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FREE DYNAMICAL QUANTUM GROUPS AND THE DYNAMICAL QUANTUM GROUP $\mathrm{SU}_Q^{\mathrm{dyn}}(2)$

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Abstract. We introduce dynamical analogues of the free orthogonal and free unitary quantum groups, which are no longer Hopf algebras but Hopf algebroids or quantum groupoids. These objects are constructed on the purely algebraic level and on the level of universal C^*-algebras. As an example, we recover the dynamical $\mathrm{SU}_q(2)$ studied by Koelink and Rosengren, and construct a refinement that includes several interesting limit cases.

Contents

1. Introduction. Dynamical quantum groups were introduced by Etingof and Varchenko as an algebraic tool to study the quantum dynamical Yang-Baxter equation appearing in statistical mechanics [7, 8, 9]. Roughly, one can associate to every dynamical quantum group a monoidal category of dynamical representations, and to every solution

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R of the dynamical Yang-Baxter equation a dynamical quantum group A_R with a specific dynamical representation π such that R corresponds to a braiding on the monoidal category generated by π.

In this article, we introduce two families of dynamical quantum groups $A_{\mathrm{o}}^B(\nabla, F)$ and $A_{\mathrm{u}}^B(\nabla, F)$ which are natural generalizations of the free orthogonal and the free unitary quantum groups introduced by Wang and van Daele [17, 18]. Roughly, these dynamical quantum groups are universal with respect to the property that they possess a corepresentation v such that F becomes a morphism of corepresentations from the inverse of the transpose $v^{-\mathsf{T}}$ or from $(v^{-\mathsf{T}})^{-\mathsf{T}}$, respectively, to v.

For a specific choice of B, ∇, F, the free orthogonal dynamical quantum group turns out to coincide with the dynamical analogue of $\mathrm{SU}_q(2)$ that arises from a trigonometric dynamical R-matrix and was studied by Koelink and Rosengren [11]. We refine the definition of this variant of $\mathrm{SU}_q(2)$ so that the resulting global dynamical quantum group includes the classical $\mathrm{SU}(2)$, the non-dynamical $\mathrm{SU}_q(2)$ of Woronowicz [21], the dynamical $\mathrm{SU}_q(2)$ and further interesting limit cases which can be recovered from the global object by suitable base changes.

In the non-dynamical case, free orthogonal and free unitary quantum groups are most conveniently constructed on the level of universal C^*-algebras, where Woronowicz's theory of compact matrix quantum groups applies [20]. We shall, however, start on the purely algebraic level and then pass to the level of universal C^*-algebras, where the main problem is to identify a good definition of a dynamical quantum group.

These new classes of dynamical quantum groups give rise to several interesting questions, for example, whether it is possible to obtain a classification similar as in [19], to determine their categories of representations as in [2] and [3], or to relate their representation theory to special functions as it was done in [11] in the special case of $\mathrm{SU}_q(2)$.

Let us now describe the organization and contents of this article in some more detail.

The first part of this article (§2) is devoted to the purely algebraic setting.

We start with a summary on dynamical quantum groups (§2.1). Roughly, these objects can be regarded as Hopf algebras, that is, as algebras A equipped with a comultiplication Δ, counit ϵ and antipode S, where the field of scalars has been replaced by a commutative algebra B equipped with an action of a group Γ. The comultiplication Δ does not take values in the ordinary tensor product $A \otimes A$, but in a product $A \tilde{\otimes} A$ that takes B and Γ into account, and the counit takes values in the crossed product algebra $B \rtimes \Gamma$ which is the unit for the product $-\tilde{\otimes}-$. If B is trivial, however, these dynamical quantum groups are just Γ-graded Hopf algebras (§2.2). In general, we shall use the term (B, Γ)-Hopf algebroid instead of dynamical quantum group to be more precise.

The free orthogonal and unitary dynamical quantum groups are defined as follows. Let B be a unital, commutative algebra with a left action of a group Γ, let $\nabla = (\gamma_1, \dots, \gamma_n)$ be an n-tuple in Γ and let $F \in \mathrm{GL}_n(B)$ such that $F_{ij} = 0$ whenever $\gamma_i \neq \gamma_j^{-1}$.

DEFINITION. The *free orthogonal dynamical quantum* $A_{\mathrm{o}}^B(\nabla, F)$ is the universal algebra with a homomorphism $r \times s \colon B \otimes B \to A_{\mathrm{o}}^B(\nabla, F)$ and a $v \in \mathrm{GL}_n(A_{\mathrm{o}}^B(\nabla, F))$ satisfying

(a) $v_{ij} r(b) s(b') = r(\gamma_i(b)) s(\gamma_j(b')) v_{ij}$ for all $b, b' \in B$ and $i, j \in \{1, \dots, n\}$,
(b) $r_n(\hat{F}) v^{-\mathsf{T}} = v s_n(F)$, where $v^{-\mathsf{T}}$ denotes the transpose of v^{-1} and $\hat{F} = (\gamma_i(F_{ij}))_{i,j}$.

THEOREM. *$A^B_{\mathrm{o}}(\nabla, F)$ can be equipped with the structure of a (B,Γ)-Hopf algebroid such that $\Delta(v_{ij}) = \sum_k v_{ik}\tilde{\otimes} v_{kj}$, $\epsilon(v_{ij}) = \delta_{i,j}\gamma_i$, and $S(v_{ij}) = (v^{-1})_{ij}$ for all i,j.*

Assume now that B is equipped with an involution and let $F \in \mathrm{GL}_n(B)$ such that $F^* = F$ and $F_{ij} = 0$ whenever $\gamma_i \neq \gamma_j$.

DEFINITION. The *free unitary dynamical quantum* $A^B_{\mathrm{u}}(\nabla, F)$ is the universal $*$-algebra with a homomorphism $r \times s\colon B \otimes B \to A^B_{\mathrm{u}}(\nabla, F)$ and a unitary $v \in \mathrm{GL}_n(A^B_{\mathrm{u}}(\nabla, F))$ satisfying the condition (a) above and (c) $\bar{v}$ is invertible and $r_n(\hat{F})\bar{v}^{-\mathsf{T}} = v s_n(F)$.

THEOREM. *$A^B_{\mathrm{u}}(\nabla, F)$ can be equipped with the structure of a (B,Γ)-Hopf $*$-algebroid such that $\Delta(v_{ij}) = \sum_k v_{ik}\tilde{\otimes} v_{kj}$, $\epsilon(v_{ij}) = \delta_{i,j}\gamma_i$, $S(v_{ij}) = (v^{-1})_{ij}$ for all i,j.*

The formulas for $\Delta(v_{ij})$ and $\epsilon(v_{ij})$ above imply that the matrices v above are corepresentations of $A^B_{\mathrm{o}}(\nabla, F)$ and $A^B_{\mathrm{u}}(\nabla, F)$, respectively, and the conditions (b) and (c) assert that F is an intertwiner from $v^{-\mathsf{T}}$ or $\bar{v}^{-\mathsf{T}}$, respectively, to v. Such intertwiner relations admit plenty functorial transformations which are studied systematically in §2.3, and yield short proofs of the results above in §2.4. There, we also consider involutions on certain quotients $A^B_{\mathrm{o}}(\nabla, F, G)$ of $A^B_{\mathrm{o}}(\nabla, F)$ which are parameterized by an additional matrix $G \in \mathrm{GL}_n(B)$.

Interestingly, the square of the antipode on the dynamical quantum groups $A^B_{\mathrm{o}}(\nabla, F)$ and $A^B_{\mathrm{u}}(\nabla, F)$ can be described in terms of a natural family of characters $(\theta^{(k)})_k$ which, like the counit ϵ, take values in $B \rtimes \Gamma$. This family is an analogue of Woronowicz's fundamental family of characters on a compact quantum group.

As a main example of the constructions above, we recover the dynamical quantum group $\mathcal{F}_R(\mathrm{SU}(2))$ of Koelink and Rosengren [11] associated to a deformation parameter $q \neq 1$ as the free orthogonal dynamical quantum group $A^B_{\mathrm{o}}(\nabla, F, G)$, where B is the meromorphic functions on the plane, $\Gamma = \mathbb{Z}$ acting by shifts, $\nabla = (1, -1)$ and $F = \begin{pmatrix} 0 & 1 \\ \tilde{f} & 0 \end{pmatrix}$, where $\tilde{f}$ is the meromorphic function $\lambda \mapsto q^{-1}(q^{2\lambda} - q^{-2})/(q^{2\lambda} - 1)$, and $G = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$. In §2.6, we show how this example can be refined such that the resulting dynamical quantum group $A^B_{\mathrm{o}}(\nabla, F, G)$ includes $\mathcal{F}_R(\mathrm{SU}(2))$ and, simultaneously, a number of interesting limit cases which can be recovered from the global object by suitable base changes.

The second part of this article (§3) extends the definition of dynamical quantum groups to the level of universal C^*-algebras. Here, B is assumed to be a unital, commutative C^*-algebra and Γ acts via automorphisms. The main tasks is to find a C^*-algebraic analogue of the product $-\tilde{\otimes}-$ that describes the target of the comultiplication. As in the algebraic setting, we construct this product in two steps, by first forming a cotensor product with respect to the Hopf C^*-algebra $C^*(\Gamma)$ naturally associated to the group Γ (§3.1), and then taking a quotient with respect to B (§3.2). Given the monoidal product, all definitions carry over from the algebraic setting to the setting of universal C^*-algebras easily (§3.3).

2. The purely algebraic level. Throughout this section, we assume all algebras and homomorphisms to be unital over a fixed common ground field, and B to be a commutative algebra equipped with a left action of a group Γ.

2.1. Preliminaries on dynamical quantum groups. This subsection summarizes the basics of dynamical quantum groups used in this article. We introduce the monoidal category of (B, Γ)-algebras, then define (B, Γ)-Hopf algebroids, and finally consider base changes and the setting of $*$-algebras. Except for the base change, most of this material is contained in [8] and [11] in slightly different guise. We omit all proofs because they are straightforward.

Let $B^{\mathrm{ev}} = B \otimes B$. A B^{ev}-*algebra* is an algebra with a homomorphism $r \times s \colon B^{\mathrm{ev}} \to A$, or equivalently, with homomorphisms $r_A = r, s_A = s \colon B \to A$ whose images commute. A morphism of B^{ev}-algebras is a B^{ev}-linear homomorphism. Write $\Gamma^{\mathrm{ev}} = \Gamma \times \Gamma$ and let $e \in \Gamma$ be the unit. Given a Γ^{ev}-graded algebra A, we write $\partial_a = (\partial_a^r, \partial_a^s) = (\gamma, \gamma')$ whenever $a \in A_{\gamma,\gamma'}$.

2.1.1. Definition. A (B, Γ)-*algebra* is a Γ^{ev}-graded B^{ev}-algebra such that $(r \times s)(B^{\mathrm{ev}}) \subseteq A_{e,e}$ and $ar(b) = r(\partial_a^r(b))a$, $as(b) = s(\partial_a^s(b))a$ for all $b \in B$, $a \in A$. A *morphism* of (B, Γ)-algebras is a morphism of Γ^{ev}-graded B^{ev}-algebras. We denote by $\mathbf{Alg}_{(B,\Gamma)}$ the category of all (B, Γ)-algebras.

2.1.2. Example. Denote by $B \rtimes \Gamma$ the crossed product, that is, the universal algebra containing B and Γ such that $e = 1_B$ and $b\gamma \cdot b'\gamma' = b\gamma(b')\gamma\gamma'$ for all $b, b' \in B$, $\gamma, \gamma' \in \Gamma$. This is a (B, Γ)-algebra, where $\partial_{b\gamma} = (\gamma, \gamma)$ and $r(b) = s(b) = b$ for all $b \in B$, $\gamma \in \Gamma$.

The category of all (B, Γ)-algebras can be equipped with a monoidal structure [12] as follows. Let A and C be (B, Γ)-algebras. Then the subalgebra

$$A \overset{\Gamma}{\otimes} C := \sum_{\gamma,\gamma',\gamma'' \in \Gamma} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''} \subset A \otimes C$$

is a (B, Γ)-algebra, where $\partial_{a \otimes c} = (\partial_a^r, \partial_c^s)$ for all $a \in A$, $c \in C$ and $(r \times s)(b \otimes b') = r_A(b) \otimes s_C(b')$ for all $b, b' \in B$. Let $I \subseteq A \overset{\Gamma}{\otimes} C$ be the ideal generated by $\{s_A(b) \otimes 1 - 1 \otimes r_C(b) : b \in B\}$. Then $A \tilde{\otimes} C := A \overset{\Gamma}{\otimes} C/I$ is a (B, Γ)-algebra again, called the *fiber product* of A and C. Write $a \tilde{\otimes} c$ for the image of an element $a \otimes c$ in $A \tilde{\otimes} C$.

