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FINITE CLOSED COVERINGS OF COMPACT QUANTUM SPACES

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Abstract. We consider the poset of all non-empty finite subsets of the set of natural numbers, use the poset structure to topologise it with the Alexandrov topology, and call the thus obtained topological space $\mathbb{P}^\infty$ the *universal partition space*. Then we show that it is a classifying space for finite closed coverings of compact quantum spaces in the sense that any such a covering is functorially equivalent to a sheaf over this partition space. In technical terms, we prove that the category of finitely supported flabby sheaves of algebras is equivalent to the category of algebras with a finite set of ideals that intersect to zero and generate a distributive lattice. In particular, the Gelfand transform allows us to view finite closed coverings of compact Hausdorff spaces as flabby sheaves of commutative unital C*-algebras over $\mathbb{P}^\infty$.

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1. Introduction

1.1. Motivation. In the day-to-day practice of the mathematical art, one can see a recurrent theme of reducing a complicated mathematical construct into its simpler constituents, and then putting these constituents together using gluing datum that prescribes how these pieces consistently fit each other. The (now) classical manifestation of such gluing arguments in various flavours of geometry is the concept of a sheaf on a topological space, or more generally on a topos. Another manifestation of such gluing arguments appeared in noncommutative geometry as the description of a noncommutative space via a finite closed covering. Here a covering is defined as a distinguished finite set of ideals that intersect to zero and generate a distributive lattice [9].

1.2. Main result. Following [9], we express the gluing datum of a compact Hausdorff space as a sheaf of algebras over a certain universal topological space, and extend it to the noncommutative setting. This universal topological space is explicitly constructed from the poset of non-empty finite subsets of the set of natural numbers. It is endowed with the Alexandrov topology, and denoted by $\mathbb{P}^\infty$. The advantages of our main theorem over its predecessor [9, Corollary 4.3] are twofold. First, it considers coverings rather than topologically unnatural ordered coverings. To this end, we need to construct more refined morphisms between sheaves than natural transformations. Next, as $\mathbb{P}^\infty$ is the colimit of all finite $\mathbb{P}^N$'s (the universal N-partition spaces obtained from the poset of all non-empty subsets of $\{0, \cdots, N\}$), it takes care of all finite coverings at once.

THEOREM 3.13. *The category of finite coverings of algebras is equivalent to the category of finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty$ whose morphisms are obtained by taking a certain quotient of the usual class of morphisms enlarged by the actions of a specific family of endofunctors.*

1.3. Sheaves, patterns, and *P*-algebras. The idea of using lattices to study closed coverings of noncommutative spaces has already been widely employed (see [10]). To afford a good C*-algebraic description, one considers closed rather than open coverings. Therefore, a natural framework for coverings uses sheaf-like objects defined on the lattice of closed subsets of a topological space, or more generally, topoi modelled upon finite closed coverings of topological spaces. Interestingly, the original definition of sheaves by Leray was given in terms of the lattice of closed subspaces of a topological space [11, p. 303]. For various reasons, this definition changed in the subsequent years into the nowadays standard open-set formulation.

Recently, however, a closed-set approach reappeared in the form of sheaf-like objects called *patterns* [13]. We show in Proposition 2.20 that for our combinatorial models based on finite Alexandrov spaces, the distinction between sheaves and patterns is immaterial. Another reformulation of sheaves over Alexandrov spaces is given by the concept of a P-*diagram*. It is widely known among commutative algebraists (e.g. see [3, Proposition 6.6] and [17, p. 174]) that any sheaf on an Alexandrov space P can be recovered from its P-diagram (cf. Theorem 2.22 concerning P-algebras). See also [7] for a different approach.

1.4. Outline. Section 1 is of preliminary nature. It is focused on explaining the emergence of the universal partition space $\mathbb{P}^\infty$ as the classifying space of finite coverings. We show how finite closed coverings of compact Hausdorff spaces naturally yield finite universal partition spaces $\mathbb{P}^N$ with the Alexandrov topology. Then we take the colimit of $\mathbb{P}^N$'s with $N \to \infty$. We continue with analysing in detail the topological properties of $\mathbb{P}^\infty$ to be ready for studying sheaves of algebras over $\mathbb{P}^\infty$. These are the key objects of Section 2 that is devoted to the main result of this paper.

1.5. Notation and conventions. Throughout the article we fix a ground field k of an arbitrary characteristic. We assume that all algebras are over k and are associative and unital but not necessarily commutative. We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the set of natural numbers (zero included) and the set of integers, respectively. The finite set $\{0, \ldots, N\}$ is denoted by $\underline{N}$ for any natural number N. We use $\mathbf{2}^N$ to denote the set of all subsets of $\underline{N}$. If $\underline{x}$ is a sequence of elements from a set X, we write $\kappa(\underline{x})$ to denote the underlying set of elements of $\underline{x}$. The symbol $|X|$ stands for the cardinality of a set X.

2. Primer on lattices and Alexandrov topology. We first recall definitions and simple facts about ordered sets and lattices to fix notation. Our main references on the subject are [2, 4, 16].

A set P together with a binary relation $\leqslant$ is called *a partially ordered set*, or *a poset* in short, if the relation $\leqslant$ is (i) reflexive, i.e. $p \leqslant p$ for any $p \in P$, (ii) transitive, i.e. $p \leqslant q$ and $q \leqslant r$ implies $p \leqslant r$ for any $p, q, r \in P$, and (iii) anti-symmetric, i.e. $p \leqslant q$ and $q \leqslant p$ implies $p = q$ for any $p, q \in P$. If only the conditions (i)-(ii) are satisfied we call $\leqslant$ a *preorder*. For every preordered set $(P, \leqslant)$ there is an opposite preordered set $(P, \leqslant)^{\mathrm{op}}$ given by $P^{\mathrm{op}} := P$ and $p \leqslant^{\mathrm{op}} q$ if and only if $q \leqslant p$ for any $p, q \in P$.

A poset $(P, \leqslant)$ is called *a semi-lattice* if for every $p, q \in P$ there exists an element $p \vee q$ such that (i) $p \leqslant p \vee q$, (ii) $q \leqslant p \vee q$, and (iii) if $r \in P$ is an element which satisfies $p \leqslant r$ and $q \leqslant r$ then $p \vee q \leqslant r$. The binary operation $\vee$ is called *join*. A poset is called *a lattice* if both $(P, \leqslant)$ and $(P, \leqslant)^{\mathrm{op}}$ are semi-lattices. The join operation in P^{op} is called *meet*, and traditionally denoted by $\wedge$. One can equivalently define a lattice P as a set with two binary associative commutative and idempotent operations $\vee$ and $\wedge$. These operations satisfy two absorption laws: $p = p \vee (p \wedge q)$ and $p = p \wedge (p \vee q)$ for any $p, q \in P$. A lattice $(P, \vee, \wedge)$ is called *distributive* if one has $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for any $p, q, r \in P$. Note that one can prove that the distributivity of meet over join we have here is equivalent to the distributivity of join over meet.

Let $(P, \leqslant)$ be a preordered set, and let $\uparrow\! p := \{q \in P |\ p \leqslant q\}$ for any $p \in P$. As a natural extension of notation, we define $\uparrow\! U := \bigcup_{p \in U} \uparrow\! p$ for all $U \subseteq P$. The sets $U \subseteq P$ that satisfy $U = \uparrow\! U$ are called *upper sets* or *dual order ideals*. The topological space we obtain from a preordered set using the upper sets as open sets is called an *Alexandrov space*. Note that a set U is open in the Alexandrov topology if and only if for any $u \in U$ one has $\uparrow\! u \subseteq U$. Observe also that reversing the preorder exchanges the closed and open sets:

Lemma 2.1. *Let $(P, \leqslant)$ be a preordered set. A subset $C \subseteq P$ is closed in the Alexandrov topology of P if and only if C is open in the Alexandrov topology of the opposite preordered set $(P, \leqslant)^{op}$.*

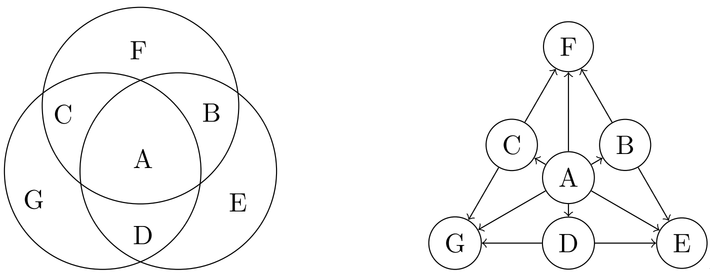
Proof. Since $(P, \leqslant) = ((P, \leqslant)^{\mathrm{op}})^{\mathrm{op}}$ and the statement is symmetric, we need to prove only one implication. Assume C is closed and let $x \in C$. In order to prove that C is open in the opposite Alexandrov topology, we need to show that $y \in C$ for any $y \leqslant x$. Suppose the contrary that $y \leqslant x$ and $y \in C^c := P \setminus C$. Since C^c is open in the Alexandrov topology of $(P, \leqslant)$ and $y \leqslant x$, we must have $x \in C^c$, which is a contradiction. ∎

2.1. Universal partition spaces. In [15], Sorkin defined and investigated the order structure on the spaces we call here partition spaces. For the lattice of subsets covering a space, the partition spaces play a role analogous to the set of meet-irreducible elements of an arbitrary finite distributive lattice, i.e. they are much smaller than lattices themselves while encoding important lattice properties. Sorkin's primary objective was to develop finite approximations for topological spaces via their finite open coverings (see also [1, 5]). Here we will investigate spaces with finite closed rather than open coverings. See also [17, 18] for a more algebraic approach. We begin by analysing properties of partition spaces.

Definition 2.2. Let X be a set and let $\mathcal{C} := \{C_0, \ldots, C_N\}$ be a finite covering of X, i.e. let $\bigcup_i C_i = X$. For any $x \in X$, we define its support $\mathit{supp}_{\mathcal{C}}(x) := \{C \in \mathcal{C} \mid x \in C\}$. A preorder $\preccurlyeq_{\mathcal{C}}$ on X is defined by $x \preccurlyeq_{\mathcal{C}} y$ if and only if $\mathit{supp}_{\mathcal{C}}(x) \supseteq \mathit{supp}_{\mathcal{C}}(y)$. We also define an equivalence relation $\sim_{\mathcal{C}}$ by letting $x \sim_{\mathcal{C}} y$ if and only if $\mathit{supp}_{\mathcal{C}}(x) = \mathit{supp}_{\mathcal{C}}(y)$. We call the quotient space $X/{\sim_{\mathcal{C}}}$ the *partition space* associated to the finite covering $\mathcal{C}$ of X. This space is partially ordered by the relation induced from $\preccurlyeq_{\mathcal{C}}$.

Definition 2.3. Let X and $\mathcal{C}$ be as before. We use $(X, \preccurlyeq_{\mathcal{C}})$ to denote the set X with its Alexandrov topology induced from the preorder relation $\preccurlyeq_{\mathcal{C}}$ defined above.

Example 2.4. Consider a region on the 2-dimensional Euclidean plane covered by three disks in a generic position, and the corresponding poset, as described below:



. (1)

Here an arrow $\to$ indicates the existence of an order relation between the source and the target.

Definition 2.5. Let X be a set and $\mathcal{C} := \{C_0, \ldots, C_N\}$ be a finite covering of X. The covering $\mathcal{C}$ viewed as a subbasis for closed sets induces a topology on X. The space X together with the topology induced from $\mathcal{C}$ is denoted by $(X, \mathcal{C})$.

