STABILITY IN PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 90
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

STABLE RANDOM FIELDS AND GEOMETRY

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Abstract. Let (M, d) be a metric space with a fixed origin $\mathbf{O}$. P. Lévy defined Brownian motion $\{X(a); a \in M\}$ as

0. $X(\mathbf{O}) = 0$.
1. $X(a) - X(b)$ is subject to the Gaussian law of mean 0 and variance $d(a, b)$.

He gave an example for $M = S^m$, the m-dimensional sphere. Let $\{Y(B); B \in \mathcal{B}(S^m)\}$ be the Gaussian random measure on S^m, that is,

1. $\{Y(B)\}$ is a centered Gaussian system,
2. the variance of $Y(B)$ is equal of $\mu(B)$, where μ is the uniform measure on S^m,
3. if $B_1 \cap B_2 = \emptyset$ then $Y(B_1)$ is independent of $Y(B_2)$.
4. for B_i, $i = 1, 2, \ldots$, $B_i \cap B_j = \emptyset, i \neq j$, we have $Y(\cup B_i) = \sum Y(B_i)$, a.e.

Set $S_a = H_a \triangle H_{\mathbf{O}}$, where H_a is the hemisphere with center a, and $\triangle$ means symmetric difference. Then

$$\{X(a) = Y(S_a); a \in S^m\}$$

is Lévy's Brownian motion.

In the case of $M = R^m$, m-dimensional Euclidean space, N. N. Chentsov showed that $\{X(a) = Y(S_a)\}$ is an R^m-parameter Brownian motion in the sense of P. Lévy. Here S_a is the set of hyperplanes in R^m which intersect the line segment $\overline{\mathbf{O}a}$. The Gaussian random measure $\{Y(\cdot)\}$ is defined on the space of all hyperplanes in R^m and the measure μ is invariant under the dual action of Euclidean motion group $Mo(m)$.

Replacing the Gaussian random measure with an SαS (Symmetric α Stable) random measure, we can easily obtain stable versions of the above examples. In this note, we will give further examples:

1. For hyperbolic space, taking as S_a a self-similar set in R^m, we obtain stable motion on the hyperbolic space.

2010 *Mathematics Subject Classification*: 60G52, 60G60, 60G51, 53C65.
Key words and phrases: stable random fields, multi-parameter additive processes, subordination.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc90-0-15

2. Take as S_a the set of all spheres in R^m of arbitrary radii which separate the origin O and the point $a \in R^m$; then we obtain a self-similar SαS random field as $\{X(a) = Y(S_a)\}$.

Along these lines, we will consider a multi-dimensional version of Bochner's subordination.

1. Multi-parameter Brownian motion of P. Lévy

1.1. Definition and construction by Lévy. In the famous book "Mouvement Brownien" ([7], [8]), P. Lévy defined a notion of Brownian motion $\{X(u); u \in M\}$ on a metric space $(M, d(\cdot,\cdot))$ with a fixed origin $\mathbf{O}$:

Definition 1.1. A Gaussian system $\{X(u)\}$ is called a Brownian motion on a metric space $(M, d(\cdot,\cdot))$ if it satisfies

1. $X(\mathbf{O}) \equiv 0$.
2. $X(u) - X(v)$ is subject to the Gaussian law of mean 0 and variance $d(u, v)$.

In the case of $M = S^m$, he constructed a Brownian motion from Gaussian random measure on the sphere S^m. Let us start with the definition of random measure.

Definition 1.2. A centered Gaussian system $\mathcal{Y} = \{Y(B); B \in \mathcal{B}, \mu(B) < \infty\}$ is called a Gaussian random measure controlled by a measure space $(E, \mathcal{B}, \mu)$ if

1. $Y(B)$ is subject to the Gaussian law of mean 0 and variance $\mu(B)$.
2. If $B_1 \cap B_2 = \emptyset$ then the random variables $Y(B_1)$ and $Y(B_2)$ are mutually independent.
3. For any sequence of mutually disjoint family of measurable sets $B_1, B_2, B_3, \ldots$,
$$Y(\cup_n B_n) = \sum_n Y(B_n), \text{ a.e.}$$

Let $\mathcal{Y}$ be a Gaussian random measure controlled by (S^m, μ), where μ is the uniform measure. For a point u of the sphere S^m, define a set $S_u = H_u \triangle H_{\mathbf{O}}$, where $H_u = \{v \in S^m; d(v, u) \geq \frac{\pi}{2}\} \in \mathcal{B}$. Then
$$X(u) = Y(S_u) = Y((H_u \cap H_O^c) \cup (H_u^c \cap H_O)),$$
is a Brownian motion on S^m.

$$\begin{aligned}
&X(u) - X(v) = Y(S_u) - Y(S_v)\\
&= Y((H_u \cap H_O^c) \cup (H_u^c \cap H_O)) - Y((H_v \cap H_O^c) \cup (H_v^c \cap H_O))\\
&= Y(H_u \cap H_v^c \cap H_{\mathbf{O}}^c) + Y(H_u^c \cap H_v \cap H_{\mathbf{O}}) - Y(H_u^c \cap H_v \cap H_{\mathbf{O}}^c) - Y(H_u \cap H_v^c \cap H_{\mathbf{O}}).
\end{aligned}$$

The variance is
$$\begin{aligned}
\mu(H_u \cap H_v^c \cap H_{\mathbf{O}}^c) + \mu(H_u^c \cap H_v \cap H_{\mathbf{O}}) + \mu(H_u^c \cap H_v \cap H_{\mathbf{O}}^c) + \mu(H_u \cap H_v^c \cap H_{\mathbf{O}})&\\
= \mu(H_u \triangle H_v)&,
\end{aligned}$$
that is, it is proportional to the geodesic distance $d(u, v)$ of $u, v \in S^m$.

1.2. Construction of Brownian motion on the Euclidean space. For m-dimensional Euclidean space R^m, N. N. Chentsov gave the following construction ([3]). Let E be the set of all hyperplanes of co-dimension 1 in R^m, and μ be the measure on E which is invariant under the (dual) action of Euclidean motion group $Mo(m)$. The dual action g^*

of $g \in Mo(m)$ is defined as $(g\mathbf{x}, \mathbf{y}) = (\mathbf{x}, g^*\mathbf{y})$, using the homogeneous coordinate $g^* = {}^t g$ (see the next subsection). Let us represent an element of E by the canonical form

$$\{\mathbf{x} \in R^m; \mathbf{a} \cdot \mathbf{x} + r = 0\}, \qquad \mathbf{a} \in S^{m-1}, r \in R_+ = [0, \infty),$$

and take a parameter $(\mathbf{a}, r) \in S^{m-1} \times R_+$ for the above plane. The invariant measure mentioned above is $d\mu(r \times \mathbf{a}) = d\mathbf{a}dr$.

