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REINSURANCE—A NEW APPROACH

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Abstract. We describe a new model of multiple reinsurance. The main idea is that the reinsurance premium is paid conditionally. It is motivated by some analysis of the ultimate price of the reinsurance contract. For simplicity we assume that the underlying risk pricing functional is the L_2-norm. An unexpected relation to the general theory of sample regularity of stochastic processes is given.

1. Introduction. The aim of this paper is to present a model of multiple reinsurance in which the premium is paid conditionally (to be explained later). We formulate a theorem on stability of the insurance system, i.e. for a wide class of such reinsurance systems the total price of the insurance is approximately the same.

For clarity, randomness is modelled by considering the probability space $[0,1)$ with the Lebesgue measure λ as the probability. If a random variable $R\colon [0,1) \to [0,\infty)$ is interpreted as the insurance risk then its L_2-norm $\|R\| = \sqrt{\mathbb{E}R^2}$ is considered as the price of the insurance. The use of the L_2-norm is assumed only for simplicity since the model presented in this paper can be easily generalized.

A first of our ideas can be explained as follows. With the risk function R given, one could consider a stop-loss reinsurance with the premium given by the following formula

$$P(R) = \left(\mathbb{E}\left[((R-k)^+)^2|\mathcal{F}\right]\right)^{1/2},$$

where k is the stop-loss level and $\mathcal{F}$ is a σ-field representing the information available at the time of purchase. The premium $P(R)$ is a random variable and it is paid conditionally.

We also suggest (and it is our second idea) that in this setting a sequence of reinsurers can be considered. Namely the 0th reinsurer is the insurance buyer himself, the 1st

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reinsurer is the insurer who accepts the risk of $R-(R-k_1)^+$, with k_1 being the appropriate stop-loss level. Analogously $(R-k_{i-1})^+ - (R-k_{i_1})^+$ expresses the risk accepted by the i-th reinsurer. This leads to the ultimate reinsurance price of the form

$$\text{Price} = P_0\left((R-0)^+ - (R-k_1)^+ + P_1\left((R-k_1)^+ - (R-k_2)^+ + P_2\left(\ldots\right)\right)\right),$$

where P_i is the conditional premium paid by the i-th reinsurer.

We shall explain the idea with the following example. Set

$$\|R\|_{\mathcal{F}} := \left(\mathbb{E}(R^2|\mathcal{F})\right)^{1/2}. \tag{1}$$

If we consider an insurer and reinsurers described by 'reinsurance levels' $k_1 < k_2$, and by σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{B}[0,1)$ and if $\mathcal{F}_0 = \{\emptyset, [0,1)\}$ then

$$P_2\left((R-k_2)^+\right) = \|(R-k_2)^+\|_{\mathcal{F}_2} \tag{2}$$

is the premium charged by the 2nd reinsurer,

$$\begin{aligned} P_1 &\left((R-k_1)^+ - (R-k_2)^+ + \|(R-k_2)^+\|_{\mathcal{F}_2}\right) \\ &= \left\|(R-k_1)^+ - (R-k_2)^+ + \|(R-k_2)^+\|_{\mathcal{F}_2}\right\|_{\mathcal{F}_2} \end{aligned} \tag{3}$$

is the premium charged by the 1st reinsurer,

$$\begin{aligned} P_0 &\left(\left\|(R-k_1)^+ - (R-k_2)^+ + \|(R-k_2)^+\|_{\mathcal{F}_2}\right\|_{\mathcal{F}_2}\right) \\ &= \left\|\left\|(R-k_1)^+ - (R-k_2)^+ + \|(R-k_2)^+\|_{\mathcal{F}_2}\right\|_{\mathcal{F}_2}\right\| \end{aligned} \tag{4}$$

is the premium charged by the insurer.

In order to describe our conditional pricing scheme it is necessary to introduce a filtration of information available to consecutive reinsurers. For simplicity we start with considering the filtration $\mathcal{F}_k$, $k \geq 0$ associated with dyadic partitions of the unit interval (into 2^k equal-length parts):

$$\mathcal{F}_k = \sigma\left(\left[n2^{-k}, (n+1)2^{-k}\right) : 0 \leq n < 2^k\right), \quad k \geq 0. \tag{5}$$

The main result of the paper states that the ultimate premium paid by the insurance buyer is (almost) independent of the model parameters (cf. Theorem 2.1). The system of conditional reinsurance pricing is described by some subfiltration $(\mathcal{F}_{k_i})_{i\geq 0}$ (satisfying rather weak requirements (12)). This implies that the final result is considerably general.

Section 2 specifies the model and presents the main theorem. It is interesting that the cited fundamental Lemma 2.2 was used to solve a classical problem of characterization of all orthogonal a.e. convergent sequences ([3], Theorem 1.6). In section 3, a short overview of the theory and related results is presented.

The last Section 4 features examples, which show among other things that some assumptions on the sequence of 'reinsurance levels' are necessary. These examples require some technicalities. We also discuss the relations between our model of premium with other models of reinsurance pricing. Models are compared with respect to the value of the ultimate premium paid.

It is natural that further generalizations of the insurance pricing at a single step should be considered, e.g. described by an arbitrary Orlicz norm instead of the L_2-norm. It turns out that the reasoning presented in this paper finds a clear generalization at least to the case of L_p-norm, $p \geq 1$. This should however be presented elsewhere since some parts of the paper are, even in the present setting, quite technical.

2. Reinsurance based on conditional premium. In this section we present the main result on the ultimate premium paid by the insurance buyer in the model suggested in our paper (Theorem 2.1). Namely, we assume that we have a sequence of reinsurers of 'increasing levels' $i = 0, 1, \ldots, j$. **The premium of (higher) level** $i+1$ **is an** $\mathcal{F}_i$**-measurable random variable** (see (5)) **paid by the reinsurer of level** i. We give an upper and a lower bound for the ultimate price of insurance in the system. As mentioned before we assume that each reinsurance is based on the stop-loss scheme. An additional assumption on the reinsurance sequence is suggested by formula (12). This assumption is in fact not very restrictive but, as implied by examples given in Section 4, an assumption on the reinsurance sequence could not be omitted (for validity of Theorem 2.1).