The product $(A, C) \mapsto A \tilde{\otimes} C$ is functorial, associative and unital in the following sense.

2.1.3. Lemma. *i) For all morphisms of (B, Γ)-algebras $\pi^1 \colon A^1 \to C^1$, $\pi^2 \colon A^2 \to C^2$, there exists a morphism $\pi^1 \tilde{\otimes} \pi^2 \colon A^1 \tilde{\otimes} A^2 \to C^1 \tilde{\otimes} C^2$, $a_1 \tilde{\otimes} a_2 \mapsto \pi^1(a_1) \tilde{\otimes} \pi^2(a_2)$.*

ii) For all (B, Γ)-algebras A, C, D, there is an isomorphism $(A \tilde{\otimes} C) \tilde{\otimes} D \to A \tilde{\otimes} (C \tilde{\otimes} D)$, $(a \tilde{\otimes} c) \tilde{\otimes} d \mapsto a \tilde{\otimes} (c \tilde{\otimes} d)$.

iii) For each (B, Γ)-algebra A, there exist isomorphisms $(B \rtimes \Gamma) \tilde{\otimes} A \to A$ and $A \tilde{\otimes} (B \rtimes \Gamma) \to A$, given by $b\gamma \tilde{\otimes} a \mapsto r(b)a$ and $a \tilde{\otimes} b\gamma \mapsto s(b)a$, respectively.

Of course, the isomorphisms above are compatible in a natural sense.

2.1.4. Remark. The product $-\tilde{\otimes}-$ is related to the left and right Takeuchi products $-{}_B\times-$ and $-\times_B-$ as follows. Given a B^{ev}-algebra A, we write ${}_\bullet A$ or $A_\bullet$ when we regard A as a B-bimodule via $b \cdot a \cdot b' := r(b)s(b')a$ or $b \cdot a \cdot b' := ar(b)s(b')$, respectively. Then the left and right Takeuchi products of B^{ev}-algebras A and C are the B^{ev}-algebras

$$A_B\times C := \Big\{ \sum_i a_i \underset{B}{\otimes} c_i \in {}_\bullet A \underset{B}{\otimes} {}_\bullet C \;\Big|\; \forall b \in B : \sum_i a_i s_A(b) \underset{B}{\otimes} c_i = \sum_i a_i \underset{B}{\otimes} c_i r_C(b) \Big\},$$
$$A \times_B C := \Big\{ \sum_i a_i \underset{B}{\otimes} c_i \in A_\bullet \underset{B}{\otimes} C_\bullet \;\Big|\; \forall b \in B : \sum_i s_A(b) a_i \underset{B}{\otimes} c_i = \sum_i a_i \underset{B}{\otimes} r_C(b) c_i \Big\},$$

where the multiplication is defined factorwise and the embedding of B^{ev} is given by $b \otimes b' \mapsto r_A(b) \underset{B}{\otimes} s_C(b')$. The assignments $(A, C) \mapsto A_B\times C$ and $(A, C) \mapsto A\times_B C$ extend to bifunctors on the category of B^{ev}-algebras and turn it into a lax monoidal category [6]. The obvious forgetful functor U from (B, Γ)-algebras to B^{ev}-algebras is compatible with these products in the sense that for every pair of (B, Γ)-algebras A, C, the inclusion $A \overset{\Gamma}{\otimes} C \hookrightarrow A \otimes C$ factorizes to inclusions of $A \tilde{\otimes} C$ into $A_B\times C$ and $A\times_B C$, yielding natural transformations from $U(-\tilde{\otimes}-)$ to $U(-)_B\times U(-)$ and $U(-) \times_B U(-)$, respectively.

Briefly, a (B, Γ)-Hopf algebroid is a coalgebra in $\mathbf{Alg}_{(B,\Gamma)}$ equipped with an antipode. To make this definition precise, we need two involutions on $\mathbf{Alg}_{(B,\Gamma)}$. Given an algebra A, we denote by A^{op} its opposite, that is, the same vector space with reversed multiplication.

2.1.5. LEMMA. *There exist automorphisms $(-)^{\mathsf{op}}$ and $(-)^{\mathsf{co}}$ of $\mathbf{Alg}_{(B,\Gamma)}$ such that for each (B, Γ)-algebra A and each morphism $\phi\colon A \to C$, we have $A^{\mathsf{co}} = A$ as an algebra and*

$$\begin{aligned} &(A^{\mathsf{op}})_{\gamma,\gamma'} = A_{\gamma^{-1},\gamma'^{-1}} \text{ for all } \gamma, \gamma' \in \Gamma, && r_{A^{\mathsf{op}}} = r_A, && s_{A^{\mathsf{op}}} = s_A, && \phi^{\mathsf{op}} = \phi,\\ &(A^{\mathsf{co}})_{\gamma,\gamma'} = A_{\gamma',\gamma} \text{ for all } \gamma, \gamma' \in \Gamma, && r_{A^{\mathsf{co}}} = s_A, && s_{A^{\mathsf{co}}} = r_A, && \phi^{\mathsf{co}} = \phi.\end{aligned}$$

Furthermore, $(-)^{\mathsf{op}} \circ (-)^{\mathsf{op}} = \mathrm{id}$, $(-)^{\mathsf{co}} \circ (-)^{\mathsf{co}} = \mathrm{id}$, $(-)^{\mathsf{op}} \circ (-)^{\mathsf{co}} = (-)^{\mathsf{co}} \circ (-)^{\mathsf{op}}$.

2.1.6. REMARK. The automorphisms above are compatible with the monoidal structure as follows. Given (B, Γ)-algebras A, C, there exist isomorphisms $(A\tilde{\otimes}C)^{\mathsf{op}} \to (A^{\mathsf{op}}\tilde{\otimes}C)^{\mathsf{op}}$ and $(A\tilde{\otimes}C)^{\mathsf{co}} \to C^{\mathsf{co}}\tilde{\otimes}A^{\mathsf{co}}$ given by $a\tilde{\otimes}c \mapsto a\tilde{\otimes}c$ and $a\tilde{\otimes}c \mapsto c\tilde{\otimes}a$, respectively. Moreover, $(B \rtimes \Gamma)^{\mathsf{co}} = B \rtimes \Gamma$ and there exists an isomorphism $S^{B\rtimes\Gamma}\colon B \rtimes \Gamma \to (B \rtimes \Gamma)^{\mathsf{op}}$, $b\gamma \mapsto \gamma^{-1}b$, and all of these isomorphisms and the isomorphisms in Lemma 2.1.3 are compatible in a natural sense.

2.1.7. DEFINITION. A (B, Γ)-*Hopf algebroid* is a (B, Γ)-algebra A equipped with morphisms $\Delta\colon A \to A\tilde{\otimes}A$, $\epsilon\colon A \to B \rtimes \Gamma$, and $S\colon A \to A^{\mathsf{co,op}}$ such that the diagrams below commute,

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\tilde{\otimes}A \arrow[d, "\Delta\tilde{\otimes}\,\mathrm{id}"] \\ A\tilde{\otimes}A \arrow[r, "\mathrm{id}\,\tilde{\otimes}\Delta"'] & A\tilde{\otimes}A\tilde{\otimes}A, \end{tikzcd} \qquad \begin{tikzcd} A\tilde{\otimes}A \arrow[d, "\epsilon\tilde{\otimes}\,\mathrm{id}"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"] & A\tilde{\otimes}A \arrow[d, "\mathrm{id}\,\tilde{\otimes}\epsilon"] \\ (B \rtimes \Gamma)\tilde{\otimes}A \arrow[r, "\cong"] & A & A\tilde{\otimes}(B \rtimes \Gamma), \arrow[l, "\cong"'] \end{tikzcd}$$

$$\begin{tikzcd} A\tilde{\otimes}A \arrow[d, "S\tilde{\otimes}\,\mathrm{id}"'] & & A \arrow[ll, "\Delta"'] \arrow[rr, "\Delta"] \arrow[d, "\epsilon"] & & A\tilde{\otimes}A \arrow[d, "\mathrm{id}\,\tilde{\otimes}S"] \\ A^{\mathsf{co,op}}\tilde{\otimes}A \arrow[r, "\check{m}"] & A & B \rtimes \Gamma \arrow[l, "\check{s}"'] \arrow[r, "\hat{r}"] & A & A\tilde{\otimes}A^{\mathsf{co,op}} \arrow[l, "\hat{m}"'] \end{tikzcd}$$

where the linear maps $\hat{m}, \check{m}, \hat{r}, \check{s}$ are given by

$$\hat{m}(a\tilde{\otimes}a') = aa' = \check{m}(a\tilde{\otimes}a'), \qquad \hat{r}(b\gamma) = r(b), \qquad \check{s}(\gamma b) = s(b)$$

for all $a, a' \in A, b \in B, \gamma \in \Gamma$.

A morphism of (B,Γ)-Hopf algebroids $(A, \Delta_A, \epsilon_A, S_A)$, $(C, \Delta_C, \epsilon_C, S_C)$ is a morphism of (B,Γ)-algebras $\pi\colon A \to C$ such that $\Delta_C \circ \pi = (\pi\tilde{\otimes}\pi) \circ \Delta_A$, $\epsilon_C \circ \pi = \epsilon_A$, $S_C \circ \pi = \pi^{\mathsf{co},\mathsf{op}} \circ S_A$. We denote the category of all (B,Γ)-Hopf algebroids by $\mathbf{Hopf}_{(B,\Gamma)}$.

A (B,Γ)-Hopf algebroid reduces to an $\mathfrak{h}$-Hopf algebroid in the sense of [11] when $\mathfrak{h}$ is a commutative Lie algebra, B is the algebra of meromorphic functions on the dual $\mathfrak{h}^*$, and $\Gamma = \mathfrak{h}^*$ acts by shifting the argument. Let us note that the axioms above can be weakened, see [11, Proposition 2.2], but our examples shall automatically satisfy the apparently stronger conditions above.

2.1.8. Example. The (B,Γ)-algebra $B \rtimes \Gamma$ is a (B,Γ)-Hopf algebroid, where $\Delta(b\gamma) = b\gamma\tilde{\otimes}\gamma = \gamma\tilde{\otimes}b\gamma$, $\epsilon(b\gamma) = b\gamma$, and $S(b\gamma) = \gamma^{-1}b$ for all $b \in B, \gamma \in \Gamma$.

Let us comment on some straightforward properties of (B,Γ)-Hopf algebroids:

2.1.9. Remarks. Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid.

i) If $\gamma \neq \gamma'$, then $\epsilon(A_{\gamma,\gamma'}) = 0$ because $(B \rtimes \Gamma)_{\gamma,\gamma'} = 0$.
ii) We have $\Delta(A)(1\tilde{\otimes}A_{e,*}) = A\tilde{\otimes}A = (A_{*,e}\tilde{\otimes}1)\Delta(A)$, where $A_{e,*} = \sum_\gamma A_{e,\gamma}$ and $A_{*,e} = \sum_\gamma A_{\gamma,e}$. Indeed, by [15, Proposition 1.3.7],

$$\sum(xS(y_{(1)})\tilde{\otimes}1)\Delta(y_{(2)}) = x\tilde{\otimes}y = \sum \Delta(x_{(1)})(1\tilde{\otimes}S(x_{(2)})y)$$

for all $x \in A_{\gamma,\gamma'}, y \in A_{\gamma',\gamma''}$, $\gamma, \gamma', \gamma'' \in \Gamma$, where $\sum x_{(1)}\tilde{\otimes}x_{(2)} = \Delta(x)$ and $\sum y_{(1)}\tilde{\otimes}y_{(2)} = \Delta(y)$.

(B,Γ)-Hopf algebroids fit into the general definition of Hopf algebroids [4] as follows.

