$. The function g and its derivative g' are strictly increasing on $\mathbb{R}$, and

$$g(z) \sim g'(z) \sim \Gamma\left(\frac{d}{2}\right) e^z \quad \text{as} \quad z \to -\infty, \tag{44}$$

see (54) and (56)–(57). Using the monotonicity of g together with the positivity of Ψ, we get from (42)

$$c \le g^{-1}\left(\frac{M}{\varrho|\Omega|}\right). \tag{45}$$

Moreover, for any $\Psi \in L^\infty(\Omega)$ there exists a unique value $c = c(\Psi)$ satisfying (42). The function c is Lipschitz continuous:

$$|c(\Psi_1) - c(\Psi_2)| \le |\Psi_1 - \Psi_2|_\infty$$

by (42), cf. [19, (1.11)]. Thus, looking for a solution of the problem (41)–(42) is reduced to finding a fixed point of the integral operator

$$\mathcal{T}(\Psi) = \varrho\vartheta^{-1}(-\Delta)^{-1}\left(g(\Psi + c(\Psi))\right). \tag{46}$$

Here $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ with an appropriate boundary condition, defined by an integral operator with the kernel equal either to the Green function of Ω or to the fundamental solution of the Laplacian. Its norm, as the operator considered in $L^\infty(\Omega)$, is denoted by A.

Applying the mean value theorem for the function g and monotonicity properties of g and g', we see that the Lipschitz constant $L(M,\vartheta)$ of $\mathcal{T}$ on the unit ball $B(0,1) \subset L^\infty(\Omega)$ satisfies the estimate

$$L(M,\vartheta) \le 2A\varrho\vartheta^{-1}\, g'\left(1 + g^{-1}\left(\frac{M}{\varrho|\Omega|}\right)\right). \tag{47}$$

Moreover, $\mathcal{T}$ maps the ball $B(0,1)$ into itself provided

$$A\varrho\vartheta^{-1} g\left(1 + g^{-1}\left(\frac{M}{\varrho|\Omega|}\right)\right) \le 1, \quad \text{i.e.} \quad M \le \varrho|\Omega|\, g\left(g^{-1}\left(\frac{\vartheta}{A\varrho}\right) - 1\right). \tag{48}$$

Since $\vartheta/\varrho = \frac{2}{\mu}\vartheta^{1-d/2}$, we conclude from (44), (47) that for sufficiently large ϑ: $M/\vartheta \ll 1$, say $M/\vartheta \le m_0$, the conditions

$$L(M,\vartheta) \le C\frac{M}{\vartheta} < \frac{1}{2}, \tag{49}$$

and (48) are satisfied. Applying the Contraction Mapping Principle we get, for each positive M, ϑ such that (48)–(49) are satisfied, a solution $\Psi = \Psi_{M,\vartheta}$, unique in the ball $B(0,1)$. The norm of that solution $\Psi = \mathcal{T}(\Psi)$ satisfies, by (49) and $|\mathcal{T}(0)|_\infty \le C\frac{M}{\vartheta}$,

$$|\Psi_{M,\vartheta}|_\infty \le 2C\frac{M}{\vartheta} \le 1. \tag{50}$$

Stationary solutions constructed above have positive energy. Indeed, for fixed $M > 0$ the energy E defined in (43) for the solution $-\vartheta\Psi = -\vartheta\Psi_{M,\vartheta}$ of (39) with mass M in (40) at the temperature ϑ, satisfies, by (29) and $|\Psi|_\infty \le 1$,

$$E \ge \frac{d}{2}\vartheta M - \frac{1}{2}\vartheta M|\Psi|_\infty \ge \frac{d-1}{2}\vartheta M > 0,$$

thus

$$\sup_{\vartheta \ge M/m_0} E = \infty. \tag{51}$$

Since both $\Psi_{M,\cdot}$ and E are continuous functions of ϑ along the branch of solutions constructed above, we obtain the following

PROPOSITION 4.3. *For $d \geq 2$ and $M > 0$ there exists $E_0 = E_0(\Omega, M) > -\infty$ such that if $E > E_0$, then there is at least one solution of (39), (40), (4), with one of the conditions (13), (14).* ∎

REMARK. For $d = 3$ steady states exist for each $M > 0$ and $\vartheta > 0$, see Propositions 4.1, 4.2. However, the corresponding values of the energy E are not arbitrary: for instance, by (31) the quantity $E/M^{7/3}$ is always bounded from below.