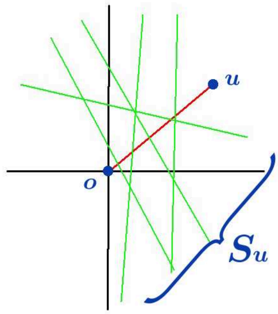

Fig. 1

Set

$$S_u = \{h \in E; h \text{ separates the origin } \mathbf{O} \text{ and } u\}.$$

Then

$$X(u) = Y(S_u)$$

is a Brownian motion on the Euclidean space $(R^m, |\cdot|)$, where $\mathcal{Y} = \{Y(\cdot)\}$ is the Gaussian random measure controlled by $(E, d\mu)$.

Note that, as we will see in the next subsection, these two constructions of Brownian motions share the same idea coming from elementary geometry.

1.3. Projective geometry

1.3.1. *m-dimensional projective space and the homogeneous coordinates.* m-dimensional projective space is defined as $P^m = (R^{m+1} \setminus \{\mathbf{O}\})/(R \setminus \{0\})$, that is, using the homogeneous coordinates,

$$P^m \ni \mathbf{x} = (x_1, x_2, \ldots, x_m, x_0) = (\underline{\mathbf{x}}, x_0),$$

P^m is nothing but $R^{m+1} \setminus \{\mathbf{O}\}$ identified by the equivalence relation

$$\mathbf{x} \sim c \times \mathbf{x}, \;\; c \in R \setminus \{0\}.$$

Let us take a representative $|\mathbf{x}| = 1$ for an element $\mathbf{x} \in P^m$. Then P^m can be considered as the manifold obtained from the sphere S^m by identifying any point x and its antipodal point $-x$. $P^m \setminus \{\mathbf{x}; x_0 = 0\}$—the rest of the infinite plane $\{x_0 = 0\}$—can be considered as $\{\underline{\mathbf{x}}/x_0\} = R^m$. This is a local coordinate system around the origin $\overline{\mathbf{O}} = (0, 0, \ldots, 0, 1)$. The plane which is perpendicular to a vector $\mathbf{x}$ is

$$H_{\mathbf{x}} = \{\mathbf{y}; y_1 x_1 + \cdots + y_0 x_0 = 0\}.$$

On the sphere this set is the great circle with respect to $\mathbf{x}$. The corresponding set of $S_{\mathbf{x}}$ in R^m (see 1.1) is the connected component of $R^d \setminus H_{\mathbf{x}}$ which does not contain the origin $\mathbf{O}$.

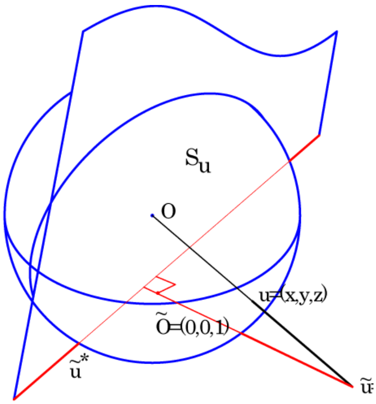

Fig. 2

Let us introduce the duality mapping

$$\mathbf{x} \Longleftrightarrow \mathbf{x}^* = \mathbf{H}_{\mathbf{x}} = \{\mathbf{y}; (\mathbf{y}, \mathbf{x}) = \mathbf{0}\}, \ \mathbf{H}^*_{\mathbf{x}} = \mathbf{x},$$

and consider $\mathbf{x}$ as a coordinate of $H_{\mathbf{x}}$. Then the set $S_{\mathbf{x}}$ coincides with the set of all hyperplanes which separate a point $\mathbf{x}$ and the origin $\mathbf{O}$.

1.3.2. *Group action and invariant measure.* Let $L(m)$, $Mo(m)$ and $SO(m+1)$ be the Lorentz group, the Euclidean motion group and the rotation group respectively. Then the hyperbolic space $\mathcal{H}_2$, the Euclidean space R^m and the sphere S^m are considered as symmetric spaces $\mathcal{H}_2 = L(m)/SO(m)$, $R^m = Mo(m)/SO(m)$ and $S^m = SO(m+1)/SO(m)$ respectively. There exist invariant measures on their dual spaces.

Let us recall the construction of Brownian motions on the sphere and Euclidean space. Consider a metric space (M, d) and a measure space $(E, \mathcal{B}, \mu)$. Assume also that the metric d and the measure μ are both invariant under the group action, and moreover that the relation

$$M \ni u \mapsto S_u \in \mathcal{B}$$

is compatible under the above group action, that is,

$$S_{g \cdot u} = g \cdot S_u.$$

Then the random field defined by $\{X(u) = Y(S_u)\}$ becomes an (M, d)-parameter Brownian motion in the sense of P. Lévy, where $\mathcal{Y} = \{Y(\cdot)\}$ is the Gaussian random measure controlled by (E, μ).

1.4. Hyperbolic space ([26]). Consider the two-sheeted hyperbolic space

$$\mathcal{H}_2 = \{|\underline{\mathbf{x}}|^2 - x_0^2 = -1\},$$

and the dual space, the 1-sheeted hyperbolic space

$$\mathcal{H}_1 = \{|\underline{\mathbf{x}}|^2 - x_0^2 = 1\}.$$

The m-dimensional Lorentz group acts on $\mathcal{H}_2$ and $\mathcal{H}_1$. There exist an invariant metric d and an invariant measure μ on these two spaces respectively. The dual space $\mathcal{H}_1$ can be considered the set of all hyperplanes of co-dimension 1 as in the Euclidean case.

Define

$$X(u) = Y(S_u), \qquad u \in \mathcal{H}_2,$$

where $S_u = \{h \in E; h \text{ separates the origin } \mathbf{O} \text{ and } u\}$, and $\mathcal{Y} = \{Y(\cdot)\}$ is the random measure controlled by the measure space $(\mathcal{H}_1, \mu)$. Then $X(u)$ is a Brownian motion on $\mathcal{H}_2$.

Thus we obtain Brownian motions on the sphere S^m, on the Euclidean space E^m and on the hyperbolic space $\mathcal{H}_2$ by a unified method. Here these three spaces are considered as symmetric spaces with constant curvatures, $+1, 0, -1$, respectively.

2. Stable random fields

2.1. Stable Random measures. Similarly as a generalization of Gaussian random measure, let us define symmetric stable random measures.

