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IDENTITIES IN LAW BETWEEN QUADRATIC FUNCTIONALS OF BIVARIATE GAUSSIAN PROCESSES, THROUGH FUBINI THEOREMS AND SYMMETRIC PROJECTIONS

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Abstract. We present three new identities in law for quadratic functionals of conditioned bivariate Gaussian processes. In particular, our results provide a two-parameter generalization of a celebrated identity in law, involving the path variance of a Brownian bridge, due to Watson (1961). The proof is based on ideas from a recent note by J.-R. Pycke (2005) and on the stochastic Fubini theorem for general Gaussian measures proved in Deheuvels *et al.* (2004).

1. Introduction. Let $b(s)$, $s \in [0,1]$, be a standard Brownian bridge on $[0,1]$, from 0 to 0, and let b_1 and b_2 be two independent copies of b. The aim of this note is to prove several bivariate generalizations of the following identity in law for the path variance of b,

$$\int_0^1 \left(b(s) - \int_0^1 b(u)du \right)^2 ds \overset{law}{=} \frac{1}{4} \int_0^1 [b_1(s)^2 + b_2(s)^2] ds, \tag{1}$$

known as *Watson's (duplication) identity* (see [12]; the reader is also referred to [9] for a detailed probabilistic discussion of (1)). More specifically, our aim is to establish a result analogous to (1) for the path variance of a *bivariate tied-down Brownian bridge* $\mathbf{B}_0$ on $[0,1]^2$, i.e. a process having the law of a standard Brownian sheet $\mathbf{W}$ conditioned to

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vanish on the edges of the square $[0,1]^2$. As discussed below, our bivariate generalizations of (1) involve four different types of "bridges" naturally attached to a given Brownian sheet $\mathbf{W}$. These four processes, along with the laws of their quadratic functionals, have been recently studied in [1].

The proof of our main result uses extensively the general *stochastic Fubini theorem*, for quadratic functionals of Gaussian processes, proved in [1] (but see also [3]), and has been inspired by the strikingly simple proof of Watson's identity given in [6] (see also the discussion contained in [7]). Such a proof is mainly based on a decomposition of the path of the random function $t \mapsto b(t)$ into the orthogonal sum of its symmetric and antisymmetric parts, around the value $t = 1/2$. We will see how this kind of decomposition can be naturally extended in the framework of bivariate functions.

The present paper is organized as follows. In Section 2 we introduce some notation. In Section 3, we state a version of the stochastic Fubini Theorem which is well adapted to the framework of this paper and we provide an alternative proof of such a result, based on the calculation of cumulants for double Wiener integrals. In Section 4 the main Theorem is stated and proved. Eventually, in Section 5 we apply our results to calculate: (a) the explicit Laplace transform of some quadratic functionals of bivariate Gaussian processes, and (b) the explicit Fourier transform of some double stochastic integrals with respect to conditioned bivariate processes. This completes some of the results obtained in [1] and [4].

2. General notation. For the rest of the paper, we will study Gaussian processes which may be expressed as suitable transformations of a standard Brownian motion or of a standard Brownian sheet. In particular, we will adopt the following notation:

- $W = \{W(t) : t \in [0,1]\}$ is a standard Brownian motion on $[0,1]$, initialized at 0;
- $b = \{b(t) : t \in [0,1]\}$ is a standard Brownian bridge on $[0,1]$, from 0 to 0;
- $\mathbf{W} = \{\mathbf{W}(t_1,t_2) : (t_1,t_2) \in [0,1]^2\}$ is a standard Brownian sheet on $[0,1]^2$ vanishing on the axes, that is, $\mathbf{W}$ is a centered Gaussian process such that, for every (t_1,t_2), $(s_1,s_2) \in [0,1]^2$,
$$\mathbb{E}[\mathbf{W}(t_1,t_2)\mathbf{W}(s_1,s_2)] = (t_1 \wedge s_1) \times (t_2 \wedge s_2);$$
- $\mathbf{B}^{(\mathbf{W})} = \{\mathbf{B}^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1]^2\}$ is the canonical *bivariate Brownian bridge* associated to $\mathbf{W}$, i.e.
$$\mathbf{B}^{(\mathbf{W})}(t_1,t_2) = \mathbf{W}(t_1,t_2) - t_1 t_2 \mathbf{W}(1,1);$$
- $\mathbf{B}_0^{(\mathbf{W})} = \{\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1]^2\}$ is the canonical *bivariate tied down Brownian bridge* associated to $\mathbf{W}$, i.e.
$$\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) = \mathbf{W}(t_1,t_2) - t_1 \mathbf{W}(1,t_2) - t_2 \mathbf{W}(t_1,1) + t_1 t_2 \mathbf{W}(1,1);$$
- $\mathbf{K}^{(\mathbf{W},i)} = \{\mathbf{K}^{(\mathbf{W},i)}(t_1,t_2) : (t_1,t_2) \in [0,1]^2\}$, $i = 1,2$, are the two canonical *Kiefer fields* (or *asymmetric bivariate bridges*) associated to $\mathbf{W}$, i.e.
$$\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2) = \mathbf{W}(t_1,t_2) - t_1 \mathbf{W}(1,t_2),$$
$$\mathbf{K}^{(\mathbf{W},2)}(t_1,t_2) = \mathbf{W}(t_1,t_2) - t_2 \mathbf{W}(t_1,1).$$

We assume that all the previous processes are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

REMARKS. (i) Conditionally on the event $\{\mathbf{W}(1,1)=0\}$, $\mathbf{W}$ is distributed as the unconditioned process $\mathbf{B}^{(\mathbf{W})}$. Moreover, for every (t_1,t_2), $(s_1,s_2)\in[0,1]^2$,

$$\mathbb{E}[\mathbf{B}^{(\mathbf{W})}(t_1,t_2)\mathbf{B}^{(\mathbf{W})}(s_1,s_2)] = (t_1\wedge s_1)\times(t_2\wedge s_2) - t_1s_1t_2s_2. \tag{2}$$

(ii) Conditionally on the event $\{\mathbf{W}(1,t)=\mathbf{W}(t,1)=0, \quad \forall t\in[0,1]\}$, $\mathbf{W}$ is distributed as the unconditioned process $\mathbf{B}_0^{(\mathbf{W})}$. In particular, for (t_1,t_2), $(s_1,s_2)\in[0,1]^2$,

$$\begin{aligned}\mathbb{E}[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)\mathbf{B}_0^{(\mathbf{W})}(s_1,s_2)] &= \mathbb{E}[b(t_1)b(s_1)]\times\mathbb{E}[b(t_2)b(s_2)] \\ &= (t_1\wedge s_1 - t_1s_1)\times(t_2\wedge s_2 - t_2s_2).\end{aligned} \tag{3}$$

