

GEOMETRIC INFINITE DIVISIBILITY, STABILITY, AND SELF-SIMILARITY: AN OVERVIEW

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Abstract. The concepts of geometric infinite divisibility and stability extend the classical properties of infinite divisibility and stability to geometric convolutions. In this setting, a random variable X is geometrically infinitely divisible if it can be expressed as a random sum of N_p components for each $p \in (0, 1)$, where N_p is a geometric random variable with mean $1/p$, independent of the components. If the components have the same distribution as that of a rescaled X , then X is (strictly) geometric stable. This leads to broad classes of probability distributions closely connected with their classical counterparts. We review fundamental properties of these distributions and discuss further extensions connected with geometric sums, including multivariate and operator geometric stability, discrete analogs, and geometric self-similarity.

1. Introduction. The concepts of geometric stability and infinite divisibility, which goes back to [70], extends the classical properties of stability and infinite divisibility (ID) to geometric convolutions

$$S_p = \sum_{j=1}^{N_p} X_j, \quad (1)$$

where N_p is a geometric variable given by the probability density function (PDF)

$$P(N_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \dots \quad (2)$$

while the $\{X_j\}$ are independent and identically distributed (IID) variables independent of N_p . In this context, the distribution of X is geometrically infinitely divisible if for each

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$p \in (0, 1)$, X has the same distribution as S_p (where of course the distribution of the $\{X_j\}$ depends on p). Similarly, the distribution of X is (strictly) geometric stable if X has the same distribution as $a_p S_p$ for some deterministic a_p , where this time the $\{X_j\}$ in (1) have the same distribution as X . This leads to broad classes of probability distributions, closely connected with their classical counterparts. A sharp peak at the mode, similar to that of the Laplace distribution (which is geometric stable), is one characteristic feature of these distributions, in addition to their heavy tails and infinite divisibility. These properties, coupled with their relation to geometric sums, which appear quite naturally in many applied problems (see, e.g., [68]), make these distributions applicable in many diverse areas.

This is a brief review of geometric stable and infinitely divisible distributions, as well as other interesting and important classes of distributions that arise in connection with geometric random sums (1). We emphasize general mechanisms generating these distributions, their fundamental properties that are relevant in applications, and their relation to other well-known classes, most notably, univariate and multivariate as well as operator stable laws. Our journey begins with Sections 2 and 3, where we introduce the concepts of geometric stability and infinite divisibility. It continues with Section 4, where we review general geometric stable laws, which are the limiting distributions of geometric sums (1). Further extensions connected with multivariate geometric sums (1) normalized by linear operators are discussed in Section 5. Our final destination is Section 6, devoted to quite recent concepts and stochastic models connected with geometric stability and self-similarity of random processes.

2. Geometric stability. The origins of (strictly) *geometric stable* distributions are connected with invariance properties of renewal process with respect to *geometric thinning*. Suppose that each point of a renewal process $\{R(t), t \geq 0\}$ is retained with probability $p \in (0, 1)$ and deleted with probability $1 - p$, independently of other points, leading to the new point process $\{R_p(t), t \geq 0\}$, known as the p -thinning of R (see, e.g., [43]) or rarefaction (see [189]). Gnedenko [37] and Rényi [165], among others, characterized those renewal processes where R and R_p have the same distribution for each p (up to the change of time scale). Since the inter-arrival of R_p is the geometric sum (1) of N_p inter-arrivals X_j of R , we can express this invariance as the equality in distribution

$$X \stackrel{d}{=} a_p \sum_{j=1}^{N_p} X_j, \quad p \in (0, 1), \quad (3)$$

where $a_p > 0$ and X, X_1, X_2, \dots are IID, independent of N_p . Perhaps the best-known solution of (3), presented in [7], is the exponential distribution given by the Laplace transform (LT)

$$\psi(t) = \frac{1}{1 + \lambda t^\alpha}, \quad t > 0, \quad (4)$$

with $\alpha = 1$ and $a_p = p$ in (3), and the corresponding renewal process R is Poisson (see [165]). As shown in [37], the only other solution of (3) supported on the positive real line is that given by the LT (4) with some $0 < \alpha \leq 1$ and $a_p = p^{1/\alpha}$ in (3). This

is the *Mittag-Leffler* distribution (see, e.g., [61, 62, 63, 80, 89, 90, 126, 156, 192]), also known as the *positive Linnik* law (see [128, 151]), which plays a crucial role in heavy-tail renewal theory arising in connection with anomalous (fractional) diffusion (see, e.g., [45] and references therein). The latter terminology has to do with the fact that distributions with the *characteristic function* (ChF) given by

$$\psi(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha}, \quad t \in R, \quad (5)$$

with $0 < \alpha \leq 2$, known as (symmetric) Linnik distributions (see, e.g., [4, 5, 23, 67, 78, 79, 125, 127, 129, 150]), solve (3) within the class of symmetric distributions on the entire real line. Since the special case $\alpha = 2$ yields the familiar Laplace distribution with its characteristic peak at the mode (see [77] and references therein), Linnik laws (5) are also known as α -Laplace distributions (see [154]).

Note that the ChFs of all these distributions are related to those of stable distributions via the relation

$$\psi(t) = \frac{1}{1 - \log \phi(t)}, \quad (6)$$

where ϕ is the ChF of a degenerate distribution when ψ is exponential, stable subordinator when ψ is Mittag-Leffler, symmetric stable when ψ is symmetric Linnik, and normal when ψ is Laplace. These are all examples of *strictly geometric stable* (SGS) random variables, introduced in [70] as those variables that satisfy the stability relation (3). As shown in [70], the general class of SGS distributions is given by the ChF (6), where ϕ is the ChF of a strictly stable distribution (see, e.g., [172]). The same concept applies to distributions on R^d (see, e.g., [118]) or in general Banach spaces (see [160]). Let us also note that the relation (6) has an interpretation in terms of subordination, or randomly stopped Lévy processes: the ChF ψ corresponds to the variable $X(Z)$, where $\{X(t), t \geq 0\}$ is a Lévy process with $X(1)$ having the ChF ϕ , and Z is an independent exponential variable. It is clear in this connection that the distribution given by ψ is infinitely divisible (ID), with the ChF ψ^u corresponding to the random variable $X(Z(u))$, with $X(t)$ as before and $\{Z(t), t \geq 0\}$ being the gamma Lévy process (so that $Z(1)$ is exponential). Special cases of random variables with ChF ψ^u include generalized Mittag-Leffler (see, e.g., [2]), generalized Linnik (see, e.g., [32, 152]), and generalized Laplace (or Bessel function) distributions (see, e.g., [77]).