DEFINITION 2.1. A symmetric α-stable (SαS) system $\mathcal{Y} = \{Y(B); B \in \mathcal{B}, \mu(B) < \infty\}$, $0 < \alpha < 2$, is called an SαS random measure controlled by the measure $(E, \mathcal{B}, \mu)$ if

1. $Y(B)$ is subject to the SαS law with strength (power of scale parameter) $\mu(B)$, that is, $E[e^{izY(B)}] = e^{-\mu(B)|z|^\alpha}$.
2. For any disjoint sets $B_1, B_2, B_3, \ldots$, the random variables $Y(B_1), Y(B_2), Y(B_3), \ldots$ form an independent family.
3. For any disjoint sets $B_1, B_2, B_3, \ldots$,

$$Y(\cup_n B_n) = \sum_n Y(B_n), \qquad \text{a.e.}$$

2.2. Stable random fields on spaces of constant curvatures ([26]). The results in 1.1–1.3 for the Gaussian system can be extended to stable cases.

DEFINITION 2.2. An SαS system $\{X(u)\}$ is called an SαS Lévy motion on a metric space $(M, d(\cdot,\cdot))$ if

1. $X(\mathbf{O}) \equiv 0$, where $\mathbf{O}$ is the origin of M.
2. $X(u) - X(v)$ is subject to the SαS law of strength $d(u, v)$.

The constructions of random fields used in the last section are also valid for stable cases.

parameter space M	group	measure space E	measure μ
sphere S^m	$SO(m+1, R)$	sphere	$\frac{d\underline{\mathbf{x}}}{(\lvert(\underline{\mathbf{x}})\rvert^2+1)^{(m+1)/2}}$
Euclidean space R^m	motion group $Mo(m)$	cylinder	$\frac{d\underline{\mathbf{x}}}{(\lvert(\underline{\mathbf{x}})\rvert^{m+1}}$
hyperbolic space $\mathcal{H}_2^m$	Lorentz group $L(m)$	$\mathcal{H}_1^m$	$\frac{d\underline{\mathbf{x}}}{(\lvert(\underline{\mathbf{x}})\rvert^2-1)^{(m+1)/2}}$

Let $\mathcal{Y}$ be the SαS random measure controlled by the measure (E, μ), and define

$$X(u) = Y(S_u).$$

Then X is an SαS Lévy motion on the metric space M. This random field $X(u)$ has independent increments along any geodesic lines. That is, for any geodesic line $L = L(t)$ of M, the 1-parameter stochastic process

$$X_L(t) = X(L(t)) - X(L(0))$$

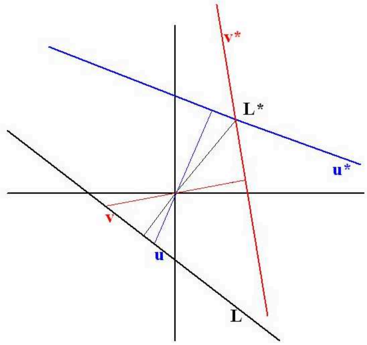

Fig. 3

is an additive process with stationary and independent increments. This fact can be derived from the following simple geometrical relation:

$$L \ni \forall \mathbf{u} \to \mathbf{u}^* \ni L^*.$$

That is, all points $\mathbf{u}$ of L, boundaries of the set $S_{\mathbf{u}}$, share a point of L^*, the dual of L. That is, as we see, when $\mathbf{u}$ moves to $\mathbf{v}$ along L, the boundary v^* rotates around L^*. This means that the set $S_{\mathbf{u}} \triangle S_{\mathbf{v}}$ increases monotonically. That is, $X(\cdot)$ has independent increments along L (also see fig. 7 in 3.4).

2.3. Chentsov type random fields. In general, suppose there exist a parameter space M, a measure space $(E, \mathcal{B}, \mu)$ and a mapping $S_u : M \ni u \mapsto S_u \in \mathcal{B}$. Let us call an S$\alpha$S random field X defined by

$$X(u) = Y(S_u)$$

a random field of Chentsov type, where $\mathcal{Y} = \{Y(\cdot)\}$ is the SαS, $0 < \alpha < 2$, random measure controlled by (E, μ).

2.3.1. *n-dimensional characteristic functions.* For n points $(u_1, u_2, \ldots, u_n)$ of parameter space M, the n-dimensional characteristic function is

$$\begin{aligned} &E[\exp\{i(z_1 X(u_1) + z_2 X(u_2) + \cdots + z_n X(u_n))\}] \\ &\qquad = E[\exp\{i(z_1 Y(S_{u_1}) + z_2 Y(S_{u_2}) + \cdots + z_n Y(S_{u_n})\}] \end{aligned}$$

Let us decompose the sets $S_{u_k},\ k = 1, \ldots, n$ into mutually disjoint sets, so that

$$X(u_1), X(u_1), \ldots, X(u_n)$$

are decomposed into their independent components. Then the above equals

$$\begin{aligned} &= E\Big[\exp\Big(i\Big\{ \sum_{\{1,2,\ldots,n\}\supset A,\ A\neq\emptyset} \Big(\sum_{k\in A} z_k\Big) Y\Big(\bigcap_{k\in A} S_{u_k} \cap \bigcap_{j\notin A} S^c_{u_j}\Big)\Big\}\Big)\Big] \\ &= \exp\Big(-\Big\{\sum_A \Big|\sum_{k\in A} z_k\Big|^\alpha \mu\Big(\bigcap_{k\in A} S_{u_k} \cap \bigcap_{j\notin A} S^c_{u_j}\Big)\Big\}\Big). \end{aligned}$$

The above means that we have a characterization of the spectral measure ν of a Chentsov type random vector $X(u_1), X(u_2), \dots, X(u_n)$,

$$E[\exp\{i(\mathbf{z}, \mathbf{X})\}] = \exp\Big(-\Big\{\int_{S^{n-1}} |(\mathbf{z}, \mathbf{s})|^\alpha \nu(\mathbf{s})\, d\mathbf{s}\Big\}\Big).$$

The spectral measures of Chentsov type random vectors concentrate on the symmetric $2 \times (2^n - 1)$ points on S^{n-1},

$$\pm(1, 0, \dots, 0), \pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\Big), \ \dots.$$

2.4. Determinism ([4], [16], [20])

2.4.1. *Consistency laws.* In general, there exist consistency laws for the above spectral measures. For instance, consider 3 SαS random variables (X_1, X_2, X_3) and compare the spectral measures of (X_1, X_2, X_3) and (X_1, X_2):

$$E[\exp\{i(z_1 X_1 + z_2 X_2 + z_3 X_3)\}] = \exp\Big(-\int_{S^2} |z_1 s_1 + z_2 s_2 + z_3 s_3|^\alpha \nu(s_1, s_2, s_3)\, ds_1\, ds_2\, ds_3\Big)$$

$$E[\exp\{i(z_1 X_1 + z_2 X_2)\}] = \exp\Big(-\int_{S^1} |z_1 s_1 + z_2 s_2|^\alpha \nu_{1,2}(s_1, s_2)\, ds_1\, ds_2\Big).$$