(iii) Conditionally on $\{\mathbf{W}(1,t)=0, \quad \forall t\in[0,1]\}$, $\mathbf{W}$ is distributed as the unconditioned process $\mathbf{K}^{(\mathbf{W},1)}$, and moreover, for (t_1,t_2), $(s_1,s_2)\in[0,1]^2$,

$$\begin{aligned}\mathbb{E}[\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)\mathbf{K}^{(\mathbf{W},1)}(s_1,s_2)] &= \mathbb{E}[b(t_1)b(s_1)]\times\mathbb{E}[W(t_2)W(s_2)] \\ &= (t_1\wedge s_1 - t_1s_1)\times(t_2\wedge s_2).\end{aligned} \tag{4}$$

(iv) Conditionally on $\{\mathbf{W}(t,1)=0, \quad \forall t\in[0,1]\}$, $\mathbf{W}$ is distributed as the unconditioned process $\mathbf{K}^{(\mathbf{W},2)}$, and moreover, for (t_1,t_2), $(s_1,s_2)\in[0,1]^2$,

$$\begin{aligned}\mathbb{E}[\mathbf{K}^{(\mathbf{W},2)}(t_1,t_2)\mathbf{K}^{(\mathbf{W},2)}(s_1,s_2)] &= \mathbb{E}[W(t_1)W(s_1)]\times\mathbb{E}[b(t_2)b(s_2)] \\ &= (t_1\wedge s_1)\times(t_2\wedge s_2 - t_2s_2).\end{aligned} \tag{5}$$

3. Stochastic Fubini identities. The following stochastic Fubini identity (8) will be useful for the proof of our main results. As shown in [3] and [1], stochastic Fubini identities for general Gaussian measures can be easily proved by means of a Laplace transform argument. Here, we shall present an alternative proof, which is based on the so called *diagram formulae* (see e.g. [11]) for the cumulants of double Wiener integrals. Note that, in what follows, we will write $d\lambda^m$, $m\geq 1$, to indicate Lebesgue measure on $\Re^m$.

THEOREM 1 (Stochastic Fubini Theorem). *Under the above assumptions and notation, for every $\phi\in L^2([0,1]^4, d\lambda^4)$ there exist two measurable random functions*

$$\left\{\int_{[0,1]^2}\phi(t_1,t_2,x_1,x_2)\mathbf{W}(dx_1,dx_2) : (t_1,t_2)\in[0,1]^2\right\} \tag{6}$$

and

$$\left\{\int_{[0,1]^2}\phi(x_1,x_2,t_1,t_2)\mathbf{W}(dx_1,dx_2) : (t_1,t_2)\in[0,1]^2\right\}. \tag{7}$$

Moreover, the following distributional identity holds:

$$\begin{aligned}&\int_{[0,1]^2}\left[\int_{[0,1]^2}\phi(t_1,t_2,x_1,x_2)\mathbf{W}(dx_1,dx_2)\right]^2 dt_1dt_2 \\ &\qquad\overset{law}{=}\int_{[0,1]^2}\left[\int_{[0,1]^2}\phi(x_1,x_2,t_1,t_2)\mathbf{W}(dx_1,dx_2)\right]^2 dt_1dt_2.\end{aligned} \tag{8}$$

Proof. The existence of the two measurable random functions (6) and (7) follows from standard arguments. To obtain (8), start by defining the two kernels (contractions) on $L^2([0,1]^4)$

$$\Phi_1(t_1,t_2;s_1,s_2) = \int_{[0,1]^2} dx_1 dx_2 \phi(x_1,x_2,t_1,t_2)\phi(x_1,x_2,s_1,s_2),$$
$$\Phi_2(t_1,t_2;s_1,s_2) = \int_{[0,1]^2} dx_1 dx_2 \phi(t_1,t_2,x_1,x_2)\phi(s_1,s_2,x_1,x_2).$$

Then, a simple application of the multiplication formula for Wiener integrals (see for instance [2, p. 211]) shows that

$$\begin{aligned} \int_{[0,1]^2}\left[\int_{[0,1]^2} \phi(t_1,t_2,x_1,x_2)\mathbf{W}(dx_1,dx_2)\right]^2 dt_1dt_2 &= \|\phi\|^2 + I_2^{\mathbf{W}}(\Phi_1), \\ \int_{[0,1]^2}\left[\int_{[0,1]^2} \phi(x_1,x_2,t_1,t_2)\mathbf{W}(dx_1,dx_2)\right]^2 dt_1dt_2 &= \|\phi\|^2 + I_2^{\mathbf{W}}(\Phi_2), \end{aligned} \tag{9}$$

where $I_2^{\mathbf{W}}(\cdot)$ stands for a standard double Wiener integral with respect to $\mathbf{W}$ (see again [2]). Now define $\chi_m(Y)$, $m \geq 1$, to be the m-th cumulant of a given real valued random variable Y (see e.g. [11]). We recall that the law of a double Wiener integral is determined by its cumulants (see e.g. [10]). Moreover, we can apply the well known *diagram formulae* for cumulants of multiple stochastic integrals (as presented, for instance, in [8, Proposition 9 and Corollary 1]) to obtain that for every $m \geq 2$ there exists a combinatorial coefficient $c_m > 0$ (independent of ϕ) such that

$$\begin{aligned} \chi_m(I_2^{\mathbf{W}}(\Phi_1)) &= c_m \int_{[0,1]^{2m}} (d\lambda^2)^{\otimes m}\ \Phi_1(x_1^{(1)},x_2^{(1)};x_1^{(2)},x_2^{(2)}) \\ &\quad \times\Phi_1(x_1^{(2)},x_2^{(2)};x_1^{(3)},x_2^{(3)}) \times \cdots \times \Phi_1(x_1^{(m)},x_2^{(m)};x_1^{(1)},x_2^{(1)}) \\ &= c_m \int_{[0,1]^{2m}} (d\lambda^2)^{\otimes m}\ \Phi_2(x_1^{(1)},x_2^{(1)};x_1^{(2)},x_2^{(2)}) \\ &\quad \times\Phi_2(x_1^{(2)},x_2^{(2)};x_1^{(3)},x_2^{(3)}) \times \cdots \times \Phi_2(x_1^{(m)},x_2^{(m)};x_1^{(1)},x_2^{(1)}) \\ &= \chi_m(I_2^{\mathbf{W}}(\Phi_2)), \end{aligned} \tag{10}$$

where the second equality can be proved by using a standard (deterministic) Fubini theorem, as well as the definition of Φ_1 and Φ_2. Since (10) holds for every m, we obtain that $I_2^{\mathbf{W}}(\Phi_1) \overset{law}{=} I_2^{\mathbf{W}}(\Phi_2)$, and the proof of (8) is therefore concluded, due to (9). ∎