REMARK 2.1. In case of exponential distribution, where in (3) we have $a_p = p$, relation (3) written in terms of the exponential ChF ψ takes on the form

$$\psi(t) = \frac{p\psi(pt)}{1 - (1-p)\psi(pt)}, \quad (7)$$

or equivalently, $\psi(t) = \psi(pt)(p + (1-p)\psi(t))$. This shows that an exponential variable X solves the equation $X \stackrel{d}{=} W_1 X_1 + pX_2$, where X , X_1 , and X_2 are IID and W_1 has a Bernoulli distribution with parameter $1 - p = P(W_1 = 1)$, and is independent of X_1 and X_2 . More general equations of the form $X \stackrel{d}{=} W_1 X_1 + W_2 X_2$, with (W_1, W_2) such that $E(W_1 + W_2) = 1$, were investigated in [9].

REMARK 2.2. The stability property (3) of the exponential distribution under geometric summation has been observed empirically in a variety of settings (see references in [108]), most notably in connection with studying *growth periods* of currency exchange rates (see [99, 109]). Within a growth period, all consecutive log-returns $X_j = \log R_{j+1} - \log R_j$, where R_j is the exchange rate on day j , are positive. Moreover, the *cumulative* log-return over the growth period is given by the random sum S_p in (1), where the number of terms N_p is the *duration* of the growth period. A remarkable property of the exchange rates, reported in [99, 109], shows that cumulative log-return S_p *has the same distribution* (up to a scale factor) as a single log-return X_j (over one day) given that X_j is positive. This is essentially the stability property (3). An interesting mixed bivariate model (S_p, N_p) , where the N_p is geometric and the $\{X_j\}$ in (1) are IID exponential, was developed in this connection in [99] (see [11, 12, 100, 101, 102] for further extensions and applications of this model).

REMARK 2.3. The concept of geometric stability as defined in (3) needs a modification in order to be applicable to discrete distributions. Such modification appeared in [2, 14], where the multiplication by a_p was replaced by the *binomial thinning operator* \odot , introduced in [188] in connection with extending the concepts of self-decomposability and stability to this setting. Recall that for any discrete random variable N supported on the set of non-negative integers $Z_+ = \{0, 1, \dots\}$ and any $a \in (0, 1)$, we have

$$a \odot N = \sum_{j=1}^N I_j, \quad (8)$$

where the $\{I_j\}$ are IID indicator random variables with parameter a , independent of N . In this setting, a Z_+ -valued random variable X is *discrete strictly geometric stable* (DSGS) if for each $p \in (0, 1)$ there exist an $a_p \in (0, 1)$ such that $X \stackrel{d}{=} a_p \odot S_p$, where S_p is the sum (1) whose terms are IID copies of X , independent of the geometric number of terms N_p . As shown in [2], there is a correspondence between DSGS distributions and *discrete stable* laws (see [24, 188]), which can be stated as

$$G(s) = \frac{1}{1 - \log H(s)}, \quad (9)$$

where G is the probability generating function (PGF) of a DSGS distribution and H is the PGF of the corresponding discrete stable distribution. Since the PGF of any discrete stable distribution is of the form $H(s) = \phi(1 - s)$, where ϕ is the LT of a stable subordinator (see [188]), the PGF of a DSGS distribution is of the form $\psi(1 - s)$, where ψ is the Mittag-Leffler LT given in (4). Note that this distribution, called *discrete Mittag-Leffler* in [60, 157], describes a standard Poisson process $\{N(t), t \geq 0\}$, randomly stopped at a Mittag-Leffler distributed Z , which is independent of the process. See [2, 3, 14, 15, 20, 57, 151, 179, 180] for further properties and extensions, including autoregressive schemes with discrete stationary distributions.

REMARK 2.4. The problem of *random stability* in the spirit of (3) was considered in many works, including [5, 16, 17, 41, 42, 54, 55, 67, 70, 72, 75, 125]. It is now well-known that the geometric distribution is somewhat unique in that it allows the existence of distributions

satisfying the stability relation (3). The key property of the geometric distribution that is behind this is the fact that the class $\{G_p, p \in (0, 1)\}$ of geometric probability generating functions (PGF) forms a commutative semi-group under the operation of convolution: $G_p(G_q(s)) = G_q(G_p(s))$ for all $p, q \in (0, 1)$, where

$$G_p(s) = \frac{ps}{1 - (1-p)s} \quad (10)$$

is the PGF of the geometric distribution (2). Indeed, a notion of stability with respect to a general family of integer-valued random variables $\{N_p, p \in \Delta \subset (0, 1)\}$ requires that the corresponding PGFs commute (see [17, 42, 75]). Perhaps the easiest way to see this, is to note that if the distribution of X is stable with respect to N_p and N_q for some $p, q \in \Delta$, so that for some $a_p, a_q > 0$ we have

$$X \stackrel{d}{=} a_p \sum_{j=1}^{N_p} X_j \quad \text{and} \quad X \stackrel{d}{=} a_q \sum_{j=1}^{N_q} X_j, \quad (11)$$

where the $\{X_j\}$ are IID copies of X , then we will also have

$$X \stackrel{d}{=} a_p a_q \sum_{i=1}^{N_p} \sum_{j=1}^{N_q} X_{ij} \stackrel{d}{=} a_q a_p \sum_{i=1}^{N_q} \sum_{j=1}^{N_p} X_{ij}, \quad (12)$$

where the $\{X_{ij}\}$ are again IID copies of X , which essentially shows that $pq \in \Delta$ and X is stable with respect to N_{pq} , whose PGF is $G_{pq}(s) = N_p(N_q(s)) = N_q(N_p(s))$. There are very few such families with commuting PGFs, all related to the geometric distribution (see [17]). Thus, for most cases other than geometric, we do not have distributions with random stability in the spirit of (3). For example, no such stable distributions exist when N_p is a Poisson distribution with mean $-\log p$, $p \in (0, 1)$.

3. Geometric infinite divisibility. Geometric infinitely divisible random variables, introduced in [70], are those variables that can be decomposed into the sum of N_p IID random variables for each $p \in (0, 1)$, where N_p is a geometric random variable.

DEFINITION 3.1. A random variable in R^d is geometrically infinitely divisible (GID) if for each $p \in (0, 1)$ we have

$$X \stackrel{d}{=} X_1^{(p)} + X_2^{(p)} + \cdots + X_{N_p}^{(p)}, \quad (13)$$

where N_p is a geometric variable (2) and the $\{X_j^{(p)}\}$ are IID variables independent of N_p .

Writing equation (13) in terms of the ChFs we obtain

$$\psi(t) = \frac{p\psi_p(t)}{1 - (1-p)\psi_p(t)}, \quad (14)$$

where ψ and ψ_p are the ChFs of X and $X_1^{(p)}$, respectively. This shows that in order for X to be GID, the function $\psi_p(t) = \psi(t)[p + (1-p)\psi(t)]^{-1}$ must be a genuine ChF for each $p \in [0, 1]$. Clearly, all strictly geometric stable distributions given by the ChF (6) are GID. As shown in [70], all GID distributions are given by the ChF of the form (6) with ϕ being the ChF of an ID distribution in the classical sense. Thus, there is a one-to-one correspondence between ID and GID distributions, as there is between SGS and strictly

stable laws. Here, we again have an interpretation in terms of randomly stopped Lévy processes: the ChF ψ of a GID distribution corresponds to the variable $X(Z)$, where $\{X(t), t \geq 0\}$ is a Lévy process with $X(1)$ having the ChF ϕ , and Z is an independent exponential variable. This shows that all GID distributions are also ID in the classical sense.