On the other hand, $E[\exp\{i(z_1 X_1 + z_2 X_2 + z_3 X_3)\}]|_{z_3=0} = E[\exp\{i(z_1 X_1 + z_2 X_2)\}]$. We have a consistency law for these spectral measures

$$\nu_{1,2}(s_1, s_2) = \int \nu(s_1, s_2, s_3)\, ds_3,$$

or in spherical coordinates (θ, φ),

$$\nu_{1,2}(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu(\theta, \varphi) \cos(\varphi) d\varphi.$$

Similar relations hold for $\nu_{2,3}$ and $\nu_{3,1}$.

Let us return to our case. Consider a stable family (not necessarily Chentsov type) of 3 variables (X_1, X_2, X_3) such that all marginal characteristic functions of the pairs (X_1, X_2), (X_2, X_3), (X_3, X_1) are of Chentsov type, that is, their spectral measures concentrate on the points $\pm(1, 0), \pm(\frac{1}{2}, \frac{1}{2}), \pm(0, 1)$. Then, from the above consistency laws, the 3-dimensional spectral measure should be of Chentsov type. The same facts hold for any higher dimensional case. Thus,

Theorem 2.3 ([21]). *If all 2-dimensional marginal characteristic functions of an SαS family $\{X(t); t \in T\}$ are of Chentsov type, then the family X itself has Chentsov type spectral measure.*

2.4.2. *Lack of point mass from geometry ([4], [16]).* Consider 3 sets A, B, C of a measure space (E, μ) and suppose $A \cap B \cap C^c = \emptyset$. Then

$$\mu(A \cap B \cap C) = \mu(A \cap B).$$

This relation means that we can calculate any 3-dimensional measure from their 2-dimensional marginal measures. In the above case, we have

$$\mu(A \cap B^c \cap C) = \mu(A \cap C) - \mu(A \cap B \cap C) = \mu(A \cap C) - \mu(A \cap B)$$

(see the left hand side of fig. 4). It is easy to show

Proposition 2.4. *Consider a Chentsov type SαS family $\{X_1, X_2, \dots, X_n\}$. If there exists an integer k such that for any k-dimensional marginals there exists at least one null set related to the point masses of spectral measures, then the whole distribution of this family can be calculated from its $(k-1)$-dimensional marginals.*

Definition 2.5. We say that the above family has k-dimensional determinism.

Theorem 2.6. *Suppose a Chentsov type SαS family $\{X(u); u \in T\}$ has k (≥ 2) dimensional determinism. If another family $\{Z(u); u \in T\}$ shares the same k-dimensional marginal distributions with X, then $\{Z\}$ is also of Chentsov type and shares the same finite dimensional distributions with X.*

Gaussian families have 2-dimensional determinism in this sense. So it should be interesting to consider the stochastic process of fields which have k (> 2) dimensional determinism and do not have 2-dimensional determinism.

2.5. Examples

2.5.1. *Stationary fields on R^m.* Take $\mathbf{u} \in R^m$, and take the corresponding measure space $(E, d\mu) = (R^m, d\mathbf{x})$. Set

$$S_{\mathbf{u}} = \{\mathbf{y} \in R^m; ||\mathbf{y} - \mathbf{u}|| \leq 1\},$$

and define SαS random field $X(\mathbf{u}) = Y(S_{\mathbf{u}})$. Note that in 2-dimensional Euclidean space, any 4 circles divide the whole space into at most 14 subregions (not 16) and this fact holds in higher dimensions. In m-dimensional Euclidean space, any $m+2$ spheres divide the space into at most $7 \times 2^{m-1}$ subregions. Using this fact the above random field has $m+2$-dimensional but not $m+1$-dimensional determinism.

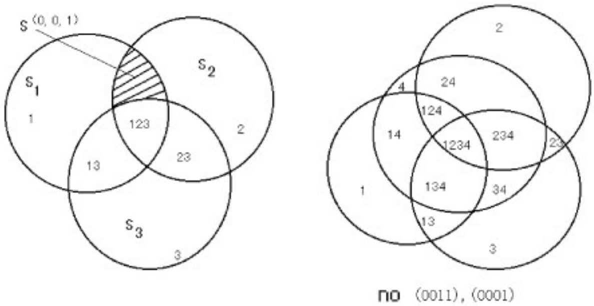

Fig. 4

2.5.2. *Lévy motions.* As we saw in 2.2, the Lévy motions on spaces of constant curvatures are Chentsov type random fields. We can consider the sets $S_{\mathbf{u}}$ as half spaces. The boundaries of $S_{\mathbf{u}}$ are hyperplanes of co-dimension 1. Let us count the number of subregions into which the space is divided by k hyperplanes. R^2 is divided by 2 lines into 4 regions, but into $7 < 2^3$ regions by 3 lines. In the same manner, it is easy to show that $m+1$ hyperplanes divide the whole space R^m into only $7 \times 2^{(k-3)} < 2^k$ regions. Thus,

Theorem 2.7. *Any m-parameter Lévy motion has $m+1$-dimensional determinism but does not have m-dimensional determinism.*

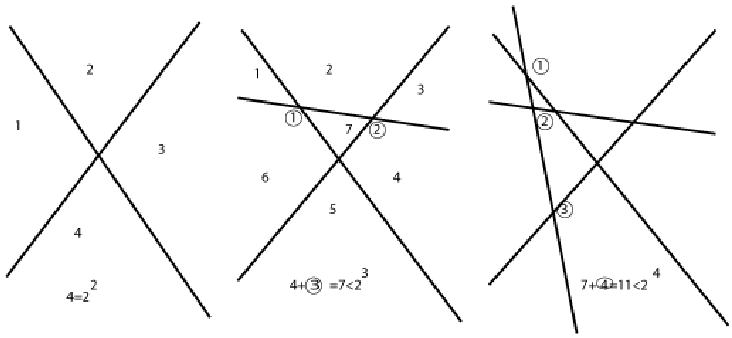

Fig. 5

2.5.3. *Self-similar stable fields.* A SαS random field $\{X(\mathbf{u}); \mathbf{u} \in R^m\}$ is called H-self-similar if

$$X_c(\mathbf{u}) = X(c \cdot \mathbf{u}) \sim c^H X(\mathbf{u}), \qquad \forall c > 0.$$

If $0 < \alpha \leq 2$, $0 < H < \frac{1}{\alpha}$, set

$$(E, \mu) = (R_+ \times R^m, d\mu(x_0, \mathbf{x}) = x_0^{\alpha H - 1 - m} dx_0 d\mathbf{x}).$$

The set E can be considered as the set of balls in R^m, that is, $(x_0, \mathbf{x}) \sim \{(\mathbf{v}, x_0); \mathbf{v} \in R^m, \|\mathbf{v} - \mathbf{x}\| \leq x_0\}$.