As shown in [1], by specializing (8) to the kernels

$$\begin{aligned} \phi^{(1)}(t_1,t_2;x_1,x_2) &= \mathbf{1}_{[0,t_1]}(x_1)\mathbf{1}_{[0,t_2]}(x_2) - t_1t_2, \\ \phi^{(2)}(t_1,t_2;x_1,x_2) &= \mathbf{1}_{[0,t_1]}(x_1)\mathbf{1}_{[0,t_2]}(x_2) - t_1\mathbf{1}_{[0,t_2]}(x_2) - t_2\mathbf{1}_{[0,t_1]}(x_1) + t_1t_2, \\ \phi^{(3)}(t_1,t_2;x_1,x_2) &= \mathbf{1}_{[0,t_1]}(x_1)\mathbf{1}_{[0,t_2]}(x_2) - t_1\mathbf{1}_{[0,t_2]}(x_2), \\ \phi^{(4)}(t_1,t_2;x_1,x_2) &= \mathbf{1}_{[0,t_1]}(x_1)\mathbf{1}_{[0,t_2]}(x_2) - t_2\mathbf{1}_{[0,t_1]}(x_1), \end{aligned}$$

we obtain the following

COROLLARY 2. *Let the above notation and assumptions hold. Then,*

$$\int_{[0,1]^2} \mathbf{B}^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \int_{[0,1]^2} \left[\mathbf{W}(t_1,t_2) - \int_{[0,1]^2} \mathbf{W}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2, \quad (11)$$

$$\int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \int_{[0,1]^2} \left[\mathbf{W}(t_1,t_2) - \int_{[0,1]} \mathbf{W}(t_1,u_2) du_2 \right.$$
$$\left. - \int_{[0,1]} \mathbf{W}(u_1,t_2) du_1 + \int_{[0,1]^2} \mathbf{W}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2, \quad (12)$$

$$\int_{[0,1]^2} \mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \int_{[0,1]^2} \left[\mathbf{W}(t_1,t_2) - \int_{[0,1]} \mathbf{W}(u_1,t_2) du_1 \right]^2 dt_1 dt_2, \quad (13)$$

$$\int_{[0,1]^2} \mathbf{K}^{(\mathbf{W},2)}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \int_{[0,1]^2} \left[\mathbf{W}(t_1,t_2) - \int_{[0,1]} \mathbf{W}(t_1,u_2) du_2 \right]^2 dt_1 dt_2. \quad (14)$$

4. Bivariate Watson's identities

4.1. *Main results.* The next Theorem, which contains the announced bivariate versions of Watson's duplication identity (1), is the main result of the section. Note that each of the three parts of the statement involves a different notion of path variance for the process $\mathbf{B}_0^{(\mathbf{W})}$.

THEOREM 3. *Let* $\mathbf{W}$ *be a standard Brownian sheet on* $[0,1]^2$*, and let* $\mathbf{W}_i$*,* $i = 1,2,3,4$*, be four independent copies of* $\mathbf{W}$*. Then,*

1.

$$\int_{[0,1]^2} \left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2$$
$$\overset{law}{=} \frac{1}{16} \int_{[0,1]^2} [\mathbf{B}^{(\mathbf{W}_1)}(t_1,t_2)^2 + \mathbf{K}^{(\mathbf{W}_2,1)}(t_1,t_2)^2 + \mathbf{K}^{(\mathbf{W}_3,2)}(t_1,t_2)^2 + \mathbf{B}_0^{(\mathbf{W}_4)}(t_1,t_2)^2] dt_1 dt_2.$$

2.

$$\int_{[0,1]^2} \left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1,u_2) du_2 \right]^2 dt_1 dt_2$$
$$\overset{law}{=} \frac{1}{4} \int_{[0,1]^2} [\mathbf{B}_0^{(\mathbf{W}_1)}(t_1,t_2)^2 + \mathbf{B}_0^{(\mathbf{W}_2)}(t_1,t_2)^2] dt_1 dt_2.$$

3.

$$\int_{[0,1]^2} \left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1,u_2) du_2 \right.$$
$$\left. - \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1,t_2) du_1 + \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2$$
$$\overset{law}{=} \frac{1}{16} \int_{[0,1]^2} \sum_{i=1}^{4} \mathbf{B}_0^{(\mathbf{W}_i)}(t_1,t_2)^2 dt_1 dt_2.$$

As anticipated, our proof of the above results is inspired by a proof of (1) recently given by J.-R. Pycke in [6], where the author uses a decomposition of the elements of $L^2([0,1],dx) = L^2([0,1])$ into the orthogonal sum of a symmetric and an antisymmetric function, around the value $x = 1/2$. Before proving Theorem 3, we shall discuss in some detail the content of [6].

To this end, define for any $f \in L^2([0,1])$ the two operators

$$Sf(x) = \frac{1}{2}(f(x) + f(1-x)) \quad \text{and} \quad Af(x) = \frac{1}{2}(f(x) - f(1-x)), \quad x \in [0,1],$$

and observe that $f(x) = (A+S)f(x)$, $Sf(x) = Sf(1-x)$ and $Af(x) = -Af(1-x)$. Moreover, for any $f, g \in L^2([0,1])$,

$$\int_0^1 Af(x)Sg(x)dx = 0. \tag{15}$$

Note also that if f is constant, then $Sf = f$ and $Af = 0$.

Remark. Let H_s be the closed subspace of $L^2([0,1])$ generated by functions f satisfying $f(x) = f(1-x)$ for almost every x, and let H_a be the subspace generated by functions g such that $g(x) = -g(1-x)$ for almost every x. Then, (15) implies that $H_s \perp H_a$, where $\perp$ indicates orthogonality in $L^2([0,1])$, and also $L^2([0,1]) = H_s \oplus H_a$. Moreover for every $f \in L^2([0,1])$, Sf and Af equal, respectively, the orthogonal projection of f on H_s, and the orthogonal projection of f on H_a.