REMARK 3.2. Note that the relation (14) can be written as

$$\psi(t) = \psi_p(t)\{p + (1-p)\psi(t)\}, \quad (15)$$

which in terms of random variables can be stated as

$$X \stackrel{d}{=} IX + X^{(p)}, \quad (16)$$

where X and $X^{(p)}$ are random variables with the ChFs ψ and ψ_p , respectively, I is a Bernoulli variable with $P(I = 1) = 1 - p$, and all the variables on the right-hand-side of (16) are mutually independent. Thus, the solution of the problem of finding all random variables X satisfying the relation (16), posed by Zolotarev (cf. [70]), is provided by the class of GID random variables. It is worth noting in this connection that only (and any) GID distribution can appear as the marginal distribution of X_n in stationary autoregressive schemes of the form

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p \\ \varepsilon_n + X_{n-1} & \text{with probability } 1 - p, \end{cases} \quad (17)$$

with ε_n having the ChF ψ_p . Such models appeared in [5, 58, 59, 61, 122, 123, 184, 185, 186, 187], see also [112].

REMARK 3.3. When defining the concept of GID for discrete distributions supported on the set of non-negative integers Z_+ , one may require that the distribution of the variables $\{X_j^{(p)}\}$ in (13) be also supported on Z_+ (see [2]). In this case, the relation (6) is usually stated as in (9) in terms of the PGFs rather than ChFs (see [2]), where G is the PGF of a discrete-GID distribution with ChF ψ and H is the PGF of the corresponding discrete-ID distribution with ChF ϕ . Since all discrete-ID distributions are compound Poisson, so that $H(s) = \exp\{-\lambda[1 - Q(s)]\}$ with some $\lambda > 0$ and PGF Q , every discrete-GID PGF admits the representation

$$G(s) = \frac{1}{1 + \lambda[1 - Q(s)]}. \quad (18)$$

Since the right-hand-side of (18) is the PGF of $\sum_{j=1}^{N_p-1} W_j$, where N_p is geometric (2) with the parameter $p = 1/(1 + \lambda)$ and the $\{W_j\}$ are IID with common PGF Q , and are independent of the N_p , we see that every discrete-GID distribution is compound geometric, as established in [2].

It is worth noting that the geometric random variable (2) is not GID (see [99]), while a shifted geometric random variable $N_p - 1$ is GID. Further, if X with the ChF ψ is GID, then the variable $X^{(p)}$ with the ChF ψ_p appearing in (14) is also GID for each $p \in (0, 1)$. In addition, the class of GID distributions is closed under geometric summation, that is the geometric sum $S = \sum_{j=1}^{N_q} X_j$ is GID whenever the X_j are IID and GID and N_q is geometric with parameter q , and independent of the $\{X_j\}$. This can be seen by noting

that whenever the ChF of X_j is of the form (6) with some ID ϕ then the ChF of S is also of this form but with ϕ replaced with $\phi^{1/q}$ (which is still ID). Let us also note that any mixture of a GID distribution with a point mass at zero is always GID, that is for each $q \in (0, 1)$ the ChF $q + (1 - q)\psi(t)$ is GID whenever ψ is GID, although the converse does not hold (the mixture can still be GID with ψ not being GID), see [114] for examples and further information. Finally, it should be noted that the class of GID distributions arises as the only limiting laws of *geometric* sums for triangular arrays, in a similar way as classical ID distributions arise in connection with deterministic sums. We refer the reader to [69, 146, 155, 158, 114, 173, 174, 175] for more information on GID, and to [2, 17, 42, 56, 75, 177, 178, 181] for generalizations to divisibility with respect to integer-valued distributions other than geometric.

4. Geometric stable distributions. With analogy to classical stable distributions, one can define a class of limiting distributions in the *geometric summation scheme* as follows (see [81, 142]).

DEFINITION 4.1. A random vector Y in R^d is said to be geometric stable (GS) with respect to the summation scheme if there exists a sequence of IID random vectors X_1, X_2, \dots , a geometric random variable N_p independent of all X_i , and constants $a_p > 0$ and $b_p \in R^d$ such that

$$a_p \sum_{i=1}^{N_p} (X_i + b_p) \xrightarrow{d} Y, \quad \text{as } p \rightarrow 0. \quad (19)$$

As shown in [71], GS distributions can be described in terms of their ChF via (6), where ϕ is a stable ChF. If the stable ChF ϕ corresponds to a strictly stable random vector, then Y is called *strictly* geometric stable (SGS), and appears as a weak limit in (19) with $b_p = 0$. If the distribution of a GS random variable Y is symmetric, then Y is called *symmetric* GS, and is necessarily SGS.

REMARK 4.2. One dimensional strictly GS laws defined as limiting distributions in (19) with $b_p = 0$ coincide with those mentioned in Section 2 in connection with the stability of geometric sums.

REMARK 4.3. The characteristic function representation of GS distributions follows from classical transfer theorems for random sums that go back to Robbins [167, 168], see, e.g., [27, 38, 39, 40, 42, 95, 96, 120, 169, 171, 190]. These results imply that whenever the $\{X_i\}$ are in the domain of attraction of a stable distribution with the ChF ϕ and the integer-valued random variables N_p converge to infinity in probability (as p converges to zero) such that pN_p converges in distribution to a non-negative random variable Z with cumulative distribution function (CDF) F and LT φ , and we have the convergence in (19), then the ChF of the limit is the convolution mixture

$$\psi(t) = \int_0^\infty \phi^z(t) dF(z) = \varphi(-\log \phi(t)). \quad (20)$$

In the geometric case, pN_p converges to the standard exponential distribution with the LT $\varphi(t) = 1/(1+t)$, which produces the basic relation (6). Different families of distributions

arise for other integer-valued variables $\{N_p, p \in \Delta \in (0, 1)\}$, whose generating functions do not need to be commutative, which is required to ensure the stability as discussed in Section 2. More information related to this can be found in [42, 75, 95, 96].

REMARK 4.4. If the convergence in (19) is along a particular subsequence, the resulting limiting distributions are called semi-geometric stable, with analogy to the semi-stable case. Their ChFs are still of the form (6), where this time ϕ is the ChF of a semi-stable law (see, e.g., [119, 153]). See [13, 26, 51, 60, 61, 63, 134, 135, 146, 154, 175] for further details.