Set

$$S_{\mathbf{u}} = \{\text{ball which contains only one of } \mathbf{O},\ \mathbf{u}\} \qquad (\text{see Fig. 6}).$$

Then,

THEOREM 2.8 ([20]). *$X(u) = Y(S_u)$ is an H-self-similar SαS random field.*

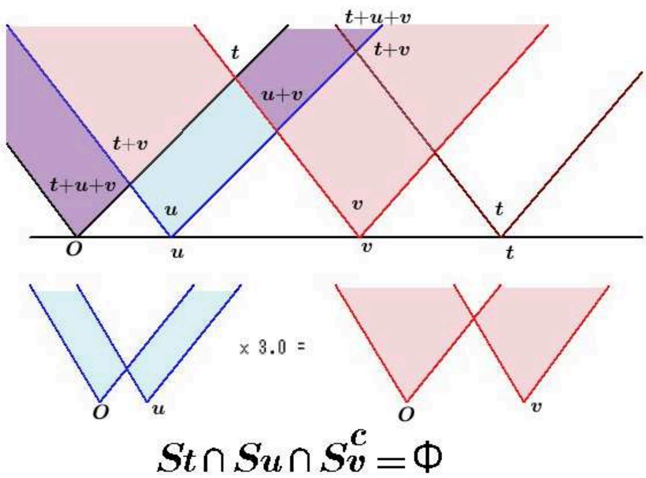

Fig. 6. $S.$ for self-similar processes

As we see in the above figure, there are no point masses in any $1+2$ dimensional marginals, in the 1-dimensional case. In the m-dimensional case, there are no point masses in any $m+2$ marginals.

THEOREM 2.9 ([14]). *The above H-self-similar processes have $m+2$-dimensional determinism.*

Note that there exist self-similar processes with more complex determinism ([16]).

3. Multi-parameter additive processes. At the meeting in Tokyo held in October 2000, Professor K. Sato proposed to investigate multi-parameter additive processes. This section is an answer to his proposal.

3.1. Linearly additive stochastic processes

DEFINITION 3.1. An R^m-parameter stochastic process $\{X(\mathbf{t}); t \in R^m\}$ is called a *linearly additive process* if for any (straight) line $L(s) = \{s\mathbf{v}+\mathbf{v}_0; s \in R^1\}$ the 1-parameter process obtained by parameter restriction $X_L(s) \equiv X(s\mathbf{v}+\mathbf{v}_0)$ has independent increments, that is, it is an additive process.

The following theorem of T. Mori is the final result on the structure of these processes.

THEOREM 3.2 ([11]). *Let $\{X(\mathbf{t})\}$ be an R^m-parameter linearly additive stochastic process which is subject to an infinitely divisible law. Then there exists a unique measure μ on the space E of all hyperplanes of co-dimension 1 in R^m and the process has a (Chentsov type) representation*

$$X(\mathbf{t}) = Y(S_{\mathbf{t}}),$$

where $S_{\mathbf{t}}$ is the connected component of $R^m \setminus \mathbf{t}^*$ which does not contain the origin, and $\{Y(B); B$ is a measurable set in $E\}$ is the random measure controlled by the measure space (E, μ).

3.2. Multi-parameter additive processes

3.2.1. *Convex cones*

DEFINITION 3.3. A set $V \subset R^m$ is called a *convex cone* if

1. $\forall \mathbf{v} \in V,\ (\mathbf{v}, \mathbf{v}_0) \geq 0$, for a fixed $\mathbf{v}_0$.
2. V is convex, that is for any $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $0 \leq c \leq 1$, $c\mathbf{v}_1 + (1-c)\mathbf{v}_2 \in V$.
3. for any $\mathbf{v} \in V$, and for any positive c, $c\mathbf{v} \in V$.

DEFINITION 3.4. A curve $\ell(t), 0 \leq t$ is called a *time-like curve* (with respect to V) if

1. $\ell(0) = \mathbf{O}$,
2. $\ell(t) \in V + \ell(s)$, for any $t > s$.

Here, we interpret the cone V as the future and $-V$ as the past.

DEFINITION 3.5. The dual cone V^* of a convex cone V is defined as

$$V^* = \{\mathbf{u} \in R^m; \mathbf{u} \cdot \mathbf{v} \leq 0,\ \forall \mathbf{v} \in V\}$$

V^* is a convex cone too, and $(V^*)^* = \overline{V}$ (the topological closure of V).

3.2.1. *Examples.*

- For $V = (R_+)^m$, $V^* = (R_-)^m$.
- For $V_{\mathbf{v}_0,c} = \{\frac{x\cdot\mathbf{v}_0}{\|x\|} \geq c\}$, $0 \leq c < 1$, $V^* = \{y; \frac{y\cdot(-\mathbf{v}_0)}{\|y\|} \geq \frac{1}{c}\}$: V is called the light cone in physics.

3.3. V**-parameter additive processes.** Let us fix a convex cone V.

DEFINITION 3.6. A random field $\{X(\mathbf{t}); \mathbf{t} \in V\}$ is called a *V-parameter additive process* if the restriction $\{X_\ell(t) = X(\ell(t))\}$ to any time-like curve ℓ is an additive process.

If V-parameter additive processes are also linearly additive, then the following representation theorem holds true:

THEOREM 3.7 ([23], [24], [25]). *Let $\{X(\mathbf{t}); \mathbf{t} \in R^m\}$ be a linearly additive $S\alpha S$ process. If the parameter restricted process $\{X(\mathbf{t}); \mathbf{t} \in V\}$ becomes a V-parameter additive process, then there exists a unique measure μ supported in the dual cone such that $X(\cdot)$ has the Chentsov type representation*

$$X(\mathbf{t}) = Y(S(\mathbf{t})),$$

where $\{Y(\cdot)\}$ is the $S\alpha S$ random measure controlled by μ.

