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ITERATIONS FOR NONLOCAL ELLIPTIC PROBLEMS

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Abstract. Convergence of an iteration sequence for some class of nonlocal elliptic problems appearing in mathematical physics is studied.

1. Introduction. We consider the following nonlocal elliptic problem:

$$-\Delta\varphi = M\frac{f(\varphi)}{(\int_\Omega f(\varphi))^p} \quad \text{in} \quad \Omega, \tag{1}$$

with the homogeneous boundary Dirichlet condition

$$\varphi|_{\partial\Omega} = 0. \tag{2}$$

Here $\varphi : \Omega \to \mathbb{R}$ is an unknown function from a bounded domain Ω of $\mathbb{R}^n$ into $\mathbb{R}$, $n \geq 2$, $f : \mathbb{R} \to \mathbb{R}^+$ is a given C^1 function and $M > 0$, $p > 0$ are given parameters.

The physical motivations for the study of nonlocal elliptic problems come from statistical mechanics ([2], [5], [6]), theory of electrolytes ([4]), and theory of thermistors ([7], [13]).

If the parameter p equals 1 and the nonlinearity $f(\varphi)$ has the exponential form $e^{-\varphi}$ (e^{φ}, resp.) then (1) is the well-known Poisson-Boltzmann equation and φ can be interpreted as the electric (gravitational, resp.) potential of system of particles in thermodynamical equilibrium interacting via Coulomb (gravitational, resp.) potential. In this interpretation, the parameter M is the total charge (mass, resp.) of the particles of the system.

The problem (1)–(2) with given $f(\varphi)$ and $p = 2$ appears in modelling the stationary temperature φ, which results when an electric current flows through a material with temperature-dependent electrical resistivity $f(\varphi)$, subject to a fixed potential difference $\sqrt{M}$ ([7], [13]).

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The existence and uniqueness of solutions of (1)–(2) depend on the parameters M, p, the geometry of the domain Ω, and on some properties of the function f. The existence results can be proved using either the technique of sub- and supersolutions ([3]), or variational methods ([6], [8]), or topological methods ([4], [10], [12], [15]), whereas the nonexistence results are a consequence of the Pohozaev identity ([3]), or construction of some special subsolutions ([3]).

The existence and uniqueness of solutions for the Poisson-Boltzmann problem with $f(\varphi) = e^{-\varphi}$ and arbitrary $M > 0$ have been proved in [8] and [10]. When f has the form e^{φ}, the solutions do not exist for large M, and in general, are not unique. Moreover, the existence and uniqueness depend largely on the geometry of Ω, see [11].

2. Picard iterations for (1)–(2). Our aim is to study the convergence of iteration schemes for nonlocal elliptic problems (1)–(2).

We start with the local elliptic problem

$$-\Delta\varphi = \lambda f(\varphi) \quad \text{in} \quad \Omega, \quad \varphi|_{\partial\Omega} = 0, \tag{3}$$

where $\lambda > 0$ is a given constant.

We transform (3) to an integral form

$$\varphi(x) = \lambda \int_\Omega G(x,y) f(\varphi(y))dy, \tag{4}$$

where $G(x,y)$ is the Green function corresponding to $-\Delta$ and the homogeneous boundary data. The right hand side of (4) defines the operator $T(\varphi)(x) = \lambda \int_\Omega G(x,y)f(\varphi(y))dy$ on the space $C^0(\bar{\Omega})$ of continuous functions on $\bar{\Omega}$. It is known that certain assumptions on the function f guarantee that the sequence of iterations $T^n(0)$ is convergent in the supremum norm to a minimal solution of (3) ([1], [14]).

The integral form of (1)–(2) is

$$\varphi(x) = \frac{M}{(\int_\Omega f(\varphi))^p} \int_\Omega G(x,y) f(\varphi(y))dy, \tag{5}$$

so we introduce the operator T on the space $C^0(\bar{\Omega})$ as

$$T(\varphi)(x) = \frac{M}{(\int_\Omega f(\varphi))^p} \int_\Omega G(x,y) f(\varphi(y))dy. \tag{6}$$

Any fixed point of T is a solution of (5). We define the Picard iteration scheme for (1)–(2) $\varphi_n = T^n(\varphi_0)$ by

$$-\Delta\varphi_n = M\frac{f(\varphi_{n-1})}{(\int_\Omega f(\varphi_{n-1}))^p} \quad \text{in} \quad \Omega, \quad \varphi_n|_{\partial\Omega} = 0, \quad n = 1, 2, \ldots ,$$

and look for a fixed point of T as the limit of the sequence $T^n(\varphi_0)$.