Let us recall that we consider the probability space $([0,1), \lambda)$, with λ being the Lebesgue measure. We also consider the filtration (5) and conditional L_2-norms

$$\|g\|_{\mathcal{F}_k} = \left(\mathbb{E}(g^2|\mathcal{F}_k)\right)^{1/2}, \quad g \in L_\infty[0,1),\ k \geq 0,$$

with $\mathbb{E}(\cdot|\mathcal{F}_k)$ being the conditional expectation in $([0,1), \lambda)$.

Let us fix a non-negative risk function R, a sequence of non-empty events $(\Delta_k)_{k\geq 0}$ and a sequence of integer levels $\mathbf{k} = (k_0, \ldots, k_{j(\mathbf{k})})$ satisfying

$$0 = k_0 < k_1 < \ldots < k_{j(\mathbf{k})} =: l. \tag{6}$$

and

$$[0,1) = \Delta_0 \supset \Delta_1 \supset \ldots, \quad \Delta_k \in \mathcal{F}_k, \quad k \geq 0, \tag{7}$$

$$R\colon [0,1) \to [0,l], \quad l = k_{j(\mathbf{k})}, \tag{8}$$

$$\{R > k_i\} \subset \Delta_{k_i} \subset \{R \geq k_i\}, \quad 0 \leq i \leq j(\mathbf{k}). \tag{9}$$

Let us define operators V_k, $k \geq 1$ on $L_\infty[0,1)$ by (cf. [3], Section 2a or [2])

$$V_k g = \mathbb{I}_{\Delta_k} + \|g\|_{\mathcal{F}_k}, \quad g \in L_\infty[0,1). \tag{10}$$

We also define a sequence of operations in $L_\infty[0,1)$ associated with the risk function R and the sequence $\mathbf{k}$ by

$$P_{k_i}^{R\mathbf{k}} g = (R - k_{i-1})^+ - (R - k_i)^+ + \|g\|_{\mathcal{F}_{k_i}}, \quad g \in L_\infty[0,1), \tag{11}$$

for $1 \leq i \leq j(\mathbf{k})$, according to ideas given in (2), (3), (4).

THEOREM 2.1. *Let conditions* (7), (8), (9) *be satisfied for some* R *and* $(\Delta_i)_{i\ge 0}$. *If the 'reinsurance system'* $\mathbf{k} = (k_0, \dots, k_{j(\mathbf{k})})$ *satisfies* (6) *and*

$$\{2^0, 2^1, \dots\} \cap \{1, \dots, l\} \subset \{k_0, \dots, k_{j(\mathbf{k})}\} \tag{12}$$

then

$$\frac{1}{2}\|V_1 \dots V_l 0\| \le \|P^{R\mathbf{k}}_{k_1} \dots P^{R\mathbf{k}}_{k_{j(\mathbf{k})}} 0\| \le 2\|V_1 \dots V_l 0\| + 1. \tag{13}$$

The quantity $\|V_1 \dots V_l 0\|$ is determined by the sequence of events $\Delta_1 \subset \dots \subset \Delta_l$, which may be interpreted as a description of randomness associated with insurance loss and the information available to the (re-)insurance buyer. The 'pricing' operators $P^{R\mathbf{k}}_{k_i}$, $i \ge 0$ are on the other hand dependent on the risk function R and the assumed levels of reinsurance k_i, thus on the assumed 'reinsurance system' $\mathbf{k}$. Thus **Theorem 2.1 is in fact a theorem on stability of the price of insurance in the model suggested in our paper**. Despite the constants $\frac{1}{2}$, 1, 2 in the formula (13) the result is, as implied by the examples of Section 4, Theorems 4.1, 4.4, still meaningful.

The proof of Theorem 2.1 is connected with the properties of the operators V_k, $k \ge 1$ described in [3], Section 2a. Namely, taking additionally

$$\mathcal{V}_i g = 2^{i-1}\mathbb{I}_{\Delta_{2^i}} + \|g\|_{2^i}, \quad g \in L_\infty[0,1), \quad i \ge 0 \tag{14}$$

we have the following estimate for any fixed $l \ge 1$, with $j(l)$ given by

$$2^{j(l)-1} < l \le 2^{j(l)}. \tag{15}$$

LEMMA 2.2. *For any sequence* $(\Delta_k)_{k\ge 1}$ *satisfying* (7) *the following estimate is valid:*

$$\mathcal{V}_0 \dots \mathcal{V}_{j(l)} 0 \le V_1 \dots V_l 0 \le 2\mathcal{V} \dots \mathcal{V}_{j(l)} 0 + 1.$$

The proof can be found in [3], Section 2.4, Step II-III.

In order to prove Theorem 2.1 it is convenient to generalize our notation. Let us recall that we have fixed the sequence of sets (7). In the following reasoning we will be considering classes of risk functions, rather than a single risk function R.

For any increasing sequence $\mathbf{k} = \big(0 = k_0, k_1, \dots, k_{j(\mathbf{k})}\big)$ of integers we define operations (cf. (11))

$$P^{Q\mathbf{k}}_{k_i} g = (Q - k_{i-1})^+ - (Q - k_i)^+ + \|g\|_{\mathcal{F}_{k_i}}, \quad 1 \le i \le j(\mathbf{k}). \tag{16}$$

By definitions (14), (16) we have immediately the following lemma.

LEMMA 2.3. *For* $j \ge 0$, $\mathbf{k} = \mathbf{d} := (0, 2^0, 2^1, \dots, 2^j)$, $Q^- = \mathbb{I}_{\Delta_{2^0}} + \sum_{i=1}^{j} 2^{i-1}\mathbb{I}_{\Delta_{2^i}}$ *we have:*

$$P^{Q^-\mathbf{d}}_{2^i} = \mathcal{V}_i, \quad 1 \le i \le j,$$

$$P^{Q^-\mathbf{d}}_{2^0} = \frac{1}{2}\mathbb{I}_{\Delta_{2^0}} + \mathcal{V}_0.$$

Proof. This follows simply from the definition since $(Q - 2^{i-1})^+ - (Q - 2^i)^+ = 2^{i-1}\mathbb{I}_{\Delta_{2^i}}$, $i \ge 1$. ∎

The following two general lemmas are straightforwardly implied by monotonicity and subadditivity of conditional L_2-norms. For our increasing sequences $\mathbf{k}$ we have:

Lemma 2.4. *For sequences* $\mathbf{k} \subset \mathbf{k}' = (0 = k'_0 < \ldots < k'_{j(\mathbf{k}')} = k_{j(\mathbf{k})})$ *and for* $Q\colon [0,1) \to [0, k_{j(\mathbf{k})}]$ *satisfying (cf.* (9)*)*

$$\forall_{0 \le i \le j(\mathbf{k}')} \exists_{\Delta_{k'_i} \in \mathcal{F}_{k'_i}} \{Q > k'_i\} \subset \Delta_{k'_i} \subset \{Q \ge k'_i\},$$

we have

$$P^{Q\mathbf{k}}_{k_1} \ldots P^{Q\mathbf{k}}_{k_{j(\mathbf{k})}} 0 \le P^{Q\mathbf{k}'}_{k'_1} \ldots P^{Q\mathbf{k}'}_{k'_{j(\mathbf{k}')}} 0. \tag{17}$$

Proof. The lemma can be obtained by repeated use of (17) with only one additional index k' in $\mathbf{k}'$:

$$\mathbf{k}' = (k_0, \ldots, k_{i_0-1}, k', k_{i_0}, \ldots, k_{j(\mathbf{k})}). \tag{18}$$

By (16) and monotonicity of conditional norms, it is enough to show that (18) implies

$$\|P^{Q\mathbf{k}}_{k_{i_0}} g\|_{\mathcal{F}_{k'}} \le \|P^{Q\mathbf{k}'}_{k'} P^{Q\mathbf{k}'}_{k_{i_0}} g\|_{\mathcal{F}_{k'}},$$

for g supported on $\Delta_{k_{i_0}}$. This can be done by using the condition $\{Q > k'_i\} \subset \Delta_{k'_i} \subset \{Q \ge k'_i\}$, namely

$$\begin{aligned} P^{Q\mathbf{k}}_{i_0} g &= \mathbb{I}_{[0,1)\setminus\Delta_{k'}} P^{Q\mathbf{k}}_{i_0} g + \mathbb{I}_{\Delta_{k'}} P^{Q\mathbf{k}}_{i_0} g \\ &= \left[(Q - k_{i_0-1})^+ - (Q - k')^+\right] \mathbb{I}_{[0,1)\setminus\Delta_{k'}} + \\ &\quad \left[k' - k_{i_0-1} + (Q - k')^+ - (Q - k_{i_0})^+ + \|g\|_{\mathcal{F}_{k_0}}\right] \mathbb{I}_{\Delta_{k'}}, \end{aligned}$$

thus by subadditivity of conditional norm

$$\begin{aligned} \|P^{Q\mathbf{k}}_{i_0} g\|_{\mathcal{F}_{k'}} \le{}& \left\|(Q - k_{i_0-1})^+ - (Q - k')^+\right\|_{\mathcal{F}_{k'}} \mathbb{I}_{[0,1)\setminus\Delta_{k'}} + \\ & \left\|k' - k_{i_0-1} + (Q - k')^+ - (Q - k_{i_0})^+ + \|g\|_{\mathcal{F}_{k_0}}\right\|_{\mathcal{F}_{k'}} \mathbb{I}_{\Delta_{k'}} \\ \le{}& \left\|(Q - k_{i_0-1})^+ - (Q - k')^+ + \left\|(Q - k')^+ - (Q - k_{i_0})^+ + \|g\|_{\mathcal{F}_{k_0}}\right\|_{\mathcal{F}_{k'}}\right\|_{\mathcal{F}_{k'}}. \ \blacksquare \end{aligned}$$

Lemma 2.5. *For* $Q \le Q'$, $Q, Q'\colon [0,1) \to [0,\infty)$ *we have*

$$P^{Q\mathbf{k}}_{k_1} \ldots P^{Q\mathbf{k}}_{k_{j(\mathbf{k})}} 0 \le P^{Q'\mathbf{k}}_{k_1} \ldots P^{Q'\mathbf{k}}_{k_{j(\mathbf{k})}}.$$

Let us once again fix a risk function $0 \le R \le l$, satisfying (9), in which $\mathbf{k}$ satisfies (6) and (12). With $j(l)$ given by (15):

$$2^{j(l)-1} < l \le 2^{j(l)},$$

let us consider the following two functions

$$Q^- = \mathbb{I}_{\Delta_{2^0}} + \sum_{i=1}^{j(l)-1} 2^{i-1} \mathbb{I}_{\Delta_{2^i}}, \qquad Q^+ = \mathbb{I}_{[0,1)} + \sum_{i=0}^{j(l)-1} 2^i \mathbb{I}_{\Delta_{2^i}}. \tag{19}$$

Their useful properties are given in the following lemma.

Lemma 2.6. *For functions* Q^-, Q^+ *given in* (19) *and our risk function* $0 \le R \le l$, *satisfying* (9), *where* $\mathbf{k}$ *satisfies* (6), (12) *we have*

$$Q^- \le R \le Q^+.$$

Proof. By (9), (12) we have

$$\begin{aligned} Q^- &= \mathbb{I}_{\Delta_{2^0}} + \sum_{i=1}^{j(l)-1} 2^{i-1}\mathbb{I}_{\Delta_{2^i}} \\ &\le (R-0)^+ - (R-2^0)^+ + \sum_{i=1}^{j(l)-1} \left[(R-2^{i-1})^+ - (R-2^i)^+\right]\mathbb{I}_{\Delta_{2^{i-1}}} \\ &\le (R-0)^+ - (R-2^0)^+ + \sum_{i=1}^{j(l)} \left[(R-2^{i-1})^+ - (R-2^i)^+\right] = R \\ &\le \mathbb{I}_{[0,1)} + \sum_{i=1}^{j(l)} 2^{i-1}\mathbb{I}_{\Delta_{2^{i-1}}} = Q^+. \ \blacksquare \end{aligned}$$

Proof of Theorem 2.1. Let $0 \le R \le l$ be a risk function satisfying (9), where $\mathbf{k}$ satisfies (6), (12). Let us observe that for $\mathbf{e} := (0, 2^0, \ldots, 2^{j(l)-1}, l) \subset \mathbf{k}$ and for $\mathbf{d} := (0, 2^0, \ldots, 2^{j(l)})$ we have $P_l^{R\mathbf{e}}0 = (R - 2^{j(l)-1})^+ = P_{2^{j(l)}}^{R\mathbf{d}}0$. By Lemma 2.4 we immediately have

$$P_{2^0}^{R\mathbf{d}} \ldots P_{2^{j(l)}}^{R\mathbf{d}}0 = P_{2^0}^{R\mathbf{e}} \ldots P_l^{R\mathbf{e}}0 \le P_{k_1}^{R\mathbf{k}} \ldots P_{k_{j(\mathbf{k})}}^{R\mathbf{k}}0.$$