3.4. Proof. Let us consider points $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ in the convex cone V. The differences are

$$\begin{aligned} X(\mathbf{u}_1) &= Y(S(\mathbf{u}_1)), \\ X(\mathbf{u}_2 + \mathbf{u}_1) - X(\mathbf{u}_1) &= Y(S(\mathbf{u}_2 + \mathbf{u}_1)) - Y(S(\mathbf{u}_1)), \\ X(\mathbf{u}_3 + \mathbf{u}_2 + \mathbf{u}_1) - X(\mathbf{u}_2 + \mathbf{u}_1) &= Y(S(\mathbf{u}_3 + \mathbf{u}_2 + \mathbf{u}_1)) - Y(S(\mathbf{u}_2 + \mathbf{u}_1)), \\ &\cdots \end{aligned}$$

If the corresponding sets $S(\cdot) \cap V^*$ for the increasing sequence $\mathbf{u}_1, \mathbf{u}_1+\mathbf{u}_2, \mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3, \ldots$, form an increasing sequence of sets, then the corresponding random variables

$$\begin{aligned} X(\mathbf{u}_1) &= Y(S(\mathbf{u}_1)), \\ X(\mathbf{u}_2 + \mathbf{u}_1) - X(\mathbf{u}_1) &= Y(S(\mathbf{u}_2 + \mathbf{u}_1) \setminus S(\mathbf{u}_1)), \\ X(\mathbf{u}_3 + \mathbf{u}_2 + \mathbf{u}_1) - X(\mathbf{u}_2 + \mathbf{u}_1) &= Y(S(\mathbf{u}_3 + \mathbf{u}_2 + \mathbf{u}_1) \setminus S(\mathbf{u}_2 + \mathbf{u}_1)), \\ &\cdots \end{aligned}$$

form an independent family.

Let us prove this fact. Set $S(\mathbf{u}_1) = \{\mathbf{u}_1\cdot\mathbf{x} \leq -1\}$ and $S(\mathbf{u}_1+\mathbf{u}_2) = \{(\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{x} \leq -1\}$, and consider the boundary of the intersection of the two sets, $B = \{\mathbf{x}; \mathbf{u}_1 \cdot \mathbf{x} = -1, (\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{x} = -1\}$. Then $\forall \mathbf{z} \in B$, $\mathbf{z} \cdot \mathbf{u}_2 = 0$.

Recall the definition of the dual cone $V^* = \{\mathbf{u}; \mathbf{u} \cdot \mathbf{v} \leq 0, \ \forall \mathbf{v} \in V\}$. This means that the set B is located outside of the set V^*. Moreover the distances of two boundaries from the origin are $1/\|\mathbf{u}_1\|$, $1/\|\mathbf{u}_1 + \mathbf{u}_2\|$, and $\|\mathbf{u}_1\| < \|\mathbf{u}_1 + \mathbf{u}_2\|$. Thus,

$$(S(\mathbf{u}_1 + \mathbf{u}_2) \cap V^*) \supset (S(\mathbf{u}_1) \cap V^*),$$

that is, the difference $X(\mathbf{u}_1 + \mathbf{u}_2) - X(\mathbf{u}_1)$ is independent of $X(\mathbf{u}_1)$.

Conversely, if the support of the measure is not contained in the dual cone, there exists an increasing sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ such that the above B and V^* have nonempty

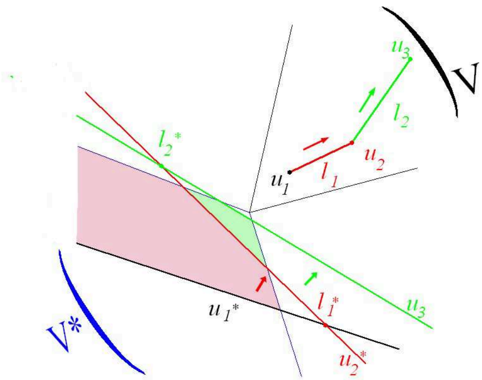

Fig. 7

intersection. So, the related process is not additive on the line $\overline{\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3}$. This completes the proof.

3.4.1. *Determinism.* As a special case of linearly additive SαS processes, multi-parameter additive processes have $m+1$-dimensional determinism.

4. Subordination. Let $Y(t)$, $0 \le t$, be a positive stable motion of index β, $0 < \beta < 1$, that is, $Y(\cdot)$ has stationary and independent increments. And let $X(t)$, $0 \le t$, be a symmetric stable motion of index α, $0 < \alpha \le 2$. Then the following result is well known as Bochner's subordination:

The random time change $Z(t) = X(Y(t))$ of the process $X(\cdot)$ becomes a symmetric stable motion with index $\alpha \cdot \beta$.

In this note, we will show an extension of the above result. The parameter of $X(\cdot)$ will be extended to a multi-dimensional space and $Y(\cdot)$ will be considered a vector-valued process.

4.1. 1-dimensional case

4.1.1. *Definitions and Bochner's subordination.* Let $X(t)$, $0 \le t$, be a symmetric stable process of index α with stationary independent increments, that is,

1. $E[e^{iz\cdot X(t)}] = e^{-at\cdot|z|^\alpha}$, $a > 0$,
2. $X(t_n) - X(t_{n-1}),\ X(t_{n-1}) - X(t_{n-2}),\ \ldots X(t_1) - X(t_0)$ is an independent system for $t_n \ge t_{n-1} \ge \ldots \ge t_0$,
3. $X(t+h) - X(t) \sim X(h)$ (equality of laws).

A positive stable process $Y(t)$ is called a subordinator of index β if

1. $E[e^{iz\cdot Y(t)}] = e^{-bt\cdot|z|^\beta(1-i\cdot\mathrm{sign}(z)\tan(\frac{\pi\beta}{2})}$, $b > 0$, that is , $Y(t)$ is subject to a positive stable distribution of index β.
2. $Y(t)$ has independent and stationary increments.