2.1.10. Remark. Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid. Denote by ${}_\bullet\epsilon$ and $\epsilon_\bullet$ the compositions of $\epsilon\colon A \to B \rtimes \Gamma$ with the linear maps $B \rtimes \Gamma \to B$ given by $b\gamma \mapsto b$ and $\gamma b \mapsto b$, respectively, and denote by ${}_\bullet\Delta$ and $\Delta_\bullet$ the compositions of Δ with the natural inclusions $A\tilde{\otimes}A \to {}_\bullet A \underset{B}{\otimes} {}_\bullet A$ and $A\tilde{\otimes}A \to A_\bullet \underset{B}{\otimes} A_\bullet$, respectively (see Remark 2.1.4).

i) The maps ${}_\bullet\epsilon, \epsilon_\bullet\colon A \to B$ will in general not be homomorphisms, but satisfy

$${}_\bullet\epsilon(ar({}_\bullet\epsilon(a'))) = {}_\bullet\epsilon(aa') = {}_\bullet\epsilon(as({}_\bullet\epsilon(a'))), \quad \epsilon_\bullet(r(\epsilon_\bullet(a))a') = \epsilon_\bullet(aa') = \epsilon_\bullet(s(\epsilon_\bullet(a))a')$$

for all $a, a' \in A$. Indeed, since $\epsilon(a) = {}_\bullet\epsilon(a)\partial_a$ for all homogeneous $a' \in A$,

$${}_\bullet\epsilon(aa')\partial_{aa'} = {}_\bullet\epsilon(a)\partial_a \cdot {}_\bullet\epsilon(a')\partial_{a'} = \partial_a({}_\bullet\epsilon(a')){}_\bullet\epsilon(a)\partial_a\partial_{a'} = {}_\bullet\epsilon(ar({}_\bullet\epsilon(a')))\partial_{aa'}$$

for all homogeneous $a, a' \in A$, and the remaining equations follow similarly.
ii) One easily verifies that $({}_\bullet A, {}_\bullet\Delta, {}_\bullet\epsilon)$ and $(A_\bullet, \Delta_\bullet, \epsilon_\bullet)$ are B-corings, ${}_\bullet\mathcal{A} := (A, {}_\bullet\Delta, {}_\bullet\epsilon)$ is a left B-bialgebroid, and $\mathcal{A}_\bullet := (A^{\mathsf{co}}, \Delta_\bullet, \epsilon_\bullet)$ is a right B-bialgebroid in the sense of [4]. Using the relations $\check{s} \circ \epsilon = s \circ \epsilon_\bullet$ and $\hat{r} \circ \epsilon = r \circ {}_\bullet\epsilon$, one furthermore finds that $(\mathcal{A}_\bullet, {}_\bullet\mathcal{A}, S)$ is a Hopf algebroid over B. To make the match with Definition 4.1 in [4], one has to take $H, s_L, t_L, \Delta_L, \epsilon_L, s_R, t_R, \Delta_R, \epsilon_R, S$ equal to $A, s, r, {}_\bullet\Delta, {}_\bullet\epsilon, r, s, \Delta_\bullet, \epsilon_\bullet, S$, respectively.

Let B and C be commutative algebras with a left action of Γ and let $\phi\colon B \to C$ be a Γ-equivariant homomorphism. We then obtain base change functors $\phi_*\colon \mathbf{Alg}_{(B,\Gamma)} \to \mathbf{Alg}_{(C,\Gamma)}$ and $\phi_*\colon \mathbf{Hopf}_{(B,\Gamma)} \to \mathbf{Hopf}_{(C,\Gamma)}$ as follows. Let A be a (B,Γ)-algebra. Regard C as a B-module via ϕ, and A as a B-bimodule, where $b\cdot a\cdot b' = r(b)as(b')$ for all $b, b' \in B$, $a \in A$. Then the vector space $\phi_*(A) := C \underset{B}{\otimes} A \underset{B}{\otimes} C$ carries the structure of a (C,Γ)-algebra such that

$$(c \underset{B}{\otimes} a \underset{B}{\otimes} d)(c' \underset{B}{\otimes} a' \underset{B}{\otimes} d') = c\partial^r_a(c') \underset{B}{\otimes} aa' \underset{B}{\otimes} (\partial^s_{a'})^{-1}(d)d',$$

$$\partial_{c \underset{B}{\otimes} a \underset{B}{\otimes} d} = \partial_a, \quad (r \times s)(c \otimes c') = c \underset{B}{\otimes} 1 \underset{B}{\otimes} c' \quad \text{for all } c, c', d, d' \in C, a, a' \in A.$$

Every morphism of (B,Γ)-algebras $\pi\colon A \to A'$ evidently yields a morphism of (C,Γ)-algebras $\phi_*(\pi)\colon \phi_*(A) \to \phi_*(A')$, $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto c \underset{B}{\otimes} \pi(a) \underset{B}{\otimes} c'$, and the assignments $A \mapsto \phi_*(A)$ and $\phi \mapsto \phi_*(\pi)$ form a functor $\phi_*\colon \mathbf{Alg}_{(B,\Gamma)} \to \mathbf{Alg}_{(C,\Gamma)}$.

2.1.11. Lemma. *i) There exists a morphism of (C,Γ)-algebras $\phi^{(0)}\colon \phi_*(B \rtimes \Gamma) \to C \rtimes \Gamma$, $c \underset{B}{\otimes} b\gamma \underset{B}{\otimes} c' \mapsto c\phi(b)\gamma c' = c\phi(b)\gamma(c')\gamma$.*

ii) For all (B,Γ)-algebras A, D, there exists a unique morphism $\phi^{(2)}_{A,D}\colon \phi_*(A\tilde{\otimes}D) \to \phi_*(A)\tilde{\otimes}\phi_*(D)$, $c \underset{B}{\otimes} (a\tilde{\otimes}d) \underset{B}{\otimes} c' \mapsto (c \underset{B}{\otimes} a \underset{B}{\otimes} 1)\tilde{\otimes}(1 \underset{B}{\otimes} d \underset{B}{\otimes} c')$.

2.1.12. Proposition. *Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid. Then $\phi_*(A)$ is (C,Γ)-Hopf algebroid with respect to the morphisms*

i) $\Delta'\colon \phi_*(A) \xrightarrow{\phi_*(\Delta)} \phi_*(A\tilde{\otimes}A) \xrightarrow{\phi^{(2)}_{A,A}} \phi_*(A)\tilde{\otimes}\phi_*(A)$, given by $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto \sum_i (c \underset{B}{\otimes} a'_i \underset{B}{\otimes} 1)\tilde{\otimes}(1 \underset{B}{\otimes} a''_i \underset{B}{\otimes} c')$ whenever $\Delta(a) = \sum a'_i\tilde{\otimes}a''_i$;

ii) $\epsilon'\colon \phi_*(A) \xrightarrow{\phi_*(\epsilon)} \phi_*(B \rtimes \Gamma) \xrightarrow{\phi^{(0)}} C \rtimes \Gamma$, given by $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto \sum_i c\phi(b_i)\gamma_i c'$ whenever $\epsilon(a) = \sum_i b_i\gamma_i$;

iii) $S'\colon \phi_*(A) \to (\phi_*A)^{\mathsf{co,op}}$ given by $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto c' \underset{B}{\otimes} S(a) \underset{B}{\otimes} c$.

The assignments $(A, \Delta, \epsilon, S) \mapsto (\phi_*(A), \Delta', \epsilon', S')$ as above and $\pi \mapsto \phi_*(\pi)$ evidently form a functor $\phi_*\colon \mathbf{Hopf}_{(B,\Gamma)} \to \mathbf{Hopf}_{(C,\Gamma)}$.

The preceding definitions and results extend to $*$-algebras as follows. Assume that B is a $*$-algebra and that Γ preserves its involution.

A (B,Γ)-$*$-algebra is a (B,Γ)-algebra with an involution that is compatible with the grading and the involution on B, and a morphism of (B,Γ)-$*$-algebras is a morphism of (B,Γ)-algebra that preserves the involution. We denote by $*$-$\mathbf{Alg}_{(B,\Gamma)}$ the category of all (B,Γ)-$*$-algebras. This subcategory of $\mathbf{Alg}_{(B,\Gamma)}$ is monoidal because the crossed product $B \rtimes \Gamma$ is a (B,Γ)-$*$-algebra with respect to the involution given by $(b\gamma)^* = \gamma^{-1}b^*$, and for all (B,Γ)-$*$-algebras A, C, the fiber product $A\tilde{\otimes}C$ is a (B,Γ)-$*$-algebra with respect to the involution given by $(a\tilde{\otimes}c)^* = a^*\tilde{\otimes}c^*$.

2.1.13. Definition. A (B,Γ)-Hopf $*$-algebroid is a (B,Γ)-Hopf algebroid (A, Δ, ϵ, S) where A is a (B,Γ)-$*$-algebra and Δ and ϵ are morphisms of (B,Γ)-$*$-algebras. A morphism of (B,Γ)-Hopf $*$-algebroids is a morphism of the underlying (B,Γ)-Hopf algebroid and (B,Γ)-$*$-algebras. We denote by $\mathbf{Hopf}^*_{(B,\Gamma)}$ the category of all (B,Γ)-Hopf $*$-algebroids.

2.1.14. REMARK. If (A,Δ,ϵ,S) is a (B,Γ)-Hopf $*$-algebroid, then $*\circ S\circ *\circ S=\mathrm{id}$; see [11, Lemma 2.9].

We denote by $\overline{A}$ the conjugate algebra of a complex algebra A; this is the set A with conjugated scalar multiplication and the same addition and multiplication. Thus, the involution of a $*$-algebra A is an automorphism $A\to\overline{A}^{\mathsf{op}}$.

2.1.15. LEMMA. *The category* $*$-$\mathbf{Alg}_{(B,\Gamma)}$ *has an automorphism* $\overline{(-)}$ *such that for every* (B,Γ)-$*$-*algebra* A *and every morphism of* (B,Γ)-$*$-*algebras* $\phi\colon A\to C$,

$$(\overline{A})_{\gamma,\gamma'}=\overline{A_{\gamma,\gamma'}} \text{ for all } \gamma,\gamma'\in\Gamma, \qquad r_{\bar{A}}=r_A\circ *, \qquad s_{\bar{A}}=s_A\circ *, \qquad \overline{\phi}=\phi.$$

Furthermore, $\overline{(-)}\circ\overline{(-)}=\mathrm{id}$, $\overline{(-)}\circ(-)^{\mathsf{op}}=(-)^{\mathsf{op}}\circ\overline{(-)}$, $\overline{(-)}\circ(-)^{\mathsf{co}}=(-)^{\mathsf{co}}\circ\overline{(-)}$.

2.1.16. REMARK. There exists an isomorphism $B\rtimes\Gamma\to\overline{B\rtimes\Gamma}$, $b\gamma\mapsto b^*\gamma$, and for each pair of (B,Γ)-$*$-algebras A,C, there exists an isomorphism $\overline{A\tilde{\otimes}C}\to\overline{A}\tilde{\otimes}\overline{C}$, $a\tilde{\otimes}c\mapsto a\tilde{\otimes}c$.

Let also C be a commutative $*$-algebra with a left action of Γ and let $\phi\colon B\to C$ be a Γ-equivariant $*$-homomorphism. Then for every (B,Γ)-$*$-algebra A, the (C,Γ)-algebra $\phi_*(A)$ is a (C,Γ)-$*$-algebra with respect to the involution given by $(c\underset{B}{\otimes}a\underset{B}{\otimes}c')^*=(\partial^r_a)^{-1}(c)^*\underset{B}{\otimes}a^*\underset{B}{\otimes}\partial^s_a(c')^*$, and we obtain a functor $\phi_*\colon *\text{-}\mathbf{Alg}_{(B,\Gamma)}\to\mathbf{Alg}^*_{(C,\Gamma)}$. Likewise, we obtain a functor $\phi_*\colon\mathbf{Hopf}^*_{(B,\Gamma)}\to\mathbf{Hopf}^*_{(C,\Gamma)}$.

2.2. The case of a trivial base algebra. Assume for this subsection that $B=\mathbb{C}$ equipped with the trivial action of Γ. Then the category of all $(\mathbb{C},\Gamma)$-Hopf algebroids is equivalent to the comma category of all Hopf algebras over $\mathbb{C}\Gamma$ as follows.