Similarly by Lemmas 2.5 and 2.6

$$P_{2^0}^{Q^-\mathbf{d}} \ldots P_{2^{j(l)}}^{Q^-\mathbf{d}}0 \le P_{2^0}^{R\mathbf{d}} \ldots P_{2^{j(l)}}^{R\mathbf{d}}0,$$

and by Lemma 2.3

$$P_{2^0}^{Q^-\mathbf{d}} \ldots P_{2^{j(l)}}^{Q^-\mathbf{d}}0 = \frac{1}{2} + \mathcal{V}_0 \ldots \mathcal{V}_{j(l)}0.$$

Finally, Lemma 2.2 implies

$$\frac{1}{2}V_1 \ldots V_l 0 \le \frac{1}{2} + \mathcal{V}_0 \ldots \mathcal{V}_{j(l)}0,$$

which further yields the first inequality in (13).

By Lemmas 2.5 and 2.6 we have

$$P_{k_1}^{R\mathbf{k}} \ldots P_{k_{j(\mathbf{k})}}^{R\mathbf{k}}0 \le P_{k_1}^{Q^+\mathbf{k}} \ldots P_{k_{j(\mathbf{k})}}^{Q^+\mathbf{k}}0. \tag{20}$$

Let us define a new sequence of sets

$$\Delta_k^+ = \Delta_{2^i} \quad \text{for } 2^i \le k < 2^{i+1},\ i \ge 0; \qquad \Delta_0^+ = [0,1).$$

Then we have $\Delta_{2^i}^+ = \Delta_{2^i}$, $i \ge 0$ and $\Delta_k \in \mathcal{F}_k$, $k \ge 0$.

We should note that the function Q^+ satisfies the assumptions of Lemma 2.4 with $\mathbf{k}' = \mathbf{n} := (0, 1, 2, \ldots, l)$ and with Δ_k^+ instead of Δ_k, $k \le l$, thus

$$P_{k_1}^{Q^+\mathbf{k}} \ldots P_{k_{j(\mathbf{k})}}^{Q^+\mathbf{k}}0 \le P_1^{Q^+\mathbf{n}} \ldots P_l^{Q^+\mathbf{n}}0.$$

We also define a new set of operators

$$\mathcal{V}_i^+ g = 2^{i-1}\mathbb{I}_{\Delta_{2^i}^+} + \|g\|_{\mathcal{F}_{2^i}}, \quad g \in L_\infty[0,1), \quad i \ge 0,$$

and

$$V_k^+ g = \mathbb{I}_{\Delta_k^+} + \|g\|_{\mathcal{F}_k}, \quad g \in L_\infty[0,1), \quad k \ge 0.$$

By definition for $i \geq 0$ we immediately have $\mathcal{V}_i^+ = \mathcal{V}_i$ since $\Delta_{2^i} = \Delta_{2^i}^+$. We also have (cf. (19))

$$Q^+ = \mathbb{I}_{[0,1)} + \sum_{i=0}^{j(l)-1} 2^i \mathbb{I}_{\Delta_{2^i}} = \sum_{k=0}^{2^{j(l)}-1} \mathbb{I}_{\Delta_k^+}$$

thus $P_k^{Q^+\mathbf{n}} = V_k^+$, $1 \leq k \leq l$ (cf. (15)).

It is now clear that

$$P_1^{Q^+\mathbf{n}} \dots P_l^{Q^+\mathbf{n}} 0 = V_1^+ \dots V_l^+ 0$$

and by Lemma 2.2 we have

$$V_1^+ \dots V_l^+ 0 \leq 1 + 2\mathcal{V}_0^+ \dots \mathcal{V}_{j(l)}^+ 0 = 1 + 2\mathcal{V}_0 \dots \mathcal{V}_{j(l)} 0.$$

Finally, we have

$$1 + 2\mathcal{V}_0 \dots \mathcal{V}_{j(l)} 0 \leq 1 + 2V_1 \dots V_l 0,$$

by Lemma 2.2 once again. By (20) the second inequality in (13) is proved. ∎

3. Relations to maximal functions of stochastic processes. Let us note that the operators $\mathcal{V}_i$, $i \geq 0$ and V_k, $k \geq 1$ turned out to be a very useful tool in investigation of continuity of processes with orthogonal increments in L_2 and bounded increments in L_p. In particular Paszkiewicz in [3] used both sets of operators to prove the following theorem. Let us say that a process $(X_t)_{t \in T}$ satisfying $X_t \in L_2$, $t \in T$ is a process with orthogonal increments if

$$\|X_t - X_s\|^2 = |t - s|, \quad s, t \in T.$$

THEOREM 3.1. *Let $T \subset [0,1]$. Every separable process $(X_t)_{t \in T}$, $X_t \in L^2$, $t \in T$ with orthogonal increments on T is sample continuous if and only if the set T possesses the following property: for $\Delta_k = \bigcap \{Z \in \mathcal{F}_k : T \subset Z\}$*

$$\lim_{k \to \infty} \|V_1 \dots V_k 0\| < \infty,$$

according to (10).

The proof of the theorem is unexpectedly hard; especially the construction of a discontinuous orthogonal process in case when $\lim_{k \to \infty} \|V_1 \dots V_k 0\| = \infty$ requires a number of combinatorial lemmata. In [2] a variation of the operators of V_i, $i \geq 1$, was used to investigate sample continuity of processes in L_p satisfying a weaker condition. Below, $|\cdot|$ is the maximum norm in R^n and $\|\cdot\|_p$ denotes the L_p-norm. We use the standard dyadic σ-fields $\mathcal{F}_k^{\otimes n}$ in $[0,1]^n$ (cf. (5)).