In the case of dimension $d = 2$ we can prove a stronger result

PROPOSITION 4.4. *For $d = 2$ there exists $\ell_0 = \ell_0(\Omega) > 0$ such that if $E/M^2 > \ell_0$, then there is at least one solution of (39), (40), (4), with one of the conditions (13), (14).*

Proof. By Propositions 4.1 and 4.2, for $\vartheta = 1$ and each $M > 0$ the equation (39) has a solution (N, Φ). Using the scaling properties of (39) for $d = 2$, it is easy to see that for each $\vartheta > 0$ the couple $(\vartheta N, \vartheta\Phi)$ solves (39) at the temperature ϑ.

Observe that by the formula (7)

$$p(\vartheta^{d/2} n, \vartheta) = \vartheta^{d/2+1} p(n, 1), \tag{52}$$

so for $d = 2$ we have $p(\vartheta N, \vartheta) = \vartheta^2 p(N, 1)$. Together with the relation $(\vartheta N)(\vartheta\Phi) = \vartheta^2 N\Phi$, this leads to the same scaling of both (kinetic and potential) parts of the energy, so $E(\vartheta N, \vartheta) = \vartheta^2 E(N, 1)$ for $\vartheta > 0$.

As we have already established in the proof of Proposition 4.4, the problem (41)–(42) for fixed M and $0 < m = M/\vartheta \leq m_0$ can be solved by a contraction argument. These solutions $\Psi_{m,1} = \Psi_{M,\frac{M}{m}}$ form a continuous branch in the space $L^\infty(\Omega)$. They lead to solutions $(-\frac{M}{m}\Delta\Psi_{m,1}, -\frac{M}{m}\Psi_{m,1})$ of (39)–(40) whose energy varies continuously with m. Since $\int_\Omega p(-\Delta\Psi_m, 1)\, dx \leq Cm$ by (17), their energies satisfy

$$\inf_{0<m\leq m_0} \frac{E}{M^2} \leq \inf_{0<m\leq m_0} \frac{1}{m^2} \int_\Omega p(-\Delta\Psi_m, 1)\, dx \leq \inf_{0<m\leq m_0} \frac{1}{m^2} Cm = \frac{C}{m_0}. \tag{53}$$

Together with (51) this proves Proposition 4.4.

Note that for $d = 2$ the ratio $E/M^2 \geq \ell_1 > -\infty$ is bounded from below by (32) for any density N and any temperature ϑ (not necessarily for steady states). ∎

REMARK. In $d \geq 3$ dimensional case, the kinetic and potential parts of the energy scale in a different way. As a consequence, a sufficient condition for the existence of steady states with given E and M reads

$$\min\left\{\frac{E}{M^2}, \frac{E}{M^{1+2/d}}\right\} > \ell_0$$

for some $\ell_0 = \ell_0(\Omega) > 0$, see a paper in preparation by the third named author.

REMARK. Results in Theorems 2.1 and 2.2 show that for steady states in d-dimensional star-shaped domains, $d \geq 4$ (which exist for $\vartheta > 0$ and each $0 \leq M < M(\vartheta)$), the ratio E/M^2 is also bounded from below.

5. Appendix: properties of Fermi functions. In this section we collect some formulas and estimates related to Fermi integrals.

Let us recall from [6] that $I_0(\lambda) = \log(1 + 1/\lambda)$, because $\frac{d}{d\lambda} I_0(\lambda) = -\frac{1}{\lambda(1+\lambda)}$.

For $\alpha > -1$ we have the formula for the derivative of I_α

$$\frac{d}{d\lambda} I_\alpha(\lambda) = -\int_0^\infty \frac{y^\alpha e^y\, dy}{(1+\lambda e^y)^2} \tag{54}$$
$$= \int_0^\infty \frac{y^\alpha}{\lambda} \frac{d}{dy}\left(\frac{1}{1+\lambda e^y}\right) dy = -\frac{\alpha}{\lambda} I_{\alpha-1}(\lambda),$$

the second line being valid for $\alpha > 0$ only. Moreover,

$$\frac{d^2}{d\lambda^2} I_\alpha(\lambda) = 2\int_0^\infty \frac{y^\alpha e^{2y}\, dy}{(1+\lambda e^y)^3}. \tag{55}$$

Therefore, I_α is a decreasing convex function of λ.

It is important that the Fermi integrals (8) have the following asymptotics:

$$I_\alpha(\lambda) \sim \frac{1}{\alpha+1} (-\log\lambda)^{\alpha+1} \quad \text{as} \quad \lambda \searrow 0, \tag{56}$$

and

$$I_\alpha(\lambda) \sim \frac{\Gamma(\alpha+1)}{\lambda} \quad \text{as} \quad \lambda \nearrow \infty. \tag{57}$$

We refer the reader to [6, Sec. 2] for a proof.