Recall that the group algebra $\mathbb{C}\Gamma$ is a Hopf $*$-algebra with involution, comultiplication, counit and antipode given by $\gamma^*=\gamma^{-1}$, $\Delta_{\mathbb{C}\Gamma}(\gamma)=\gamma\otimes\gamma$, $\epsilon_{\mathbb{C}\Gamma}(\gamma)=1$, $S_{\mathbb{C}\Gamma}(\gamma)=\gamma^{-1}$ for all $\gamma\in\Gamma\subset\mathbb{C}\Gamma$. Objects of the comma category $\mathbf{Hopf}_{\mathbb{C}\Gamma}$ are pairs consisting of a Hopf algebra A and a morphism of Hopf algebras $A\to\mathbb{C}\Gamma$, and morphisms from (A,π_A) to (C,π_C) are all morphisms $A\xrightarrow{\phi}C$ such that $\pi_C\circ\phi=\pi_A$. Likewise, we define the comma category $\mathbf{Hopf}^*_{\mathbb{C}\Gamma}$ of Hopf $*$-algebras over $\mathbb{C}\Gamma$.

Note that a $(\mathbb{C},\Gamma)$-algebra is just a $\Gamma\times\Gamma$-graded algebra and $A\tilde{\otimes}C=A\overset{\Gamma}{\otimes}C\subseteq A\otimes C$ for all $(\mathbb{C},\Gamma)$-algebras A,C. Moreover, $\mathbb{C}\rtimes\Gamma=\mathbb{C}\Gamma$, and for every $(\mathbb{C},\Gamma)$-algebra A, the isomorphisms $(\mathbb{C}\rtimes\Gamma)\tilde{\otimes}A\to A$ and $A\tilde{\otimes}(\mathbb{C}\rtimes\Gamma)\to A$ are equal to $\epsilon_{\mathbb{C}\Gamma}\otimes\mathrm{id}$ and $\mathrm{id}\otimes\epsilon_{\mathbb{C}\Gamma}$.

2.2.1. LEMMA. *Let* (A,Δ,ϵ,S) *be a* $(\mathbb{C},\Gamma)$-*Hopf algebroid and let* $\epsilon':=\epsilon_{\mathbb{C}\Gamma}\circ\epsilon\colon A\to\mathbb{C}$. *The* (A,Δ,ϵ',S) *is a Hopf algebra and* $\epsilon\colon A\to\mathbb{C}\Gamma$ *is a morphism of Hopf algebras.*

Proof. The preceding observations easily imply that (A,Δ,ϵ',S) is a Hopf algebra. To see that ϵ is a morphism of Hopf algebras, use the fact that Δ,ϵ,S are $\Gamma\times\Gamma$-graded. ∎

2.2.2. LEMMA. *Let* (A,Δ,ϵ,S) *be a Hopf algebra with a morphism* $\pi\colon A\to\mathbb{C}\Gamma$. *Then* A *is a* $(\mathbb{C},\Gamma)$-*algebra with respect to the grading given by* $A_{\gamma,\gamma'}=\{a\in A:(\pi\otimes\mathrm{id}\otimes\pi)(\Delta(a))=\gamma\otimes a\otimes\gamma'\}$ *for all* $\gamma,\gamma'\in\Gamma$, *and* (A,Δ,π,S) *is a* $(\mathbb{C},\Gamma)$-*Hopf algebroid.*

Proof. The formula above evidently defines a $\Gamma \times \Gamma$-grading on A. Coassociativity of Δ implies that $\Delta(A) \subseteq A\tilde{\otimes}A$. The remark preceding Lemma 2.2.1 and the relation $\epsilon_{\mathbb{C}\Gamma} \circ \pi = \epsilon$ imply $(\pi \tilde{\otimes} \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \tilde{\otimes} \pi) \circ \Delta$. Finally, in the notation of Definition 2.1.7, $\check{m} \circ (S \tilde{\otimes} \mathrm{id}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = \check{s} \circ \pi$ and similarly $\hat{m} \circ (\mathrm{id} \tilde{\otimes} S) \circ \Delta = \hat{r} \circ \pi$. ∎

Putting everything together, one easily verifies:

2.2.3. PROPOSITION. *There exists an equivalence of categories* $\mathbf{Hopf}_{(\mathbb{C},\Gamma)} \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathbf{Hopf}_{\mathbb{C}\Gamma}$, *where* $\mathbf{F}(A, \Delta, \epsilon, S) = ((A, \Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S), \epsilon)$, $\mathbf{F}\phi = \phi$ *and* $\mathbf{G}((A, \Delta, \epsilon, S), \pi) = (A, \Delta, \pi, S)$ *with the grading on A defined as in Lemma 2.2.2, and* $\mathbf{G}\phi = \phi$. *Likewise, there exists an equivalence* $\mathbf{Hopf}^*_{(\mathbb{C},\Gamma)} \rightleftarrows \mathbf{Hopf}^*_{\mathbb{C}\Gamma}$.

Let us next consider the base change from $\mathbb{C}$ to a commutative algebra C along the unital inclusion $\phi\colon \mathbb{C} \to C$ for a $(\mathbb{C}, \Gamma)$-Hopf algebroid (A, Δ, ϵ, S).

2.2.4. REMARK. The action of Γ on C and the morphism $\epsilon\colon A \to \mathbb{C}\Gamma$ turn C into a left module algebra over the Hopf algebra $(A, \Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S)$, and $\phi_*(A, \Delta, \epsilon, S)$ coincides with the Hopf algebroids considered in [4, §3.4.6] and [10, Theorem 3.1], and is closely related to the quantum transformation groupoid considered in [14, Example 2.6].

Assume that C is an algebra of functions on Γ on which Γ acts by left translations.

2.2.5. PROPOSITION. *Define* $m\colon A \to \mathrm{End}(A)$ *and* $m_r, m_s\colon C \to \mathrm{End}(A)$ *by* $m(a')a = a'a$, $m_r(c)a = c(\partial^r_a)a$, $m_s(c)a = c(\partial^s_a)a$ *for all* $a, a' \in A$, $c \in C$. *Then there exists a homomorphism* $\lambda\colon \phi_*(A) \to \mathrm{End}(A)$, $c \otimes a \otimes c' \mapsto m_r(c)m(a)m_s(c')$, *and λ is injective if* $aA_{\gamma,\gamma'} \neq 0$ *for all non-zero* $a \in A$ *and all* $\gamma, \gamma' \in \Gamma$.

Proof. First, note that $m(a')m_r(c)a = a'c(\partial^r_a)a = c((\partial^r_{a'})^{-1}\partial^r_{a'a})a'a = m_r(\partial^r_{a'}(c))m(a)a$ and likewise $m(a')m_s(c) = m_s(\partial^s_{a'}(c))m(a')$ for all $a, a' \in A$, $c \in C$. The existence of λ follows. Assume that $aA_{\gamma,\gamma'} \neq 0$ for all non-zero $a \in A$ and all $\gamma, \gamma' \in \Gamma$. Let $d := \sum_i c_i \otimes a_i \otimes c'_i \in \phi_*(A)$ be non-zero, where all a_i are homogeneous. Identifying $C \otimes A \otimes C$ with a space of A-valued functions on $\Gamma \times \Gamma$ and using the assumption, we first find $\gamma, \gamma' \in \Gamma$ such that $a := \sum_i c_i(\partial^r_{a_i}\gamma)a_i c'_i(\gamma')$ is non-zero, and then an $a' \in A_{\gamma,\gamma'}$ such that $\lambda(d)a' = aa' \neq 0$. ∎

2.2.6. REMARK. Regard elements of C as functionals on $\mathbb{C}\Gamma$ via $c(\sum_i b_i\gamma_i) = \sum_i b_i c(\gamma_i)$. Then $m_r(c)a = (c \circ \epsilon \otimes \mathrm{id})(\Delta(a))$, $m_s(c)a = (\mathrm{id} \otimes c \circ \epsilon)(\Delta(a))$ for all $c \in C$, $a \in A$.

2.2.7. EXAMPLE. Let G be a compact Lie group, $\mathcal{O}(G)$ its Hopf algebra of representative functions [16, §1.2] and $T \subseteq G$ a torus of rank d. We now apply Proposition 2.2.5, where

- $A = \mathcal{O}(G)$, regarded as a Hopf $(\mathbb{C}, \hat{T})$-algebroid as in Lemma 2.2.2 using the homomorphism $\pi\colon \mathcal{O}(G) \to \mathcal{O}(T)$ induced from the inclusion $T \subseteq G$, and the isomorphism $\mathcal{O}(T) \cong \mathbb{C}\hat{T}$,
- $C = U\mathfrak{t}$ is the enveloping algebra of the Lie algebra $\mathfrak{t}$ of T, regarded as a polynomial algebra of functions on $\hat{T}$ such that $X(\chi) = \frac{d}{dt}\big|_{t=0}\chi(e(tX))$, where $e\colon \mathfrak{t} \to T$ denotes the exponential map.

If we regard $U\mathfrak{t}$ as functionals on the algebra $\mathbb{C}\hat{T} \cong \mathcal{O}(T)$ as in Remark 2.2.6, then $X(f) = \frac{d}{dt}\big|_{t=0} f(e(tX))$ and hence $m_r, m_s \colon U\mathfrak{t} \to \mathrm{End}(\mathcal{O}(G))$ are given by

$$(m_r(X)a)(x) = \frac{d}{dt}\Big|_{t=0} a(e(tX)x), \qquad (m_s(X)a)(x) = \frac{d}{dt}\Big|_{t=0} a(xe(tX))$$

for all $X \in \mathfrak{t}$, $a \in \mathcal{O}(G)$, $x \in G$. Thus $\lambda(\mathcal{O}(G)) \subseteq \mathrm{End}(\mathcal{O}(G))$ is the algebra generated by multiplication operators for functions in $\mathcal{O}(G)$ and by left and right differentiation operators along $T \subseteq G$.

If G is connected, then $\mathcal{O}(G)$ has no zero-divisors and hence λ is injective as soon as for all $\chi, \chi' \in \hat{T}$, there exists some non-zero $a \in \mathcal{O}(G)$ such that $a(xyz) = \chi(x)a(y)\chi'(z)$ for all $x, z \in T$ and $y \in G$.

2.3. Intertwiners for (B, Γ)-algebras. In this subsection, we study relations of the form used to define the free orthogonal and free unitary dynamical quantum groups $A^B_{\mathrm{o}}(\nabla, F)$ and $A^B_{\mathrm{u}}(\nabla, F)$, and show that such relations admit a number of natural transformations. Conceptually, these relations express that certain matrices are intertwiners or morphisms of corepresentations, and the transformations correspond to certain functors of corepresentation categories. Although elementary, these observations provide short and systematic proofs for the main results in the following subsection.

Regard $M_n(B)$ as a subalgebra of $M_n(B \rtimes \Gamma)$, and let A be a (B, Γ)-algebra. Given a linear map $\phi \colon A \to C$ between algebras, we denote by $\phi_n \colon M_n(A) \to M_n(C)$ its entry-wise extension to $n \times n$-matrices.

2.3.1. Definition. A matrix $u \in M_n(A)$ is *homogeneous* if there are $\gamma_1, \ldots, \gamma_n \in A$ such that $u_{ij} \in A_{\gamma_i, \gamma_j}$ for all i, j. In that case, let $\partial_{u,i} := \gamma_i$ for all i and $\partial_u := \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in M_n(B \rtimes \Gamma)$. An *intertwiner* for homogeneous matrices $u, v \in M_n(A)$ is an $F \in \mathrm{GL}_n(B)$ satisfying $\partial_v F \partial_u^{-1} \in M_n(B)$ and $r_n(\partial_v F \partial_u^{-1})u = vs_n(F)$. We write such an intertwiner as $u \xrightarrow{F} v$ and let $\hat{F} := \partial_v F \partial_u^{-1}$ if u, v are understood.

If $u \xrightarrow{F} v$ and $v \xrightarrow{G} w$ are intertwiners, then evidently so are $v \xrightarrow{F^{-1}} u$ and $u \xrightarrow{GF} w$.

2.3.2. Definition. We denote by $\mathcal{R}_n(A)$ the category of all homogeneous matrices in $M_n(A)$ together with their intertwiners as morphisms, and by $\mathcal{R}_n^{\times}(A)$ and $\mathcal{R}_n^{\times\mathsf{T}}(A)$ the full subcategories formed by all homogeneous v in $\mathrm{GL}_n(A)$ or $\mathrm{GL}_n(A)^{\mathsf{T}}$, respectively.