Proposition 2.6. *Let X be a set and let $\mathcal{C}$ be a finite covering. The Alexandrov topology defined by the preorder $\preccurlyeq_{\mathcal{C}}$ coincides with the topology in Definition 2.5.*

Proof. We need to prove that a subset L is closed in $(X, \mathcal{C})$ if and only if it is closed in $(X, \preccurlyeq_{\mathcal{C}})$. By Lemma 2.1 and the definition of Alexandrov topology, we see that L is closed in $(X, \preccurlyeq_{\mathcal{C}})$ if and only if $L = \bigcup_{x\in L} {\downarrow}x$, where ${\downarrow}x := \{x' \in X |\ x' \preccurlyeq_{\mathcal{C}} x\}$. On the other hand, let $C_x := \bigcap_{C\in supp_{\mathcal{C}}(x)} C$. We have $x' \preccurlyeq_{\mathcal{C}} x$ if and only if x' is covered by the same sets from $\mathcal{C}$, or more. In other words, $x' \preccurlyeq_{\mathcal{C}} x$ if and only if $x' \in C_x$, so that $C_x = {\downarrow}x$. Finally, note that L is closed in $(X, \mathcal{C})$ if and only if $L = \bigcup_{x\in L} C_x$. The result follows. ∎

COROLLARY 2.7. *The canonical quotient map* $\pi\colon (X, \mathcal{C}) \to (X/{\sim_{\mathcal{C}}},\ \preccurlyeq_{\mathcal{C}})$ *is a continuous map which is both open and closed.*

Proof. The above proposition allows us to replace $(X, \mathcal{C})$ by $(X, \preccurlyeq_{\mathcal{C}})$ thus converting topological properties to preorder properties. Since π is surjective and $x \preccurlyeq_{\mathcal{C}} y$ if and only if $\pi(x) \preccurlyeq_{\mathcal{C}} \pi(y)$, one easily verifies that π is continuous and open. To conclude that it is also closed, we apply Lemma 2.1. ∎

LEMMA 2.8. *Let* $\mathcal{C}$ *be a finite covering of a set* X. *Let* $X/\sim_{\mathcal{C}}$ *be the partition space associated with the covering* $\mathcal{C}$ *and* $\pi\colon X \to X/\sim_{\mathcal{C}}$ *be the canonical surjection on the quotient space. Denote by* $\Lambda_{\mathcal{C}}$ *the lattice of subsets of* X *generated by the covering* $\mathcal{C}$ *and by* $\Lambda_{\pi(\mathcal{C})}$ *the lattice of subsets of* $X/{\sim_{\mathcal{C}}}$ *generated by* $\pi(\mathcal{C}) := \{\pi(C) |\ C \in \mathcal{C}\}$. *The following assignments*

$$\hat{\pi}\colon \Lambda_{\mathcal{C}} \longrightarrow \Lambda_{\pi(\mathcal{C})}, \qquad \lambda \longmapsto \pi(\lambda),$$
$$\hat{\pi}^{-1}\colon \Lambda_{\pi(\mathcal{C})} \longrightarrow \Lambda_{\mathcal{C}}, \qquad \lambda \longmapsto \pi^{-1}(\lambda),$$

define mutually inverse lattice isomorphisms.

Proof. Inverse images preserve set unions and intersections. Hence $\hat{\pi}^{-1}$ is a lattice morphism. On the other hand, though in general images preserve only unions, here we have

$$\pi(x) \in \pi(C_i) \quad \Leftrightarrow \quad x \in C_i \tag{2}$$

for any i. It follows that

$$\begin{aligned} \pi(x) \in \pi(C_{i_1}) \cap \cdots \cap \pi(C_{i_k}) &\Leftrightarrow x \in C_{i_1} \cap \cdots \cap C_{i_k} \\ &\Rightarrow \pi(x) \in \pi(C_{i_1} \cap \cdots \cap C_{i_k}). \end{aligned} \tag{3}$$

In other words, $\pi(C_{i_1}) \cap \cdots \cap \pi(C_{i_k})$ is a subset of $\pi(C_{i_1} \cap \cdots \cap C_{i_k})$. As the containment in the other direction always holds, it follows that $\hat{\pi}$ is also a lattice morphism. Finally, since π is surjective and $\pi^{-1}(\pi(C_i)) = C_i$ for all i, one sees that $\hat{\pi}^{-1}$ and $\hat{\pi}$ are the inverse of each other. ∎

We are ready now to introduce the universal partition spaces as natural partition spaces associated with projective spaces. The projective space over a field $\Bbbk$ is denoted by $\mathbb{P}^N(\Bbbk)$. It is defined as the space $\Bbbk^{N+1} \setminus \{0\}$ divided by the diagonal action of the non-zero scalars $\Bbbk^\times := \Bbbk \setminus \{0\}$. By [9, Example 4.2], the partition space associated to a certain closed refinement of the affine covering of the complex projective space $\mathbb{P}^N(\mathbb{C})$ coincides with

$$\boxed{\mathbb{P}^N := \mathbb{P}^N(\mathbb{Z}/2) := \left\{(z_0, \ldots, z_N) \in (\mathbb{Z}/2)^{N+1} |\ \exists i \in \underline{N},\ z_i = 1\right\}}\,. \tag{4}$$

Since we are not interested in the natural discrete topology of the projective space $\mathbb{P}^N(\mathbb{Z}/2)$ but in its poset structure inherited from the affine covering of $\mathbb{P}^N(\mathbb{C})$ (or any other projective space $\mathbb{P}^N(\Bbbk)$), we abbreviate the notation to $\mathbb{P}^N$. We call $\mathbb{P}^N$ the *universal N-partition space*. We topologise it with the Alexandrov topology coming from its poset structure, which can be easily described as follows. For any $a := (a_i)_{i\in\underline{N}}$ and $b := (b_i)_{i\in\underline{N}}$ in $\mathbb{P}^N$ we write $a \leqslant b$ if and only if $a_i \leqslant b_i$ for any $i \in \underline{N}$. Remembering that the affine covering of $\mathbb{P}^N(\mathbb{C})$ generates a free lattice [8, Proposition 1.2], it is not surprising that the topological space $\mathbb{P}^N$ enjoys a property justifying calling it universal. As a direct generalization of [9, Proposition 4.1], we obtain:

THEOREM 2.9. *Let $\underline{\mathcal{C}} := (C_0, \ldots, C_N)$ be a finite covering of X with a fixed ordering on the elements of the covering. Let χ_a be the characteristic function of a subset $a \subseteq \underline{N}$ viewed as an element of $\mathbb{P}^N$. Then the map $\xi\colon X \ni x \mapsto \chi_{s(x)} \in \mathbb{P}^N$, where $s(x) := \{i \in \underline{N} |\ x \in C_i\}$, yields a morphism of preordered sets $\xi\colon (X, \preccurlyeq_{\mathcal{C}})^{op} \to (\mathbb{P}^N,\ \leqslant)$ and, consequently, a continuous map between Alexandrov spaces. Moreover, ξ is both open and closed, and it factors as $\xi = \hat{\xi} \circ \pi$, where $\hat{\xi} : (X/{\sim_{\mathcal{C}}}, \preccurlyeq_{\mathcal{C}})^{op} \to (\mathbb{P}^N,\ \leqslant)$ is an embedding of Alexandrov topological spaces.*

2.2. Topological properties of partition spaces. Both $\mathbf{2}^N$ (the set of all subsets of $\underline{N}$) and $\mathbf{2}^N \setminus \{\emptyset\}$ are posets with respect to the inclusion relation $\subseteq$. For any non-empty subset $a \subseteq \underline{N}$, one has a sequence $(a_0, \ldots, a_N)$ where

$$a_i := \begin{cases} 1 & \text{if } i \in a, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In other words, the sequence $(a_0, \ldots, a_N)$ is the characteristic function χ_a of the subset $a \subseteq \underline{N}$. The assignment $a \mapsto \chi_a$ determines a bijection between the set of non-empty subsets of $\underline{N}$ and the universal N-partition space $\mathbb{P}^N$. Its inverse is defined as

$$\nu(z) := \{i \in \underline{N} |\ z_i = 1\}, \quad z := (z_i)_{i\in\underline{N}} \in \mathbb{P}^N. \tag{6}$$

With this bijection, one has $(a_i)_{i\in\underline{N}} \leqslant (b_i)_{i\in\underline{N}}$ if and only if $\nu((a_i)_{i\in\underline{N}}) \subseteq \nu((b_i)_{i\in\underline{N}})$. In other words, we have the following:

PROPOSITION 2.10. *The map $\nu\colon \mathbb{P}^N \to \mathbf{2}^N \setminus \{\emptyset\}$ is an isomorphism of posets, and thus a homeomorphism of Alexandrov spaces.*

DEFINITION 2.11. For any $i \in \underline{N}$ and any non-empty subset $a \subseteq \underline{N}$, we define open sets

$$\mathbb{A}_i^N := \{(z_0, \ldots, z_N) \in \mathbb{P}^N \mid z_i = 1\} = \uparrow\!\chi_{\{i\}} \quad \text{and} \quad \mathbb{A}_a^N := \bigcap_{i\in a} \mathbb{A}_i^N = \uparrow\!\chi_a\,.$$

Note that the sets $\mathbb{A}_i^N$ form a subbasis for the Alexandrov topology of $\mathbb{P}^N$. For brevity, when there is no risk of confusion, we omit the superscripts and write $\mathbb{A}_i$ and $\mathbb{A}_a$ instead of $\mathbb{A}_i^N$ and $\mathbb{A}_a^N$.

LEMMA 2.12. *For all $N \geqslant 0$, the map $\phi_N\colon \mathbb{P}^N \longrightarrow \mathbb{P}^{N+1}$ defined by*

$$\phi_N(z_0, \ldots, z_N) := (z_0, \ldots, z_N, 0)$$

is an embedding of topological spaces.

Proof. The fact that the maps ϕ_N are injective is obvious. They are also continuous since we have

$$\phi_N^{-1}(\mathbb{A}_i^{N+1}) = \begin{cases} \mathbb{A}_i^N & \text{if } i \leqslant N, \\ \emptyset & \text{if } i = N+1. \end{cases} \tag{7}$$

Finally, ϕ_N's yield homeomorphisms between their domains and their images because

$$\phi_N(\mathbb{P}^N) \cap \mathbb{A}_i^{N+1} = \begin{cases} \phi_N(\mathbb{A}_i^N) & \text{if } i \in \underline{N}, \\ \emptyset & \text{otherwise,} \end{cases} \tag{8}$$

for the open subsets in the subbasis of the Alexandrov topology. ■

The maps $\phi_N \colon \mathbb{P}^N \to \mathbb{P}^{N+1}$ form a direct system of continuous maps of Alexandrov topological spaces. Hence we can define the infinite universal partition space $\mathbb{P}^\infty$ as a direct limit:

Definition 2.13. $\boxed{\mathbb{P}^\infty := \varinjlim_{N \geqslant 0} \mathbb{P}^N}$.