THEOREM 3.2. *Let $T \subset [0,1]^n$, $q > 1$. Every separable process $(X_t)_{t \in T}$, $X_t \in L_p$, $t \in T$, satisfying*

$$\|X_t - X_s\|_p \leq \sqrt[q]{|s - t|}, \quad s, t \in T$$

is path continuous if and only if T satisfies: for $\Delta_k = \bigcap \{Z \in \mathcal{F}_k^{\otimes n} : T \subset Z\}$, $V_k^p h = 2^{k/q - in/p} \mathbb{I}_{\Delta_k} + \sqrt[p]{\mathbb{E}(g^p | \mathcal{F}_k^{\otimes n})}$, $g \in L_p$,

$$\lim_{k \to \infty} \|V_1 \dots V_k 0\| < \infty.$$

Path continuity of processes with bounded increments in a more general setting was also investigated by means of so called majorizing measures. In particular Talagrand in [4] showed for a large class of Young functions Φ that whenever for a compact metric space (T,d) we have

$$\mathcal{M}(T) := \inf\left\{\sup_{t\in T}\int_0^{\operatorname{diam}(T)} \Phi^{-1}\left(\frac{1}{m\{s\in T\colon d(s,t)<\varepsilon\}}\right) \mathrm{d}\varepsilon \colon m\in P(T)\right\} < \infty \quad (21)$$

(with $P(T)$ denoting the set of all Borel probability measures) then all separable processes $(X_t)_{t\in T}$ satisfying $\|X_t - X_s\|_\Phi \le d(t,s)$ (i.e. $\mathbb{E}\Phi\left(|X_t - X_s|/d(t,s)\right) \le 1$), $s,t\in T$ are path continuous. Any $m\in P(T)$ for which the supremum in (21) is finite is called a majorizing measure. The condition $\mathcal{M}(T)<\infty$ is also necessary if Φ grows faster than the function $\Phi_\gamma(x) = x^{\gamma\ln\ln(x+e)}$, for some $\gamma>0$. In [1] it was also shown that (21) is also necessary for, roughly speaking, metrics being a root of a metric, i.e. d^q is also a metric in T, for some $q>0$ (see [1] for details).

The following surprising theorem can be deduced from our reasoning and from [3], [2]. It connects our reinsurance model with properties of stochastic processes on a set T and with majorizing measures for $\|\cdot\|_\Phi = \|\cdot\|$, $d(s,t) = \sqrt{|s-t|}$. Thus we put $\mathcal{M}(T,\|\cdot\|) := \inf_{m\in P(T)}\sup_{t\in T}\int_0^1 \sqrt{1/m\{s\in T\colon |s-t|<\varepsilon^2\}}\,\mathrm{d}\varepsilon$.

As a crucial assumption we need the following

$$\Delta_k = \bigcap\{Z\in\mathcal{F}_k\colon T\subset Z\}, \quad k\ge 0 \quad (22)$$

for our sequence (7).

Theorem 3.3. *Let $\emptyset\ne T\subset[0,1]$ be a finite set. Let Δ_k be defined in* (22)*. Let $\mathbf{k}$ be a 'reinsurance system' $0=k_0<k_1<\ldots<k_{j(\mathbf{k})}$, satisfying*

$$\{2^0,2^1,\ldots\}\cap\{1,2,3,\ldots,k_{j(\mathbf{k})}\}\subset\mathbf{k},$$
$$2^{-k_{j(\mathbf{k})}} < \min_{s,t\in A}|s-t|.$$

Let $R\colon[0,1)\to[0,k_{j(\mathbf{k})})$ be a risk function satisfying (9)*. For the operators $P^{R\mathbf{k}}_{k_i}$, $i\ge 0$ defined by* (11) *we have*

$$\|P^{R\mathbf{k}}_{k_0}\ldots P^{R\mathbf{k}}_{k_{j(\mathbf{k})}}0\| \sim \mathcal{M}(T,\|\cdot\|) \sim \sup_{\substack{X \text{ process}\\ \text{with bounded inc.}}} \mathbb{E}\max_{s,t\in A}(X_t-X_s) \sim \sup_{\substack{X \text{ process}\\ \text{with orthog. inc.}}} \mathbb{E}\max_{s,t\in A}(X_t-X_s). \quad (23)$$

The relation $\sim$ in the above theorem denotes equality modulo some universal constant, i.e.

$$\mathcal{S}\sim\mathcal{P} \equiv \exists_{a\in\mathbb{R}}\left[\frac{1}{a}\mathcal{S}-a\le\mathcal{P}\le a\mathcal{S}+a\right],$$

for any quantities $\mathcal{S}$, $\mathcal{P}$.

4. Important examples. In this section we present some technical examples which compare the conditional reinsurance scheme with two other natural schemes. As it turns out, **the total price of the insurance with conditional reinsurance can be considerably larger than the price of the insurance without reinsurance** (Theorem 4.1); **it can also be considerably smaller than the price of insurance where all reinsurers are paid unconditionally** (Theorem 4.4).

The theorems of this section imply that our insurance system with conditional reinsurance premiums is optimal if there is no single insurer willing to price the risk R at $\|R\|$. Our conditional pricing scheme can also produce the ultimate price any given factor lower than the premium in a system where all the reinsurance premiums are paid naively, summing up to

$$\|(R-k_0)^+ - (R-k_1)^+\| + \ldots + \|(R-k_{j(\mathbf{k})-1})^+ - (R-k_{j(\mathbf{k})})^+\|.$$

THEOREM 4.1. *For any $C>0$ there exists a sequence $[0,1)=\Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_{2^j}$, $\Delta_k \in \mathcal{F}_k$, $0 \le k \le 2^j$ and a function $R\colon [0,1) \to [0,2^j)$, for some $j \ge 1$, satisfying*

$$\{R > 2^i\} \subset \Delta_{2^i} \subset \{R \ge 2^i\}, \quad 0 \le i \le j,$$

and for $\mathbf{d} = (0, 2^0, 2^1, \ldots, 2^j)$, according to (11), we have

$$C\|R\| < \|P_{2^0}^{R\mathbf{d}} \ldots P_{2^j}^{R\mathbf{d}} 0\|.$$

Before we proceed to the demonstration of the theorem first let us fix the following notation. Let us notice that each σ-field $\mathcal{F}_{2^i}$, $i \ge 1$ can be interpreted as the product $\mathcal{F}_{2^{i-1}} \otimes \mathcal{F}_{2^{i-1}}$. Thus for any $i \ge 1$ there exists an event $A_i \in \mathcal{F}_{2^i}$ independent of $\mathcal{F}_{2^{i-1}}$ with $\lambda(A_i)$ being any given multiple of $2^{-2^{i-1}}$. Let us fix a number $h \ge 1$ and a sequence $(A_i)_{i>h}$ for which for every integer $i > h$ we have $A_i \in \mathcal{F}_{2^i}$ and the set A_i is independent of $\mathcal{F}_{2^{i-1}}$. Moreover A_i satisfies

$$\frac{1}{4}\left(\frac{i}{i+1}\right)^2 \le \lambda(A_i) \le \frac{1}{4}\left(\frac{i}{i+1}\right)^{3/2}, \quad i > h. \tag{24}$$