Evidently, $\mathcal{R}_n(A)$ is a groupoid, and the assignment $A \mapsto \mathcal{R}_n(A)$ extends to a functor from (B, Γ)-algebras to groupoids.

We shall make frequent use of the following straightforward relations.

2.3.3. Lemma. *Let $u, v \in M_n(A)$ be homogeneous, $F \in M_n(B)$ and $\hat{F} = \partial_v F \partial_u^{-1}$. Then*

$$\hat{F} \in M_n(B) \;\Leftrightarrow\; (F_{ij} = 0 \text{ whenever } \partial_{v,i} \neq \partial_{u,j}).$$

Assume that these condition holds. Then $\hat{F} = (\partial_{v,i}(F_{ij}))_{i,j} = (\partial_{u,j}(F_{ij}))_{i,j}$ and

$$\hat{F}^{\mathsf{T}} = \partial_u F^{\mathsf{T}} \partial_v^{-1}, \qquad (\partial_v F)^{-\mathsf{T}} = F^{-\mathsf{T}} \partial_u^{-1}, \qquad (F\partial_u^{-1})^{-\mathsf{T}} = \partial_v F^{-\mathsf{T}}.$$

If B is a $*$-algebra and Γ preserves the involution, then $\overline{\partial_v F} = \overline{F}\partial_u^{-1}$ and $\overline{F\partial_u} = \partial_v^{-1}\overline{F}$.

Given $u, v \in M_n(A)$ such that $\partial^s_{u_{ik}} = \partial^r_{v_{kj}}$ for all i, k, j, let $u \tilde{\boxtimes} v := (\sum_k u_{ik} \tilde{\otimes} v_{kj})_{i,j} \in M_n(A \tilde{\otimes} A)$.

2.3.4. LEMMA. *There exist functors*

$$\begin{aligned}
\boldsymbol{\epsilon}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(B \rtimes \Gamma), \qquad & u \mapsto \partial_u, \qquad && (u \xrightarrow{F} v) \mapsto (\partial_u \xrightarrow{F} \partial_v),\\
\boldsymbol{\Delta}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A\tilde{\otimes}A), \qquad & u \mapsto u\tilde{\boxtimes}u, \qquad && (u \xrightarrow{F} v) \mapsto (u\tilde{\boxtimes}u \xrightarrow{F} v\tilde{\boxtimes}v),\\
(-)^{\mathsf{op}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{op}}), \qquad & u \mapsto u^{\mathsf{op}} := u, \qquad && (u \xrightarrow{F} v) \mapsto (u^{\mathsf{op}} \xrightarrow{\hat{F}} v^{\mathsf{op}}),
\end{aligned}$$

and $\partial_{u\tilde{\boxtimes}u} = \partial_u$, $\partial_{u^{\mathsf{op}}} = \partial_u^{-1}$ *for all* $u \in \mathcal{R}_n(A)$.

Proof. For each $u \in \mathcal{R}_n(A)$, the matrices $\partial_u, u\tilde{\boxtimes}u, u^{\mathsf{op}}$ evidently are homogeneous, and for every intertwiner $u \xrightarrow{F} v$, Lemma 2.3.3 implies

$$\begin{aligned}
r_n(\hat{F})u\tilde{\boxtimes}u &= vs_n(F)\tilde{\boxtimes}u = v\tilde{\boxtimes}r_n(\hat{F})u = v\tilde{\boxtimes}vs_n(F),\\
r_n(\partial_{v^{\mathsf{op}}}\hat{F}\partial_{u^{\mathsf{op}}}^{-1})u^{\mathsf{op}} &= r_n(F)^{\mathsf{op}}u^{\mathsf{op}} = (r_n(\hat{F})u)^{\mathsf{op}} = (vs_n(F))^{\mathsf{op}} = v^{\mathsf{op}}s_n(\hat{F})^{\mathsf{op}}.
\end{aligned}$$

Functoriality of the assignments is evident. ∎

2.3.5. LEMMA. *There exist contravariant functors*

$$\begin{aligned}
(-)^{\mathsf{T},\mathsf{co}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{co}}), \qquad & u \mapsto u^{\mathsf{T},\mathsf{co}} := u^{\mathsf{T}}, \qquad && (u \xrightarrow{F} v) \mapsto (v^{\mathsf{T},\mathsf{co}} \xrightarrow{F^{\mathsf{T}}} u^{\mathsf{T},\mathsf{co}}),\\
(-)^{-\mathsf{co}}\colon \mathcal{R}_n^{\times}(A) \to \mathcal{R}_n(A^{\mathsf{co}}), \qquad & u \mapsto u^{-\mathsf{co}} := u^{-1}, \qquad && (u \xrightarrow{F} v) \mapsto (v^{-\mathsf{co}} \xrightarrow{\hat{F}^{-1}} u^{-\mathsf{co}}),
\end{aligned}$$

and $\partial_{u^{\mathsf{T},\mathsf{co}}} = \partial_u$ *and* $\partial_{u^{-\mathsf{co}}} = \partial_u^{-1}$ *for all* u.

Proof. If $u \in \mathcal{R}_n(A)$, then $u^{\mathsf{T},\mathsf{co}}$ evidently is homogeneous as claimed. Assume $u \in \mathcal{R}_n^{\times}(A)$. We claim that $u^{-\mathsf{co}}$ is homogeneous and $\partial_{u^{-\mathsf{co}}} = \partial_u^{-1}$. For each i,j, let w_{ij} be the homogeneous part of $(u^{-1})_{ij}$ of degree $(\partial_{u,j}^{-1}, \partial_{u,i}^{-1})$. Then $\sum_l u_{il}w_{lj}$ is homogeneous of degree $(\partial_{u,i}\partial_{u,j}^{-1}, e)$ and coincides with the homogeneous part of the sum $\sum_l u_{il}(u^{-1})_{lj}$ of the same degree for each i,j. Hence, $uw = uu^{-1}$ and the claim follows.

Let $u \xrightarrow{F} v$ be an intertwiner. Using Lemma 2.3.3, one easily verifies that

$$\begin{aligned}
s_n(\partial_{u^{\mathsf{T},\mathsf{co}}}F^{\mathsf{T}}\partial_{v^{\mathsf{T},\mathsf{co}}}^{-1})v^{\mathsf{T}} &= s_n(\hat{F})^{\mathsf{T}}v^{\mathsf{T}} = (vs_n(F))^{\mathsf{T}} = (r_n(\hat{F})u)^{\mathsf{T}} = u^{\mathsf{T}}r_n(F^{\mathsf{T}}),\\
s_n(\partial_{u^{-\mathsf{co}}}\hat{F}^{-1}\partial_{v^{-\mathsf{co}}}^{-1})v^{-1} &= s_n(F^{-1})v^{-1} = u^{-1}r_n(\hat{F}^{-1}).
\end{aligned}$$

Finally, functoriality of the assignments is easily checked. ∎

Forming suitable compositions, we obtain further co- or contravariant functors

$$(-)^{-\mathsf{T}} = (-)^{\mathsf{T},\mathsf{co}} \circ (-)^{-\mathsf{co}}\colon \mathcal{R}_n^{\times}(A) \to \mathcal{R}_n^{\times\mathsf{T}}(A), \qquad \begin{cases} u \mapsto u^{-\mathsf{T}} := (u^{-1})^{\mathsf{T}},\\ (u \xrightarrow{F} v) \mapsto (u^{-\mathsf{T}} \xrightarrow{\hat{F}^{-\mathsf{T}}} v^{-\mathsf{T}}),\end{cases}$$

$$(-)^{-\perp} = (-)^{-\mathsf{co}} \circ (-)^{\mathsf{T},\mathsf{co}}\colon \mathcal{R}_n^{\times\mathsf{T}}(A) \to \mathcal{R}_n^{\times}(A), \qquad \begin{cases} u \mapsto u^{-\perp} := (u^{\mathsf{T}})^{-1},\\ (u \xrightarrow{F} v) \mapsto (u^{-\perp} \xrightarrow{\hat{F}^{-\mathsf{T}}} v^{-\perp})\end{cases}$$

and

$$(-)^{-\mathsf{co},\mathsf{op}} = (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}}\colon \mathcal{R}_n^{\times}(A) \to \mathcal{R}_n(A^{\mathsf{co},\mathsf{op}}), \quad \begin{cases} u \mapsto (u^{-\mathsf{co}})^{\mathsf{op}},\\ (u \xrightarrow{F} v) \mapsto (v^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} u^{-\mathsf{co},\mathsf{op}}),\end{cases}$$

where $\partial_{u^{-\mathsf{co},\mathsf{op}}} = \partial_u$ and $\partial_{u^{-\mathsf{T}}} = \partial_{u^{-\perp}} = \partial_u^{-1}$ for all u.

2.3.6. Lemma. *The following relations hold:*

i) $(-)^{\mathsf{op}} \circ (-)^{-\mathsf{T}} = (-)^{-\perp} \circ (-)^{\mathsf{op}}$, ii) $(-)^{-\mathsf{T}} \circ (-)^{\mathsf{op}} = (-)^{\mathsf{op}} \circ (-)^{-\perp}$,

iii) $(-)^{-\mathsf{T}} \circ \boldsymbol{\Delta} = \boldsymbol{\Delta} \circ (-)^{-\mathsf{T}}$, iv) $(-)^{-\mathsf{T}} \circ (-)^{-\mathsf{co},\mathsf{op}} = (-)^{-\mathsf{co},\mathsf{op}} \circ (-)^{-\mathsf{T}}$.

Proof. i) We first check that the compositions agree on objects. Let us write v^{op} if we regard $v \in M_n(A)$ as an element of $M_n(A^{\mathsf{op}})$. Then map $M_n(A) \to M_n(A^{\mathsf{op}})$ given by $v \mapsto (v^{\mathsf{T}})^{\mathsf{op}} = (v^{\mathsf{op}})^{\mathsf{T}}$ is an antihomomorphism and hence $(v^{-\mathsf{T}})^{\mathsf{op}} = (v^{\mathsf{T},\mathsf{op}})^{-1} = (v^{\mathsf{op}})^{-\perp}$ for all $v \in \mathrm{GL}_n(A)$. The compositions also agree on morphisms because for every intertwiner $u \xrightarrow{F} v$, we have $\partial_{v^{-\mathsf{T}}}(\partial_v F \partial_u^{-1})^{-\mathsf{T}} \partial_{u^{-\mathsf{T}}}^{-1} = \partial_v^{-\mathsf{T}} \partial_v F^{-\mathsf{T}} \partial_u^{-1} \partial_u = F^{-\mathsf{T}}$.

ii) This equation follows similarly like i).

iii) Let $u \in \mathcal{R}_n^{\times}(A)$. Then $(u \tilde{\boxtimes} u)^{-\mathsf{T}} = u^{-\mathsf{T}} \tilde{\boxtimes} u^{-\mathsf{T}}$ because

$$\sum_k (u\tilde{\boxtimes}u)_{ik}(u^{-\mathsf{T}}\tilde{\boxtimes}u^{-\mathsf{T}})_{jk} = \sum_{k,l,m} u_{il}(u^{-1})_{mj} \tilde{\otimes} u_{lk}(u^{-1})_{km} = \delta_{i,j} 1 \tilde{\otimes} 1.$$

and similarly $\sum_k (u^{-\mathsf{T}}\tilde{\boxtimes}u^{-\mathsf{T}})_{ki}(u\tilde{\boxtimes}u)_{kj} = \delta_{i,j} 1\tilde{\otimes}1$. For morphisms, we have nothing to check because $\partial_{u\tilde{\boxtimes}u} = \partial_u$.

iv) This equation follows from the relation $(-)^{-\mathsf{T}} \circ (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}} = (-)^{\mathsf{op}} \circ (-)^{-\perp} \circ (-)^{-\mathsf{co}} = (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}} \circ (-)^{\mathsf{T},\mathsf{co}} \circ (-)^{-\mathsf{co}}$. ■

Assume for a moment that (A, Δ, ϵ, S) is a Hopf (B, Γ)-algebroid.

2.3.7. Definition. A *matrix corepresentation* of (A, Δ, ϵ, S) is a $v \in \mathcal{R}_n(A)$ for some $n \in \mathbb{N}$ satisfying $\Delta_n(v) = v\tilde{\boxtimes}v$, $\epsilon_n(v) = \partial_v$, $S_n(v) = v^{-1}$.