4.1. GS characteristic function. A substitution of the stable ChF as given in [172] into (6) produces a GS ChF (see, e.g., [118]),

$$\psi(t) = \left[1 + \int_{S_d} |\langle t, s \rangle|^\alpha \omega_{\alpha,1}(\langle t, s \rangle) \gamma(ds) - i \langle t, m \rangle \right]^{-1}, \quad (21)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, $0 < \alpha \leq 2$, γ is a finite measure on the unit sphere S_d of R^d , $m \in R^d$, and

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sign}(x) \tan(\pi\alpha/2), & \text{if } \alpha \neq 1, \\ 1 + i\beta \frac{2}{\pi} \text{sign}(x) \log|x|, & \text{if } \alpha = 1. \end{cases} \quad (22)$$

As in the stable case, the parameter α is the *index of stability* that controls the tails, while γ is the *spectral measure* that reflects the dependence structure. For example, the association of the the components of a GS random vector implies certain conditions on the spectral measure (see [124]). However, in contrast with the stable case, the components can never be independent and the location parameter m is not a shift parameter, although it coincides with the mean whenever it exists ($\alpha > 1$). If $d = 1$, the ChF can be written as (see, e.g., [117])

$$\psi(t) = (1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t)^{-1}, \quad (23)$$

where $\beta \in [-1, 1]$ is a skewness parameter, while $\mu \in R$ and $\sigma \geq 0$ control location and scale, respectively. There are many other parameterizations of GS distributions, which parallel those for the stable case (see, e.g., [117]). Let us note that univariate GS distributions are supported on the entire real line, with the exception of the case $\alpha < 1$ and $|\beta| = 1$, where we obtain a distribution concentrated on $(0, \infty)$ for $\beta = 1$, $\mu > 0$ and on $(-\infty, 0)$ for $\beta = -1$, $\mu < 0$. More information on the univariate case can be found in [10, 36, 73, 69, 83, 84, 87, 88, 117].

4.2. Basic properties. The following fundamental representation, taken from [97], shows that GS distributions are location and scale mixtures of stable distributions.

PROPOSITION 4.5. *If Y is GS with index α , spectral measure γ and a location parameter m , and X is α -stable with the same spectral measure and location parameter 0, then*

$$Y \stackrel{d}{=} \begin{cases} mZ + Z^{1/\alpha}X, & \text{if } \alpha \neq 1, \\ mZ + (\frac{2}{\pi}Z \log Z)g + ZX, & \text{if } \alpha = 1, \end{cases} \quad (24)$$

where Z is a standard exponential random variable, independent of X , and

$$g = (g_1, \dots, g_d) = \int_{S_d} s\gamma(ds). \quad (25)$$

Many properties of GS distributions, such as their densities and distribution functions, moments, and tail behavior, follow from the stable case via (24). In particular, for any non-negative p_1, \dots, p_d , the joint moment $E\{|Y_1|^{p_1} \dots |Y_d|^{p_d}\}$ is finite if and only if $p_1 + \dots + p_d < \alpha$, whenever $\alpha \in (0, 2)$ and the coordinates of $Y = (Y_1, \dots, Y_d)$ are d -fold dependent (see, e.g., [172]), that is $\gamma\{s \in S_d : s_i \neq 0, i = 1, \dots, d\} > 0$. In one dimension, we simply have $E\{|Y|^p\} < \infty$ if and only if $p < \alpha$. Moreover, for any Borel set $B \subset S_d$, we have $x^\alpha P(\|Y\| > x, Y/\|Y\| \in B) \rightarrow c_\alpha \gamma(B)$ as $x \rightarrow \infty$, where c_α is an appropriate constant. The individual coordinates of Y have power tails as well, $x^\alpha P(\pm Y_k > x) \rightarrow c_\alpha (1 \pm \beta_k) \sigma_k^\alpha / 2$ as $x \rightarrow \infty$, where β_k and σ_k are the skewness and the scale parameters of Y_k , $k = 1, \dots, d$. The PDFs and distribution functions of general GS distributions do not have explicit forms, but admit certain integral representations (obtained via (24)), involving their stable counterparts. It should be noted that all marginals and linear transformations of GS random vectors are GS. In particular, the quantity $\langle b, Y \rangle$ is GS for all $b \in R^d$ whenever Y is GS. The converse is also true when for all $b \in R^d$ the quantity $\langle b, Y \rangle$ is strictly GS, symmetric GS, or GS with $\alpha \in [1, 2)$; the case $\alpha = 2$ is still open.

Series representations in one dimension, analogous to LePage representations of stable random variables (see [172]), were developed in [104]. Information on simulation and parameter estimation connected with this model can be found in [82, 88, 98]. There are also alternative representations (see [10, 25, 78, 86, 87, 152]), in particular, one showing that in one-dimension we always have $Y \stackrel{d}{=} ZT$, where Z and T are independent, Y is GS, Z is standard exponential, and T has an *explicit* density (see [69]). Thus, all univariate GS distributions are exponential mixtures, and thus unimodal with the mode at the origin. Note that this property implies that all multivariate GS distributions are (linearly) unimodal with the mode at zero. Let us note that the unimodality of this class does not follow from their self-decomposability, as is the case with the stable laws, since not all GS distributions are in the class L of self-decomposable distributions (see, e.g., [163] for examples). It is also important to note that the multivariate GS densities in the symmetric case are unbounded at the mode, as are one-dimensional strictly GS densities with $\alpha < 1$ (see [69]). This behavior at the mode is one characteristic feature of this class that distinguishes these distributions from the stable laws, which have the same tail behavior and domains of attraction. Since high peak at the mode and power tails are often found in empirical distributions of financial data, GS-based models offer an attractive alternative to normal and stable distributions in this area (see, e.g., [19, 81, 89, 92, 97, 103, 107, 109, 116, 130, 131, 143, 144, 159, 161, 162]).

4.3. Special cases and examples. Below we discuss important special cases of GS laws.

4.3.1. Improper GS laws. If the ChF ϕ that appears in (6) corresponds to a degenerate stable law, we obtain an *improper* GS distribution with the ChF

$$\psi(t) = [1 - i \langle t, m \rangle]^{-1}, \quad (26)$$

In one dimension, this distribution is the familiar exponential, while in R^d we have $Y \stackrel{d}{=} mZ \stackrel{d}{=} D(m)W$, where Y is a GS variable with the ChF (26), Z is standard exponential, $D(m)$ is a diagonal matrix with the elements of m on its main diagonal, and

$W = (W_1, \dots, W_d)$ has a Marshall-Olkin exponential distribution, given by the survival function

$$P(W_1 > x_1, \dots, W_d > x_d) = e^{-\max(x_1, \dots, x_d)}.$$

Since Y satisfies the stability relation (3) with $a_p = p$, the improper GS distribution is strictly geometric stable.

4.3.2. Strictly GS laws. Strictly GS (SGS) distributions arise as the limiting laws in (19) with $b_p = 0$. Equivalently, SGS random vectors are characterized via the stability property (3), where necessarily $a_p = p^{1/\alpha}$ (see, e.g., [84, 85]). As shown in [142], the ChF of a SGS random vector is of the form (6) with ϕ being the ChF of a strictly stable distribution. Consequently, all SGS distributions have the ChFs of the form (21), where either $\gamma \equiv 0$ (an improper GS law with the ChF (26)) or $\gamma \neq 0$, in which case we must have $\alpha \neq 1$ and $m = 0$ or $\alpha = 1$ and $\int_{S_d} s_k \gamma(ds) = 0$ for all $k = 1, \dots, d$. In one dimension, SGS distributions are described by the ChF (23) with $\alpha \neq 1$ and $\mu = 0$ or $\alpha = 1$ and $\beta = 0$, although certain alternative parameterizations are often used in this case (see, e.g., [117]). These distributions are also known in the literature as the (non-symmetric) Linnik distributions (see, e.g., [29, 30, 31, 78]).