We can represent the points of $\mathbb{P}^\infty$ as infinite sequences $\{(z_i)_{i\in\mathbb{N}} \mid z_i \in \mathbb{Z}/2\}$ where the number of non-zero terms is finite and greater than zero. We can also view $\mathbb{P}^\infty$ as the colimit of all finite $\mathbb{P}^N$'s. The canonical morphisms of the colimit $i_N \colon \mathbb{P}^N \to \mathbb{P}^\infty$ send a finite sequence $(z_0, \ldots, z_N)$ to the infinite sequence $(z_0, \ldots, z_N, 0, 0, \ldots)$ obtained from the finite sequence by padding it with countably many 0's. The topology on the colimit is the topology induced by the maps $\{i_N\}_{N\in\mathbb{N}}$.

We also have a natural poset structure on $\mathbb{P}^\infty$. Here $(a_i)_{i\in\mathbb{N}} \leqslant (b_i)_{i\in\mathbb{N}}$ if and only if $a_i \leqslant b_i$ for any $i \in \mathbb{N}$. This poset structure coincides with the poset structure of the set of all finite subsets of $\mathbb{N}$. Denote the set of all finite subsets of $\mathbb{N}$ by $\texttt{Fin}$. One can extend the bijection $\nu \colon \mathbb{P}^N \to \mathbf{2}^N \setminus \{\emptyset\}$ (see (6)) to a bijection $\nu \colon \mathbb{P}^\infty \to \texttt{Fin} \setminus \{\emptyset\}$. The inverse of ν is given by the assignment $a \mapsto \chi_a := (a_i)_{i\in\mathbb{N}}$ that is defined as

$$a_i := \begin{cases} 1 & \text{if } i \in a, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

for any $a \in \texttt{Fin}$. The map $\nu \colon \mathbb{P}^\infty \to \texttt{Fin} \setminus \{\emptyset\}$ is an isomorphism of posets, and therefore the Alexandrov topological spaces $\mathbb{P}^\infty$ and $\texttt{Fin} \setminus \{\emptyset\}$ are homeomorphic.

Thus we have two possibly different topologies on $\mathbb{P}^\infty$: one coming from the preorder structure and the other coming from the colimit. However, we check that they coincide.

Theorem 2.14. *The following statements hold:*

1. *The Alexandrov topology and the colimit topology on $\mathbb{P}^\infty$ are the same.*
2. *The spaces $\mathbb{P}^N$ are T_0 but not T_1 for any $N = 1, \ldots, \infty$.*
3. *$\mathbb{P}^N$ is a connected topological space for any $N = 0, 1, \ldots, \infty$.*
4. *The topology on $\mathbb{P}^\infty$ is compactly generated.*

Proof. For any $i \in \mathbb{N}$ and $a \in \texttt{Fin} \setminus \{\emptyset\}$, we define

$$\mathbb{A}_i^\infty := \uparrow\!\chi_{\{i\}} \quad \text{and} \quad \mathbb{A}_a^\infty := \bigcap_{i\in a} \mathbb{A}_i^\infty = \uparrow\!\chi_a \tag{10}$$

which are open in the Alexandrov topology.

Proof of (1): Let $i_N \colon \mathbb{P}^N \to \mathbb{P}^\infty$ be the structure maps of the colimit. We need to prove that an open set in one topology is open in the other, and vice versa. The set $\{\mathbb{A}_a^\infty \mid a \in \texttt{Fin} \setminus \{\emptyset\}\}$ is a basis for the Alexandrov topology since each $\mathbb{A}_a^\infty$ is an upper set. Then

$$i_N^{-1}(\mathbb{A}_a^\infty) = \begin{cases} \mathbb{A}_a^N & \text{if } a \subseteq \underline{N}, \\ \emptyset & \text{if } a \nsubseteq \underline{N} \end{cases} \tag{11}$$

is an open set in $\mathbb{P}^N$ for any $N \geqslant 0$ and $a \in \mathbf{2}^N \setminus \{\emptyset\}$. Therefore every open set in the Alexandrov topology is open in the colimit topology. Now, let $U \subseteq \mathbb{P}^\infty$ be open in the colimit topology. We can assume that every sequence in $\mathbb{P}^\infty$ is of the form χ_a for a unique $a \in \mathtt{Fin} \setminus \{\emptyset\}$ since $z = \chi_{\nu(z)}$ for all $z \in \mathbb{P}^\infty$. Next, let $\chi_a \in U$ and $\chi_a \leqslant \chi_b$ for some $\chi_b \in \mathbb{P}^\infty$. We need to show that $\chi_b \in U$. Since $a \subseteq b \in \mathtt{Fin} \setminus \{\emptyset\} \subset \mathbf{2}^\mathbb{N}$, there is a natural number $N := \max(b) \geqslant \max(a)$. Moreover, we have the inequality

$$i_N^{-1}(\chi_a) = \chi_a \leqslant \chi_b = i_N^{-1}(\chi_b) \tag{12}$$

in $\mathbb{P}^N$. As $i_N^{-1}(U)$ is open in the Alexandrov topology of $\mathbb{P}^N$, it follows that $\chi_b \in i_N^{-1}(U)$, which in turn implies that $\chi_b \in U$.

Proof of (2): Let $p, q \in \mathbb{P}^N$, $p \neq q$. Then $\nu(p) \neq \nu(q)$. Let us suppose without the loss of generality that $i \in \nu(p)$ and $i \notin \nu(q)$. Then $q \notin \uparrow p$, which proves that $\mathbb{P}^N$ is T_0. On the other hand, if $p \leqslant q$ then for any open set $U \subseteq \mathbb{P}^N$ such that $p \in U$ also $q \in U$ (as U is an upper set). It follows that $\mathbb{P}^N$ is not T_1.

Proof of (3): Suppose there exists a non-empty subset $V \subsetneq \mathbb{P}^N$ that is both open and closed. Let $\chi_a \in V$ and $\chi_b \in \mathbb{P}^N \setminus V$. Then, because V and $\mathbb{P}^N \setminus V$ are open, we have $\chi_{a\cup b} \in V$ and $\chi_{a\cup b} \in \mathbb{P}^N \setminus V$, which is a contradiction.

Proof of (4): In order to prove our assertion, we need to show that for any $a \in \mathtt{Fin} \setminus \{\emptyset\}$ the set $\mathbb{A}_a^\infty$ is compact. Let $a \in \mathtt{Fin} \setminus \{\emptyset\}$ and suppose that $\mathcal{U} := \{U_i\}_{i\in I}$ is an open covering of $\mathbb{A}_a^\infty$. Since $\chi_a \in \mathbb{A}_a^\infty$ and $\mathcal{U}$ is a covering, there exists $j \in I$ such that $\chi_a \in U_j$. As U_j is open in the Alexandrov topology, we obtain $\uparrow\chi_a = \mathbb{A}_a^\infty \subseteq U_j$. Consequently, for any finite subset α of $\mathtt{Fin} \setminus \{\emptyset\}$, the set $\bigcup_{a\in\alpha} \mathbb{A}_a^\infty$ is compact. The result follows. ■

2.3. Continuous maps between partition spaces. In what follows in this subsection, unless explicitly stated otherwise, N will be a natural number or ∞. Accordingly, the set $\{0, \ldots, N\}$ will be a finite set or will be $\mathbb{N}$ if $N = \infty$. For example, a permutation $\sigma : \{0, \ldots, N\} \to \{0, \ldots, N\}$ is either a finite permutation or an arbitrary bijection $\mathbb{N} \to \mathbb{N}$.

Let $\mathtt{Op}(\mathbb{P}^N)$ be the lattice of open subsets of $\mathbb{P}^N$. It is obvious that any continuous map $f : \mathbb{P}^N \to \mathbb{P}^M$ defines a morphism between lattices of open sets of the form

$$\mathfrak{X}_f : \mathtt{Op}(\mathbb{P}^M) \ni U \longmapsto \mathfrak{X}_f(U) := f^{-1}(U) \in \mathtt{Op}(\mathbb{P}^N). \tag{13}$$

Conversely, we have the following:

Proposition 2.15. *Let M and N be finite natural numbers or ∞. Let $\mathfrak{X} : \mathtt{Op}(\mathbb{P}^M) \to \mathtt{Op}(\mathbb{P}^N)$ be a lattice morphism with the property that*

$$\bigcup_{i\in\{0,\ldots,M\}} \mathfrak{X}(\mathbb{A}_i^M) = \mathbb{P}^N, \tag{14a}$$

$$\bigcap_{i\in a} \mathfrak{X}(\mathbb{A}_i^M) = \emptyset \quad \textit{for all infinite} \quad a \subseteq \{0, \ldots, M\}. \tag{14b}$$

Then there exists a unique continuous function $f_{\mathfrak{X}} : \mathbb{P}^N \to \mathbb{P}^M$ such that, for all open subsets $U \subseteq \mathbb{P}^M$, we have $\mathfrak{X}(U) = f_{\mathfrak{X}}^{-1}(U)$.

Proof. We define a map $f_{\mathfrak{X}} : \mathbb{P}^N \to \mathbb{P}^M$ as

$$f_{\mathfrak{X}} : z \longmapsto \chi_a, \quad \text{where} \quad a := \{i \in \{0, \dots, M\} \mid z \in \mathfrak{X}(\mathbb{A}_i^M)\}. \tag{15}$$

We observe that a is non-empty due to the condition (14a), and finite due to the condition (14b). By definition,

$$z \in f_{\mathfrak{X}}^{-1}(\mathbb{A}_i^M) \Leftrightarrow f_{\mathfrak{X}}(z) \in \mathbb{A}_i^M \Leftrightarrow i \in \nu(f_{\mathfrak{X}}(z)) \Leftrightarrow z \in \mathfrak{X}(\mathbb{A}_i^M). \tag{16}$$

This proves the continuity of $f_{\mathfrak{X}}$ because the sets $\mathbb{A}_i^M$ form a subbasis of the Alexandrov topology. The uniqueness follows from combining (16) with the fact that knowing for all i's whether or not $z' \in \mathbb{A}_i^M$ determines $z' \in \mathbb{P}^M$. ■

Note that the conditions (14a) and (14b) are satisfied for $\mathfrak{X}_f$ for any continuous f because $\bigcap_{i\in a} \mathbb{A}_i = \emptyset$ for any infinite a, and f^{-1} preserves infinite unions and intersections.

Finally, in order to characterize in Theorem 2.17 the homeomorphisms between the universal N-partition spaces $\mathbb{P}^N$, we will need the following technical lemma.

LEMMA 2.16. *Let N and M be finite natural numbers or ∞. Let $f\colon \mathbb{P}^N \to \mathbb{P}^M$ be a continuous map of Alexandrov spaces.*

1. *If f is injective, then $|\nu(z)| \leqslant |\nu(f(z))|$ for any $z \in \mathbb{P}^N$.*
2. *If f is a homeomorphism, then $|\nu(z)| = |\nu(f(z))|$ for any $z \in \mathbb{P}^N$.*

Proof. Observe that for any $z \in \mathbb{P}^N$ one can compute $|\nu(z)|$ as

$$|\nu(z)| = \max\{n \in \underline{N} \mid a_1 < \cdots < a_n = z,\ a_i \in \mathbb{P}^N\}. \tag{17}$$

Here the symbol $x < y$ means $x \leqslant y$ and $x \neq y$. On the other hand, any map between spaces equipped with preorders is continuous with respect to the Alexandrov topologies induced by these preorders if and only if it is monotone, i.e. it preserves the preorders. Therefore, if f is continuous (i.e. preorder preserving) and injective, then (17) implies that $|\nu(z)| \leqslant |\nu(f(z))|$ for any $z \in \mathbb{P}^N$. Finally, if f is a homeomorphism, then we also have $|\nu(f(z))| \leqslant |\nu(f^{-1}(f(z)))| = |\nu(z)|$, so that $|\nu(z)| = |\nu(f(z))|$ for any $z \in \mathbb{P}^N$. ■

Note that any continuous bijection between any two finite homeomorphic topological spaces (not necessarily Hausdorff) is always a homeomorphism. Hence, for any finite N, a continuous bijection from $\mathbb{P}^N$ to $\mathbb{P}^N$ is automatically a homeomorphism, so that it enjoys the property (2) of the lemma above.