2.3.8. Lemma. *If $v \xrightarrow{F} w$ is a morphism in $\mathcal{R}_n(A)$ and v is a matrix corepresentation, then so is w.*

Proof. Applying the morphisms Δ, ϵ, S and the functors $\boldsymbol{\Delta}, \boldsymbol{\epsilon}, (-)^{-\mathsf{co},\mathsf{op}}$ to $v \xrightarrow{F} w$ or its inverse, we get intertwiners $w\tilde{\boxtimes}w \xrightarrow{F^{-1}} v\tilde{\boxtimes}v = \Delta_n(v) \xrightarrow{F} \Delta_n(w)$, $\partial_w \xrightarrow{F^{-1}} \partial_v = \epsilon_n(v) \xrightarrow{F} \epsilon_n(w)$ and $w^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} v^{-\mathsf{co},\mathsf{op}} = S_n(v) \xrightarrow{F} S_n(w)$. ■

Let us now discuss the involutive case.

Given a $*$-algebra C and a matrix $v \in M_n(C)$, we write $\overline{v} := (v_{ij}^*)_{i,j} = (v^*)^{\mathsf{T}}$.

Assume that B is a $*$-algebra, that Γ preserves the involution, and that A is a (B,Γ)-algebra. Then there exists an obvious functor $\mathcal{R}_n(A) \to \mathcal{R}_n(\bar{A})$, given by $u \mapsto u$ and $(u \xrightarrow{F} v) \mapsto (u \xrightarrow{\overline{F}} v)$. Composition with $(-)^{\mathsf{op}}$ gives a functor

$$(-)^{\overline{\mathsf{op}}} \colon \mathcal{R}_n(A) \to \mathcal{R}_n(\overline{A}^{\mathsf{op}}), \qquad u \mapsto u^{\overline{\mathsf{op}}} := u^{\mathsf{op}}, \qquad (u \xrightarrow{F} v) \mapsto (u^{\overline{\mathsf{op}}} \xrightarrow{\hat{\overline{F}}} v^{\overline{\mathsf{op}}}),$$

and $\partial_{u^{\overline{\mathsf{op}}}} = \partial_u^{-1}$ for all u. For later use, we note the following relation.

2.3.9. Lemma. *Let $u^{-\mathsf{T}} \xrightarrow{F} v$ be an intertwiner in $\mathcal{R}_n^{\times}(A) \cap \mathcal{R}_n^{\times,\mathsf{T}}(A)$. Then $(v^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F^*} u^{\overline{\mathsf{op}}}$ is an intertwiner in $\mathcal{R}_n^{\times}(\overline{A}^{\mathsf{op}}) \cap \mathcal{R}_n^{\times,\mathsf{T}}(\overline{A}^{\mathsf{op}})$.*

Proof. Subsequent applications of the functors $(-)^{\overline{\mathsf{op}}}$, $(-)^{-\mathsf{T}}$ yield intertwiners $(v^{-\mathsf{T}})^{\overline{\mathsf{op}}} = (v^{\overline{\mathsf{op}}})^{-\perp} \xrightarrow{\hat{\overline{F}}^{-1}} u^{\overline{\mathsf{op}}}$ and $(u^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{\overline{F}^{-\mathsf{T}} = F^{-*}} v^{\overline{\mathsf{op}}}$. ■

Finally, assume that A is a (B,Γ)-$*$-algebra. Then there exists a functor

$$(-)^{*,\mathsf{co}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{co}}), \qquad u \mapsto u^{*,\mathsf{co}} := u^*, \qquad (u \xrightarrow{F} v) \mapsto (v^{*,\mathsf{co}} \xrightarrow{\hat{F}^*} u^{*,\mathsf{co}}),$$

because $s_n(F^*)v^* = u^* r_n(\hat{F}^*)$ for every intertwiner $u \xrightarrow{F} v$, and $\partial_{u^{*,\mathsf{co}}} = \partial_u^{-1}$. Composing with $(-)^{\mathsf{T},\mathsf{co}}$ for A^{co} and with $(-)^{-\mathsf{T}}$, respectively, we get functors

$$\overline{(-)}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A), \qquad u \mapsto \overline{u} = (u_{ij}^*)_{i,j}, \qquad (u \xrightarrow{F} v) \mapsto (\overline{u} \xrightarrow{\overline{\hat{F}}} \overline{v}), \tag{1}$$

$$\mathcal{R}_n(A) \to \mathcal{R}_n(A), \qquad u \mapsto \overline{\overline{u}} = \overline{u}^{-\mathsf{T}} = \overline{u^{-\perp}}, \qquad (u \xrightarrow{F} v) \mapsto (\overline{u}^{-\mathsf{T}} \xrightarrow{F^{-*}} \overline{v}^{-\mathsf{T}}). \tag{2}$$

2.4. The free orthogonal and free unitary dynamical quantum groups. Using the preparations of the last subsection, we now show that the algebras $A_{\mathrm{o}}^B(\nabla, F)$ and $A_{\mathrm{u}}^B(\nabla, F)$ are (B,Γ)-Hopf algebroids as claimed in the introduction.

Let B be a commutative algebra with an action of a group Γ as before, and let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\nabla = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in M_n(B \rtimes \Gamma)$.

Let $F \in \mathrm{GL}_n(B)$ be ∇-*odd* in the sense that $\nabla F \nabla \in M_n(B)$. The first definition and theorem in the introduction can be reformulated as follows.

2.4.1. Definition. The *free orthogonal dynamical quantum group over B with parameters* (∇, F) is the universal (B,Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F)$ with a $v \in \mathcal{R}_n^\times(A_{\mathrm{o}}^B(\nabla, F))$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ is an intertwiner.

2.4.2. Theorem. *The (B,Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F)$ can be equipped with a unique structure of a (B,Γ)-Hopf algebroid such that v becomes a matrix corepresentation.*

Proof. The existence of morphisms $\Delta\colon A \to A\tilde{\otimes}A$, $\epsilon\colon A \to B \rtimes \Gamma$, $S\colon A \to A^{\mathsf{co},\mathsf{op}}$ satisfying $\Delta_n(v) = v\tilde{\boxtimes}v$, $\epsilon_n(v) = \nabla$, $S_n(v) = v^{-1}$ follows from the universal property of A and the relations

$$\boldsymbol{\Delta}(v^{-\mathsf{T}} \xrightarrow{F} v) = ((v\tilde{\boxtimes}v)^{-\mathsf{T}} \xrightarrow{F} v\tilde{\boxtimes}v), \quad \boldsymbol{\epsilon}(v^{-\mathsf{T}} \xrightarrow{F} v) = (\nabla^{-\mathsf{T}} \xrightarrow{F} \nabla),$$
$$(v^{-\mathsf{T}} \xrightarrow{F} v)^{-\mathsf{co},\mathsf{op}} = (v^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} (v^{-\mathsf{co},\mathsf{op}})^{-\mathsf{T}});$$

see Lemma 2.3.5 and 2.3.6. Straightforward calculations show that (A, Δ, ϵ, S) is a (B,Γ)-Hopf algebroid. ∎

2.4.3. Remarks. i) In the definition of $A_{\mathrm{o}}^B(\nabla, F)$, we may evidently assume that Γ is generated by the diagonal components $\gamma_1, \dots, \gamma_n$ of ∇.

ii) Denote by $B_0 \subseteq B$ the smallest Γ-invariant subalgebra containing the entries of F and F^{-1}, and by $\iota\colon B_0 \to B$ the inclusion. Then there exists an obvious isomorphism $A_{\mathrm{o}}^B(\nabla, F) \cong \iota_* A_{\mathrm{o}}^{B_0}(\nabla, F)$.

iii) Let $H \in \mathrm{GL}_n(B)$ be ∇-even and $\hat{H} = \nabla H \nabla^{-1}$. Then there exists an isomorphism $A_{\mathrm{o}}^B(\nabla, HF\hat{H}^{\mathsf{T}}) \to A_{\mathrm{o}}^B(\nabla, F)$ of (B,Γ)-Hopf algebroids whose extension to matrices sends $v \in A_{\mathrm{o}}^B(\nabla, HF\hat{H}^{\mathsf{T}})$ to $w := r_n(\hat{H}) v s_n(H)^{-1} \in A_{\mathrm{o}}^B(\nabla, F)$. Indeed, there exists such a morphism of (B,Γ)-algebras because in $A_{\mathrm{o}}^B(\nabla, F)$, we have intertwiners $v \xrightarrow{H} w$, $v^{-\mathsf{T}} \xrightarrow{\hat{H}^{-\mathsf{T}}} w^{-\mathsf{T}}$ and $v^{-\mathsf{T}} \xrightarrow{F} v$, whence $w^{-\mathsf{T}} \xrightarrow{HF\hat{H}^{\mathsf{T}}} w$, and this morphism is compatible with Δ, ϵ, S because w is a matrix corepresentation by Lemma 2.3.8. A similar argument yields the inverse of this morphism.

Assume that B carries an involution which is preserved by Γ, and let $F \in \mathrm{GL}_n(B)$ be self-adjoint and ∇-*even* in the sense that $\nabla F \nabla^{-1} \in M_n(B)$. The second definition and theorem in the introduction can be reformulated as follows.

2.4.4. DEFINITION. The *free unitary dynamical quantum group over B with parameters* (∇, F) is the universal (B,Γ)-$*$-algebra $A^B_{\mathrm{u}}(\nabla, F)$ with a unitary $u \in \mathcal{R}^\times_n(A^B_{\mathrm{u}}(\nabla, F))$ such that $\partial_v = \nabla$ and $(v^{-\mathsf{T}})^{-\mathsf{T}} \xrightarrow{F} v$ is an intertwiner.

2.4.5. THEOREM. *The $*$-algebra $A^B_{\mathrm{u}}(\nabla, F)$ can be equipped with a unique structure of a (B,Γ)-Hopf $*$-algebroid such that v becomes a matrix corepresentation.*

To prove this result, we introduce an auxiliary (B,Γ)-algebra which does not involve the involution on B.

2.4.6. DEFINITION. We denote by $A^B_{\mathrm{u}'}(\nabla, F)$ the universal (B,Γ)-algebra with $v, w \in \mathcal{R}^\times_n(A)$ such that $\partial_v = \nabla$, $\partial_w = \nabla^{-1}$ and $v^{-\mathsf{T}} \xrightarrow{1} w$, $w^{-\mathsf{T}} \xrightarrow{F} v$ are intertwiners.

Using the same techniques as in the proof of Theorem 2.4.2, one finds:

2.4.7. PROPOSITION. *The (B,Γ)-algebra $A^B_{\mathrm{u}'}(\nabla, F)$ can be equipped with a unique structure of a (B,Γ)-Hopf algebroid such that v and w become matrix corepresentations.*

2.4.8. PROPOSITION. *The (B,Γ)-algebra $A^B_{\mathrm{u}'}(\nabla, F)$ can be equipped with an involution such that it becomes a (B,Γ)-Hopf $*$-algebroid and $w = \bar{v}$.*

Proof. Let $A := A^B_{\mathrm{u}'}(\nabla, F)$. By Lemma 2.3.9, we have intertwiners $(w^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{1} v^{\overline{\mathsf{op}}}$ and $(v^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F^*=F} w^{\overline{\mathsf{op}}}$. The universal property of A yields a homomorphism $j\colon A \to \overline{A}^{\mathsf{op}}$ satisfying $j_n(v) = w^{\overline{\mathsf{op}}}$ and $j_n(w) = v^{\overline{\mathsf{op}}}$. Composition of j with the canonical map $\bar{A}^{\mathsf{op}} \to A$ yields the desired involution, which is easily seen to be compatible with the comultiplication and counit. ∎

Theorem 2.4.5 now is an immediate corollary to the following result:

2.4.9. THEOREM. *There exists a unique $*$-isomorphism $A^B_{\mathrm{u}}(\nabla, F) \to A^B_{\mathrm{u}'}(\nabla, F)$ whose extension to matrices sends u to v.*

Proof. One easily verifies that the universal properties of $A := A^B_{\mathrm{u}}(\nabla, F)$ and $A' := A^B_{\mathrm{u}'}(\nabla, F)$ yield homomorphisms $A \to A'$ and $A' \to A$ whose extensions to matrices satisfy $u \mapsto v$ and $v \mapsto u$, $w \mapsto \bar{u}$, respectively. ∎

The following analogues of Remarks 2.4.3 apply to $A^B_{\mathrm{u}}(\nabla, F)$:

2.4.10. REMARKS. i) We may assume that Γ is generated by the diagonal components of ∇, and if $\iota\colon B_0 \hookrightarrow B$ denotes the inclusion of the smallest Γ-invariant $*$-subalgebra containing the entries of F and F^{-1}, then $A^B_{\mathrm{u}}(\nabla, F) \cong \iota_* A^{B_0}_{\mathrm{u}}(\nabla, F)$.

ii) Let $H \in \mathrm{GL}_n(B)$ be ∇-even and unitary, and let $\hat{H} = \nabla H \nabla^{-1}$. Then there exists an isomorphism $A^B_{\mathrm{u}}(\nabla, HFH^*) \to A^B_{\mathrm{u}}(\nabla, F)$ of (B,Γ)-Hopf algebroids whose extension to matrices sends $u \in A^B_{\mathrm{u}}(\nabla, HFH^*)$ to $z := r_n(\hat{H}) u s_n(H)^{-1} \in A^B_{\mathrm{u}}(\nabla, F)$. Indeed, there exists such a morphism of (B,Γ)-algebras because z is a product of unitaries and in $A^B_{\mathrm{u}}(\nabla, F)$, we have intertwiners $u \xrightarrow{H} z$, $\bar{u}^{-\mathsf{T}} \xrightarrow{H^{-*}} \bar{z}^{-\mathsf{T}}$ by (2), and $\bar{u}^{-\mathsf{T}} \xrightarrow{F} u$, whence $\bar{z}^{-\mathsf{T}} \xrightarrow{HFH^*} z$, and this morphism is compatible with Δ, ϵ, S

because z is a matrix corepresentation by Lemma 2.3.8. A similar argument yields the inverse of this morphism.