Such h (and sets A_i) exist since for the bounds in (24) we have $\frac{1}{4}\left(\frac{i}{i+1}\right)^{3/2} - \frac{1}{4}\left(\frac{i}{i+1}\right)^2 > 2^{-2^{i-1}}$ for i large enough.

We can additionally define

$$A_0, \ldots, A_h = [0,1). \tag{25}$$

Now we can define

$$\Delta_{2^i} = A_0 \cap \ldots \cap A_i, \quad i \ge 0 \tag{26}$$

and also

$$\Delta_0 = [0,1),$$
$$\Delta_k = \Delta_{2^i}, \quad 2^i \le k < 2^{i+}, \, i \ge 0.$$

Thus $\Delta_k \in \mathcal{F}_k$, $k \ge 0$.

Let $j \in \mathbb{N}$, $j > h$. We define a risk function R_j by

$$R_j = \mathbb{I}_{\Delta_{2^0}} + 2^0 \mathbb{I}_{\Delta_{2^1}} + \ldots + 2^{j-1}\mathbb{I}_{\Delta_{2^j}}. \tag{27}$$

Now for the notions defined above we can formulate the following lemmas.

LEMMA 4.2. *With R_j as above we have $R_j\colon [0,1) \to [0,2^j]$ and*

$$\{R_j \ge 2^i\} = \Delta_{2^i}, \quad 0 \le i \le j,$$
$$\{R_j > 2^i\} = \Delta_{2^{i+1}}, \quad 0 \le i < j,$$

in particular (9) *holds for* $(k_0, \ldots, k_{j(\mathbf{k})}) = (0, 2^0, \ldots, 2^j)$. *Moreover* $(R_j - 2^{i-1})^+ - (R_j - 2^i)^+ = 2^{i-1}\mathbb{I}_{\Delta_{2^i}}$, $1 \le i \le j$ *and* $R_j - (R_j - 2^0)^+ = \mathbb{I}_{\Delta_{2^0}}$.

By independence of $A_{i'}$ and A_i, $i \neq i'$ and by (26) we have

Lemma 4.3. *For integers* $i > i' \geq 0$ *we have*

$$\left\|\mathbb{I}_{\Delta_{2^i}}\right\|_{\mathcal{F}_{2^{i'}}} = \sqrt{\mathbb{E}(\mathbb{I}_{\Delta_{2^{i'}}}\mathbb{I}_{A_{i'+1}}\ldots\mathbb{I}_{A_i}|\mathcal{F}_{2^{i'}})} = \sqrt{\mathbb{E}(\mathbb{I}_{A_{i'+1}}|\mathcal{F}_{2^{i'}})\cdot\ldots\cdot\mathbb{E}(\mathbb{I}_{A_i}|\mathcal{F}_{2^{i'}})}\mathbb{I}_{\Delta_{2^{i'}}}$$
$$= \sqrt{\lambda\left(A_{i'+1}\right)\cdot\ldots\cdot\lambda\left(A_i\right)}\mathbb{I}_{\Delta_{2^{i'}}}.$$

This yields two almost immediate corollaries.

Corollary 1. *For integers* $0 \le i' < i \le j$ *we have*

$$\left\|(R-2^{i-1})^+ - (R-2^i)^+\right\|_{\mathcal{F}_{i'}} = \|2^{i-1}\mathbb{I}_{\Delta_{2^i}}\|_{\mathcal{F}_{i'}} = \sqrt{\lambda\left(A_{i'+1}\right)\cdot\ldots\cdot\lambda\left(A_i\right)}2^{i-1}\mathbb{I}_{\Delta_{2^{i'}}}.$$

Corollary 2. *For* $\mathbf{d}(j) = (0, 2^0, 2^1, \ldots 2^j)$ *we have*

$$\left\|P_{2^0}^{R_j\mathbf{d}(j)}\ldots P_{2^j}^{R_j\mathbf{d}(j)}0\right\| = \left\|R-(R-2^0)^+\right\| + \left\|(R-2^0)^+ - (R-2^1)^+\right\| + \ldots$$
$$+ \left\|(R-2^{j-2})^+ - (R-2^{j-1})^+\right\| + \left\|(R-2^{j-1})^+\right\|.$$

Proof. Let us notice that $\|P_{2^j}^{R_j\mathbf{d}(j)}0\|_{\mathcal{F}_{j-1}} = \|(R-2^{j-1})^+\|_{\mathcal{F}_{j-1}}$. Moreover for every $0 \le i \le j$ we have

$$\|P_{2^i}^{R_j\mathbf{d}(j)}\ldots P_{2^j}^{R_j\mathbf{d}(j)}0\|_{\mathcal{F}_{2^{i-1}}} = \sum_{m=i}^{j}\|(R-2^{m-1})^+ - (R-2^m)^+\|_{\mathcal{F}_{2^{i-1}}},$$

assuming a convenient notation $2^{-1} := 0$. Indeed, if the above is true for some $0 < i \le j$ then

$$\begin{aligned}
&\|P_{2^{i-1}}^{R_j\mathbf{d}(j)}\ldots P_{2^j}^{R_j\mathbf{d}(j)}0\|_{\mathcal{F}_{2^{i-2}}} \\
&= \left\|(R-2^{i-2})^+ - (R-2^{i-1})^+ + \|P_{2^i}^{R_j\mathbf{d}(j)}\ldots P_{2^j}^{R_j\mathbf{d}(j)}0\|_{\mathcal{F}_{2^{i-1}}}\right\|_{\mathcal{F}_{2^{i-2}}} \\
&= \left\|2^{i-2}\mathbb{I}_{\Delta_{2^{i-1}}} + \sum_{m=i}^{j}\|(R-2^{m-1})^+ - (R-2^m)^+\|_{\mathcal{F}_{2^{i-1}}}\right\|_{\mathcal{F}_{2^{i-2}}} \\
&= \sum_{m=i-1}^{j}\|(R-2^{m-1})^+ - (R-2^m)^+\|_{\mathcal{F}_{2^{i-2}}},
\end{aligned}$$