Bochner considered the composition $Z(t) = X(Y(t))$ and obtained

THEOREM 4.1 (Bochner's subordination [2]). *$X(Y(t))$ is a symmetric stable motion of index $\alpha \cdot \beta$.*

4.2. Sketch of the proof

4.2.1. *Approximation of subordinator.* Let $T_{c^{-n}\lambda}(t)$, $n \in \mathbf{Z}$ be a sequence of independent Poisson processes with intensities $c^{-n}\lambda$, for a constant $c > 1$. Consider the sum

$$T(t) = \sum_{-\infty}^{\infty} c^{n\gamma} T^n_{c^{-n}\lambda}(t).$$

$T(t)$ is a semi-self-similar process, that is,

$$T(c^k \cdot t) \sim c^{k\gamma} T(t), \ \forall k \in \mathbf{N},$$

and the characteristic function is

$$\varphi_T(z) = E[\exp(i \cdot zT(t))] = \exp\Big(-\lambda \cdot t \sum_{n=-\infty}^{\infty} c^{-n}(1 - \exp(ic^\gamma z))\Big).$$

The series in the above equation converges if $1 < \gamma$. Let us replace the semi-self-similar constant c and the intensity constant λ by $c_p = c\frac{1}{2^p}$, $\lambda_p = \frac{\lambda}{2^p}$, $p = 1, 2, \ldots$. The above series converges to

$$\exp\Big(-\lambda t \int_0^\infty \frac{1 - \exp(ix^{-\gamma} z)}{c'} dx\Big), \qquad c' = c - 1.$$

The above integral is equal to

$$\exp\left(\frac{\lambda \cdot t}{\gamma c'} \int_0^\infty (1 - \cos x) x^{-\frac{\gamma+1}{\gamma}} dx |z|^{\frac{1}{\gamma}} (1 - i\mathrm{sign}(z) \tan\Big(\frac{\pi/\gamma}{2}\Big)\right).$$

That is, $T(t)$ converges to the subordinator $Y(t)$ of index $\beta = \frac{1}{\gamma}$.

4.2.2. *Subordination by a Poisson process.* Let $X(t)$, $0 \le t$ be a symmetric α-stable motion, that is,

$$E[e^{izX(t)}] = \exp(-t(|z|^\alpha)).$$

Consider the time-changed process $X(aT_\lambda(t))$, by a Poisson process $T_\lambda(t)$ which is independent of $X(t)$. The characteristic function is

$$E[e^{izX(aT_\lambda(t))}] = \exp(-\lambda t(1 - e^{-a|z|^\alpha})).$$

Consider the characteristic function of the process $X(aT_\lambda(t) + bT_\mu(t))$ which is obtained by the time change using two independent Poisson processes with different means and different jumps:

$$\begin{aligned} E[e^{iz(X(aT_\lambda(t)+bT_\mu(t)))}] &= e^{-(\lambda+\mu)t} \sum_{j,k} \frac{e^{(aj+bk)|z|^\alpha}}{(\lambda t)^j (\mu t)^k} \\ &= \exp(-t(\lambda(1 - e^{-a|z|^\alpha}) + \mu(1 - e^{-b|z|^\alpha}))). \end{aligned}$$

Thus the characteristic function of the process $X(T(t))$ in 4.2.1 is

$$E[e^{izX(T(t))}] = \exp\Big(-\lambda t \sum_n c^{-n}(1 - e^{-c^{n\gamma}|z|^\alpha})\Big)$$

and, as in 4.2.1, the above sum converges to the following integral:

$$\exp\left(-\lambda t\int_0^\infty \frac{1-e^{x^{-\gamma}|z|^\alpha}}{c'}\,dx\right)=\exp\left(-\lambda t\int_0^\infty \frac{1-e^{(x|z|^{-\alpha/\gamma})^{-\gamma}}}{c'}\,dx\right)$$
$$=\exp\left(-\lambda t|z|^{\frac{\alpha}{\gamma}}\int_0^\infty \frac{1-e^{y^{-\gamma}}}{c'}\,dy\right).$$

Thus the limit process $X(Y(t))$ of $X(T(t))$ has the symmetric stable law of index $\alpha\cdot\beta$, $\beta=\frac{1}{H}$.

4.2.3. *Increments.* Let us consider the 2-dimensional characteristic function. For $t\geq s\geq 0$,

$$\begin{aligned}
&E[\exp(i(z_1(X(aT_\lambda(t)+bT\mu(t))-X(aT_\lambda(s)+bT_\mu(s)))+z_2X(aT_\lambda(s)+bT_\mu(s))))]\\
&=\sum_{k_t\geq k_s\geq 0,\ell_t\geq\ell_s\geq 0} E[\exp(i(z_1(X(ak_t+b\ell_t)-X(ak_s+b\ell_s))+z_2X(ak_s+b\ell_s)))\\
&\qquad\cdot P(T_\lambda(t)=k_t,T_\lambda(s)=k_s,T_\mu(t)=\ell_t,T_\mu(s)=\ell_s)\\
&=\sum E[\exp(iz_1(X(ak_t+b\ell_t)-X(ak_s+b\ell_s)))]P(T_\lambda(t)-T_\lambda(s)=k_t-k_s)\\
&\quad\cdot P(T_\mu(t)-T_\mu(s)=\ell_t-\ell_s)\cdot P(T_\lambda(t)=k_t,T_\lambda(s)=k_s,T_\mu(t)=\ell_t,T_\mu(s)=\ell_s)\\
&=\sum_{k_t-k_s,\ell_t-\ell_s} E[\exp(iz_1(X((ak_t-ak_s)+(b\ell_t-b\ell_s))))]\\
&\quad\cdot P(T_\lambda(t-s)=k_t-k_s)\cdot P(T_\mu(t-s)=\ell_t-\ell_s)\\
&\quad\cdot\sum_{k_s,\ell_s} E[\exp(iz_2(X(ak_s+b\ell_s)))]P(T_\lambda(s)=k_s)P(T_\mu(s)=\ell_s)\\
&=E[\exp(i(z_1(X(aT_\lambda(t)+bT\mu(t))-X(aT_\lambda(s)+bT_\mu(s)))))]\cdot E[z_2X(aT_\lambda(s)+bT_\mu(s))].
\end{aligned}$$

This means that the processes $X(T(t))$ and $X(Y(t))$ have independent increments. Along these lines we can prove that the process $X(Y(t))$ is an S$\alpha\beta$S Lévy motion, that is, a process having stationary and independent increments.

4.3. Multi-dimensional case. First, we need the concept of multi-dimensional random time (subordinator). Let us fix a future cone V.

4.3.1. *Multi-dimensional subordinator [27].* Let ν be a measure on $V\cap S^{m-1}$. There is one-to-one correspondence between the measure ν and an R^m-valued positive stable process $\mathbf{Y}(t)(=\mathbf{Y}_\nu),0\leq t$, with index $0<\beta<1$ which satisfies the following properties:

1. $\mathbf{Y}(\cdot;\omega)$ is a time-like curve for a.e. ω.
2. $\mathbf{Y}(t)-\mathbf{Y}(s),t>s$ is independent of $\mathbf{Y}(t)$, and $\mathbf{Y}(t)-\mathbf{Y}(s)\sim\mathbf{Y}(t-s)$.