We finally consider involutions on certain quotients of $A_{\mathrm{o}}^{B}(\nabla, F)$.

Assume that $F, G \in \mathrm{GL}_n(B)$ are ∇-odd and $GF^* = FG^*$. Let $Q := G(\nabla \bar{G} \nabla)$.

2.4.11. DEFINITION. The *free orthogonal dynamical quantum group* over B with parameters (∇, F, G) is the universal (B, Γ)-algebra $A_{\mathrm{o}}^{B}(\nabla, F, G)$ with a $v \in \mathcal{R}_n^{\times}(A)$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ and $v \xrightarrow{Q} v$ are intertwiners.

The algebra $A_{\mathrm{o}}^{B}(\nabla, F, G)$ depends only on Q and not on G, but shall soon be equipped with an involution that does depend on G.

Evidently, there exists a canonical quotient map $A_{\mathrm{o}}^{B}(\nabla, F) \to A_{\mathrm{o}}^{B}(\nabla, F, G)$, and

$$A_{\mathrm{o}}^{B}(\nabla, F, G) \cong A_{\mathrm{o}}^{B}(\nabla, F)/(r(q) - s(q)) \quad \text{if } Q = \mathrm{diag}(q, \ldots, q),$$
$$A_{\mathrm{o}}^{B}(\nabla, F, G) \cong A_{\mathrm{o}}^{B}(\nabla, F)/(r(q_i) - s(q_j) | i, j = 1, \ldots, n) \quad \text{if } Q = \mathrm{diag}(q_1, \ldots, q_n),$$

because in the second case $(r_n(\hat{Q})v)_{ij} = r_n(\gamma_i(q_i))v_{ij} = v_{ij} r_n(q_i)$ and $(v s_n(Q))_{ij} = v_{ij} s_n(q_j)$ in $A_{\mathrm{o}}^{B}(\nabla, F)$ for all i, j.

2.4.12. THEOREM. *The (B, Γ)-algebra $A_{\mathrm{o}}^{B}(\nabla, F, G)$ can be equipped with a unique structure of a (B, Γ)-Hopf $*$-algebroid such that $\bar{v} \xrightarrow{G} v$ becomes an intertwiner and v a matrix corepresentation.*

Proof. The existence of Δ, ϵ, S follows similarly as in the case of $A_{\mathrm{o}}^{B}(\nabla, F)$; one only needs to observe that additionally, application of the functors $\boldsymbol{\Delta}, \boldsymbol{\epsilon}$ and $(-)^{\mathsf{co,op}}$ to the intertwiner $(v \xrightarrow{Q} v)$ yield intertwiners $(v \tilde{\boxtimes} v \xrightarrow{Q} v \tilde{\boxtimes} v)$, $(\nabla \xrightarrow{Q} \nabla)$ and $((v^{-1})^{\mathsf{co,op}} \xrightarrow{Q} (v^{-1})^{\mathsf{co,op}})$

Let us prove existence of the involution. Let $w := r_n(\nabla G \nabla)^{-1} v s_n(G)$. Then there exist intertwiners

$$w \xrightarrow{G} v, \qquad (v \xrightarrow{G^{-1}} w) \circ (v \xrightarrow{Q} v) = (v \xrightarrow{\nabla \bar{G} \nabla} w),$$
$$(v \xrightarrow{\nabla \bar{G} \nabla} w)^{\overline{\mathsf{op}}} \circ (w \xrightarrow{G} v)^{\overline{\mathsf{op}}} = \ (v^{\overline{\mathsf{op}}} \xrightarrow{G} w^{\overline{\mathsf{op}}}) \circ (w^{\overline{\mathsf{op}}} \xrightarrow{\nabla \bar{G} \nabla} v^{\overline{\mathsf{op}}}) = (w^{\overline{\mathsf{op}}} \xrightarrow{Q} w^{\overline{\mathsf{op}}}),$$
$$(v^{-\mathsf{T}} \xrightarrow{F} v) \circ (w \xrightarrow{(\nabla \bar{G} \nabla)^{-1}} v)^{-\mathsf{T}} = (v^{-\mathsf{T}} \xrightarrow{F} v) \circ (w^{-\mathsf{T}} \xrightarrow{G^*} v^{-\mathsf{T}}) = (w^{-\mathsf{T}} \xrightarrow{FG^* = GF^*} v),$$
$$((v \xrightarrow{G^{-1}} w) \circ (w^{-\mathsf{T}} \xrightarrow{GF^*} v))^{\overline{\mathsf{op}}} = (w^{-\mathsf{T}} \xrightarrow{F^*} w)^{\overline{\mathsf{op}}} = ((w^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F} w^{\overline{\mathsf{op}}});$$

where we used Lemma 2.3.5 in the last line. The universal property of $A := A_{\mathrm{o}}^{B}(\nabla, F, G)$ therefore yields a homomorphism $j \colon A \to \overline{A}^{\mathsf{op}}$ such that $j_n(v) = w^{\overline{\mathsf{op}}}$, and this j corresponds to a conjugate-linear antihomomorphism $A \to A$, $a \mapsto a^*$. To see that the map $a \mapsto a^*$ is involutive, we only need to check $\overline{w} = v$. The functor $\overline{(-)}$ of (1) applied to $w \xrightarrow{G} v$ yields $\overline{w} \xrightarrow{\bar{\bar{G}} = \nabla \bar{G} \nabla} \overline{v} = w$, and composition with $w \xrightarrow{G} v$ gives $\overline{w} \xrightarrow{Q} v$. Hence, $\overline{w} = v$.

Finally, the involution is compatible with the comultiplication and counit because w is a matrix corepresentation by Lemma 2.3.8. ∎

2.4.13. REMARKS. i) The canonical quotient map $A_{\mathrm{o}}^{B}(\nabla, F) \to A_{\mathrm{o}}^{B}(\nabla, F, G)$ is a morphism of (B, Γ)-Hopf algebroids.

ii) Analogues of Remarks 2.4.3 and 2.4.10 apply to $A_{\mathrm{o}}^{B}(\nabla, F, G)$.
iii) Note that $A_{\mathrm{o}}^{B}(\nabla, F, G)$ is the universal (B,Γ)-$*$-algebra with a $v \in \mathcal{R}_n^{\times}(A)$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ and $\overline{v} \xrightarrow{G} v$ are intertwiners. Indeed, the composition of $\overline{v} \xrightarrow{G} v$ with its image under the functor $\overline{(-)}$ in (1) yields $v \xrightarrow{Q} v$.
iv) If $F = G$, then $\overline{v} = v^{-\mathsf{T}}$ and hence v is unitary. In general, assume that $H \in \mathrm{GL}_n(B)$ satisfies $\nabla H \nabla^{-1} \in M_n(B)$ and $\overline{H} H^{\mathsf{T}} \in \mathbb{C}\cdot G^{-1}F$. Then $u := r_n(H^{-1}) v s_n(\nabla^{-1} H \nabla)$ is a unitary matrix corepresentation whose entries generate $A_{\mathrm{o}}^{B}(\nabla, F, G)$ as a (B,Γ)-algebra. Indeed, $u \xrightarrow{\nabla^{-1}H\nabla} v$ is an intertwiner, and applying $\overline{(-)}$ and $(-)^{-\mathsf{T}}$, respectively, we get $\overline{u} \xrightarrow{\overline{H}} \overline{v} \xrightarrow{G} v \xrightarrow{F^{-1}} v^{-\mathsf{T}} \xrightarrow{H^{\mathsf{T}}} u^{-\mathsf{T}}$ which is scalar by assumption so that $\overline{u} = u^{-\mathsf{T}}$.

We finally consider a simple example; a more complex one is considered in §2.6.

2.4.14. Example. Equip $\mathbb{C}[X]$ with an involution such that $X^* = X$ and an action of $\mathbb{Z}$ such that $X \overset{k}{\mapsto} X - k$ for all $k \in \mathbb{Z}$, and let $\gamma_1 = 1$, $\gamma_2 = -1$, $\nabla = \mathrm{diag}(\gamma_1, \gamma_2)$ and $F = G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $A_{\mathrm{o}}^{\mathbb{C}[X]}(\nabla, F, G) \cong \iota_*(A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G))$, where $\iota\colon \mathbb{C} \to \mathbb{C}[X]$ is the canonical map.

The algebra $A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G)$ equipped with $\Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S$ is a Hopf $*$-algebra by Lemma 2.2.1. It is generated by the entries of a unitary matrix v which satisfies $\overline{v} = G^{-1} v G$ and therefore has the form $v = \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. The relations $vv^* = 1 = v^*v$ then imply that $\alpha, \alpha^*, \gamma, \gamma^*$ commute and $\alpha\alpha^* + \gamma\gamma^* = 1$. Therefore, $A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G)$ is isomorphic to the Hopf $*$-algebra $\mathcal{O}(\mathrm{SU}(2))$ of representative functions on $\mathrm{SU}(2)$.

The algebra $A_{\mathrm{o}}^{\mathbb{C}[X]}(\nabla, F, G) \cong \iota_*(A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G))$ can be identified with the subalgebra of $\mathrm{End}(\mathcal{O}(\mathrm{SU}(2)))$ generated by multiplication operators associated to elements of $\mathcal{O}(\mathrm{SU}(2))$ and left or right invariant differentiation operators along the diagonal torus in $\mathrm{SU}(2)$; see Example 2.2.7.

2.5. The square of the antipode and the scaling character groups. The square of the antipode on the free dynamical quantum groups $A_{\mathrm{o}}^{B}(\nabla, F)$, $A_{\mathrm{u}}^{B}(\nabla, F)$, $A_{\mathrm{o}}^{B}(\nabla, F, G)$ can be described in terms of certain character groups as follows.

Recall the isomorphisms of Lemma 2.1.3 iii) and the anti-automorphism $S^{B \rtimes \Gamma}$ of $B \rtimes \Gamma$ given by $b\gamma \mapsto \gamma^{-1} b$.

2.5.1. Definition. Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid. A *character group* on A is a family of morphisms $\theta = (\theta^{(k)}\colon A \to B \rtimes \Gamma)_{k \in \mathbb{Z}}$ satisfying $(\theta^{(k)} \tilde{\otimes} \theta^{(l)}) \circ \Delta = \theta^{k+l}$, $\theta^{(0)} = \epsilon$ and $\theta^{(k)} \circ S = S^{B \rtimes \Gamma} \circ \theta^{(-k)}$ for all $k, l \in \mathbb{Z}$. We call a character group θ *scaling* if $S^2 = (\theta^{(1)} \tilde{\otimes} \mathrm{id} \tilde{\otimes} \theta^{(-1)}) \circ \Delta^{(2)}$, where $\Delta^{(2)} = (\Delta \tilde{\otimes} \mathrm{id}) \circ \Delta = (\mathrm{id} \tilde{\otimes} \Delta) \circ \Delta$.