since all summands under the outer norm are collinear (by Corollary 1). ∎

Proof of Theorem 4.1. With an arbitrary integer j fixed let us observe that we have by (26), (27)

$$\begin{aligned}
\|R_j\|^2 &= (2^0)^2\lambda(\Delta_{2^0}\setminus\Delta_{2^1}) + \ldots + (2^{j-1})^2\lambda(\Delta_{2^{j-1}})\setminus\Delta_{2^j}) + (2^j)^2\lambda(\Delta_{2^j}) \\
&\le (2^0)^2\lambda(\Delta_{2^0}) + \ldots + (2^j)^2\lambda(\Delta_{2^j}) \\
&= (2^0)^2\lambda(A_0) + \ldots + (2^j)^2\lambda(A_0)\cdot\ldots\cdot\lambda(A_j) \le K
\end{aligned}$$

for some $K > 0$ as, by (24), for $i > h$,

$$(2^i)^2\lambda(A_0)\cdot\ldots\cdot\lambda(A_i) \le 4^i\left(\frac{1}{4}\right)^{i-h}\left(\frac{h+1}{h+2}\cdot\ldots\cdot\frac{i}{i+1}\right)^{3/2} = \frac{4^h(h+1)^{3/2}}{(i+1)^{3/2}}.$$

On the other hand by Corollary 2 and Lemma 4.2

$$\begin{aligned}
&\left\| P_{2^0}^{R_j \mathbf{d}(j)} \ldots P_{2^j}^{R_j \mathbf{d}(j)} 0 \right\| \\
&> 2^h \sqrt{\lambda(\Delta_{2^{h+1}})} + \ldots + 2^{j-1} \sqrt{\lambda(\Delta_{2^j})} \\
&\geq 2^h \sqrt{\lambda(A_{h+1})} + \ldots + 2^{j-1} \sqrt{\lambda(A_{h+1} \cap \ldots \cap A_j)} \qquad \text{(by (25), (26))} \\
&\geq 2^h (h+1) \left(\frac{1}{h+2} + \ldots + \frac{1}{j} \right) \qquad \text{(by (24))}.
\end{aligned}$$

We can conclude that $\|P_{2^0}^{R_j \mathbf{d}} \ldots P_{2^j}^{R_j \mathbf{d}} 0\|$ tends to infinity as $j \to \infty$, whereas $\|R_j\| < K$ for some constant K and all integers $j > h$. ■

THEOREM 4.4. *For any $C > 0$ there exists a sequence $[0,1) = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_{2^j}$, $\Delta_k \in \mathcal{F}_k$, $0 \leq k \leq 2^j$ and a function R: $[0,1) \to [0, 2^j]$, for some integer j, satisfying*

$$\{R > 2^i\} \subset \Delta_{2^i} \subset \{R \geq 2^i\}, \quad 0 \leq i \leq j$$

and for $\mathbf{d} = (0, 2^0, 2^1, \ldots 2^j)$, according to (16), we have

$$\begin{aligned}
&C \cdot \|P_{2^0}^{R\mathbf{d}} \ldots P_{2^j}^{R\mathbf{d}} 0\| < \\
&\quad \|R - (R - 2^0)^+\| + \|(R - 2^0)^+ - (R - 2^1)^+\| + \ldots + \|(R - 2^{j-1})^+ - (R - 2^j)^+\|.
\end{aligned}$$

Denote $j(s) = 2^s$ for integer s (for typographical reasons). Then we have

LEMMA 4.5. *There exists a non-increasing sequence of sets Δ_{2^i}, $i \geq j(4)$, satisfying*

$$\Delta_{2^i} \text{ is an interval with left end } 0,$$

$$\frac{1}{4^i i^2} \leq \lambda(\Delta_{2^i}) \leq \frac{2}{4^i i^2}, \tag{28}$$

$$\Delta_{2^i} \in \mathcal{F}_{2^{j(s)}}, \quad \text{for } i < j(s+1). \tag{29}$$

Proof. For $s \geq 4$ we have

$$\begin{aligned}
2^{j(s)} &> 2j(s+1) + 2(s+1), \\
2^{2^{j(s)}} &> 4^{j(s+1)} j(s+1)^2, \\
2^{2^{j(s)}} &> 4^i i^2,
\end{aligned}$$

if only $i < j(s+1)$. Thus there exist numbers $n(i)$ satisfying

$$\frac{1}{4^i i^2} \leq \frac{n(i)}{2^{2^{j(s)}}} \leq \frac{2}{4^i i^2}, \quad j(s) \leq i < j(s+1).$$

It is enough to set

$$\Delta_{2^i} = [0, n(i) 2^{-2^{j(s)}}), \quad j(s) \leq i < j(s+1),\ s \geq 4. \ ■$$

Now we can complete our definition of the sequence $(\Delta_k)_{k \geq 0}$. Let

$$\Delta_{2^i} = \Delta_0 = [0,1), \quad 0 \leq i < j(4),$$

$$\Delta_{2^i} = \Delta_{2^i+1} = \ldots = \Delta_{2^{i+1}-1}, \quad i \geq 0.$$

Let us consider an integer $t > 4$. We define a sequence

$$\mathbf{d}_t = (0, 2^0, 2^1, \ldots, 2^{j(t)}) \tag{30}$$

and a risk function (cf. (27))

$$R_t = \mathbb{I}_{\Delta_0} + 2^0\mathbb{I}_{\Delta_{2^0}} + \ldots + 2^{j(t)-1}\mathbb{I}_{\Delta_{2^{j(t)-1}}}. \tag{31}$$

By definition of the sequence $(\Delta_k)_{k\geq 0}$ and the function R_t we immediately have a lemma on particularly strong measurability of increments of R_t. Namely

LEMMA 4.6. *With notation as above we have* $R_t\colon [0,1) \to [0, 2^{j(t)}]$ *and*

$$R_t - (R_t - 2^{j(s+1)})^+ \text{ is } \mathcal{F}_{2^{j(s)}}\text{-measurable,}$$

for all $s \geq 4$.