First, we note that for a contraction g on the Banach space X, i.e. a mapping $g : X \to X$ such that $||g(x) - g(y)|| \le \alpha||x - y||$ for some constant $\alpha \in [0, 1)$ and for all $x, y \in X$, g maps the ball $B_R(0) \subset X$ into itself whenever $R > ||g(0)||/(1 - \alpha)$.

Indeed, let $R > ||g(0)||/(1-\alpha)$. Then for $x \in B_R(0)$ we have $||g(x)|| \le ||g(x)-g(0)||+ ||g(0)|| < \alpha||x|| + ||g(0)|| < \alpha R + ||g(0)|| < R$.

For the operator T we obtain the following

LEMMA 1. *Assume that the function f is Lipschitz continuous and $0 < a \le f \le b$. Then T is a contraction on $C^0(\bar{\Omega})$ for all sufficiently small $M > 0$.*

Proof. Let $\mu = (\int_\Omega f(\varphi))^{-p}$. For $\varphi_1, \varphi_2 \in C^0(\bar{\Omega})$ we have

$$\begin{aligned}|T(\varphi_1)(x) - T(\varphi_2)(x)| &= M\left|\mu_1 \int_\Omega G(x,y) f(\varphi_1(y))dy - \mu_2 \int_\Omega G(x,y) f(\varphi_2(y))dy\right| \\ &\le M\mu_1 \int_\Omega G(x,y)|f(\varphi_1(y)) - f(\varphi_2(y))|dy \\ &+ Mc(p,\Omega,f)\mu_1\mu_2 \int_\Omega G(x,y) f(\varphi_2(y))dy \int_\Omega |f(\varphi_1(y)) - f(\varphi_2(y))|dy \\ &\le MC(p,\Omega,f)|\varphi_1 - \varphi_2|_\infty.\end{aligned}$$

We used the fact that $\sup_{x\in\Omega} \int_\Omega G(x,y)dy < \infty$ whenever the boundary $\partial\Omega$ is sufficiently smooth, see [9].

For sufficiently small M we have $\alpha_M = MC(p,\Omega,f) < 1$.

LEMMA 2. *Assume that $f \in C^1(\mathbb{R})$. For a given $R > 0$ there exists $\bar{M}$ such that for $M < \bar{M}$, $T : B_R(0) \to B_R(0)$ and T is a contraction.*

Proof. Let us define the function f_R as $f_R(\varphi) = f(\varphi)$ for $|\varphi| < R$ and $f_R(\varphi) = f(R)$ for $|\varphi| \ge R$. We denote by T_R the operator (6) with $f = f_R$.

It follows from Lemma 1 that for sufficiently small M the operator T_R is a contraction with the contraction constant less than 1/2. For such M we have $T_R : B_R(0) \to B_R(0)$. It remains to note that $T_R = T$ on $B_R(0)$.

From Lemmas 1 and 2 we get

THEOREM 1. *If the solutions of (1)–(2) satisfy an a priori estimate $|\varphi|_\infty < R$ for $M < M_0$, then for sufficiently small M and $\varphi_0 \in C^0(\bar{\Omega})$ with $|\varphi_0|_\infty < R$ the Picard iteration sequence $T^n(\varphi_0)$ converges to the unique solution of (1)–(2).*

As we have seen, the key point of the above reasoning is the existence of an a priori estimate of solutions of (1)–(2). This condition is satisfied for the Poisson-Boltzmann problem of gravitational type in any bounded domain Ω of $\mathbb{R}^2$ and $M < 4\pi$.

In [12] such an a priori estimate of solutions of (1)–(2) has been proved under the assumptions that f is a positive decreasing differentiable function such that $\sup|f'/f| < +\infty$ and $0 < p \le 1$.

3. The iteration scheme for (1)–(2). We have for the local elliptic problem (3) the following fact from the general theory of PDE ([1]).