The next Lemma is proved in [6], and is based on a simple computation of covariances.

Lemma 4. *Let b be a standard Brownian bridge on $[0,1]$, from 0 to 0. Then, the two processes*

$$Ab = \left\{ Ab(t) : t \in \left[0, \frac{1}{2}\right] \right\} \quad \text{and} \quad Sb = \left\{ Sb(t) : t \in \left[0, \frac{1}{2}\right] \right\}$$

are stochastically independent, and moreover

$$Ab \overset{law}{=} \left\{ \frac{b(2t)}{2} : t \in \left[0, \frac{1}{2}\right] \right\} \quad \text{and} \quad Sb \overset{law}{=} \left\{ \frac{W(2t)}{2} : t \in \left[0, \frac{1}{2}\right] \right\}. \tag{16}$$

Lemma 4 yields an immediate proof of Watson's duplication identity (1). As a matter of fact, one can write, due to (15) and symmetry,

$$\int_0^1 \left(b(s) - \int_0^1 b(u)du \right)^2 = 2 \int_0^{\frac{1}{2}} \left[\left(Sb(t) - 2\int_0^{\frac{1}{2}} Sb(u)du \right)^2 + (Ab(t))^2 \right] dt,$$

and then use the relations

$$2\int_0^{\frac{1}{2}} (Ab(t))^2 dt \overset{law}{=} \frac{1}{2}\int_0^{\frac{1}{2}} b(2t)^2 dt = \frac{1}{4}\int_0^1 b(v)^2 dv$$

where the identity in law stems from the first part of (16), and

$$\begin{aligned} 2\int_0^{\frac{1}{2}} \left(Sb(t) - 2\int_0^{\frac{1}{2}} Sb(u)du \right)^2 dt &\overset{law}{=} \frac{1}{2}\int_0^{\frac{1}{2}} \left(W(2t) - 2\int_0^{\frac{1}{2}} W(2u)du \right)^2 dt \\ &= \frac{1}{4}\int_0^1 \left(W(v) - \int_0^1 W(z)dz \right)^2 dv \overset{law}{=} \frac{1}{4}\int_0^1 b(v)^2 dv \end{aligned}$$

where the first identity in law derives again from (16), and the second follows from a stochastic Fubini identity such as the one proved e.g. in [3].

In the next subsection we show that the content of Lemma 4 provides some key elements to achieve the proof of Theorem 3.

4.2. *Proof of Theorem 3.* To prove Theorem 3 we start by defining, for every function F on $[0,1]^2$, the four operators

$$\begin{aligned} S_1F(x_1,x_2) &= \frac{1}{2}[F(x_1,x_2)+F(1-x_1,x_2)], \\ S_2F(x_1,x_2) &= \frac{1}{2}[F(x_1,x_2)+F(x_1,1-x_2)], \\ A_1F(x_1,x_2) &= \frac{1}{2}[F(x_1,x_2)-F(1-x_1,x_2)], \\ A_2F(x_1,x_2) &= \frac{1}{2}[F(x_1,x_2)-F(x_1,1-x_2)], \end{aligned} \tag{17}$$

where $(x_1,x_2)\in[0,1]^2$, as well as

$$\begin{aligned} T^{(1)}F(x_1,x_2) &= S_1S_2F(x_1,x_2) = S_2S_1F(x_1,x_2), \\ T^{(2)}F(x_1,x_2) &= S_1A_2F(x_1,x_2) = A_2S_1F(x_1,x_2), \\ T^{(3)}F(x_1,x_2) &= A_1S_2F(x_1,x_2) = S_2A_1F(x_1,x_2), \\ T^{(4)}F(x_1,x_2) &= A_1A_2F(x_1,x_2) = A_2A_1F(x_1,x_2). \end{aligned} \tag{18}$$

Note that $F=\sum_{i=1,\dots,4}T^{(i)}F$, and also note the following symmetric and antisymmetric properties: for every $(x_1,x_2)\in[0,1]^2$,

$$\begin{aligned} T^{(1)}F(x_1,x_2) &= T^{(1)}F(1-x_1,x_2) = T^{(1)}F(x_1,1-x_2), \\ T^{(2)}F(x_1,x_2) &= T^{(2)}F(1-x_1,x_2) = -T^{(2)}F(x_1,1-x_2), \\ T^{(3)}F(x_1,x_2) &= -T^{(3)}F(1-x_1,x_2) = T^{(3)}F(x_1,1-x_2), \\ T^{(4)}F(x_1,x_2) &= -T^{(4)}F(1-x_1,x_2) = -T^{(4)}F(x_1,1-x_2). \end{aligned}$$

This implies that, if F is constant, then $T^{(1)}F=F$, and $T^{(i)}F=0$ for each $i=2,3,4$. By using (15) we have moreover that, for $i\neq j$ and $F,G\in L^2([0,1]^2,dx_1dx_2)=L^2([0,1]^2)$,

$$\int_{[0,1]^2} T^{(i)}F(x_1,x_2)T^{(j)}G(x_1,x_2)dx_1dx_2 = 0,$$

so that

$$\int_{[0,1]^2} F(x_1,x_2)^2dx_1dx_2 = 4\sum_{i=1}^{4}\int_{[0,\frac{1}{2}]^2} T^{(i)}F(x_1,x_2)^2dx_1dx_2. \tag{19}$$

Remark. Let us introduce four closed subspaces of $L^2([0,1]^2)$: (i) $H^{(1)}$ is the space generated by functions that are symmetric around the two axes $x_1=1/2$ and $x_2=1/2$; (ii) $H^{(2)}$ is the space generated by functions that are symmetric around the axis $x_1=1/2$ and antisymmetric around $x_2=1/2$; (iii) $H^{(3)}$ is the space generated by functions F that are antisymmetric around $x_1=1/2$ and symmetric around $x_2=1/2$; (iv) $H^{(4)}$ is the space generated by functions F that are antisymmetric around the two axes $x_1=1/2$

and $x_2 = 1/2$. Then, the above relations imply that such spaces are mutually orthogonal in $L^2([0,1]^2)$, and $L^2([0,1]^2) = \oplus_i H^{(i)}$. Moreover, for $i = 1, ..., 4$, $T^{(i)}$, as defined in (18), coincides with the orthogonal projection operator on $H^{(i)}$. To conclude, observe that, by using standard tensor product notation

$$H^{(1)} = H_s \otimes H_s, \qquad H^{(2)} = H_s \otimes H_a,$$
$$H^{(3)} = H_a \otimes H_s, \qquad H^{(4)} = H_a \otimes H_a,$$

so that $L^2([0,1]^2) = (H_s \oplus H_a) \otimes (H_s \oplus H_a)$, where the spaces H_s, $H_a \subset L^2([0,1])$ have been defined in the previous subsection.