REMARK 4.6. Since all SGS random vectors X are GID and the relation (14) holds with $\psi_p(t) = \psi(p^{1/\alpha}t)$, where ψ is the ChF of X , we also have (15), which in this case can be written as

$$\psi(t) = \psi(\rho t)\psi^{(\rho)}(t),$$

with $\psi^{(\rho)}(t) = \rho^\alpha + (1 - \rho^\alpha)\psi(t)$ and $\rho = p^{1/\alpha} \in (0, 1)$. What this shows is that all SGS distributions belong to the class L of the self-decomposable (SD) distributions, and lead to stationary, non-Gaussian autoregressive processes $X_n = \rho X_{n-1} + \varepsilon_n$, where the $\{X_n\}$ are SGS with common ChF ψ and the IID $\{\varepsilon_n\}$ have common ChF $\psi^{(\rho)}$ (see [112]). Many such processes with general SD stationary distributions have been developed, including GS special cases of exponential processes (see, e.g., [35]), Laplace processes (see [1, 22, 121]), Linnik processes (see [5, 58]), and Mittag-Leffler processes (see [61]).

4.3.3. Symmetric GS laws. A GS random vector is called *symmetric geometric stable* if $P(Y \in A) = P(-Y \in A)$ for all Borel sets A in R^d . It is well-known (see, e.g., [69]) that the ChF of symmetric GS random vectors with index $0 < \alpha < 2$ is of the form

$$\psi(t) = \left[1 + \int_{S_d} |\langle t, s \rangle|^\alpha \gamma(ds) \right]^{-1}, \quad (27)$$

where γ is a unique, finite, and symmetric measure on S_d . When $\alpha = 2$, the relation (27) holds as well, although in this case γ is not unique. In one dimension, these are symmetric Linnik distributions with the ChF (5).

4.3.4. Multivariate Linnik distributions. If the ChF ϕ in (6) corresponds to a sub-Gaussian stable law on R^d (see, e.g., [172]), then the corresponding GS ChF is of the form

$$\psi(t) = \left[1 + \left(\frac{1}{2} t' \Sigma t \right)^{\alpha/2} - i \langle m, t \rangle \right]^{-1}, \quad (28)$$

where t' denotes the transpose of t and Σ is a non-negative definite symmetric matrix. In the symmetric case $m = 0$ we obtain the *multivariate Linnik distribution* studied in [4, 150], which generalizes the one-dimensional case (5).

4.3.5. Laplace distributions. If $\alpha = 2$ then the GS ChF (21), as well as as the multivariate Linnik ChF (28), become

$$\psi(t) = \left[1 + \frac{1}{2} t' \Sigma t - i \langle t, m \rangle \right]^{-1}, \quad (29)$$

where Σ is a $d \times d$ non-negative definite symmetric matrix. If the matrix Σ is positive-definite, then the distribution is truly d -dimensional, and its PDF is expressible in terms of the modified Bessel functions of the third kind (see [106]). This multivariate asymmetric Laplace (AL) distribution (see [77, 106]), along with the improper one, stand out within the class of GS distributions in that they have a finite covariance matrix (with the AL covariance matrix being $\Sigma + mm'$). In case $d = 2r + 3$, where r is a non-negative integer, the AL density takes on an explicit form (see [77]), which is particularly simple when $d = 3$ ($r = 0$):

$$g(y) = \frac{e^{-y' \Sigma^{-1} m - \sqrt{2 + m' \Sigma^{-1} m} \sqrt{y' \Sigma^{-1} y}}}{2\pi \sqrt{y' \Sigma^{-1} y} |\Sigma|^{1/2}}, \quad y \neq 0. \quad (30)$$

It is worth noting that when $d > 1$ all AL PDFs blow out at the origin, as is the case with multivariate symmetric GS densities as well. All marginal distributions of multivariate AL laws are also AL, as are their linear transformations. In addition, according to Proposition 4.5, all AL laws are location and scale mixtures of Gaussian distributions, as the variable X in (24) in this case is Gaussian (with mean 0 and covariance Σ). In the symmetric case $m = 0$, mentioned in [4], AL distributions are elliptically contoured, and belong to the class of *G-type* distributions. The latter are scale mixtures of normal laws with infinitely divisible stochastic variance (see [132, 133, 136, 170]).

When $d = 1$, we obtain a univariate skew Laplace distribution (see [77, 103, 105, 107, 111, 112]) with the ChF

$$\psi(t) = \left(\frac{1}{1 + i \frac{\sigma \kappa}{\sqrt{2}} t} \right) \left(\frac{1}{1 - i \frac{\sigma}{\sqrt{2 \kappa}} t} \right), \quad t \in R, \quad \kappa, \sigma > 0, \quad (31)$$

which reduces to the classical symmetric Laplace distribution with mean zero and variance σ^2 when the skewness parameter κ is equal to one. The above factorization of the AL ChF ψ shows that an AL random variable Y admits stochastic representations

$$Y \stackrel{d}{=} \frac{1}{\sqrt{2}} \left(\frac{1}{\kappa} Z_1 - \kappa Z_2 \right) \stackrel{d}{=} \frac{1}{\sqrt{2}} I Z,$$

where Z , Z_1 and Z_2 are IID standard exponential variables and I , which is independent of Z , takes on the values $1/\kappa$ and $-\kappa$ with probabilities $1/(1 + \kappa^2)$ and $\kappa^2/(1 + \kappa^2)$, respectively. Consequently, the PDF of Y is of the form

$$g(x) = \frac{2\kappa}{1 + \kappa^2} \cdot \begin{cases} f(x\kappa) & \text{for } x \geq 0 \\ f\left(\frac{x}{\kappa}\right) & \text{for } x < 0, \end{cases} \quad (32)$$

where $f(x) = \exp\{-\sqrt{2}|x|/\sigma\}/(\sqrt{2}\sigma)$ is the PDF of the symmetric Laplace distribution

with variance σ^2 . Thus, the AL model emerges from a general scheme (32) of incorporating skewness into *any* symmetric PDF f , proposed in [34] (see also [33]). Many standard distributions were extended to their skew counterparts through this scheme, including normal (see [149]), student-t (see [6, 34, 44]), Weibull (see [65, 66]), and exponential power (see [8, 33, 76]).

With their characteristic peak at the mode and semi-heavy tails, skewness, and relation to limit results for geometric sums, multivariate and univariate Laplace distributions have appeared in many applications due to their simplicity and flexibility, see [77] for more information and additional references.