4.3.2. *Multi-parameter additive process with stationary increments.* Let $X(\mathbf{t})$, $t\in V$ be a V-parameter additive process (cf. 3.3). Suppose X has stationary increments, that is,

$$E[e^{i(X(\mathbf{t})-X(\mathbf{s})z}]=e^{-\sigma(\mathbf{t}-\mathbf{s})\|z\|^\alpha}$$

and $\sigma(\mathbf{t})=|\mathbf{t}|\sigma(\frac{\mathbf{t}}{|\mathbf{t}|})$. Then theorem 3.7 can be modified as

THEOREM 4.2 ([23], [24], [25]). *There is one-to-one correspondence between measures on $V^* \cap S^{m-1}$ and additive, stationary increments processes on time-like curves. Here the measure μ on V^* has the form $d\mu(r \cdot \mathbf{q}) = d\mu_{S^{m-1}}(\mathbf{q})\frac{dr}{r^{m+1}}, \mathbf{x} = r \times \mathbf{q}, r \geq 0, \mathbf{q} \in S^{m-1}$.*

4.4. Subordination in multi-dimensional case. We can easily rewrite the proofs in 4.2.1 - 4.2.3, and obtain an extension of Bochner's subordination.

THEOREM 4.3. *The time-changed process $X(\mathbf{Y}(t))$ is an S$\alpha\beta$S Lévy motion.*

4.4.1. *Subordination by a Poisson process of direction* $\mathbf{a} \in V$. Consider the time changed process $X(\mathbf{a}T_\lambda(t))$ by a Poisson process of direction $\mathbf{a} \in V$. The characteristic function is

$$E[e^{izX(\mathbf{a}T_\lambda(t))}] = \exp(-\lambda t(1 - e^{-\sigma(\mathbf{a})|z|^\alpha})),$$

where $\sigma(\mathbf{a})$ is the strength of the SαS Lévy motion $X|_{t\mathbf{a}} = X(\mathbf{a}t)$ along the line $\{t\mathbf{a} : t \geq 0\}$. The characteristic function of the process $X(\mathbf{a}T_\lambda(t) + \mathbf{b}T_\mu(t))$ for two independent Poisson processes $T_\lambda(t), T_\mu(t)$ with different directions $\mathbf{a}$ and $\mathbf{b}$ is

$$\begin{aligned}E[e^{iz(X(\mathbf{a}T_\lambda(t)+\mathbf{b}T_\mu(t)))}] &= e^{-(\lambda+\mu)t}\sum_{j,k}\frac{e^{\sigma(j\mathbf{a}+k\mathbf{b})|z|^\alpha}}{(\lambda t)^j(\mu t)^k} = e^{-(\lambda+\mu)t}e^{-\lambda t e^{-\sigma(\mathbf{a})|z|^\alpha}}e^{-\mu t e^{-\sigma(\mathbf{b})|z|^\alpha}}\\ &= \exp(-t(\lambda(1 - e^{-\sigma(\mathbf{a})|z|^\alpha}) + \mu(1 - e^{-\sigma(\mathbf{b})|z|^\alpha}))).\end{aligned}$$

Note that the relation $\sigma(j\mathbf{a} + k\mathbf{b}) = j\sigma(\mathbf{a}) + k\sigma(\mathbf{b})$ comes from the properties that $X(\mathbf{t})$ has independent stationary increments. Thus the characteristic function of the process $X(\mathbf{a}T_1(t) + \mathbf{b}T_2(t))$ with the processes like 1.2.1 is

$$\begin{aligned}&E[e^{izX(\mathbf{a}T_1(t)+\mathbf{b}T_2(t))}]\\ &\quad= \exp\Big(-\lambda_1 t\sum_n c^{-n}(1 - e^{-c^{\sigma(\mathbf{a})nH|z|^\alpha}}) - \lambda_2 t\sum_n c^{-n}(1 - e^{-c^{\sigma(\mathbf{b})nH|z|^\alpha}})\Big).\end{aligned}$$

By arguments similar to 1.2.2, we can show that the time-changed process is subject to S$\alpha\beta$S law and the strength is proportional to the time parameter t.

4.4.2. *Increments.* Let us consider the 2-dimensional characteristic function. For the points $t \geq s \geq 0$,

$$\begin{aligned}&E[\exp(i(z_1(X(\mathbf{a}T_\lambda(t) + \mathbf{b}T\mu(t)) - X(\mathbf{a}T_\lambda(s) + \mathbf{b}T_\mu(s))) + z_2X(\mathbf{a}T_\lambda(s) + \mathbf{b}T_\mu(s))))]\\ &\quad= \sum_{k_t\geq k_s\geq 0,\ell_t\geq\ell_s\geq 0} E[\exp(i(z_1(X(\mathbf{a}k_t + \mathbf{b}\ell_t) - X(\mathbf{a}k_s + \mathbf{b}\ell_s)) + z_2X(ak_s + b\ell_s)))]\\ &\qquad\qquad \cdot P(T_\lambda(t) = k_t, T_\lambda(s) = k_s, T_\mu(t) = \ell_t, T_\mu(s) = \ell_s)\\ &\quad= \sum E[\exp(iz_1(X(\mathbf{a}k_t + \mathbf{b}\ell_t) - X(\mathbf{a}k_s + \mathbf{b}\ell_s)))]P(T_\lambda(t) - T_\lambda(s) = k_t - k_s)\\ &\qquad \cdot P(T_\mu(t) - T_\mu(s) = \ell_t - \ell_s) \cdot P(T_\lambda(t) = k_t, T_\lambda(s) = k_s, T_\mu(t) = \ell_t, T_\mu(s) = \ell_s)\\ &\quad= \sum_{k_t-k_s,\ell_t-\ell_s} E[\exp(iz_1(X((\mathbf{a}k_t - \mathbf{a}k_s) + (\mathbf{b}\ell_t - \mathbf{b}\ell_s))))]\\ &\qquad \cdot P(T_\lambda(t-s) = k_t - k_s) \cdot P(T_\mu(t-s) = \ell_t - \ell_s)\\ &\qquad \cdot \sum_{k_s,\ell_s} E[\exp(iz_2(X(\mathbf{a}k_s + \mathbf{b}\ell_s)))]P(T_\lambda(s) = k_s)P(T_\mu(s) = \ell_s)\\ &\quad= E[\exp(i(z_1(X(\mathbf{a}T_\lambda(t)+\mathbf{b}T\mu(t)) - X(\mathbf{a}T_\lambda(s)+\mathbf{b}T_\mu(s)))))] \cdot E[z_2X(\mathbf{a}T_\lambda(s)+\mathbf{b}T_\mu(s))].\end{aligned}$$

This means the process $X(\mathbf{Y}(t))$ has independent increments. Thus the process $X(\mathbf{Y}(t))$ is a SαS Lévy motion.

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