4.3. *Proof of part 1.* An easy calculation of covariances, based on the product formula (3) and Lemma 4, implies that the two bivariate processes

$$\{A_1 \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1/2] \times [0,1]\}$$
$$\text{and } \{S_1 \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1/2] \times [0,1]\}$$

are stochastically independent, and an analogous conclusion holds for the two processes

$$\{A_2 \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1] \times [0,1/2]\}$$
$$\text{and } \{S_2 \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1] \times [0,1/2]\}.$$

This entails immediately that the four (jointly) Gaussian processes

$$\{T^{(i)} \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2) \in [0,1/2]^2\}, \ \ i = 1, ..., 4,$$

are mutually independent. Now, by applying (19) to the random continuous function

$$(t_1,t_2) \mapsto \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) du_1 du_2$$

we obtain, thanks to symmetry,

$$\int_{[0,1]^2} \left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_{[0,1]^2} du_1 du_2 \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) \right]^2 dt_1 dt_2$$
$$= 4 \int_{[0,\frac{1}{2}]^2} \left[T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - 4 \int_{[0,\frac{1}{2}]^2} T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2$$
$$+ 4 \sum_{i=2}^{4} \int_{[0,\frac{1}{2}]^2} T^{(i)} \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2.$$

Since for any Brownian sheet $\mathbf{W}$

$$\mathbf{K}^{(\mathbf{W},1)}(t_2,t_1) \overset{law}{=} \mathbf{K}^{(\mathbf{W},2)}(t_1,t_2)$$

where the identity holds for the two processes as a whole, the proof of Theorem 3 is achieved once the following three identities in law are shown,

$$4 \int_{[0,\frac{1}{2}]^2} \left[T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - 4 \int_{[0,\frac{1}{2}]^2} T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(u_1,u_2) du_1 du_2 \right]^2 dt_1 dt_2$$
$$\overset{law}{=} \frac{1}{16} \int_{[0,1]^2} \mathbf{B}^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2, \qquad (20)$$

$$4\int_{[0,\frac{1}{2}]^2} T^{(2)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \frac{1}{16}\int_{[0,1]^2}\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)^2 dt_1 dt_2,$$

$$\overset{law}{=} 4\int_{[0,\frac{1}{2}]^2} T^{(3)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2, \qquad (21)$$

$$4\int_{[0,\frac{1}{2}]^2} T^{(4)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2 \overset{law}{=} \frac{1}{16}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1 dt_2. \qquad (22)$$

To prove (20), just observe that Lemma 4 and (5) entail

$$\{S_1\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2)\in[0,1/2]\times[0,1]\}$$
$$\overset{law}{=} \{2^{-\frac{1}{2}}\mathbf{K}^{(\mathbf{W},2)}(t_1,t_2) : (t_1,t_2)\in[0,1/2]\times[0,1]\} \qquad (23)$$

and therefore

$$\{T^{(1)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2)\in[0,1/2]^2\} \overset{law}{=} \{2^{-1}\mathbf{W}(t_1,t_2) : (t_1,t_2)\in[0,1/2]^2\} \qquad (24)$$

so that

$$4\int_{[0,\frac{1}{2}]^2}\left[T^{(1)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - 4\int_{[0,\frac{1}{2}]^2} T^{(1)}\mathbf{B}_0^{(\mathbf{W})}(u_1,u_2)du_1du_2\right]^2 dt_1dt_2$$
$$\overset{law}{=} \int_{[0,\frac{1}{2}]^2}\left[\mathbf{W}(t_1,t_2) - 4\int_{[0,\frac{1}{2}]^2}\mathbf{W}(u_1,u_2)du_1du_2\right]^2 dt_1dt_2$$
$$\overset{law}{=} \frac{1}{4}\int_{[0,\frac{1}{2}]^2}\left[\mathbf{W}(2t_1,2t_2) - 4\int_{[0,\frac{1}{2}]^2}\mathbf{W}(2u_1,2u_2)du_1du_2\right]^2 dt_1dt_2$$
$$= \frac{1}{16}\int_{[0,1]^2}\left[\mathbf{W}(s_1,s_2) - \int_{[0,1]^2}\mathbf{W}(v_1,v_2)dv_1dv_2\right]^2 ds_1ds_2$$
$$\overset{law}{=} \frac{1}{16}\int_{[0,1]^2}\mathbf{B}^{(\mathbf{W})}(s_1,s_2)^2 ds_1ds_2$$

where the last equality is a consequence of a stochastic Fubini theorem, and namely of relation (11) in the statement of Corollary 2.

To prove (21), we use (23), (5) and Lemma 4 to obtain that

$$\{T^{(2)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2)\in[0,1/2]^2\} \overset{law}{=} \{2^{-\frac{3}{2}}\mathbf{K}^{(\mathbf{W},2)}(t_1,2t_2) : (t_1,t_2)\in[0,1/2]^2\}$$

and eventually

$$4\int_{[0,\frac{1}{2}]^2} T^{(2)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2 dt_1dt_2 \overset{law}{=} \frac{1}{2}\int_{[0,\frac{1}{2}]^2}\mathbf{K}^{(\mathbf{W},2)}(t_1,2t_2)^2 dt_1dt_2$$
$$\overset{law}{=} \frac{1}{4}\int_{[0,\frac{1}{2}]^2}\mathbf{K}^{(\mathbf{W},2)}(2t_1,2t_2)^2 dt_1dt_2 = \frac{1}{16}\int_{[0,1]^2}\mathbf{K}^{(\mathbf{W},2)}(u_1,u_2)^2 du_1du_2.$$