More precise asymptotics for the Fermi integrals near $\lambda = 0$ are given in

Lemma 5.1. *The following asymptotic relations hold as $\lambda \searrow 0$:*

$$r_\alpha(\lambda) \equiv I_\alpha(\lambda) - \frac{(-\log\lambda)^{\alpha+1}}{\alpha+1} = \mathcal{O}\left((-\log\lambda)^{\alpha-1}\right) \quad \textit{for each} \quad \alpha \geq 1, \tag{58}$$

and for each $\alpha > 0$

$$R_\alpha(\lambda) \equiv (-\log\lambda)^{-\alpha}\left\{ I_\alpha(\lambda)(-\log\lambda) - \frac{\alpha+2}{\alpha+1} I_{\alpha+1}(\lambda)\right\} \to -\frac{\pi^2}{3}. \tag{59}$$

Proof. First we prove (58) for $1 \leq \alpha \leq 2$, and then by recurrence, using the de l'Hôpital rule, for $k \leq \alpha < k+1$ with $k = 2, 3, \dots$. Let $\ell = -\log\lambda$; evidently $\ell \to \infty$ as $\lambda \to 0$.

Rewrite r_α as

$$\begin{aligned} r_\alpha(\lambda) &= \int_0^\infty \frac{y^\alpha\, dy}{1+\lambda e^y} - \int_0^\ell y^\alpha\, dy \\ &= -\int_0^\ell \frac{\lambda e^y y^\alpha\, dy}{1+\lambda e^y} + \int_\ell^\infty \frac{y^\alpha\, dy}{1+\lambda e^y} \\ &= -\int_0^\ell \frac{(\ell-v)^\alpha\, dv}{1+e^v} + \int_0^\infty \frac{(\ell+u)^\alpha\, du}{1+e^u} \\ &= \int_0^\ell \frac{(\ell+u)^\alpha - (\ell-u)^\alpha}{1+e^u}\, du + \int_\ell^\infty \frac{(\ell+u)^\alpha}{1+e^u}\, du \equiv J_1 + J_2. \end{aligned}$$

Clearly,

$$0 \leq J_2 \leq 2^\alpha \int_\ell^\infty \frac{u^\alpha\, du}{1+e^u} \leq 2^\alpha I_\alpha(1) < \infty,$$

and even $J_2 = o(1)$ holds as $\ell \to \infty$.

Next, we note an elementary inequality

$$(1+\tau)^\alpha - (1-\tau)^\alpha - 2\alpha\tau \leq C\tau^3 \tag{60}$$

for $1 \le \alpha \le 2$, $0 \le \tau \le 1$ and some $C = C_\alpha$. This follows by showing that the difference of the left and right hand sides is a decreasing function on the interval $0 \le \tau \le 1$.

Then, we have

$$\begin{aligned} 0 \le J_1 &= \ell^\alpha \int_0^\ell \frac{\left(1+\frac{u}{\ell}\right)^\alpha - \left(1-\frac{u}{\ell}\right)^\alpha}{1+e^u}\, du \\ &\le \ell^\alpha \int_0^\ell \frac{2\alpha\frac{u}{\ell} + C\left(\frac{u}{\ell}\right)^3}{1+e^u}\, du \le \ell^{\alpha-1} 2\alpha \int_0^\infty \frac{u\, du}{1+e^u} + \ell^{\alpha-3} C \int_0^\infty \frac{u^3\, du}{1+e^u}, \end{aligned}$$

which shows the claim (58).

Similarly, we obtain

$$\begin{aligned} R_\alpha(\lambda) &= \ell^{-\alpha}\left\{ r_\alpha(\lambda)\ell - \frac{\alpha+2}{\alpha+1} r_{\alpha+1}(\lambda)\right\} \\ &= \ell^{-\alpha+1}\int_0^\ell \frac{(\ell+u)^\alpha - (\ell-u)^\alpha}{1+e^u}\, du + \ell^{-\alpha}\int_\ell^\infty \frac{\ell(\ell+u)^\alpha\, du}{1+e^u} \\ &\quad - \frac{\alpha+2}{\alpha+1}\ell^{-\alpha}\left(\int_0^\ell \frac{(\ell+u)^{\alpha+1} - (\ell-u)^{\alpha+1}}{1+e^u}\, du + \int_\ell^\infty \frac{(\ell+u)^{\alpha+1}\, du}{1+e^u}\right) \\ &= \ell \int_0^\ell \frac{2\alpha\frac{u}{\ell} - 2(\alpha+2)\frac{u}{\ell} + \mathcal{O}\left(\left(\frac{u}{\ell}\right)^3\right)}{1+e^u}\, du + \ell^{-\alpha}\mathcal{O}(1) \\ &= -4\int_0^\ell \frac{u\, du}{1+e^u} + \mathcal{O}(\ell^{-2}) + \mathcal{O}(\ell^{-\alpha}) \to -\frac{\pi^2}{3} \end{aligned}$$

as $\ell \to \infty$. Again by the recurrent use of the de l'Hôpital rule, this result extends from the parameter range $0 < \alpha \le 1$ to $k < \alpha \le k+1$.