THEOREM 2. *If f is a positive decreasing function, then the problem (3) has a unique solution for each $\lambda > 0$.*

A modified iteration process for nonlocal elliptic problems has been defined in [15]: $\varphi_n = S(\varphi_{n-1}) = S^n(\varphi_0)$, where $S(\varphi_{n-1})$ is the unique solution of

$$-\Delta\varphi_n = \lambda_{n-1} f(\varphi_n) \quad \text{in} \quad \Omega, \tag{7}$$

$$\lambda_{n-1} = M\left(\int_\Omega f(\varphi_{n-1})\right)^{-p}, \tag{8}$$

$$\varphi_n|_{\partial\Omega} = 0, \quad n = 1, 2, \ldots\ , \tag{9}$$

and φ_0 is an arbitrary element of $C^0(\overline{\Omega})$.

For this new iteration scheme we have, cf. [15]:

THEOREM 3. *If f is a positive decreasing function, $\int_0^\infty f(s)ds = A < \infty$ and $0 < p \leq 2$, then for every $\varphi_0 \in C^0(\bar{\Omega})$ the sequence of iterations $S^n(\varphi_0)$ is convergent in the supremum norm to a solution of (1)–(2).*

Now the question is how this iteration procedure works for (1)-(2) in the case of an increasing function f.

4. Iterations for (1)–(2) with $f(\varphi) = e^\varphi$. Note that for an arbitrary increasing function f the local elliptic problem (3) may have more than one solution. Therefore, we should modify slightly our process of construction of the iteration sequence for (1)–(2). The idea to use minimal solutions of (3) seems to be reasonable. It is known that the minimal solution of (3) with a positive increasing $f \in C^1(\mathbb{R})$ exists and is unique, see [14].

Here we apply the iteration procedure (7)–(9) to the particular case of (1)–(2) with $f(\varphi) = e^\varphi$, $p = 1$ and $\Omega = K_1(0) = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$.

In this case (1)–(2) reads

$$-\Delta\varphi = M\frac{e^\varphi}{\int_\Omega e^\varphi} \quad \text{in} \quad \Omega, \quad \varphi|_{\partial\Omega} = 0. \tag{10}$$

It is known that (10) has a unique radially symmetric solution for $M < 8\pi$, and has no solution for $M \geq 8\pi$ ([3]).

The local problem

$$-\Delta\varphi = \lambda e^\varphi \text{ in } \Omega, \quad \varphi|_{\partial\Omega} = 0, \tag{11}$$

has the minimal solution

$$\varphi_1(r; \lambda) = \ln \frac{8(4 - \lambda - 2\sqrt{4 - 2\lambda})}{[\lambda + r^2(4 - \lambda - 2\sqrt{4 - 2\lambda})]^2}$$

and the maximal solution

$$\varphi_2(r; \lambda) = \ln \frac{8(4 - \lambda + 2\sqrt{4 - 2\lambda})}{[\lambda + r^2(4 - \lambda + 2\sqrt{4 - 2\lambda})]^2},$$

where $\lambda \in (0, 2]$, see [1].

We have

$$\lambda \int_\Omega e^{\varphi_1} = 2\pi\lambda \int_0^1 re^{\varphi_1(r;\lambda)}dr = 2\pi(2 - \sqrt{4 - 2\lambda}) \in (0, 4\pi]$$

and

$$\lambda \int_\Omega e^{\varphi_2} = 2\pi\lambda \int_0^1 re^{\varphi_2(r;\lambda)}dr = 2\pi(2 + \sqrt{4 - 2\lambda}) \in [4\pi, 8\pi).$$

Thus for $M \in (0, 4\pi]$, $\varphi_1(r; \lambda)$ is the solution of (10), where λ satisfies

$$\lambda = M\Big(\int_\Omega e^{\varphi_1}\Big)^{-1} = \frac{\lambda M}{2\pi(2 - \sqrt{4 - 2\lambda})}.$$

For $M \in [4\pi, 8\pi)$, $\varphi_2(r;\lambda)$ is the solution of (10) with λ satisfying

$$\lambda = M\left(\int_\Omega e^{\varphi_2}\right)^{-1} = \frac{\lambda M}{2\pi(2+\sqrt{4-2\lambda})}.$$

For $\bar\varphi \in C^0(\bar\Omega)$ let $S_1(\bar\varphi)$ ($S_2(\bar\varphi)$, resp.) denote the minimal (maximal, resp.) solution of (11) with

$$\lambda = M\left(\int_\Omega e^{\bar\varphi}\right)^{-1},$$

where $M \in (0, 4\pi]$ ($M \in [4\pi, 8\pi)$, resp.).