THEOREM 2.17. *Let N be a finite natural number or ∞. A map $f\colon \mathbb{P}^N \to \mathbb{P}^N$ is a homeomorphism if and only if there exists a bijection $\sigma\colon \underline{N} \to \underline{N}$ such that $f(\chi_a) = \chi_{\sigma(a)}$ for any subset $a \subseteq \mathbb{N}$.*

Proof. We consider a bijection $\sigma\colon \underline{N} \to \underline{N}$. It induces a bijection of the form

$$f_\sigma\colon \mathbb{P}^N \ni \chi_a \longmapsto \chi_{\sigma(a)} \in \mathbb{P}^N \tag{18}$$

with the inverse $(f_\sigma)^{-1} = f_{\sigma^{-1}}$. Since $f_\sigma(\mathbb{A}_i) = \mathbb{A}_{\sigma(i)}$ for all i and the set of all $\mathbb{A}_i$'s is a subbasis for the topology of $\mathbb{P}^N$, we conclude that f_σ is a homeomorphism.

Conversely, assume that we have a homeomorphism $f\colon \mathbb{P}^N \to \mathbb{P}^N$. Consider $\ell \subseteq \underline{N}$ and $\chi_\ell \in \mathbb{P}^N$. By Lemma 2.16, the function f satisfies $|\nu(z)| = |\nu(f(z))|$ for any $z \in \mathbb{P}^N$. Therefore, applying the support map ν to both sides of the equality $f(\chi_{\{i\}}) = \chi_{\{\sigma(i)\}}$ determines a unique map $\sigma\colon \underline{N} \to \underline{N}$ satisfying this equality:

$$\nu(f(\chi_{\{i\}})) =: \nu(\chi_{\{\sigma(i)\}}) = \{\sigma(i)\}. \tag{19}$$

The inverse of the map thus defined is given by the formula $\{\sigma^{-1}(i)\} := \nu(f^{-1}(\chi_{\{i\}}))$.

Next, we proceed by induction on the cardinality of $a \subseteq \underline{N}$. Assume that we have already proven that $f(\chi_a) = \chi_{\sigma(a)}$ for all a such that $0 < |a| \leqslant n$. Pick $a \subseteq \underline{N}$ with $|a| = n$ and $j \in \underline{N} \setminus a$. Then, since the continuity of f is equivalent to f being monotone, we obtain $\chi_{\sigma(a)} = f(\chi_a) \leqslant f(\chi_{a\cup\{j\}})$ in $\mathbb{P}^N$. Hence $f(\chi_{a\cup\{j\}}) = \chi_{\sigma(a)\cup\ell}$ for some $\ell \subseteq \underline{N}$. On the other hand, by Lemma 2.16, we see that $|\nu(f(\chi_{a\cup\{j\}}))| = n+1$, so that $\ell = \{k\}$ for some $k \notin \sigma(a)$. It remains to prove that $k = \sigma(j)$. By definition $\chi_{\sigma(a)\cup\{k\}} \in \mathbb{A}_k$. Therefore, $\chi_{a\cup\{j\}} \in f^{-1}(\mathbb{A}_k) = \mathbb{A}_{\sigma^{-1}(k)}$, whence $\sigma^{-1}(k) \in a \cup \{j\}$. Combining this with $\sigma^{-1}(k) \notin a$ yields $\sigma(j) = k$, as needed. ∎

We end this subsection by introducing a monoid that acts on $\mathbb{P}^\infty$ by continuous maps and is pivotal in our classification theorem. It is a monoid that labels all finite sequences that can be formed from a given finite set.

Definition 2.18. A surjection $\alpha\colon \mathbb{N} \to \mathbb{N}$ is called *tame* if

1. $\alpha^{-1}(i)$ is finite for any $i \in \mathbb{N}$,
2. $|\alpha^{-1}(i)| > 1$ for finitely many $i \in \mathbb{N}$.

We denote the monoid of all such tame surjections by $\mathcal{M}$.

It is clear that the composition of any two tame surjections is again a tame surjection, and that the monoid is generated by bijections and the following tame surjection:

$$\partial(i) := \begin{cases} i & \text{if } i = 0, \\ i-1 & \text{if } i > 0. \end{cases} \tag{20}$$

We can view the elements of $\mathbb{P}^\infty$ as maps from $\mathbb{N}$ to $\{0,1\}$, and on such maps the monoid $\mathcal{M}$ acts by pullbacks. Moreover, the tameness property ensures that such pullbacks preserve $\mathbb{P}^\infty$ and

$$f_\alpha(\chi_a) := \alpha^*(\chi_a) = \chi_{\alpha^{-1}(a)} \qquad \text{for all } a \in \mathtt{Fin} \setminus \emptyset \tag{21}$$

guarantees that they are morphisms of posets. Thus we obtain an action of $\mathcal{M}$ on $\mathbb{P}^\infty$ by maps continuous in the Alexandrov topology. Observe that this pullback representation of the monoid $\mathcal{M}$ is faithful. Note also that Theorem 2.17 can be rephrased to link the bijections from $\underline{N}$ to $\underline{N}$ with the homeomorphisms from $\mathbb{P}^N$ to $\mathbb{P}^N$ by the formula $f(\chi_a) := \chi_{\sigma^{-1}(a)}$ for any subset $a \subseteq \mathbb{N}$. This makes Theorem 2.17 compatible with (21).

2.4. The lattice of open subsets of $\mathbb{P}^\infty$. In this subsection, we provide a direct generalization of [9, Subsection 2.2] needed to upgrade the flabby-sheaf classification of ordered N-coverings [9, Corollary 4.3] to a classification of arbitrary finite ordered coverings we arrive at in Lemma 3.9.

LEMMA 2.19. *Let* $(\mathtt{Lat}(A), \cap, +)$ *denote the lattice of all ideals in an algebra* A. *Assume that* $(I_i)_{i\in\mathbb{N}}$ *is a sequence of ideals such that only finitely many of them are different from* A *and that the lattice they generate is distributive. Then, for any open subset* $U \subseteq \mathbb{P}^\infty$, *the map given by*

$$R^{(I_i)_i}(U) := \bigcap_{a\in\nu(U)} \sum_{i\in a} I_i \tag{22}$$

defines a morphism of lattices $R^{(I_i)_i}\colon \mathtt{Op}(\mathbb{P}^\infty) \to \mathtt{Lat}(A)$.

Proof. Since the map $\nu\colon \mathbb{P}^\infty \to \mathtt{Fin}\setminus\{\emptyset\}$ given by (6) for $N = \infty$ is a bijection, we have

$$\nu(U_1 \cap U_2) = \nu(U_1) \cap \nu(U_2) \quad \text{and} \quad \nu(U_1 \cup U_2) = \nu(U_1) \cup \nu(U_2) \tag{23}$$

for any $U_1, U_2 \in \mathtt{Op}(\mathbb{P}^\infty)$. In order to prove that $R^{(I_i)_i}$ is a morphism of lattices, we need to show that

$$R^{(I_i)_i}(U_1\cap U_2) = R^{(I_i)_i}(U_1)+R^{(I_i)_i}(U_2), \quad R^{(I_i)_i}(U_1\cup U_2) = R^{(I_i)_i}(U_1)\cap R^{(I_i)_i}(U_2), \tag{24}$$

for all $U_1, U_2 \in \mathtt{Op}(\mathbb{P}^\infty)$. Note that the latter of the above equalities is trivially satisfied.

To prove the former identity, first we observe that for all upper sets $\alpha_1, \alpha_2 \subseteq \mathtt{Fin}$

$$\alpha_1 \cap \alpha_2 = \{a_1 \cup a_2 \mid a_1 \in \alpha_1, a_2 \in \alpha_2\}. \tag{25}$$

Indeed, since $a_1 \subseteq a_1 \cup a_2$ and $a_2 \subseteq a_1 \cup a_2$, we see that the left hand side contains the right hand side. The other inclusion follows from the fact that $a = a \cup a$. Next, we note that although the intersection in (22) is potentially infinite, there are only finitely many ideals different from A. This fact allows us to use the distributivity of the lattice generated by the ideals I_i. Furthermore, since ν is a homeomorphism (see below (9)) with respect to the Alexandrov topologies (open sets are upper sets), we can use (25) to conclude that

$$\forall\, U_1, U_2 \in \mathtt{Op}(\mathbb{P}^\infty): \quad \nu(U_1 \cap U_2) = \nu(U_1) \cap \nu(U_2) = \{a_1 \cup a_2 \mid a_1 \in \nu(U_1), a_2 \in \nu(U_2)\}.$$

Combining all this together, we obtain:

$$\begin{aligned}
\bigcap_{a\in\nu(U_1\cap U_2)} \sum_{i\in a} I_i &= \bigcap_{a\in\nu(U_1)\cap\nu(U_2)} \sum_{i\in a} I_i \\
&= \bigcap_{a_1\in\nu(U_1)} \bigcap_{a_2\in\nu(U_2)} \Big(\sum_{i\in a_1} I_i + \sum_{j\in a_2} I_j\Big) \\
&= \bigcap_{a_1\in\nu(U_1)} \Big(\sum_{i\in a_1} I_i + \bigcap_{a_2\in\nu(U_2)} \sum_{j\in a_2} I_j\Big) \\
&= \bigcap_{a_1\in\nu(U_1)} \sum_{i\in a_1} I_i + \bigcap_{a_2\in\nu(U_2)} \sum_{j\in a_2} I_j.
\end{aligned} \tag{26}$$

The result follows. ∎

2.5. Sheaves and patterns on Alexandrov spaces. A pattern [13] is a sheaf-like object defined on the category of closed subsets $\mathtt{Cl}(X)$ of a topological space X with inclusions. Explicitly, a pattern of sets on a topological space X is a covariant functor $F\colon \mathtt{Cl}(X)^{\mathrm{op}} \to \mathbf{Set}$ to the category of sets satisfying the property that, for any closed subset C of X and any given *finite* closed covering $\{C_\lambda\}_\lambda$ of C, the canonical diagram

$$F(C) \to \prod_\lambda F(C_\lambda) \rightrightarrows \prod_{\lambda,\mu} F(C_\lambda \cap C_\mu) \tag{27}$$

is an equalizer diagram. A pattern F on a topological space is called *global* if for any inclusion of closed sets $C' \subseteq C$ the restriction morphism $F(C) \to F(C')$ is an epimorphism.

We would like to note that for compact Hausdorff spaces Leray's definition of *faisceau continu* [11, p. 303] is equivalent to the definition of a sheaf. However, in this paper we only consider sheaves over Alexandrov spaces which are of completely different nature, and thus we cannot exchange these two concepts. On the other hand, for any finite Alexandrov space, we show below that the category of global patterns and the category of flabby sheaves are equivalent up to a natural duality.