Finally, note that $-\frac{C_\alpha}{\lambda}(\log\lambda + 1) \le R_\alpha(\lambda) \le 0$ for $\lambda \ge 1$ and some $C_\alpha > 0$, and $R_\alpha(\lambda) + 4\int_0^\ell \frac{u\,du}{1+e^u} \le 0$ for $\lambda \le 1$. ■

LEMMA 5.2. *For $\alpha > \beta$, $I_\alpha \circ I_\beta^{-1}$ is an increasing convex function on $\mathbb{R}^+$ while for $\alpha < \beta$ this is an increasing concave function on $\mathbb{R}^+$.*

Proof. This result follows from the strict monotonicity of the quotient I'_α/I'_β which is decreasing for $\alpha > \beta$, and increasing for $\alpha < \beta$, see the argument in [6, Lemma 2.2]. ■

LEMMA 5.3. *For all $\beta < \alpha + 1$ the inequality*

$$I'_{\alpha+\beta} I'_{\alpha-\beta} - (I'_\alpha)^2 \ge 0$$

holds on $\mathbb{R}^+$.

Proof. Using the integral representation (54) of the derivatives of Fermi functions we write

$$\begin{aligned} I'_{\alpha+\beta}(\lambda) I'_{\alpha-\beta}(\lambda) - (I'_\alpha(\lambda))^2 &= \int_0^\infty\int_0^\infty \frac{y^{\alpha+\beta}e^y v^{\alpha-\beta}e^v - y^\alpha e^y v^\alpha e^v}{(1+\lambda e^y)^2(1+\lambda e^v)^2}\, dy\, dv \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{y^{\alpha-\beta}e^y v^{\alpha-\beta}e^v}{(1+\lambda e^y)^2(1+\lambda e^v)^2}(y^\beta - v^\beta)^2\, dy\, dv \ge 0, \end{aligned}$$

after the symmetrization of the double integral. ■

LEMMA 5.4. *For $\alpha > \beta$ the function*

$$f(\lambda) = \frac{I_\alpha^{1/(\alpha+1)}(\lambda)}{I_\beta^{1/(\beta+1)}(\lambda)}$$

is increasing on $\mathbb{R}^+$.

Proof. Observe that by the property (54)

$$\begin{aligned}\operatorname{sgn}\{f'(\lambda)\} &= \operatorname{sgn}\left\{ I_\alpha^{1/(\alpha+1)}(\lambda) I_\beta^{-1/(\beta+1)}(\lambda) \left(\frac{1}{\alpha+1} \frac{I'_\alpha(\lambda)}{I_\alpha(\lambda)} - \frac{1}{\beta+1} \frac{I'_\beta(\lambda)}{I_\beta(\lambda)} \right) \right\} \\ &= \operatorname{sgn}\left\{ \frac{1}{\lambda} \left(-\frac{I'_\alpha(\lambda)}{I'_{\alpha+1}(\lambda)} + \frac{I'_\beta(\lambda)}{I'_{\beta+1}(\lambda)} \right) \right\} \\ &= \operatorname{sgn}\{ I'_\beta(\lambda) I'_{\alpha+1}(\lambda) - I'_\alpha(\lambda) I'_{\beta+1}(\lambda) \}.\end{aligned}$$

The expression in braces can be represented using (54) as

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{y^{\alpha+1} e^y v^\beta e^v - y^\alpha e^y v^{\beta+1} e^v}{(1+\lambda e^y)^2 (1+\lambda e^v)^2}\, dy\, dv \\ &\qquad = \frac{1}{2} \int_0^\infty \int_0^\infty \frac{y^\beta e^y v^\beta e^v}{(1+\lambda e^y)^2 (1+\lambda e^v)^2} \left(y^{\alpha-\beta} - v^{\alpha-\beta} \right) (y-v)\, dy\, dv \geq 0,\end{aligned}$$

after the symmetrization of the double integral. ■

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