The case of $T^{(3)}$ can be treated analogously by using (4). To conclude, we note that

$$\{A_1\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) : (t_1,t_2)\in[0,1/2]\times[0,1]\}$$
$$\overset{law}{=} \{2^{-1}\mathbf{B}_0^{(\mathbf{W})}(2t_1,t_2) : (t_1,t_2)\in[0,1/2]\times[0,1]\}$$

and therefore

$$\{T^{(4)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2):(t_1,t_2)\in[0,1/2]^2\}\overset{law}{=}\{2^{-2}\mathbf{B}_0^{(\mathbf{W})}(2t_1,2t_2):(t_1,t_2)\in[0,1/2]^2\},$$

so that

$$\begin{aligned}4\int_{[0,\frac{1}{2}]^2}T^{(4)}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2 &\overset{law}{=}\frac{1}{4}\int_{[0,\frac{1}{2}]^2}\mathbf{B}_0^{(\mathbf{W})}(2t_1,2t_2)^2dt_1dt_2\\ &= \frac{1}{16}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(u_1,u_2)^2du_1du_2.\end{aligned}$$

4.4. *Proof of part 2.* We write

$$\begin{aligned}&\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_0^1\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\\ &\qquad= S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_0^1 S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2+A_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2),\end{aligned}$$

where the operators S_2 and A_2 are defined in (17). Since $S_2=T^{(1)}+T^{(3)}$ and $A_2=T^{(2)}+T^{(4)}$, we can use orthogonality and symmetry to obtain

$$\begin{aligned}&\int_{[0,1]^2}\left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_0^1\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\right]^2dt_1dt_2\\ &\quad=\int_{[0,1]^2}\left[S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_0^1 S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\right]^2dt_1dt_2\\ &\qquad+\int_{[0,1]^2}A_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2\\ &\quad=2\int_{[0,1]\times[0,1/2]}\left[S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-2\int_0^{\frac{1}{2}}S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\right]^2dt_1dt_2\\ &\qquad+2\int_{[0,1]\times[0,1/2]}A_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2.\end{aligned}$$

We already know that the restrictions to $[0,1]\times[0,1/2]$ of the two processes $S_2\mathbf{B}_0^{(\mathbf{W})}$ and $A_2\mathbf{B}_0^{(\mathbf{W})}$ are stochastically independent. Moreover Lemma 4 and (4) imply the two relations

$$\begin{aligned}&\{S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2):(t_1,t_2)\in[0,1]\times[0,1/2]\}\\ &\qquad\overset{law}{=}\{2^{-\frac{1}{2}}\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2):(t_1,t_2)\in[0,1]\times[0,1/2]\},\\ &\{A_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2):(t_1,t_2)\in[0,1]\times[0,1/2]\}\\ &\qquad\overset{law}{=}\{2^{-1}\mathbf{B}_0^{(\mathbf{W})}(t_1,2t_2):(t_1,t_2)\in[0,1]\times[0,1/2]\}.\end{aligned}\tag{25}$$

As a consequence, we obtain

$$\begin{aligned}2\int_{[0,1]\times[0,1/2]}A_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2 &\overset{law}{=}\frac{1}{2}\int_{[0,1]\times[0,1/2]}\mathbf{B}_0^{(\mathbf{W})}(t_1,2t_2)^2dt_1dt_2\\ &=\frac{1}{4}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2.\end{aligned}$$

To conclude the proof, use the first part of (25) and scaling to obtain

$$\begin{aligned}
&2\int_{[0,1]\times[0,1/2]}\left[S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-2\int_0^{\frac{1}{2}}S_2\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\right]^2dt_1dt_2\\
&\quad\overset{law}{=}\frac{1}{2}\int_{[0,1]\times[0,1/2]}\left[\mathbf{K}^{(\mathbf{W},1)}(t_1,2t_2)-2\int_0^{\frac{1}{2}}\mathbf{K}^{(\mathbf{W},1)}(t_1,2u_2)du_2\right]^2dt_1dt_2\\
&\quad=\frac{1}{4}\int_{[0,1]^2}\left[\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)-\int_0^1\mathbf{K}^{(\mathbf{W},1)}(t_1,u_2)du_2\right]^2dt_1dt_2.
\end{aligned}$$

Now define $\{\lambda_i, f_i : i \geq 1\}$ and $\{\gamma_i, g_i : i \geq 1\}$ to be the sequences of eigenvalues and eigenfunctions of the Hilbert-Schmidt operators associated to the covariance function, respectively of $t \mapsto b(t)$, and of

$$t \mapsto Z(t) := W(t) - \int_0^1 W(z)dz.$$

It is well known (see e.g. [5]) that there exist two sequences $\{\xi_i : i \geq 1\}$ and $\{\zeta_i : i \geq 1\}$ of i.i.d. standard Gaussian random variables such that the Karhunen-Loève expansions of b and Z are respectively given by

$$b(t) = \sum_{i\geq 1}\xi_i\sqrt{\lambda_i}f_i(t) \quad \text{and} \quad Z(t) = \sum_{i\geq 1}\zeta_i\sqrt{\gamma_i}g_i(t),$$

and moreover (see [3]) $\gamma_i = \lambda_i$ for every $i \geq 1$. Since (4) implies that, for every (t_1,t_2), $(s_1,s_2) \in [0,1]^2$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)-\int_0^1\mathbf{K}^{(\mathbf{W},1)}(t_1,u_2)du_2\right)\left(\mathbf{K}^{(\mathbf{W},1)}(s_1,s_2)-\int_0^1\mathbf{K}^{(\mathbf{W},1)}(s_1,u_2)du_2\right)\right]\\
&\qquad=\mathbb{E}[b(t_1)b(s_1)]\times\mathbb{E}\left[\left(W(t_2)-\int_0^1W(z)dz\right)\left(W(s_2)-\int_0^1W(z)dz\right)\right]
\end{aligned}$$

we conclude immediately (by using, for instance, [1, Lemma 4.1]) that the Karhunen-Loève expansion of the bivariate Gaussian process

$$\mathbf{Z}(s,t) = \mathbf{K}^{(\mathbf{W},1)}(s,t) - \int_0^1 \mathbf{K}^{(\mathbf{W},1)}(s,u)du$$

is given by

$$\mathbf{Z}(s,t) = \sum_{i,j\geq 1}\sqrt{\lambda_i\lambda_j}\theta_{ij}f_i(s)g_i(t),$$

where $\{\theta_{ij} : i,j \geq 1\}$ is an array of i.i.d. standard Gaussian random variables. This last relation entails that

$$\begin{aligned}
&\frac{1}{4}\int_{[0,1]^2}\left[\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)-\int_0^1\mathbf{K}^{(\mathbf{W},1)}(t_1,u_2)du_2\right]^2dt_1dt_2\\
&\qquad=\frac{1}{4}\sum_{i\geq 1}\lambda_i\sum_{j\geq 1}\lambda_j\theta_{ij}^2\overset{law}{=}\frac{1}{4}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2. \qquad (26)
\end{aligned}$$

To justify the last equality in law, just observe that, thanks again to [1, Lemma 4.1] and formula (3), the Karhunen-Loève expansion of $\mathbf{B}_0$ is given by

$$\sum_{i,j\geq 1} \sqrt{\lambda_i \lambda_j}\eta_{ij} g_i(s) g_i(t)$$

where $\{\eta_{ij} : i, j \geq 1\}$ is an array of i.i.d. standard Gaussian random variables (the reader is referred to [1] for a detailed discussion of Karhunen-Loève expansions for bivariate Gaussian processes).