For $M \in (0, 4\pi]$ we define the mapping

$$F_1(\lambda) = M\left(\int_\Omega e^{\varphi_1}\right)^{-1} = \frac{\lambda M}{2\pi(2-\sqrt{4-2\lambda})}.$$

F_1 is continuous and decreasing for $\lambda \in (0,2]$, has a unique fixed point $\bar\lambda = M(8\pi - M)/(8\pi^2)$, and $\lim_{\lambda\to 0^+} F_1(\lambda) = M/\pi$, $\lim_{\lambda\to 2^-} F_1(\lambda) = M/2\pi$. We have $F_1'(\bar\lambda) = -M/(8\pi - 2M)$, hence the fixed point $\bar\lambda$ is stable for $M \in (0, 8\pi/3)$, i.e. $F_1^n(\lambda) \to \bar\lambda$ for λ from some neighbourhood of $\bar\lambda$, and unstable for $M \in (8\pi/3, 4\pi]$.

The well-defined iterations of a decreasing function may converge to a fixed point or to a periodic orbit of period two.

First, we note that the sequence of iterations $F_1^n(\lambda)$, $\lambda \in (0,2]$, is defined only for $M \in (0, 2\pi]$, see Fig. 1.

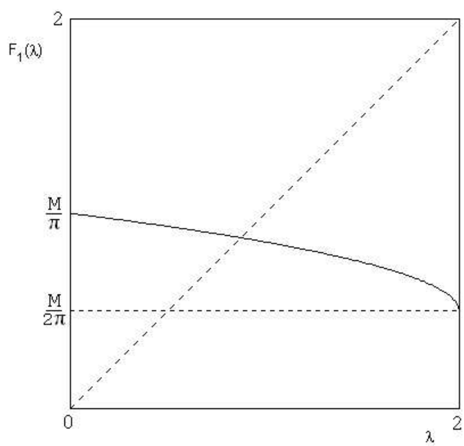

Fig. 1

After some lengthy calculations we observe that F_1^2 has a unique fixed point, and thus there exists no periodic point of F_1. Hence, taking $M \in (0, 2\pi]$ we get that the iterations $\lambda_n = F_1^n(\lambda)$ tend to a fixed point of F_1 for any $\lambda \in (0,2]$. This implies that for $\varphi_0 \in C^0(\bar\Omega)$ satisfying $M(\int_\Omega e^{\varphi_0})^{-1} \le 2$, the sequence $\varphi_{\lambda_n} \equiv \varphi_1(r;\lambda_n) = S_1^{n+1}(\varphi_0)$, $n = 0, 1, 2, \ldots$, $\lambda_n = F_1^n(\lambda_0)$, $\lambda_0 = M(\int_\Omega e^{\varphi_0})^{-1}$, is convergent in the supremum norm to a solution of (10).

In fact, using the integral form of (10) we have

$$\varphi_{\lambda_n}(x) = \frac{M}{\int_\Omega e^{\varphi_{\lambda_{n-1}}}} \int_\Omega G(x,y) e^{\varphi_{\lambda_n}(y)} dy, \quad n = 1, 2, \ldots .$$

Here, as before, $G(x,y)$ denotes the Green function corresponding to $-\Delta$ and the homogeneous boundary data. Applying the Lebesgue dominated convergence theorem we get

$$\varphi_{\bar\lambda}(x) = \frac{M}{\int_\Omega e^{\varphi_{\bar\lambda}}} \int_\Omega G(x,y) e^{\varphi_{\bar\lambda}(y)} dy,$$

which means that $\varphi_{\bar\lambda}$ is a solution of the Poisson-Boltzmann problem (10).