It follows from Lemma 2.1 that the lattice of open sets of an Alexandrov space $(P, \leqslant)$ is isomorphic to the lattice of closed sets of the dual Alexandrov space $(P, \leqslant)^{\mathrm{op}}$. Hence:

PROPOSITION 2.20. *Let $(P, \leqslant)$ be a finite preordered set. The category of (flabby) sheaves on the Alexandrov space $(P, \leqslant)$ is isomorphic to the category of (global) patterns on the opposite Alexandrov space $(P, \leqslant)^{op}$.*

Proof. Since the lattice of closed subsets of $(P, \leqslant)^{\mathrm{op}}$ is isomorphic to the lattice of open subsets of $(P, \leqslant)$, we conclude that any (flabby) sheaf on $(P, \leqslant)$ is a (global) pattern on $(P, \leqslant)^{\mathrm{op}}$ regardless of P being finite. Conversely, assume that F is a (global) pattern on $(P, \leqslant)^{\mathrm{op}}$, and let $\mathcal{U}$ be an open cover of an open subset V of $(P, \leqslant)$. As P is finite, the number of open and closed subsets of P is finite, so that $\mathcal{U}$ is a finite open cover in $(P, \leqslant)$. Thus $\mathcal{U}$ is also a finite collection of closed subsets of $(P, \leqslant)^{\mathrm{op}}$ covering the closed set V in $(P, \leqslant)^{\mathrm{op}}$. Furthermore,

$$F(V) \to \prod_{U \in \mathcal{U}} F(U) \rightrightarrows \prod_{U, U' \in \mathcal{U}} F(U \cap U') \tag{28}$$

is an equalizer diagram because F is a (global) pattern on $(P, \leqslant)^{\mathrm{op}}$. Hence F is a sheaf on the Alexandrov space $(P, \leqslant)$. ∎

The restriction that P is finite comes from the definition of a pattern. A pattern is a sheaf-like object where Diagram (28) is an equalizer only for *finite* closed coverings, as opposed to a sheaf where Diagram (28) is an equalizer for every (finite or infinite) open covering.

Next, we consider a poset $(P, \leqslant)$ as a category by letting

$$Ob(P) := P \quad \text{and} \quad \mathrm{Hom}_P(p, q) := \begin{cases} \{p \to q\} & \text{if } p \leqslant q, \\ \emptyset & \text{otherwise.} \end{cases} \tag{29}$$

Then a covariant functor $X\colon P \to \mathbf{Vect}_k$ to the category of vector spaces over k is just a collection of vector spaces $\{X_p\}_{p\in P}$ together with linear maps $T_{qp}\colon X_p \to X_q$ such that (i) $T_{pp} = \mathrm{id}_{X_p}$ and (ii) $T_{rq} \circ T_{qp} = T_{rp}$. Any such a covariant functor will be called a right *P-module*. The category of right P-modules and their morphisms will be denoted by $\mathbf{Mod}_P$. We will call a P-module flabby if each T_{pq} is an epimorphism. If $X\colon P \to \mathbf{Alg}_k$

is a functor into the category of k-algebras, then it will be referred as a right P-*algebra*. The category of right P-algebras and their morphisms will be denoted by $\mathbf{Alg}_P$.

For a topological space X and a covering $\mathcal{O}$ of X, we say that $\mathcal{O}$ is stable under finite intersections if for any finite collection $O_1, \ldots, O_n$ of sets from $\mathcal{O}$ there exists a subset $\mathcal{O}' \subseteq \mathcal{O}$ such that

$$\bigcap_{i=1}^{n} O_i = \bigcup_{O' \in \mathcal{O}'} O'. \tag{30}$$

The above property allows us to describe the sheaf condition in a different but equivalent form. A standard argument shows:

LEMMA 2.21. *Let F be a sheaf of algebras on a topological space X. Then for any open subset $U \subseteq X$ and any open covering $\mathcal{U}$ of U that is stable under finite intersections, the canonical morphism $F(U) \to \lim_{V \in \mathcal{U}} F(V)$ is an isomorphism.*

The following theorem concerning sheaves of algebras is a straightforward adaptation of a well-known result for sheaves of modules (see [3, Proposition 6.6] for a proof).

THEOREM 2.22. *Let $(P, \leqslant)$ be a poset. Then the category of sheaves of k-algebras on P with the Alexandrov topology induced by the poset structure is equivalent to the category of P-algebras.*

3. Classification of finite coverings via the universal partition space $\mathbb{P}^\infty$. The aim of this section is to establish an equivalence between the category of finite coverings of algebras and an appropriate category of finitely-supported flabby sheaves of algebras. To this end, we first define a number of different categories of coverings and sheaves. Then we explore their interrelations to assemble a path of functors yielding the desired equivalence of categories.

3.1. Categories of coverings. Let X be a topological space and $\mathcal{C}$ be a collection of subsets of X that cover X, i.e. $\bigcup_{U \in \mathcal{C}} U = X$. We allow $\emptyset \in \mathcal{C}$. Recall that such a set $\mathcal{C}$ is called a *covering* of X. A covering $\mathcal{C}$ is called finite if the set $\mathcal{C}$ is finite. A covering $\mathcal{C}$ of a topological space X is called *closed* (resp. *open*) if $\mathcal{C}$ consists of closed (resp. open) subsets of X. Let us now consider the category of pairs of the form $(X, \mathcal{C})$ where X is a topological space and $\mathcal{C}$ is a closed (or open) covering. A morphism $f\colon (X, \mathcal{C}) \to (X', \mathcal{C}')$ is a continuous map of topological spaces $f\colon X \to X'$ such that for any $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}'$ with the property that $C \subseteq f^{-1}(C')$. In the spirit of the Gelfand transform, we are going to dualize this category to the category of algebras.

Let $\Pi := \{\pi_i\colon A \to A_i\}_i$ be a finite set of epimorphisms of algebras. We allow the case $A_i = \mathbf{0}$ for some i. Denote by Λ the lattice of ideals generated by $\ker \pi_i$, where $\cap$ and $+$ denote the join and meet operations respectively. Recall from [9] that the set Π is called a *covering* if the lattice Λ is distributive and $\bigcap_i \ker(\pi_i) = \mathbf{0}$. Finally, an ordered family $\underline{\Pi} := (\pi_i\colon A \to A_i)_i$ is called an *ordered covering* if the set $\kappa(\underline{\Pi}) := \{\pi_i\colon A \to A_i\}_i$ is a covering. In such an ordered sequence $(\pi_i\colon A \to A_i)_i$ we allow repetitions.

In [9], for each natural number N, the authors defined a category $\boldsymbol{C}_N$ whose objects are pairs $(A; \pi_0, \ldots, \pi_N)$, where A is a unital algebra and the ordered sequence $(\pi_0, \ldots, \pi_N)$ is an ordered covering of A. (Note that herein we begin labelling covering elements from 0 rather than from 1 as in [9].) A morphism between two objects $f\colon (A;\pi_0,\dots,\pi_N) \to (A';\pi'_0,\dots,\pi'_N)$ is a morphism of algebras $f\colon A \to A'$ such that $f(\ker(\pi_i)) \subseteq \ker(\pi'_i)$ or, equivalently, such that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_i))$, for any $i = 0,\dots,N$. This category is called *the category of ordered* $(N+1)$*-coverings of algebras.*

For any natural number N, there is a functor $e_N\colon \boldsymbol{C}_N \to \boldsymbol{C}_{N+1}$ defined on the set of objects as $e_N(A;\pi_0,\dots,\pi_N) := (A;\pi_0,\dots,\pi_N, A \to \mathbf{0})$ for all $(A;\pi_0,\dots,\pi_N) \in Ob(\boldsymbol{C}_N)$, and as identity on the sets of morphisms. Thus e_N is a faithful functor. It is also full because, for any $(A,\underline{\Pi})$ and $(A',\underline{\Pi}')$ in $Ob(\boldsymbol{C}_N)$, we have

$$\mathrm{Hom}_{\boldsymbol{C}_{N+1}}(e_N(A,\underline{\Pi}), e_N(A',\underline{\Pi}')) = \mathrm{Hom}_{\boldsymbol{C}_N}((A,\underline{\Pi}),(A',\underline{\Pi}')). \tag{31}$$

Our next step is to introduce the category $\mathcal{OC}ov_{\mathrm{fin}}$ of pairs of the form $(A,\underline{\Pi})$ where A is again a unital algebra but $\underline{\Pi}$ is an infinite (rather than finite) sequence of epimorphisms $A \overset{\pi_i}{\to} A_i$, $i \in \mathbb{N}$, such that: (i) all but finitely many of these epimorphisms have zero codomain and (ii) the underlying set $\kappa(\underline{\Pi})$ of epimorphisms is a covering of A. A morphism

$$f\colon (A;\pi_0,\pi_1,\dots) \longrightarrow (A';\pi'_0,\pi'_1,\dots) \tag{32}$$

is a morphism of algebras $f\colon A \to A'$ with the property that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_i))$ for any $i \in \mathbb{N}$. Alternatively, we can define $\mathcal{OC}ov_{\mathrm{fin}}$ as a colimit:

Definition 3.1. The category $\mathcal{OC}ov_{\mathrm{fin}} := \mathrm{colim}_{N\in\mathbb{N}}\boldsymbol{C}_N$ is called the category of *finite ordered coverings* of algebras.

Next, recall from the beginning of this section that, in the category of topological spaces together with a prescribed finite covering, a covering is a collection of sets devoid of an ordering on the covering sets. Hence it is necessary for us to replace the ordered sequences of epimorphisms in the objects of the category $\mathcal{OC}ov_{\mathrm{fin}}$ by the *finite sets* of epimorphisms of algebras.

Definition 3.2. Let $\mathcal{C}ov_{\mathrm{fin}}$ be a category whose objects are pairs (A,Π), where A is a unital algebra and Π is a finite *set* of unital algebra epimorphisms that is a covering of the algebra A. A morphism $f\colon (A,\Pi) \to (A',\Pi')$ in this category is a morphism of algebras $f\colon A \to A'$ satisfying the condition that for any epimorphism $\pi'_i\colon A' \to A'_i$ in the covering Π' there exists an epimorphism $\pi_j\colon A \to A_j$ in the covering Π such that $\ker(\pi_j) \subseteq f^{-1}(\ker(\pi'_i))$. This category will be called *the category of finite coverings of algebras.*

If $f\colon (A,\Pi) \to (A',\Pi')$ is a morphism in $\mathcal{C}ov_{\mathrm{fin}}$, we will say that f is implemented by the morphism of algebras $f\colon A \to A'$. Note that the matching of the epimorphisms, or rather the kernels, is not part of the datum defining a morphism.

We also need the following auxiliary category.

Definition 3.3. The category $\mathcal{A}ux$ is a category whose objects are the same as the objects of $\mathcal{OC}ov_{\mathrm{fin}}$. A morphism $f\colon (A,\underline{\Pi}) \to (A',\underline{\Pi}')$ in $\mathcal{A}ux$ is a morphism of algebras $f\colon A \to A'$ satisfying the property that for every π'_j appearing in the sequence $\underline{\Pi}'$ there exists an epimorphism π_i appearing in the ordered sequence $\underline{\Pi}$ such that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_j))$. As before, the matching of the epimorphisms is not part of the datum defining a morphism.