REMARK 4.7. A related class of distributions, termed *log-Laplace* (LL) in [108, 109] and *double-Pareto* in [164], arises via $X = \exp(Y)$ with Y being AL. These distributions appeared in connection with modeling under-reported data in [47, 52]. Their densities are double-power laws of the form

$$f(x) = \frac{1}{\delta} \frac{\alpha\beta}{\alpha + \beta} \cdot \begin{cases} (x/\delta)^{\beta-1} & \text{for } 0 < x < \delta \\ (\delta/x)^{\alpha+1} & \text{for } x \geq \delta, \end{cases} \quad (33)$$

where $\delta > 0$ (scale) and $\alpha, \beta > 0$ (tail parameters). Consequently, a log-log plot of f produces a distinct “tent” shape, observed empirically in a variety of fields in connection with modeling *growth rates* in numerous applications, such as gross domestic product, stock prices, company sizes, physics and others (extensive references are available in [108, 109]). It is argued in [108, 109] that the emergence of the LL distributions in these settings is connected with their fundamental properties of stability with respect to *geometric multiplication* (see [74]), which reflect analogous properties of the Laplace distribution in the summation setting.

5. Operator geometric stable distributions. Operator geometric stable laws generalize GS distributions, as they arise as the limiting distributions in (19) when the sums are normed by linear operators on R^d rather than by scalars $a_p > 0$ (see [75, 92]).

DEFINITION 5.1. A random vector Y on R^d is *operator geometric stable* (OGS) if for N_p geometric with the PDF (2) there exist IID random vectors X_1, X_2, \dots independent of N_p , linear operators A_p , and centering constants b_p such that

$$A_p \sum_{i=1}^{N_p} (X_i + b_p) \xrightarrow{d} Y \quad \text{as } p \rightarrow 0. \quad (34)$$

If (34) holds we say that the distribution of X_1 is weakly geometrically attracted to that of Y , and the collection of such distributions is called the *generalized domain of geometric attraction* of Y .

As shown in [92], OGS distributions are all ID as well as GID, and are closely related to *operator stable* (OS) laws, which are the limiting distributions in this setting for deterministic sums (see, e.g., [64, 139]). Similarly to the GS case, any OGS ChF is of the form (6) with ϕ being operator stable ChF. Equivalently, any OGS random vector Y has the same distribution as that of $L(Z)$, where Z is a standard exponential variable while $\{L(t), t \geq 0\}$ is a stochastic process with stationary independent increments, independent

of Z , and such that $L(1)$ is OS and $L(0) = 0$ almost surely. Thus, every OGS random vector admits the stochastic representation

$$Y \stackrel{d}{=} Z^E X + a_Z, \tag{35}$$

where Z is as before and X , which is independent of Z , is OS with exponent E and distribution ω satisfying

$$\omega^t = t^E \omega * \varepsilon_{a_t} \quad \text{for all } t > 0. \tag{36}$$

Here, ε_a is the unit mass at the point $a \in R^d$, $t^E = \exp(E \log t)$ and $\exp(A) = I + A + A^2/2! + A^3/3! + \dots$ is the usual exponential operator. The representation (35) is a generalization of (24), which applies to GS random vectors.

REMARK 5.2. Using the results of [182] for subordinated Lévy processes (see also [53]), one can derive the Lévy representation of the OGS ChF. As shown in [92] the ChF of an OGS law, related to a full OS law ω via (6), is of the form

$$\psi(t) = e^{i\langle a, t \rangle} + \int_{R^d} \left\{ e^{i\langle t, x \rangle} - 1 - \frac{i\langle t, x \rangle}{1 + \|x\|^2} \right\} d\nu(x)$$

where $\nu(x) = h(x)dx$, $h(x) = \int_0^\infty g(s, x) \frac{1}{s} de^{-s} ds$,

$$a = \int_0^\infty \int_{R^d} \frac{x}{1 + \|x\|^2} g(s, x) dx \frac{1}{s} e^{-s} ds, \tag{37}$$

and $g(s, x)$ is the Lebesgue density of ω^s for any $s > 0$. The asymptotics of the Lévy measure in the one-dimensional GS case were studied in [115].

REMARK 5.3. The problems of geometric summation (and geometric stability) with operator norming have also appeared in more abstract setting of probability on groups, see [48, 49, 50, 51] for details. Further generalizations where the random summation in (34) is other than geometric can be found in [93].

5.1. Special cases and examples. Clearly, when the operators in (34) are of the form $A_p = a_p I_d$, where $a_p > 0$ and I_d is a d -dimensional identity matrix, then the limiting distributions coincide with the GS laws on R^d . Additionally, we get multivariate Laplace distributions if the $\{X_i\}$ in (34) are in the normal domain of attraction.

5.1.1. Strictly OGS laws. If the OS law given by the characteristic function ϕ in (6) is strictly OS, then the distribution given by ψ is called *strictly* OGS. For the strictly OGS distributions, the convergence in (34) holds with $b_p = 0$. In addition, the representation (35) reduces to

$$Y \stackrel{d}{=} Z^E X, \tag{38}$$

where, as before, Z is a standard exponential variable and X is strictly OS with exponent E (and independent of Z). It should be noted that strictly OGS distributions generalize the stability property of the GS laws, see [92] (and also [49, 50] for more general results on strict geometric stability on nilpotent Lie groups).

PROPOSITION 5.4. *Let Y, Y_1, Y_2, \dots be IID random variables in R^d , and let N_p be a geometrically distributed random variable (2), independent of the sequence (Y_i) . Then*

$$S_p = A_p \sum_{i=1}^{N_p} Y_i \stackrel{d}{=} Y, \quad p \in (0, 1), \quad (39)$$

with some operators A_p on R^d if and only if Y is strictly OGS, in which case Y admits the representation (38) for some OS random variable X with exponent E and $A_p = p^E$ for $p \in (0, 1)$.

5.1.2. Marginally geometric stable laws. If the operators A_p in (34) are diagonal matrices $\text{diag}(a_{p,1}, \dots, a_{p,d})$ for some positive $\{a_{p,i}\}$, then the one-dimensional marginals of the limiting OGS vector Y are GS with the ChF (23) and possibly *different values* of α . Here, the ChF of Y is given by (6) with ϕ corresponding to a marginally stable OS random vector (see, e.g., [18, 137, 145, 166]). If the values of α for all marginal distributions are less than 2, then the ChF of Y is of the form

$$\psi(t) = \left(1 + \int_{S_d} \int_0^\infty \left(e^{i\langle t, r^E s \rangle} - 1 - \frac{i\langle t, r^E s \rangle}{1 + \|r^E s\|^2} \right) \frac{dr}{r^2} \gamma(ds) - i\langle t, m \rangle \right)^{-1}, \quad (40)$$

where E is a diagonal matrix

$$E = \text{diag}(1/\alpha_1, \dots, 1/\alpha_d), \quad 0 < \alpha_i \leq 2, \quad i = 1, \dots, d, \quad (41)$$