Now for $M \in (2\pi, 4\pi]$ we observe that for $M \in (2\pi, 2\pi(\sqrt{5}-1)]$ (for $M \in (2\pi(\sqrt{5}-1), 4\pi]$, respectively) F_1 maps the interval $[\lambda^*, 2]$ ($(\lambda^*, \lambda^{**})$) into itself, see Fig. 2 and Fig. 3. Here $\lambda^* = 16\pi(M-2\pi)/M^2$ and $\lambda^{**} = 128\pi^2(M-2\pi)(M^3 - 16\pi^2 M + 32\pi^3)/M^6$, they satisfy $F_1(\lambda^*) = 2$ and $F_1(\lambda^{**}) = \lambda^*$.

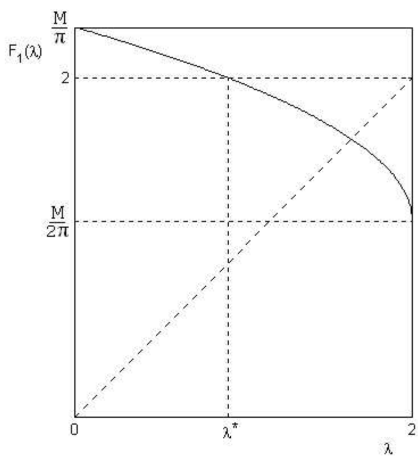

Fig. 2

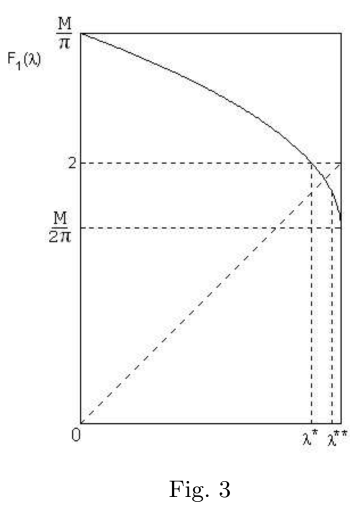

Fig. 3

When $M \in (2\pi, 2\pi(\sqrt{5}-1))$ we get, due to the stability of a fixed point $\bar{\lambda}$ and the absence of periodic points of F_1, that the sequence of iterations $\lambda_n = F_1^n(\lambda)$ tends to $\bar{\lambda}$ for $\lambda \in [\lambda^*, 2]$. For $M = 2\pi(\sqrt{5}-1)$ the convergence holds on $(\lambda^*, 2)$.

In the case $M \in (2\pi(\sqrt{5}-1), 8\pi/3)$ and $\lambda \in (\lambda^*, \lambda^{**})$, some periodic orbit occurs. Moreover, if $M \in (8\pi/3, 4\pi)$ then $\bar{\lambda}$ is an unstable fixed point of F_1, and hence φ_{λ_n} is not convergent to a solution of (10).

Thus, choosing an appropriate initial function $\varphi_0 \in C^0(\bar{\Omega})$, so that $\lambda_0 = M(\int_\Omega e^{\varphi_0})^{-1}$ lies in the domain of attraction of $\overline{\lambda}$, we can obtain the convergence of the sequence $\varphi_{\lambda_n} \equiv S_1^{n+1}(\varphi_0)$ to a solution of (10) for $M \in (2\pi, 2\pi(\sqrt{5}-1)]$.

Taking $M \in [4\pi, 8\pi)$ we get that the map

$$F_2(\lambda) = M\biggl(\int_\Omega e^{\varphi_2}\biggr)^{-1} = \frac{\lambda M}{2\pi(2+\sqrt{4-2\lambda})}$$

is a continuous increasing function of $\lambda \in (0, 2]$, has a unique fixed point in $(0, 2]$, and $\lim_{\lambda\to 0^+} F_2(\lambda) = 0$, $\lim_{\lambda\to 2^-} F_2(\lambda) = M/2\pi$, see Fig. 4. Thus the iterations $\lambda_n = F_2^n(\lambda)$ do not tend to a fixed point of F_2 for any $\lambda \in (0, 2]$.