Now we want to prove that the categories $\mathcal{A}ux$ and $\mathcal{C}ov_{\mathrm{fin}}$ are equivalent. Recall first that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *essentially surjective* if for every $X \in Ob(\mathcal{D})$ there exists an object $C_X \in Ob(\mathcal{C})$ and an isomorphism $\omega_X : F(C_X) \to X$ in $\mathcal{D}$.

THEOREM 3.4 ([12, IV. 4 Theorem 1]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor that is fully faithful and essentially surjective. Then F is an equivalence of categories.*

LEMMA 3.5. *For every object $(A; \pi_0, \pi_1, \ldots)$ and morphism $f\colon (A, \underline{\Pi}) \to (A', \underline{\Pi}')$ in the category $\mathcal{A}ux$ consider the assignment*

$$\mathfrak{Z}(A; \pi_0, \pi_1, \ldots) := (A; \{\pi_i |\ i \in \mathbb{N}\}) \in Ob(\mathcal{C}ov_{\mathrm{fin}}) \quad \text{and} \quad \mathfrak{Z}(f) := f \in Mor(\mathcal{C}ov_{\mathrm{fin}}).$$

The assignment defines a functor $\mathfrak{Z}\colon \mathcal{A}ux \to \mathcal{C}ov_{\mathrm{fin}}$ establishing the equivalence of categories.

Proof. One can see that

$$\mathrm{Hom}_{\mathcal{A}ux}((A, \underline{\Pi}),\ (B, \underline{\Theta})) = \mathrm{Hom}_{\mathcal{C}ov_{\mathrm{fin}}}((A, \kappa(\underline{\Pi})),\ (B, \kappa(\underline{\Theta}))). \tag{33}$$

This implies that $\mathfrak{Z}$ is fully faithful, and that it makes sense for the functor $\mathfrak{Z}$ to act as identity on the set of morphisms. Given an object (A, Π) in $\mathcal{C}ov_{\mathrm{fin}}$, one can choose an ordering on the finite set Π and obtain an ordered sequence of epimorphisms

$$(\pi_0\colon A \to A_0, \pi_1\colon A \to A_1, \ldots, \pi_N\colon A \to A_N), \tag{34}$$

where $N := |\Pi| - 1$. We can pad this sequence with $A \to \mathbf{0}$ to get an infinite sequence $\underline{\Pi}$ of epimorphisms where only finitely many epimorphisms are non-trivial. This infinite sequence has the property that the corresponding finite set $\kappa(\underline{\Pi})$ of epimorphisms is the set $\Pi \cup \{A \to \mathbf{0}\}$. Since the identity morphism $\mathrm{id}_A\colon A \to A$ implements an isomorphism

$$(A, \Pi \cup \{A \to \mathbf{0}\}) \longrightarrow (A, \Pi) \tag{35}$$

in $\mathcal{C}ov_{\mathrm{fin}}$, we infer that $\mathfrak{Z}$ is essentially surjective. Now the result follows from Theorem 3.4. ∎

The category $\mathcal{A}ux$ sits in between the category $\mathcal{OC}ov_{\mathrm{fin}}$ of ordered coverings and the category $\mathcal{C}ov_{\mathrm{fin}}$ of coverings:

$$\mathcal{OC}ov_{\mathrm{fin}} \hookrightarrow \mathcal{A}ux \xrightarrow{\simeq} \mathcal{C}ov_{\mathrm{fin}}. \tag{36}$$

The definitions of morphisms in the categories $\mathcal{A}ux$ and $\mathcal{C}ov_{\mathrm{fin}}$ coincide even though the classes of objects are different. On the other hand, the categories $\mathcal{OC}ov_{\mathrm{fin}}$ and $\mathcal{A}ux$ share the same objects, but there are more morphisms in $\mathcal{A}ux$ than in $\mathcal{OC}ov_{\mathrm{fin}}$:

$$\mathrm{Hom}_{\mathcal{OC}ov_{\mathrm{fin}}}((A, \underline{\Pi}), (B, \underline{\Pi}')) \subseteq \mathrm{Hom}_{\mathcal{A}ux}((A, \underline{\Pi}), (B, \underline{\Pi}')). \tag{37}$$

Explicitly, one can describe $\mathrm{Hom}_{\mathcal{A}ux}((A, \underline{\Pi}), (B, \underline{\Pi}'))$ as the set of morphisms $f\colon A \to B$ of algebras for which there exists a sequence of epimorphisms $\underline{\Pi}''$ obtained from $\underline{\Pi}$ by permutations and insertions of already existing epimorphisms, and such that f is a morphism in $\mathrm{Hom}_{\mathcal{OC}ov_{\mathrm{fin}}}((A, \underline{\Pi}''), (B, \underline{\Pi}'))$. This can be elegantly expressed by introducing another auxiliary category $\widetilde{\mathcal{A}ux}$ such that $\mathcal{A}ux$ comes out as the quotient of $\widetilde{\mathcal{A}ux}$ by an equivalence relation on the morphisms (cf. Definition 3.6 and Lemma 3.7 below).

The reason why we prefer working with ordered sequences of epimorphisms in $\mathcal{A}ux$ rather than the sets of epimorphisms in $\mathcal{C}ov_{\mathrm{fin}}$ is that we want to interpret coverings in the language of sheaves. Working with sheaves inevitably introduces order on the set of epimorphisms because of the particular nature of morphisms in the category of sheaves (cf. Lemma 3.9). Fortunately, by Lemma 3.5, our auxiliary category $\mathcal{A}ux$, where the objects are based on ordered sequences, is equivalent to $\mathcal{C}ov_{\mathrm{fin}}$, the category of finite coverings of algebras where the objects are based on finite sets of epimorphisms.

Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a tame surjection from the monoid $\mathcal{M}$ (Definition 2.18). Any such α gives rise to an endofunctor $\breve{\alpha}\colon \mathcal{OC}ov_{\mathrm{fin}} \to \mathcal{OC}ov_{\mathrm{fin}}$ defined on objects by

$$\breve{\alpha}(A, (\pi_i)_i) := (A, (\pi_{\alpha(i)})_i), \tag{38}$$

and by identity on the morphisms.

Definition 3.6. The category $\widetilde{\mathcal{A}ux}$ is a category whose objects are the same as in $\mathcal{OC}ov_{\mathrm{fin}}$ and $\mathcal{A}ux$, and whose morphisms are pairs $(f, \alpha) : (A, \underline{\Pi}) \to (A', \underline{\Pi}')$ such that $\alpha \in \mathcal{M}$ and

$$f : \breve{\alpha}(A, \underline{\Pi}) \longrightarrow (A', \underline{\Pi}')$$

is a morphism in $\mathcal{OC}ov_{\mathrm{fin}}$. The identity morphisms are simply $(\mathrm{id}_A, \mathrm{id}_{\mathbb{N}})$, and the composition of morphisms is defined as

$$(g, \beta) \circ (f, \alpha) = (g \circ (\breve{\beta} f), \alpha \circ \beta).$$

Note that we have $(\beta \circ \alpha)\breve{} = \breve{\alpha}\breve{\beta}$.

We define an equivalence relation on $\widetilde{\mathcal{A}ux}$ as follows. We say that two morphisms (f, α), (f', α') in $\mathrm{Hom}_{\widetilde{\mathcal{A}ux}}((A, \underline{\Pi}), (A', \underline{\Pi}'))$ are equivalent (here denoted by $(f, \alpha) \sim (f', \alpha')$) if $f = f'$ as morphisms of algebras. By [12, Proposition II.8.1], we know the quotient category $\widetilde{\mathcal{A}ux}/\!\sim$ exists. Moreover, it is easy to see that the relation $\sim$ preserves the compositions of morphisms. Hence, by the proof of [12, Proposition II.8.1], we do not need to extend the relation $\sim$ to form a quotient category. We are now ready for:

Lemma 3.7. *The category $\mathcal{A}ux$ and the quotient category $\widetilde{\mathcal{A}ux}/\!\sim$ are isomorphic.*

Proof. We implement the isomorphism with two functors

$$F : \widetilde{\mathcal{A}ux}/\!\sim \longrightarrow \mathcal{A}ux, \quad G : \mathcal{A}ux \longrightarrow \widetilde{\mathcal{A}ux}/\!\sim\,, \tag{39}$$

defined as identities on objects. For any equivalence class $[f, \alpha]_\sim$ of morphisms in $\widetilde{\mathcal{A}ux}/\!\sim$, we define $F([f, \alpha]_\sim) := f$. On the other hand, for any morphism $f : (A, (\pi_i)_{i\in\mathbb{N}}) \to (A', ({\pi'}_i)_{i\in\mathbb{N}})$ in $\mathcal{A}ux$, we set $G(f) := [f, \alpha]_\sim$, where α is any element of $\mathcal{M}$ satisfying:

$$\alpha(i) = \begin{cases} i - N & \text{for } i > N, \\ j, \text{ where } j \text{ is such that } \ker \pi_j \subseteq f^{-1}(\ker {\pi'}_i), & \text{for } i \leqslant N. \end{cases} \tag{40}$$

Here $N \in \mathbb{N}$ is a number such that for any $i > N$ we have ${\pi'}_i := A' \to 0$. It is obvious that $F \circ G = \mathrm{id}_{\mathcal{A}ux}$ and $G \circ F = \mathrm{id}_{\widetilde{\mathcal{A}ux}/\sim}$. One can easily see that F and G are functorial — it is enough to note that $\breve{\alpha} f = f$ as morphisms of algebras. ∎

3.2. Sheaf picture for coverings. Let $\mathrm{Sh}(\mathbb{P}^\infty)$ be the category of flabby sheaves of algebras over $\mathbb{P}^\infty$. A morphism $f\colon F \to G$ in $\mathrm{Sh}(\mathbb{P}^\infty)$ is a collection $\{f_U\colon F(U) \to G(U)\}_{U\in\mathtt{Op}(\mathbb{P}^\infty)}$ of morphisms of algebras (indexed by the open subsets of $\mathbb{P}^\infty$) that fit into the following commutative diagram

$$\begin{tikzcd} F(U) \arrow[r, "f_U"] \arrow[d, "\mathrm{Res}^U_V(F)"'] & G(U) \arrow[d, "\mathrm{Res}^U_V(G)"] \\ F(V) \arrow[r, "f_V"'] & G(V) \end{tikzcd} \tag{41}$$

for any chain $V \subseteq U$ of open subsets of $\mathbb{P}^\infty$.

DEFINITION 3.8. A flabby sheaf $F \in Ob(\mathrm{Sh}(\mathbb{P}^\infty))$ is said to have finite support if there exists $N \in \mathbb{N}$ such that $F(\mathbb{A}_n) = 0$ for any $n > N$. The full subcategory of flabby sheaves with finite support will be denoted by $\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$.

Here is an alternative way of seeing sheaves with finite support on $\mathbb{P}^\infty$. Any sheaf of algebras on $\mathbb{P}^N$ can be extended to a sheaf of algebras on $\mathbb{P}^{N+1}$ by the direct image functor

$$\mathrm{Sh}(\mathbb{P}^N) \ni F \longmapsto (\phi_N)_*(F) \in \mathrm{Sh}(\mathbb{P}^{N+1}) \tag{42}$$

with respect to the canonical embedding $\phi_N \colon \mathbb{P}^N \to \mathbb{P}^{N+1}$ defined in Lemma 2.12. Then we obtain an injective system of categories $(\mathrm{Sh}(\mathbb{P}^N), j_N)$ whose colimit can be identified with $\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$.