called the *exponent* of Y , $r^E = \text{diag}(|r_1|^{1/\alpha_1}, \dots, |r_d|^{1/\alpha_d})$, the spectral measure γ is a finite measure on the unit sphere S_d in R^d , and $m \in R^d$ is the shift parameter. As in the stable and GS cases, the measure γ determines the dependence structure. The fact that these distributions allow for a different tail behavior for their marginals makes them, along with marginally stable laws, attractive in financial portfolio analysis (see [92, 140, 144, 159]). Let us note that OGS distributions admit a stability property similar to (39) with diagonal A_p (see [92]). Two interesting examples from this class are a bivariate distribution with Laplace and Linnik marginal distribution (see [92]), given by the ChF

$$\psi(t, s) = \frac{1}{1 + \sigma^2 t^2 + \eta^\alpha |s|^\alpha}, \quad s, t \in R, \quad \alpha \in (0, 2], \quad (42)$$

and a bivariate distribution with exponential and Mittag-Leffler marginal distributions (see [91]), given by the LT

$$\psi(t, s) = \frac{1}{1 + \sigma t + \eta^\alpha s^\alpha}, \quad s, t \in R_+, \quad \alpha \in (0, 1]. \quad (43)$$

6. Geometric self-similarity. Recall that a stochastic process $X(t)$ is *self-similar* with index H if for all $c > 0$ we have $X(c \cdot) \stackrel{d}{=} c^H X(\cdot)$, so that the time scale change is (stochastically) equivalent to the space scale change (see, e.g., [28]). In the same spirit, one can consider self-similarity with respect to a *random time change*, understood as subordination (see [94, 110]). Let $\mathcal{T} = \{T_c(t), t \geq 0\}$, $c \geq 1$, be a family of random time changes, with $T_c(0) = 0$ almost surely and non-decreasing sample paths, such $E[T_c(t)] = ct$. Following [94], we define stochastic self-similarity as follows.

DEFINITION 6.1. Let $\{X(t), t \geq 0\}$ be a stochastic process on R^d and let \mathcal{T} be a family of processes described above, independent of X . The process $X(t)$ is stochastically self-similar with index H (or H -sss) with respect to the family \mathcal{T} if

$$X(T_c(\cdot)) \stackrel{d}{=} c^H X(\cdot), \quad c \geq 1. \tag{44}$$

REMARK 6.2. The above notion of stochastic self-similarity, which involves stochastic renormalization in *time*, is apparently unrelated to that considered in [46] and [191], which involves stochastic renormalization in *space*.

REMARK 6.3. Note that by iterating the relation (44), we obtain

$$X(T_{c_1} \circ T_{c_2}(\cdot)) \stackrel{d}{=} X(T_{c_2} \circ T_{c_1}(\cdot)) \stackrel{d}{=} (c_1 c_2)^H X(\cdot), \quad c_1, c_2 \geq 1, \tag{45}$$

so it is natural to require that the family \mathcal{T} be a commutative composition semigroup, that is

$$T_{c_1} \circ T_{c_2}(\cdot) \stackrel{d}{=} T_{c_1 c_2}(\cdot), \quad c_1, c_2 \geq 1, \tag{46}$$

where \circ stands for the composition of two functions.

6.1. Examples. One example of a process that is H-sss is the standard gamma process $\{G(t), t \geq 0\}$, which is stochastically self-similar (with index $H = 1$) with respect to the family

$$\mathcal{T} = \{T_c(t) = t + NB_p(t), t \geq 0\}, \quad c \geq 1. \tag{47}$$

Here, $p = 1/c \in (0, 1]$ and $\{NB_p(t), t \geq 0\}$ is a negative binomial Lévy process (NBP) studied in [113], whose marginal distributions are negative binomial with the PDF

$$P(NB_p(t) = k) = \frac{\Gamma(t+k)}{\Gamma(t)k!} p^t (1-p)^k, \quad k = 0, 1, 2, \dots \tag{48}$$

It is easy to verify that this family of negative binomial processes with drift satisfies all the required properties, including $ET_c(t) = ct$ as well as (46).

REMARK 6.4. Another family of random time changes that satisfies these properties can be constructed as follows (see [110]). For each non-negative integer k let

$$N_p^{(k)} = kNB_p(1/k), \quad p \in (0, 1),$$

where $NB_p(1/k)$ is the $1/k$ -lag increment of the NBP. In [176, 180], the distribution of $N_p^{(k)} + 1$, which is supported on the set of integers $\{1, k + 1, 2k + 1, \dots\}$, is referred to as Harris law (see also [17, 141] in this connection). Since $N_p(k)$ is ID, it leads to a Lévy processes $\{N_p^{(k)}(t), t \geq 0\}$, which in turn allows us to define the family of such processes with drift,

$$\mathcal{T}_k = \{T_c^{(k)}(t) = t + N_p^{(k)}(t), t \geq 0\}, \quad p \in (0, 1), \quad c = 1/p \geq 1. \tag{49}$$

This family satisfies all the required properties, including (46). Then the re-scaled standard gamma Lévy process $\{G^{(k)}(t), t \geq 0\}$, where $G^{(k)}(1) = kG(1/k)$ (gamma distribution with shape parameter $1/k$ and scale parameter k), is stochastically self-similar (with index $H = 1$) with respect to the family \mathcal{T}_k given by (49). See [110] for more details and further generalizations.

REMARK 6.5. As noted in [110], the Lévy process $\{G_\alpha(t), t \geq 0\}$, whose lag-1 increments have the Mittag-Leffler distribution given by the LT (4), is stochastically self-similar with index $H = 1/\alpha$ with respect to the family of negative binomial time changes (47).

Other stochastic self-similar processes can be constructed by subordinating stable processes to the gamma process or the Mittag-Leffler Lévy process $G_\alpha(t)$ discussed above. Here, one would follow the following general fact, noted in [94].

PROPOSITION 6.6. *If $\{X(t), t \geq 0\}$ is any self-similar process with index H and $\{Z(t), t \geq 0\}$ is an independent Lévy process that is stochastically self-similar with index H' with respect to a family $\mathcal{T} = \{T_c(t), t \geq 0\}$, $c \geq 1$, then the subordinated process $\{Y(t) = X(Z(t)), t \geq 0\}$ is stochastically self-similar with index HH' with respect to the same family.*

When $X(t)$ is a Brownian motion and $Z(t)$ is a gamma process, we obtain the Laplace motion discussed in [77]. This model plays an increasingly important role in mathematical finance, where it is known as the variance-gamma process (see [21, 130, 131, 183]). A more general process with *correlated* increments, termed a *fractional Laplace motion* in [94, 138], arises by subordinating a *fractional Brownian motion* to the gamma process.