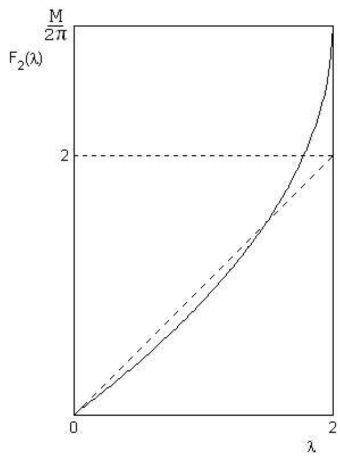

Fig. 4

The conclusion is that for the Poisson-Boltzmann problem of gravitational type we can expect the convergence of our iteration scheme to the solution of this problem only in a restricted range of parameter M.

5. Iterations for (1)–(2) with increasing f. In this section we study the iteration process for a more general class of problems (1)–(2) with an increasing function f.

The construction of the iteration scheme will be based on the theory of minimal solutions of (3). We require the following assumptions on f:

(f1) $f \in C^1(\mathbb{R})$;
(f2) f is positive;
(f3) f is increasing.

The assumptions (f1)-(f3) guarantee the following properties of the minimal solution of (3):

(P1) There exists $\Lambda \in (0, +\infty)$ such that the problem (3) has a unique minimal solution $\widetilde{\varphi}_\lambda$ for $\lambda \in (0, \Lambda)$ ([14]).

(P2) $\widetilde{\varphi}_\lambda$ is an increasing function of the parameter λ ([14]).

(P3) $\widetilde{\varphi}_\lambda$ is left continuous in λ, i.e. $\lim_{\lambda\to\mu^-} \widetilde{\varphi}_\lambda = \widetilde{\varphi}_\mu$ (see Section IV.2.3 in [1]).

(P4) $\lim_{\lambda\to 0^+} \widetilde{\varphi}_\lambda = 0$.

In order to describe the scheme of iterations for (1)–(2), we define $S(\bar{\varphi})$ for $\bar{\varphi} \in C^0(\bar{\Omega})$ as the minimal solution of (3) with

$$\lambda = M\bigg(\int_\Omega f(\bar{\varphi})\bigg)^{-p} \equiv H(\bar{\varphi}),$$

i.e. $S(\bar{\varphi}) = \widetilde{\varphi}_\lambda$, where $\lambda = H(\bar{\varphi})$. We note that H decreases because of (f3).

We put $\varphi_0 \equiv 0$. Let M be sufficiently small such that $\lambda_0 = H(0) < \Lambda$. We take $\varphi_n = S^n(0)$ as the iteration sequence for (1)–(2) satisfying

$$-\Delta\varphi_n = \lambda_{n-1} f(\varphi_n) \quad \text{in} \quad \Omega, \quad \lambda_{n-1} = H(\varphi_{n-1}), \quad \varphi_n|_{\partial\Omega} = 0, \quad n = 1, 2, \ldots\ .$$

The question is whether it converges to a solution of (1)–(2).

We have $\varphi_1 = \widetilde{\varphi}_{\lambda_0} \geq 0 = \varphi_0$, hence $\lambda_1 = H(\varphi_1) \leq H(\varphi_0) = \lambda_0$ which gives $\widetilde{\varphi}_{\lambda_1} \leq \widetilde{\varphi}_{\lambda_0}$, so $0 \leq \varphi_2 \leq \varphi_1$. The last inequality leads to $\lambda_2 = H(\varphi_2) \geq H(\varphi_1) = \lambda_1$, and thus we get $\varphi_3 = \widetilde{\varphi}_{\lambda_2} \geq \widetilde{\varphi}_{\lambda_1} = \varphi_2$. On the other hand from $\varphi_2 \geq 0 = \varphi_0$ it follows that $\lambda_2 = H(\varphi_2) \leq H(\varphi_0) = \lambda_0$, which implies $\varphi_3 = \widetilde{\varphi}_{\lambda_2} \leq \widetilde{\varphi}_{\lambda_0} = \varphi_1$. So we have $0 \leq \varphi_2 \leq \varphi_3 \leq \varphi_1$ and $\lambda_1 \leq \lambda_2 \leq \lambda_0$. In the same way, from $\varphi_3 \geq \varphi_2$ and $\varphi_3 \leq \varphi_1$ we obtain $\varphi_4 \leq \varphi_3$ and $\varphi_4 \geq \varphi_2$, respectively. Hence $0 \leq \varphi_2 \leq \varphi_4 \leq \varphi_3 \leq \varphi_1$ and $\lambda_1 \leq \lambda_3 \leq \lambda_2 \leq \lambda_0$ hold.