Proof. Let us notice that $R_t - (R_t - 2^{j(s+1)})^+ = \mathbb{I}_{\Delta_0} + \sum_{n=0}^{j(s+1)-1} 2^n \mathbb{I}_{\Delta_{2^n}}$ and (29) can be used. ∎

The definition (16) implies

LEMMA 4.7. *If* $\mathbf{k} = (k_0, k_1, \ldots k_{j(\mathbf{k})})$,

$$0 = k_0 < k_1 < \ldots < k_j < \ldots < k_{j'} < \ldots < k_{j(\mathbf{k})},$$

$R\colon [0,1) \to [0, k_{j(\mathbf{k})}]$ *and*

$$R - (R - k_{j'})^+ \text{ is } \mathcal{F}_{k_j}\text{-measurable,}$$

then

$$P^{R\mathbf{k}}_{k_{j+1}} \ldots P^{R\mathbf{k}}_{k_{j'}} 0 = (R - k_j)^+ - (R - k_{j'})^+.$$

It is also a simple exercise to show that by subadditivity of conditional norms we have

LEMMA 4.8. *If* $\mathbf{k} = (k_0, k_1, \ldots k_{j(\mathbf{k})})$,

$$0 = k_0 < k_1 < \ldots < k_j < \ldots < k_{j'} < \ldots < k_{j''} < \ldots < k_{j(\mathbf{k})},$$

$R\colon [0,1) \to [0, k_{j(\mathbf{k})}]$ *then*

$$\|P^{R\mathbf{k}}_{k_j} \ldots P^{R\mathbf{k}}_{k_{j''}} 0\| \leq \|P^{R\mathbf{k}}_{k_j} \ldots P^{R\mathbf{k}}_{k_{j'}} 0\| + \|P^{R\mathbf{k}}_{k_{j'+1}} \ldots P^{R\mathbf{k}}_{k_{j''}} 0\|.$$

Proof. Notice that for any functions $h, g \in L_\infty[0,1)$ we have $P^{R\mathbf{k}}_{k_j}(h+g) \leq (P^{R\mathbf{k}}_{k_j} h) + \|g\|_{\mathcal{F}_{k_j}}$, $1 \leq j \leq k_{j(\mathbf{k})}$. ∎

Proof of Theorem 4.4. For the risk function given in (31) and 'reinsurance system' (30) we have the estimate

$$\begin{aligned}
&\|R_t - (R_t - 2^0)^+\| + \|(R_t - 2^0)^+ - (R_t - 2^1)^+\| + \ldots + \|(R_t - 2^{j(t)-1})^+ - (R_t - 2^{j(t)})^+\| \\
&\geq \sum_{j(4) < i \leq j(t)} \|(R_t - 2^{i-1})^+ - (R_t - 2^i)^+\| = \sum_{j(4) < i \leq j(t)} 2^{i-1}\sqrt{\lambda(\{R_t > 2^{i-1}\})} \\
&= \sum_{j(4) < i \leq j(t)} 2^{i-1}\sqrt{\lambda(\Delta_{2^{i-1}})} \geq \sum_{j(4) < i \leq j(t)} 2^{i-1}\sqrt{\frac{1}{4^{i-1}(i-1)^2}} = \sum_{i=j(4)}^{j(t)-1} \frac{1}{i}.
\end{aligned}$$

It is clear that the rightmost term grows to infinity as $t \to \infty$.

On the other hand we have

$$
\begin{aligned}
&\|P_{2^0}^{R_t\mathbf{d}_t}\dots P_{2^{j(t)}}^{R_t\mathbf{d}_t}0\|\\
&\le \|P_{2^0}^{R_t\mathbf{d}_t}\dots P_{2^{j(4)}}^{R_t\mathbf{d}_t}0\| + \|P_{2^{j(4)+1}}^{R_t\mathbf{d}_t}\dots P_{2^{j(5)}}^{R_t\mathbf{d}_t}0\| + \dots + \|P_{2^{j(t-1)+1}}^{R_t\mathbf{d}_t}\dots P_{2^{j(t)}}^{R_t\mathbf{d}_t}0\|\\
&\quad \text{(by Lemma 4.8)}\\
&\le \|R_t - (R_t - 2^{j(4)})^+\| + \|(R_t - 2^{j(4)})^+ - (R_t - 2^{j(5)})^+\| + \dots\\
&\quad + \|(R_t - 2^{j(t-1)})^+ - (R_t - 2^{j(t)})^+\| \quad \text{(by Lemmas 4.6 and 4.7)}\\
&< 2^{j(4)} +\\
&\quad \sqrt{(2^{j(4)})^2\lambda(\Delta_{2^{j(4)}}) + (2^{j(4)+1})^2\lambda(\Delta_{2^{j(4)+1}}) + \dots + (2^{j(5)-1})^2\lambda(\Delta_{2^{j(5)-1}})}\\
&\quad + \dots +\\
&\quad \sqrt{(2^{j(t-1)})^2\lambda(\Delta_{2^{j(t-1)}}) + (2^{j(t-1)+1})^2\lambda(\Delta_{2^{j(t-1)+1}}) + \dots + (2^{j(t)-1})^2\lambda(\Delta_{2^{j(t)-1}})}\\
&\quad \text{(by (31))}\\
&\le 2^{j(4)} + \sqrt{2}\left(\frac{1}{j(4)^2} + \frac{1}{(j(4)+1)^2} + \dots + \frac{1}{(j(5)-1)^2}\right)^{1/2} + \dots\\
&\quad + \sqrt{2}\left(\frac{1}{j(t-1)^2} + \frac{1}{(j(t-1)+1)^2} + \dots + \frac{1}{(j(t)-1)^2}\right)^{1/2} \quad \text{(by (28))}\\
&\le 2^{2^4} + \sqrt{2}\cdot\left(2^4\frac{1}{(2^4)^2}\right)^{1/2} + \dots + \sqrt{2}\cdot\left(2^{t-1}\frac{1}{(2^{t-1})^2}\right)^{1/2}\\
&< 2^{2^4} + \frac{8}{\sqrt{2}-1}, \quad \text{since } j(s) = 2^s,
\end{aligned}
$$

which completes the proof. ∎

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