4.5. *Proof of part 3.* We first observe that

$$\begin{aligned}
0 &= T^{(2)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 = T^{(4)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 \\
&= T^{(3)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 = T^{(4)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 \\
&= T^{(i)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1, u_2) du_1 du_2, \quad i = 2, 3, 4,
\end{aligned}$$

and

$$\begin{aligned}
T^{(i)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 &= \int_0^1 T^{(i)} \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1, \quad i = 1, 2, \\
T^{(j)} \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 &= \int_0^1 T^{(j)} \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2, \quad j = 1, 3.
\end{aligned}$$

As a consequence, by orthogonality and symmetry,

$$\begin{aligned}
&\int_{[0,1]^2} \Bigg[\mathbf{B}_0^{(\mathbf{W})}(t_1, t_2) - \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 \\
&\qquad - \int_0^1 \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 + \int_{[0,\frac{1}{2}]^2} \mathbf{B}_0^{(\mathbf{W})}(u_1, u_2) du_1 du_2 \Bigg]^2 dt_1 dt_2 \\
&= 4 \int_{[0,\frac{1}{2}]^2} \Bigg[T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(t_1, t_2) - 2 \int_0^{\frac{1}{2}} T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 \\
&\quad - 2 \int_0^{\frac{1}{2}} T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 + 4 \int_{[0,\frac{1}{2}]^2} T^{(1)} \mathbf{B}_0^{(\mathbf{W})}(u_1, u_2) du_1 du_2 \Bigg]^2 dt_1 dt_2 \\
&\quad + 4 \int_{[0,\frac{1}{2}]^2} \Bigg[T^{(2)} \mathbf{B}_0^{(\mathbf{W})}(t_1, t_2) - 2 \int_0^{\frac{1}{2}} T^{(2)} \mathbf{B}_0^{(\mathbf{W})}(u_1, t_2) du_1 \Bigg]^2 dt_1 dt_2 \\
&\quad + 4 \int_{[0,\frac{1}{2}]^2} \Bigg[T^{(3)} \mathbf{B}_0^{(\mathbf{W})}(t_1, t_2) - 2 \int_0^{\frac{1}{2}} T^{(3)} \mathbf{B}_0^{(\mathbf{W})}(t_1, u_2) du_2 \Bigg]^2 dt_1 dt_2 \\
&\quad + 4 \int_{[0,\frac{1}{2}]^2} \Bigg[T^{(4)} \mathbf{B}_0^{(\mathbf{W})}(t_1, t_2) \Bigg]^2 dt_1 dt_2 \\
&\stackrel{def}{=} Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

Since we know, thanks to the previous discussion, that the Q_i's are mutually independent, it is now sufficient to show that, for $i = 1, ..., 4$,

$$Q_i \stackrel{law}{=} \frac{1}{16} \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_i)}(t_1, t_2)^2 dt_1 dt_2. \tag{27}$$

We start with Q_2 (by symmetry, the case of Q_3 is handled analogously), and recall that we have already proved that

$$\begin{aligned} Q_i &\stackrel{law}{=} \frac{1}{4} \int_{[0,\frac{1}{2}]^2} \left[\mathbf{K}^{(\mathbf{W},2)}(2t_1, 2t_2) - 2 \int_0^{\frac{1}{2}} \mathbf{K}^{(\mathbf{W},2)}(2u_1, 2t_2) du_1 \right]^2 dt_1 dt_2 \\ &= \frac{1}{16} \int_{[0,1]^2} \left[\mathbf{K}^{(\mathbf{W},2)}(v_1, v_2) - \int_0^1 \mathbf{K}^{(\mathbf{W},2)}(z, v_2) dz \right]^2 dv_1 dv_2 \end{aligned}$$

so that (27) in the case $i = 2, 3$ derives immediately from (26). Since we have proved (27) for $i = 4$ (to obtain part 1 of Theorem 3) we are now left with the case $i = 1$.

To see that (27) holds also in this case, use (24) to write, after a standard change of variables,

$$\begin{aligned} Q_1 \stackrel{law}{=} \frac{1}{16} \int_{[0,1]^2} \Bigg[&\mathbf{W}(t_1, t_2) - \int_0^1 \mathbf{W}(t_1, u_2) du_2 \\ &- \int_0^1 \mathbf{W}(u_1, t_2) du_1 + \int_{[0,1]^2} \mathbf{W}(u_1, u_2) du_1 du_2 \Bigg]^2 dt_1 dt_2 \end{aligned}$$

and then apply relation (12) in Corollary 2.

Remark. Note that the techniques used for the proof of Therorem 3 could be also applied to the case of general n-variate Gaussian processes, for $n > 2$.

5. Application: Fourier transforms of double Wiener integrals with respect to conditioned Gaussian processes. Let the above notation hold, and let $\mathbf{W}_1$ and $\mathbf{W}_2$ be two independent Brownian sheets. In this section, we are interested in finding the explicit Fourier transform of the three double Wiener integrals

$$\begin{aligned} \mathbf{I} &= \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) \mathbf{B}^{(\mathbf{W}_2)}(dt_1, dt_2) \\ &= \int_{[0,1]^2} [\mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) - \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(u_1, u_2) du_1 du_2] \mathbf{B}^{(\mathbf{W}_2)}(dt_1, dt_2); \\ \mathbf{J} &= \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) \mathbf{K}^{(\mathbf{W}_2,2)}(dt_1, dt_2) \\ &= \int_{[0,1]^2} [\mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) - \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(t_1, u_2) du_2] \mathbf{K}^{(\mathbf{W}_2,2)}(dt_1, dt_2); \\ \mathbf{Y} &= \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) \mathbf{B}_0^{(\mathbf{W}_2)}(dt_1, dt_2) \\ &= \int_{[0,1]^2} [\mathbf{B}_0^{(\mathbf{W}_1)}(t_1, t_2) - \int_0^1 \mathbf{B}_0^{(\mathbf{W}_1)}(t_1, u_2) du_2 \\ &\quad - \int_0^1 \mathbf{B}_0^{(\mathbf{W}_1)}(u_1, t_2) du_1 + \int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W}_1)}(u_1, u_2) du_1 du_2] \mathbf{B}_0^{(\mathbf{W}_2)}(dt_1, dt_2). \end{aligned}$$