Continuing, we finally have

$$\varphi_0 \leq \varphi_2 \leq \varphi_4 \leq \ldots \leq \varphi_{2k} \leq \varphi_{2k-1} \leq \ldots \leq \varphi_3 \leq \varphi_1 \quad \text{in} \quad \Omega, \tag{12}$$

and

$$\lambda_1 \leq \lambda_3 \leq \lambda_5 \leq \ldots \leq \lambda_{2k-1} \leq \lambda_{2k} \leq \ldots \leq \lambda_4 \leq \lambda_2 \leq \lambda_0. \tag{13}$$

Thus the sequences φ_{2k} and φ_{2k-1} are convergent uniformly on Ω to some functions u and v, respectively. We see that $u \leq v \leq \varphi_1 = \widetilde{\varphi}_{\lambda_0}$, which gives an a priori estimate of u and v, so $|u|_\infty < R$, $|v|_\infty < R$, where $R = |\widetilde{\varphi}_{\lambda_0}|_\infty$. Hence u and v are small for sufficiently small M, because $\lambda_0 = M(\int_\Omega f(0))^{-p} \ll 1$ implies via (P4) that $|\widetilde{\varphi}_{\lambda_0}|_\infty \ll 1$.

It follows from (13) that $H(\varphi_{2k}) = \lambda_{2k} \to \check{\lambda} = H(u)$ and $H(\varphi_{2k-1}) = \lambda_{2k-1} \to \hat{\lambda} = H(v)$. Obviously we have $\hat{\lambda} \leq \check{\lambda} \leq \lambda_0$.

Moreover, the functions u and v are solutions of (3) with the parameters $\hat{\lambda}$ and $\check{\lambda}$, respectively, so u and v satisfy

$$-\Delta u = M\frac{f(u)}{(\int_\Omega f(v))^p} \quad \text{in} \quad \Omega, \tag{14}$$

$$-\Delta v = M\frac{f(v)}{(\int_\Omega f(u))^p} \quad \text{in} \quad \Omega, \tag{15}$$

$$u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \tag{16}$$

By (P3) u is a minimal solution, i.e. $u = \widetilde{u}_{\hat{\lambda}}$. Hence to prove the convergence of φ_n to a solution of (1)–(2), it remains to show that $u = v$, i.e. the problem (14)–(16) does not have a solution u, v in the ball $B_R(0) \subset C^0(\bar{\Omega})$ such that $u \neq v$. We show that this nonexistence result holds for sufficiently small M. First, we note that for $M \ll 1$ the problem (14)–(16) has a unique solution.

Indeed, let us introduce the function space $X = C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ with the norm $||(u,v)|| = |u|_\infty + |v|_\infty$.

We define the following operator T on X

$$T(u,v)(x) = \left(\frac{M}{(\int_\Omega f(v))^p} \int_\Omega G(x,y) f(u(y))dy, \frac{M}{(\int_\Omega f(u))^p} \int_\Omega G(x,y) f(v(y))dy \right). \tag{17}$$

Any fixed point of T is a solution of (14)–(16). Proceeding as in the proof of Theorem 1, we get that T defined by (17) is a contraction on the ball $B_R(0) \subset X$ for sufficiently small M.

It remains to note that, whenever the nonlocal elliptic problem (1)–(2) has a solution φ satisfying the estimate $|\varphi|_\infty \leq CM$, the functions $u = v = \varphi$ solve (14)–(16) and for sufficiently small M it must be a unique solution of this problem.

In this way we have proved the following

THEOREM 4. *Let f satisfy (f1)-(f3). Assume that for small M the problem (1)–(2) has a solution φ with $|\varphi|_\infty \leq CM$. Then for sufficiently small M the iteration sequence $\varphi_n = S^n(0)$ is convergent in the supremum norm to a solution of (1)–(2).*

It is known that if f is a continuous positive increasing function and $\lim_{z\to+\infty} z/f(z) > M|\Omega|^{-p} \sup_{x\in\Omega} \int_\Omega G(x,y)dy$, then (1)–(2) has a solution φ which satisfies an a priori estimate $|\varphi|_\infty \leq CM$, see [12].

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