6.1.1. Fractional Laplace motion. Let $\{B_H(t), t \geq 0\}$, $H \in (0, 1)$, be a fractional Brownian motion with the covariance function

$$E[B_H(t)B_H(s)] = \frac{\sigma^2}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \geq 0, \quad (50)$$

and let $\{G(t), t \geq 0\}$ be a standard gamma process. Then

$$\{Y(t), t \geq 0\} \stackrel{d}{=} \{B_H(G(t)), t \geq 0\} \quad (51)$$

is the fractional Laplace motion (FLM) with shape parameter 1, denoted by $\mathcal{FLM}_H(\sigma)$. (When G in (51) is a gamma process with parameter $\nu > 0$, which is a Lévy process whose lag- t increments have gamma distributions with shape parameter t/ν and unit scale, we obtain a more general FLM with shape parameter $\nu > 0$.) Since the gamma process is H -sss with $H = 1$ with respect to the family (47), by Proposition 6.6, so is the FLM (51). The one-dimensional distributions of $\mathcal{FLM}_H(\sigma)$ are scale mixtures of Gaussian laws,

$$Y(t) \stackrel{d}{=} (\sigma^2[G(t)]^{2H})^{1/2}Z. \quad (52)$$

Here, Z is a standard normal random variable while the “stochastic variance” $[G(t)]^{2H}$, which is independent of Z , has the *generalized gamma* distribution given by the PDF

$$f(x) = \frac{\gamma x^{\alpha\gamma-1}}{c^{\alpha\gamma}\Gamma(\alpha)} e^{-(x/c)^\gamma}, \quad x > 0, \quad (53)$$

with scale $c = 1$ and shape parameters $\alpha = t$ and $\gamma = 0.5/H$. It is worth noting that the distribution of $Y(t)$ is ID if and only if $1/2 \leq H \leq 1$, which reflects the fact that the powers of gamma variable $[G(t)]^{1/\gamma}$ in (52) are ID whenever $0 < |\gamma| \leq 1$ but not ID for $\gamma > 1$ (interestingly, the case $\gamma < -1$ is still open). This is closely related to the tail behavior of the FLM, derived in [94].

PROPOSITION 6.7. *The tail probability of $Y(t) \sim \mathcal{FLM}_H(\sigma)$ admits the following asymptotic behavior:*

$$P(Y(t) > x) \sim cx^{2(t-1)/(1+2H)} \exp\{-bx^{2/(1+2H)}\}, \quad x \rightarrow \infty, \quad (54)$$

where $c = aH^{2H/(1+2H)}$ and

$$a = \frac{1}{\sqrt{1+2H}\Gamma(t)} H^{t/(1+2H)-1/2}, \quad b = \frac{1+2H}{2} H^{-2H/(1+2H)}. \quad (55)$$

Other basic properties of the FLM, derived in [94], reflect those of the FBM. In particular, the covariance structures of the $\mathcal{FLM}_H(\sigma)$ is asymptotically equivalent to that of the FBM. Similarly, the moment scaling of the $\mathcal{FLM}_H(\sigma)$,

$$E|Y(t)|^q = c_q \frac{\Gamma(Hq+t)}{\Gamma(t)}, \quad c_q = \sqrt{\frac{2^q}{\pi}} \Gamma\left(\frac{1+q}{2}\right), \quad (56)$$

which is linear in t when $q = 1/H$, is asymptotically the same as that of the FBM, $E|Y(t)|^q \sim c_q t^{qH}$ as $t \rightarrow \infty$. This reflects the fact that the increments of the FLM become Gaussian with an increasing lag. In analogy with the FBM, the stationary process

$$\{W_j, j = 1, 2, \dots\} \stackrel{d}{=} \{Y(\eta j) - Y(\eta(j-1)), j = 1, 2, \dots\}, \quad \eta > 0, \quad (57)$$

consisting of lag- η increments of the FLM, is called a *fractional Laplace noise* (FLN). The covariance function of the FLN, derived in [94], has the same asymptotic behavior as that of the fractional Brownian noise,

$$r(n) = E[W_j W_{j+n}] \sim \sigma^2 H(2H-1)(\eta)^{2H} n^{2H-2} \quad \text{as } n \rightarrow \infty \text{ and } H \neq 0.5,$$

so that when $H > 1/2$ the series $\sum_{n=1}^{\infty} r(n)$ diverges, and similarly to the fractional Brownian noise, the FLN exhibits a long-range dependence. More details on theory and applications of the FLM or FLN can be found in [94, 138, 147, 148].

6.2. Geometric renormalization group and self-similarity. As noted in [94, 110], stochastic self-similarity of FLM leads to random stability properties of FLN. Indeed, by taking $t = 1$ in the relation (44), we find that $(1/p)^H Y(1) \stackrel{d}{=} Y(N_p(1))$, where Y is an FLM $\mathcal{FLM}_H(\sigma)$ and $\{N_p(t) = NB_p(t) + t, t \geq 0\}$ is the negative binomial process with drift. Since $Y(N_p(1))$ is the sum of the first $N_p(1)$ lag-1 increments of Y , this relation can be written as

$$p^H \sum_{j=1}^{N_p} W_j \stackrel{d}{=} W_1,$$

where $N_p = N_p(1)$ is a geometric variable (2) and the $\{W_j\}$ are the increments (57) of Y with $\eta = 1$. Note that when $H = 1/2$, then the $\{W_j\}$ are independent and Laplace distributed, so this is a generalization of geometric stability. This motivates the following definition (see [94]).

DEFINITION 6.8. For each $p \in (0, 1)$, let $\{N_p(t), t \geq 1\}$ be a NB Lévy process with drift. A composition group of transformations $\{T_p, \circ\}, p \in (0, 1)$, of stationary sequences $W = \{W_j, j = 1, 2, \dots\}$, defined by

$$T_p : W \rightarrow T_p W = \{(T_p W)_k, k = 1, 2, \dots\},$$

where

$$(T_p W)_k = p^H (W_{N_p(k-1)+1} + \cdots + W_{N_p(k)}), \quad k = 1, 2, \dots$$

and $N_p(t)$ is independent of W , is called a geometric renormalization group with parameter $H \in (0, 1)$.

Thus, T_p transforms a stationary sequence W into another one, obtained by summing the components of W over successive blocks of *geometric* size $N_p^{(k)} = N_p(k) - N_p(k-1)$ and normalizing by $p^H = (\text{expected block size})^H$. Note that this group is commutative, since we have $N_p(N_q(k)) = N_q(N_p(k))$, $p, q \in (0, 1)$, $k = 1, 2, \dots$, where N_p and N_q are two independent negative binomial processes with drift. The fact that FLN is a *fixed point* of the transformations in in group, leads to the following definition (see [94])

DEFINITION 6.9. Let $\{T_p, \circ\}$, $p \in (0, 1)$, be a geometric renormalization group with parameter $H \in (0, 1)$. If $W = \{W_j, j = 1, 2, \dots\}$ is a stationary sequence and $T_p W \stackrel{d}{=} W$, $p \in (0, 1)$, then we say that W is geometrically self-similar with index H .

We see that a lag-1 FLN is a geometric self-similar stationary sequence. We conclude by noting that the notion of geometric self-similarity in relation to stationary sequences is quite similar to invariance properties of renewal processes with respect to geometric thinning, discussed in Section 2, since both properties are based on stability with respect to geometric summation.

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