We shall show that such computations can be made by means of Theorem 3. To this end, we introduce some notation borrowed from [1]: for every $a \in \mathbb{C}$,

1. $C(a) = \prod_{j=1}^{\infty} \cosh\left(\frac{a}{j\pi}\right)$;
2. $C_{\mathrm{odd}}(a) = \prod_{j=0}^{\infty} \cosh\left[\frac{a}{(2j+1)\pi}\right]$;
3. $C_{\mathrm{even}}(a) = \prod_{j=1}^{\infty} \cosh\left[\frac{a}{2j\pi}\right] = C\left(\frac{a}{2}\right)$;
4. $S(a) = \prod_{j=1}^{\infty} \left[\pi j \sinh\left(\frac{a}{\pi j}\right)/a\right]$;
5. $S_{\mathrm{even}}(a) = \prod_{j=1}^{\infty} \left[\pi 2j \sinh\left(\frac{a}{\pi 2j}\right)/a\right] = S(a/2)$;
6. $S_{\mathrm{odd}}(a) = \prod_{j=1}^{\infty} \left[\pi(2j-1) \sinh\left(\frac{a}{\pi(2j-1)}\right)/a\right] = C(a/2)$;
7. $\mathcal{T}(a) = \sum_{j=0}^{\infty} \left\{\tanh\left(\frac{2a}{(2j+1)\pi}\right)[(2j+1)\pi]^{-1}\right\}$.

Moreover, we recall the following result:

Proposition 5 (see [1, Proposition 4.1]). *For every $u \in \Re$*

(i) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2} \mathbf{B}^{(\mathbf{W})}(s,t)^2 ds dt\right)\right] = \left(C_{odd}(2u)\frac{4\mathcal{T}(u)}{u}\right)^{-\frac{1}{2}}$;

(ii) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2} \mathbf{B}_0^{(\mathbf{W})}(s,t)^2 ds dt\right)\right] = \{S(u)\}^{-\frac{1}{2}}$;

(iii) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2} \mathbf{K}^{(\mathbf{W},1)}(s,t)^2 ds dt\right)\right] = \{S_{odd}(2u)\}^{-\frac{1}{2}}$.

Then, we have

Theorem 6. *Under the above assumptions and notation, for every $u \in \Re$*

1.

$$\begin{aligned}
&\mathbb{E}[\exp(iu\mathbf{I})] \\
&\quad = \mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2}\left[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2) - \int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(u_1,u_2)du_1du_2\right]^2 dt_1dt_2\right)\right] \\
&\quad = \left\{C_{odd}\left(\frac{u}{2}\right)\frac{16\mathcal{T}(u/4)}{u} \times S\left(\frac{u}{4}\right)\right\}^{-\frac{1}{2}} \times S_{odd}\left(\frac{u}{2}\right),
\end{aligned} \tag{28}$$

2.

$$\begin{aligned}
&\mathbb{E}[\exp(iu\mathbf{J})] \\
&\quad = \mathbb{E}\bigg[\exp\bigg(-\frac{u^2}{2}\int_{[0,1]^2}\bigg[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\bigg]^2 dt_1dt_2\bigg)\bigg] \\
&\quad = \bigg\{S\Big(\frac{u}{2}\Big)\bigg\}^{-1},
\end{aligned} \tag{29}$$

3.

$$\begin{aligned}
&\mathbb{E}[\exp(iu\mathbf{Y})] \\
&\quad = \mathbb{E}\bigg\{\exp\bigg(\int_{[0,1]^2}\bigg[\mathbf{B}_0^{(\mathbf{W}_1)}(t_1,t_2)-\int_0^1\mathbf{B}_0^{(\mathbf{W}_1)}(t_1,u_2)du_2 \\
&\qquad .-\int_0^1\mathbf{B}_0^{(\mathbf{W}_1)}(u_1,t_2)du_1+\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W}_1)}(u_1,u_2)du_1du_2\bigg]^2 dt_1dt_2\bigg)\bigg\} \\
&\quad = \bigg\{S\Big(\frac{u}{4}\Big)\bigg\}^{-2},
\end{aligned} \tag{30}$$

Proof. The first equality in (28) follows from conditioning and independence. To obtain the second just recall that Theorem 3 implies that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(-\frac{u^2}{2}\int_{[0,1]^2}\bigg[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(u_1,u_2)du_1du_2\bigg]^2 dt_1dt_2\bigg)\bigg] \\
&\qquad = \mathbb{E}\bigg[\exp\bigg(-\frac{(u/4)^2}{2}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2\bigg)\bigg] \\
&\qquad \times \mathbb{E}\bigg[\exp\bigg(-\frac{(u/4)^2}{2}\int_{[0,1]^2}\mathbf{B}^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2\bigg)\bigg] \\
&\qquad \times \mathbb{E}\bigg[\exp\bigg(-\frac{(u/4)^2}{2}\int_{[0,1]^2}\mathbf{K}^{(\mathbf{W},1)}(t_1,t_2)^2dt_1dt_2\bigg)\bigg]^2,
\end{aligned}$$

and the conclusion follows from Proposition 5. Likewise,

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(-\frac{u^2}{2}\int_{[0,1]^2}\bigg[\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)-\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,u_2)du_2\bigg]^2 dt_1dt_2\bigg)\bigg] \\
&\qquad = \mathbb{E}\bigg[\exp\bigg(-\frac{(u/2)^2}{2}\int_{[0,1]^2}\mathbf{B}_0^{(\mathbf{W})}(t_1,t_2)^2dt_1dt_2\bigg)\bigg]^2,
\end{aligned}$$

so that the proof is completed with another application of Proposition 5. Formula (30) is proved in exactly the same way. ∎

As pointed out in the Introduction, Theorem 6 extends part of the results contained in [1, Section 4] and [4].

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