For a flabby sheaf F in $Ob(\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty))$, we will use $\mathrm{Res}_i(F)$ to denote the restriction epimorphism $F(\mathbb{P}^\infty) \to F(\mathbb{A}_i)$ for any $i \in \mathbb{N}$. Note that, since F is a sheaf with finite support, all but finitely many morphisms $\mathrm{Res}_i(F)$ are of the form $F(\mathbb{P}^\infty) \to \mathbf{0}$. The following lemma is a reformulation of [9, Corollary 4.3] in our new setting. (Cf. [17, Proposition 1.10] for a commutative version.) The proof uses Lemma 2.19 and is essentially the same as in [9, Proposition 2.2]. Note that we can apply the generalized Chinese Remainder Theorem (e.g. see [14, Theorem 18 on p. 280] and [13]) as there is always only a finite number of non-trivial congruences.

LEMMA 3.9. *For any $(A, \underline{\Pi}) \in Ob(\mathcal{OC}ov_{\mathrm{fin}})$ and $F \in \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$, the following assignments*

$$\Psi(A, \underline{\Pi}) := \big\{U \mapsto A/R^{\underline{\Pi}}(U)\big\}_{U \in \mathtt{Op}(\mathbb{P}^\infty)} \in \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty),$$
$$\Phi(F) := (F(\mathbb{P}^\infty);\ \mathrm{Res}_0(F),\ \mathrm{Res}_1(F),\ \ldots,\ \mathrm{Res}_n(F),\ \ldots) \in \mathcal{OC}ov_{\mathrm{fin}},$$

yield functors establishing an equivalence between the category $\mathcal{OC}ov_{\mathrm{fin}}$ of ordered coverings and the category $\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$ of finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty$.

We would like to extend the equivalence we constructed in Lemma 3.9 to an equivalence of categories between $\mathcal{A}ux$ (and therefore $\mathcal{C}ov_{\mathrm{fin}}$) and a suitable category of sheaves filling the following diagram:

$$\begin{tikzcd} \mathcal{OC}ov_{\mathrm{fin}} \arrow[r] \arrow[d, "\Psi"', "\simeq"] & \mathcal{A}ux \arrow[r, "\mathfrak{Z}", "\simeq"'] \arrow[d, dotted, "\simeq"] & \mathcal{C}ov_{\mathrm{fin}} \\ \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty) \arrow[r, dotted] & \mathrm{Sh}^{???}_{\mathrm{fin}}(\mathbb{P}^\infty). \end{tikzcd} \tag{43}$$

As $\mathcal{A}ux$ is isomorphic to a quotient category, we expect $\mathrm{Sh}^{???}_{\mathrm{fin}}(\mathbb{P}^\infty)$ to be a quotient of the following category of sheaves with extended morphisms:

DEFINITION 3.10. The objects of $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$ are finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty$. A morphism $[\tilde{f},\alpha^*] : P \to Q$ in $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$ is a pair consisting of a continuous map (see (21))

$$\alpha^* : \mathbb{P}^\infty \longrightarrow \mathbb{P}^\infty, \quad \chi_a \longmapsto \chi_{\alpha^{-1}(a)},$$

where $\mathcal{M} \ni \alpha : \mathbb{N} \to \mathbb{N}$ is a tame surjection (Definition 2.18), and a morphism of sheaves

$$\tilde{f} : \alpha^*_* P \to Q.$$

Composition of morphisms is given by

$$[\tilde{g},\beta^*] \circ [\tilde{f},\alpha^*] := [\tilde{g} \circ (\beta^*_* \tilde{f}), \beta^* \circ \alpha^*].$$

To define $\mathrm{Sh}^{???}_{\mathrm{fin}}(\mathbb{P}^\infty)$ as a quotient category equivalent to $\mathcal{A}ux \cong \widetilde{\mathcal{A}ux}/{\sim}$, we proceed by first proving the equivalence of $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$ and $\widetilde{\mathcal{A}ux}$.

LEMMA 3.11. *Let $\Psi : \mathcal{OC}ov_{\mathrm{fin}} \to \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$ and $\Phi : \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty) \to \mathcal{OC}ov_{\mathrm{fin}}$ be functors defined in Lemma 3.9. Then the functors*

$$\widetilde{\Psi} : \widetilde{\mathcal{A}ux} \longrightarrow \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty), \quad \widetilde{\Phi} : \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty) \longrightarrow \widetilde{\mathcal{A}ux},$$

defined on objects by

$$\widetilde{\Psi}(A,\underline{\Pi}) := \Psi(A,\underline{\Pi}), \quad \widetilde{\Phi}(P) := \Phi(P),$$

and on morphisms by

$$\widetilde{\Psi}(f,\alpha) := [\Psi f, \alpha^*], \quad \widetilde{\Phi}[\tilde{f},\alpha^*] := (\Phi\tilde{f}, \alpha),$$

establish an equivalence of categories between $\widetilde{\mathcal{A}ux}$ and $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$.

Proof. We divide the proof into several steps.

1. $(\alpha^*)^{-1}(\mathbb{A}_i) = \mathbb{A}_{\alpha(i)}$ for all $i \in \mathbb{N}$. Indeed,
$$\begin{aligned}(\alpha^*)^{-1}(\mathbb{A}_i) &= (\alpha^*)^{-1}(\{\chi_a \mid i \in a \subset \mathbb{N}\})\\ &= \{\chi_b \mid \alpha^*(\chi_b) = \chi_a \text{ and } i \in a \subset \mathbb{N}\}\\ &= \{\chi_b \mid \chi_{\alpha^{-1}(b)} = \chi_a \text{ and } i \in a \subset \mathbb{N}\}\\ &= \{\chi_b \mid i \in \alpha^{-1}(b)\}\\ &= \{\chi_b \mid \alpha(i) \in b \subset \mathbb{N}\}\\ &= \mathbb{A}_{\alpha(i)}.\end{aligned}$$
2. As α is tame by assumption, $\alpha^{-1}(a)$ is finite for any finite $a \subseteq \mathbb{N}$. Hence α^* is well defined.
3. Equality $\alpha^* = \beta^*$ implies that $\alpha = \beta$ for any surjective maps $\alpha, \beta : \mathbb{N} \to \mathbb{N}$. Hence the functor $\widetilde{\Phi}$ is well defined.
4. $\alpha^*_*\Psi = \Psi\breve{\alpha}$. Indeed, for any $(A, (\pi_i)_i) \in \widetilde{\mathcal{A}ux}$, we see that
$$\begin{aligned}(\alpha^*_*\Psi)((A,(\pi_i)_i)) &= \alpha^*_*(U \mapsto A/R^{(\pi_i)_i}(U))\\ &= U \mapsto A/R^{(\pi_i)_i}((\alpha^*)^{-1}(U)),\end{aligned}$$

$$
\begin{aligned}
(\Psi\breve{\alpha})((A,(\pi_i)_i)) &= \widetilde{\Psi}((A,(\pi_{\alpha(i)})_i)) \\
&= U \mapsto A/R^{(\pi_{\alpha(i)})_i}(U).
\end{aligned}
$$

On the other hand, the observation that for any open $U \subseteq \mathbb{P}^\infty$ we have $U = \bigcup_{a \text{ s.t. } \chi_a \in U} \bigcap_{i \in a} \mathbb{A}_i$ combined with the result from Step (1) yield:

$$
\begin{aligned}
R^{(\pi_i)_i}((\alpha^*)^{-1}(U)) &= R^{(\pi_i)_i}\Big((\alpha^*)^{-1}\big(\bigcup_{\substack{a \text{ s.t.}\\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_i\big)\Big) \\
&= R^{(\pi_i)_i}\Big(\bigcup_{\substack{a \text{ s.t.}\\ \chi_a \in U}} \bigcap_{i \in a} (\alpha^*)^{-1}(\mathbb{A}_i)\Big) \\
&= R^{(\pi_i)_i}\Big(\bigcup_{\substack{a \text{ s.t.}\\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_{\alpha(i)}\Big) \\
&= \bigcap_{\substack{a \text{ s.t.}\\ \chi_a \in U}} \Big(\sum_{i \in a} \ker \pi_{\alpha(i)}\Big) \\
&= R^{(\pi_{\alpha(i)})_i}\Big(\bigcup_{\substack{a \text{ s.t.}\\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_i\Big) \\
&= R^{(\pi_{\alpha(i)})_i}(U).
\end{aligned}
$$

5. Let $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ be maps from $\mathcal{M}$. Then $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. Indeed, for any $\chi_a \in \mathbb{P}^\infty$, we obtain:

$$
(\beta^* \circ \alpha^*)(\chi_a) = \beta^*(\chi_{\alpha^{-1}(a)}) = \chi_{(\beta^{-1} \circ \alpha^{-1})(a)} = \chi_{(\alpha \circ \beta)^{-1}(a)} = (\alpha \circ \beta)^*(\chi_a).
$$

6. $\widetilde{\Psi}$ is functorial. Indeed, take any composable morphisms (f,α) and (g,β) in $\widetilde{\mathcal{A}ux}$. Then the previous two steps and the functoriality of Ψ yield

$$
\begin{aligned}
\widetilde{\Psi}((g,\beta)\circ(f,\alpha)) &= \widetilde{\Psi}((g \circ (\breve{\beta} f), \alpha \circ \beta)) \\
&= (\Psi(g \circ (\breve{\beta} f)), (\alpha \circ \beta)^*) \\
&= (\Psi(g) \circ \Psi(\breve{\beta} f)), \beta^* \circ \alpha^*) \\
&= ((\Psi g) \circ (\beta^*_* \Psi f), \beta^* \circ \alpha^*) \\
&= [\Psi g, \beta^*] \circ [\Psi f, \alpha^*] \\
&= \widetilde{\Psi}((g,\beta)) \circ \widetilde{\Psi}((f,\alpha)).
\end{aligned}
$$

7. $\Phi\alpha^*_* = \breve{\alpha}\Phi$. Indeed, take any $P \in \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$. Using the result of Step (1), we obtain:

$$
\begin{aligned}
(\Phi\alpha^*_*)(P) &= \Phi(U \mapsto P(\alpha^{-1}(U))) \\
&= (P(\mathbb{P}^\infty), (P(\mathbb{P}^\infty) \mapsto P((\alpha^*)^{-1}(\mathbb{A}_i)))_i) \\
&= (P(\mathbb{P}^\infty), (P(\mathbb{P}^\infty) \mapsto P(\mathbb{A}_{\alpha(i)}))_i) \\
&= \breve{\alpha}((P(\mathbb{P}^\infty), (P(\mathbb{P}^\infty) \mapsto P(\mathbb{A}_i))_i)) \\
&= (\breve{\alpha}\Phi)(P).
\end{aligned}
$$

8. $\widetilde{\Phi}$ is functorial. The proof uses the result from the previous step, and is analogous to the proof of Step (6).
9. The natural isomorphism $\eta\colon \Psi\Phi \to \mathrm{id}_{\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)}$ comes from a family of isomorphisms of sheaves $\eta_P\colon \Psi\Phi P \to P$. The latter are given by the canonical isomorphisms between the image of an epimorphism and the quotient of its domain by its kernel (cf. [9, Proposition 2.2]): $\eta_{P,U} : P(\mathbb{P}^\infty)/\ker(P(\mathbb{P}^\infty) \to P(U)) \longrightarrow P(U)$. To see that $\alpha^*_*\eta_P = \eta_{\alpha^*_*P}$ for any sheaf P, note that $\alpha^*_*\eta_P : \alpha^*_*\Psi\Phi P = \Psi\Phi\alpha^*_*P \longrightarrow \alpha^*_*P$ and
$$\begin{aligned}(\alpha^*_*\eta_P)_U &:= \eta_{P,(\alpha^*)^{-1}(U)}\\ &= P(\mathbb{P}^\infty)/\mathrm{ker}(P(\mathbb{P}^\infty) \to P((\alpha^*)^{-1}(U))) \to P((\alpha^*)^{-1}(U))\\ &= \eta_{\alpha^*_*P,U}.\end{aligned}$$
Here the first equality is just the definition of the action of the direct image functor on morphisms.
10. The family of maps $\widetilde{\eta}_P := [\eta_P, \mathrm{id}^*_{\mathbb{N}}] : \widetilde{\Psi}\widetilde{\Phi}P \to P$ establishes a natural isomorphism between $\widetilde{\Psi}\widetilde{\Phi}$ and $\mathrm{id}_{\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)}$. It is clear that $\widetilde{\eta}_P$'s are isomorphisms. We know that η is a natural isomorphism. In particular, for any $\alpha \in \mathcal{M}$ and any morphism $\tilde{f} : \alpha^*_*P \to Q$ in $\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$, the following diagram is commutative:
$$\begin{tikzcd} \Psi\Phi\alpha^*_*P \arrow[r, "\eta_{\alpha^*_*P}"] \arrow[d, "\Psi\Phi\tilde{f}"'] & \alpha^*_*P \arrow[d, "\tilde{f}"] \\ \Psi\Phi Q \arrow[r, "\eta_Q"'] & Q\,. \end{tikzcd}$$
On the other hand, we need to establish the commutativity of the diagrams
$$\begin{tikzcd} \widetilde{\Psi}\widetilde{\Phi}P \arrow[r, "\widetilde{\eta}_P"] \arrow[d, "{\widetilde{\Psi}\widetilde{\Phi}[\tilde{f},\alpha^*]}"'] & P \arrow[d, "{[\tilde{f},\alpha^*]}"] \\ \widetilde{\Psi}\widetilde{\Phi}Q \arrow[r, "\widetilde{\eta}_Q"'] & Q\,. \end{tikzcd}$$
Using the commutativity of the first of the preceding two diagrams and the displayed formula in Step (9), we obtain the desired:
$$\begin{aligned}\widetilde{\eta}_Q \circ (\widetilde{\Psi}\widetilde{\Phi}[\tilde{f}, \alpha^*]) &= [\eta_Q, \mathrm{id}^*_{\mathbb{N}}] \circ [\Psi\Phi\tilde{f}, \alpha^*]\\ &= [\eta_Q \circ (\Psi\Phi\tilde{f}), \alpha^*]\\ &= [\tilde{f} \circ \eta_{\alpha^*_*P}, \alpha^*]\\ &= [\tilde{f} \circ (\alpha^*_*\eta_P), \alpha^*]\\ &= [\tilde{f}, \alpha^*] \circ [\eta_P, \mathrm{id}^*_{\mathbb{N}}]\\ &= [\tilde{f}, \alpha^*] \circ \widetilde{\eta}_P.\end{aligned}$$

11. By [9, Proposition 2.2]), we have $\Phi\Psi = \mathrm{id}_{\widetilde{\mathcal{OC}ov_{\mathrm{fin}}}}$. Hence, it is easy to see that the family of identity morphisms $(\mathrm{id}_A, \mathrm{id}_{\mathbb{N}})$ in $\widetilde{\mathcal{A}ux}$ establishes a natural isomorphism between $\widetilde{\Phi}\widetilde{\Psi}$ and $\mathrm{id}_{\widetilde{\mathcal{A}ux}}$. ■

Our next step is to define an equivalence relation on $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$. Let $[\tilde{f}, \alpha^*], [\tilde{g}, \beta^*] : P \to Q$ be morphisms in $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$. We say that they are equivalent $([\tilde{f}, \alpha^*] \sim [\tilde{g}, \beta^*])$ if $\tilde{f}_{\mathbb{P}^\infty} = \tilde{g}_{\mathbb{P}^\infty}$ as morphisms of algebras (cf. the equivalence relation on $\widetilde{\mathcal{A}ux}$, Lemma 3.7). As before, by [12, Proposition II.8.1], we know that the quotient category $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim$ exists. Moreover, it is easy to see that the relation $\sim$ preserves the compositions of morphisms. Hence, by the proof of [12, Proposition II.8.1], we do not need to extend the relation $\sim$ to form a quotient category. Note that the equivalence class of the morphism $[\tilde{f}, \alpha^*]$ in $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$ can be represented by $\tilde{f}_{\mathbb{P}^\infty}$. Therefore the quotient functor $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty) \to \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim$ is defined on morphisms as

$$[\tilde{f}, \alpha^*] \longmapsto \tilde{f}_{\mathbb{P}^\infty}. \tag{44}$$

In other words,

$$[\tilde{f}, \alpha^*]_\sim := \tilde{f}_{\mathbb{P}^\infty}. \tag{45}$$

The final step to arrive at our classification of finite coverings by finitely-supported flabby sheaves is as follows:

LEMMA 3.12. *The functors $\widetilde{\Psi} : \widetilde{\mathcal{A}ux} \to \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$ and $\widetilde{\Phi} : \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty) \to \widetilde{\mathcal{A}ux}$ send equivalent morphisms to equivalent morphisms. They descend to functors between quotient categories*

$$\begin{tikzcd}
\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty) \arrow[r, "\widetilde{\Psi}"] \arrow[d] & \widetilde{\mathcal{A}ux} \arrow[d] & \widetilde{\mathcal{A}ux} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty) \arrow[d] \\
\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim \arrow[r, "\overline{\Psi}"'] & \widetilde{\mathcal{A}ux}/\!\sim, & \widetilde{\mathcal{A}ux}/\!\sim \arrow[r, "\overline{\Phi}"'] & \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim,
\end{tikzcd} \tag{46}$$

establishing the equivalence of $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim$ and $\widetilde{\mathcal{A}ux}/\!\sim$.

Proof. Note that for any morphism f in $\mathcal{OC}ov_{\mathrm{fin}}$ and any morphism $\tilde{f}$ in $\mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty)$, we have the following equalities of algebra maps:

$$(\Psi f)_{\mathbb{P}^\infty} = f, \quad \Phi\tilde{f} = \tilde{f}_{\mathbb{P}^\infty}. \tag{47}$$

It follows that, if $(f, \alpha) \sim (g, \beta)$ in $\widetilde{\mathcal{A}ux}$, then

$$\widetilde{\Psi}(f, \alpha) = [\Psi f, \alpha^*] \sim [\Psi g, \beta^*] = \widetilde{\Psi}(g, \beta) \tag{48}$$

in $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$. Similarly, if $[\tilde{f}, \alpha^*] \sim [\tilde{g}, \beta^*]$ in $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)$, then

$$\widetilde{\Phi}[\tilde{f}, \alpha^*] = (\Phi\tilde{f}, \alpha) \sim (\Phi\tilde{g}, \beta) = \widetilde{\Phi}[\tilde{g}, \beta^*]. \tag{49}$$

This ends the proof. ■

Summarizing the foregoing results, we obtain the following commutative diagram of functors:

$$
\begin{tikzcd}
 & \mathcal{C}ov_{\mathrm{fin}} \arrow[rr] & & \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim \\
\mathcal{A}ux \arrow[ur, "\mathfrak{Z}"] \arrow[rr, "\sim"] & & \widetilde{\mathcal{A}ux}/\!\sim \arrow[ur, "\overline{\Psi}"] & \\
 & \mathrm{Sh}_{\mathrm{fin}}(\mathbb{P}^\infty) \arrow[uu] \arrow[rr] & & \widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty). \arrow[uu] \\
\mathcal{OC}ov_{\mathrm{fin}} \arrow[uu] \arrow[ur, "\Psi"] \arrow[rr] & & \widetilde{\mathcal{A}ux} \arrow[uu] \arrow[ur, "\widetilde{\Psi}"'] &
\end{tikzcd}
\tag{50}
$$

Using the above diagram, we immediately conclude the main result of this article:

THEOREM 3.13. *The assignments given for any* $(A,\Pi)\in Ob(\mathcal{C}ov_{\mathrm{fin}})$, $F\in Ob(\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim)$, $f\in Mor(\mathcal{C}ov_{\mathrm{fin}})$, $[\tilde{f},\alpha^*]_\sim\in Mor(\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim)$, *by the formulae*

$$
\begin{aligned}
(A,\Pi) &\longmapsto \big\{U\mapsto A/R^{\underline{\Pi}}(U)\big\}_{U\in\mathtt{Op}(\mathbb{P}^\infty)}\in Ob(\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim),\\
F &\longmapsto (F(\mathbb{P}^\infty),\ \{\mathrm{Res}_0(F),\ \mathrm{Res}_1(F),\ \ldots,\ \mathrm{Res}_n(F),\ \ldots\})\in Ob(\mathcal{C}ov_{\mathrm{fin}}),\\
f &\longmapsto [\Psi(f),\alpha_f^*]_\sim\in Mor(\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim),\\
[\tilde{f},\alpha^*]_\sim &\longmapsto \tilde{f}_{\mathbb{P}^\infty}\in Mor(\mathcal{C}ov_{\mathrm{fin}}),
\end{aligned}
$$

are equivalence functors between the category $\mathcal{C}ov_{\mathrm{fin}}$ *of finite coverings of algebras and the quotient category* $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim$ *of the category of finitely-supported flabby sheaves of algebras over* $\mathbb{P}^\infty$ *with extended morphisms. Here* $(A,\underline{\Pi})$ *is the image of* (A,Π) *under an equivalence functor inverse to* $\mathfrak{Z}$*, and* α_f *is a tame surjection defined as in* (40).

Observe that the equivalence functors of the above theorem are, essentially, identity on morphisms. This is because, on both sides of the equivalence, morphisms considered as input data are only algebra homomorhisms (see (45) and Definition 3.2). They do, however, satisfy quite different conditions to be considered morphisms in an appropriate category. Thus the essence of the theorem is to re-interpret the natural defining conditions for an algebra homomorphism to be a morphism of coverings to more refined conditions that make it a morphism between sheaves. What we gain this way is a functorial description of coverings by the more potent concept of a sheaf. We know now that lattice operations applied to a covering will again yield a covering.

We end this section by stating Theorem 3.13 in the classical setting of the Gelfand-Neumark equivalence [6, Lemma 1] between the category of compact Hausdorff spaces and the opposite category of commutative unital C*-algebras. Since the intersection of closed ideals in a C*-algebra equals their product, the lattices of closed ideals in C*-algebras are always distributive. Therefore, remembering that the epimorphisms of commutative unital C*-algebras can be equivalently presented as the pullbacks of embeddings of compact Hausdorff spaces, we obtain:

COROLLARY 3.14. *The category of finite closed coverings of compact Hausdorff spaces (see the beginning of this section) is equivalent to the opposite of the quotient category* $\widetilde{\mathrm{Sh}}_{\mathrm{fin}}(\mathbb{P}^\infty)/\!\sim$ *of finitely-supported flabby sheaves of commutative unital C*-algebras over* $\mathbb{P}^\infty$ *with extended morphisms.*

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