We construct scaling character groups using intertwiners of the form $u \xrightarrow{H} S_n^2(u)$ for suitable matrix corepresentations u.

2.5.2. Lemma. *Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid, let $\theta = (\theta^{(k)}\colon A \to B \rtimes \Gamma)_{k \in \mathbb{Z}}$ be a family of morphisms satisfying $(\theta^{(k)} \tilde{\otimes} \theta^{(l)}) \circ \Delta = \theta^{k+l}$ for all $k, l \in \mathbb{Z}$, and let $u \in \mathcal{R}_n^{\times}(A)$ be a matrix corepresentation.*

i) $S_n^2(u) = (u^{-\mathsf{T}})^{-\mathsf{T}}$.
ii) *Let* $H = \partial_u^{-1}\theta_n^{(1)}(u)$. *Then* $H \in \mathrm{GL}_n(B)$, $\partial_u H \partial_u^{-1} \in M_n(B)$ *and* $\theta_n^{(k)}(u) = \partial_u H^k$ *for all* $k \in \mathbb{Z}$.
iii) $\theta_n^{(0)} = \epsilon_n(u)$ *and* $\theta_n^{(k)}(S_n(u)) = S_n^{B\rtimes\Gamma}(\theta_n^{(-k)}(u))$ *for all* $k \in \mathbb{Z}$.
iv) $S_n^2(u) = ((\theta^{(1)}\tilde{\otimes}\,\mathrm{id}\,\tilde{\otimes}\theta^{(-1)}) \circ \Delta^{(2)})_n(u)$ *if and only if* $u \xrightarrow{H} S_n^2(u)$ *is an intertwiner.*

Proof. i) The map $M_n(A) \to M_n(A)$ given by $x \mapsto S_n(x)^{\mathsf{T}}$ is an antihomomorphism and therefore preserves inverses. Hence, $S_n^2(u) = S_n(u^{-\mathsf{T}})^{\mathsf{T}} = (S_n(u)^{\mathsf{T}})^{-\mathsf{T}} = (u^{-\mathsf{T}})^{-\mathsf{T}}$.

ii) Since each $\theta^{(k)}$ preserves the grading, there exists a family $(H_k)_{k\in\mathbb{Z}}$ of elements of $\mathrm{GL}_n(B)$ satisfying $\partial_u H_k \partial_u^{-1} \in M_n(B)$ and $\theta_n^{(k)}(u) = \partial_u H_k$ for all $k \in \mathbb{Z}$. The assumption on θ implies that $H_k H_l = H_{k+l}$ for all $k, l \in \mathbb{Z}$, and consequently, $H_k = H_1^k$ for all $k \in \mathbb{Z}$.

iii) By ii), $\theta_n^{(0)}(u) = \partial_u = \epsilon_n(u)$ and

$$\theta_n^{(k)}(S_n(u)) = \theta_n^{(k)}(u^{-1}) = \theta_n^{(k)}(u)^{-1} = H^{-k}\partial_u^{-1} = S_n^{B\rtimes\Gamma}(\partial_u H^{-k}) = S_n^{B\rtimes\Gamma}(\theta_n^{(-k)}(u)).$$

iv) This follows from the relation

$$\begin{aligned}((\theta^{(1)}\tilde{\otimes}\,\mathrm{id}\,\tilde{\otimes}\theta^{(-1)}) \circ \Delta^{(2)})_n(u) &= \theta_n^{(1)}(u)\tilde{\boxtimes}u\tilde{\boxtimes}\theta_n^{(-1)}(u) \\ &= \partial_u H\tilde{\boxtimes}u\tilde{\boxtimes}\partial_u H^{-1} = r_n(\partial_u H\partial_u^{-1})us_n(H^{-1}). \ \blacksquare\end{aligned}$$

We first apply the lemma above to $A_{\mathrm{o}}^B(\nabla, F)$.

2.5.3. Proposition. *Let* $F \in \mathrm{GL}_n(B)$ *be* ∇*-odd. Then* $A_{\mathrm{o}}^B(\nabla, F)$ *has an intertwiner* $v \xrightarrow{H} S^2(v)$ *and a scaling character group* θ *such that* $\theta_n^{(k)}(v) = \nabla H^k$ *for all* $k \in \mathbb{Z}$, *where* $H = (\nabla F\nabla)^{\mathsf{T}}F^{-1}$.

Proof. By Lemma 2.5.2 i), $(v^{-\mathsf{T}} \xrightarrow{F} v) \circ (v^{-\mathsf{T}} \xrightarrow{F} v)^{-\mathsf{T}} = S^2(v) \xrightarrow{H^{-1}} v$. To construct θ, let $k \in \mathbb{Z}$ and $x = \nabla H^k$. By Lemma 2.3.3, $x^{-\mathsf{T}} = (H^{-\mathsf{T}})^k\nabla^{-1}$ and hence

$$(\nabla F\nabla)x^{-\mathsf{T}} = \nabla F\nabla(\nabla^{-1}F^{-1}\nabla^{-1}F^{\mathsf{T}})^k\nabla^{-1} = \nabla(\nabla^{-1}F^{\mathsf{T}}\nabla^{-1}F^{-1})^kF = xF.$$

The universal property of $A_{\mathrm{o}}^B(\nabla, F)$ yields a morphism $\theta^{(k)}\colon A_{\mathrm{o}}^B(\nabla, F) \to B \rtimes \Gamma$ such that $\theta_n^{(k)}(v) = x$. Using Lemma 2.5.2, one easily verifies that the family $(\theta^{(k)})_k$ is a scaling character group. $\blacksquare$

Assume that B carries an involution which is preserved by Γ. We call a character group $(\theta^{(k)})_k$ on a (B, Γ)-Hopf $*$-algebroid *imaginary* if $\theta^{(k)} \circ * = * \circ \theta^{(-k)}$ for all $k \in \mathbb{Z}$.

2.5.4. Proposition. *Let* $F \in \mathrm{GL}_n(B)$ *be* ∇*-even. Then* $A_{\mathrm{u}}^B(\nabla, F)$ *has intertwiners* $u \xrightarrow{F^{-1}} S_n^2(u)$ *and* $\bar{u} \xrightarrow{(\nabla F\nabla^{-1})^{\mathsf{T}}} S_n^2(\bar{u})$, *and an imaginary scaling character group* θ *such that* $\theta_n^{(k)}(u) = \nabla F^{-k}$ *and* $\theta_n^{(k)}(\bar{u}) = F^{k\mathsf{T}}\nabla^{-1}$ *for all* $k \in \mathbb{Z}$.

Proof. By Lemma 2.5.2 i), the first intertwiner is the inverse of $S_n^2(u) = (u^{-\mathsf{T}})^{-\mathsf{T}} = \bar{u}^{-\mathsf{T}} \xrightarrow{F} u$, and the second intertwiner is the inverse of $(\bar{u}^{-\mathsf{T}} \xrightarrow{F} u)^{-\mathsf{T}}$. To construct θ, let $k \in \mathbb{Z}$ and $x = \nabla F^{-k}$, $y = F^{k\mathsf{T}}\nabla^{-1}$. Using Lemma 2.3.3, we find

$$y = x^{-\mathsf{T}}, \qquad y^{-\mathsf{T}} = x, \qquad (\nabla F\nabla^{-1})y^{-\mathsf{T}} = (\nabla F\nabla^{-1})x = \nabla F^{1-k} = xF.$$

The universal property of the algebra $A_{\mathrm{u'}}^B(\nabla, F)$ and Theorem 2.4.9 yield a morphism $\theta^{(k)}\colon A_{\mathrm{u}}^B(\nabla, F) \to B \rtimes \Gamma$ such that $\theta_n^{(k)}(u) = x$ and $\theta_n^{(k)}(\bar{u}) = y$. Using Lemma 2.5.2, one easily shows that $(\theta^{(k)})_k$ is a scaling character group. It is imaginary since by Lemma 2.3.3,

$$\overline{\theta_n^{(-k)}(u)} = \overline{\nabla F^k} = \overline{F}^k \nabla^{-1} = F^{\mathsf{T}k} \nabla^{-1} = \theta_n^{(k)}(\bar{u}) \quad \text{for all } k \in \mathbb{Z}. \; \blacksquare$$

The case $A_{\mathrm{o}}^B(\nabla, F, G)$ requires some preparation. Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd and

$$H = (\nabla F \nabla)^{\mathsf{T}} F^{-1} = \nabla^{-1} F^{\mathsf{T}} \nabla^{-1} F^{-1}, \qquad\qquad Q = G \nabla \bar{G} \nabla$$

as before. We say that a diagram with arrows labeled by matrices commutes if for all possible directed paths with the same starting and ending point in the diagram, the products of the labels along the arrows coincide.

2.5.5. Lemma. *In the diagram below, (A) commutes if and only if (D) commutes, and (B) commutes if and only (C) commutes:*

$$\begin{tikzcd}
\bullet \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "(A)"] & \bullet \arrow[r, "\nabla F^{-\mathsf{T}} \nabla"] \arrow[d, "\overline{G}^{\mathsf{T}}"] \arrow[dr, phantom, "(C)"] & \bullet \arrow[d, "G"] \\
\bullet \arrow[r, "\overline{F}^{\mathsf{T}}"] \arrow[d, "\nabla \overline{G} \nabla"'] \arrow[dr, phantom, "(B)"] & \bullet \arrow[r, "\nabla^{-1} \bar{F}^{-1} \nabla^{-1}"] \arrow[d, "\nabla^{-1} G^{\mathsf{T}} \nabla^{-1}"'] \arrow[dr, phantom, "(D)"] & \bullet \arrow[d, "\nabla \overline{G} \nabla"] \\
\bullet \arrow[r, "F"'] & \bullet \arrow[r, "\nabla F^{-\mathsf{T}} \nabla"'] & \bullet
\end{tikzcd}$$

If all squares commute, then $HQ = QH$, $\overline{G} \nabla H^{-1} = \overline{HG} \nabla$, and $QF = F \nabla Q^{\mathsf{T}} \nabla^{-1}$.

Proof. Applying the transformation $X \mapsto X^{-\mathsf{T}}$ and reversing invertible arrows, one can obtain (D) from (A) and (C) from (B). If all small squares commute, then the three asserted relations follow from the commutativity of the large square, of the lower two squares, and of the left two squares, respectively. $\blacksquare$

2.5.6. Proposition. *Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd. Assume that $FG^* = GF^*$ and $F^*(\nabla \bar{G} \nabla)^* = (\nabla \bar{G} \nabla) F$, and let $H = \nabla^{-1} F^{\mathsf{T}} \nabla^{-1} F^{-1}$. Then $A_{\mathrm{o}}^B(\nabla, F, G)$ has an intertwiner $v \xrightarrow{H} S^2(v)$ and an imaginary scaling character group $(\theta^{(k)})_k$ such that $\theta_n^{(k)}(v) = \nabla H^k$.*

Proof. We can re-use the arguments in the proof of Proposition 2.5.3 and only have to show additionally that $\nabla H^k \xrightarrow{Q} \nabla H^k$ is an intertwiner and that $\theta_n^{(-1)}(\bar{v}) = \overline{\theta_n^{(1)}(v)}$. But by the lemma above, $(\nabla Q \nabla^{-1}) \nabla H^k = \nabla Q H^k = \nabla H^k Q$ and

$$\theta_n^{(-1)}(\bar{v}) = \theta_n^{(1)}(\overline{G} v (\nabla \overline{G} \nabla)^{-1}) = \overline{G} \nabla H^{-1} \nabla^{-1} \overline{G}^{-1} \nabla^{-1} = \overline{H} \nabla^{-1} = \overline{\theta_n^{(1)}(v)}. \; \blacksquare$$

2.5.7. Remark. Applying the functor (2) to $v^{-\mathsf{T}} \xrightarrow{F} v$, $\bar{v} \xrightarrow{G} v$, $v \xrightarrow{\nabla \bar{G} \nabla} \bar{v}$, we obtain intertwiners $\bar{v}^{-\mathsf{T}} \xrightarrow{F^*} \bar{v}$, $\bar{v}^{-\mathsf{T}} \xrightarrow{G^*} v^{-\mathsf{T}}$, $v^{-\mathsf{T}} \xrightarrow{(\nabla \bar{G} \nabla)^*} \bar{v}^{-\mathsf{T}}$, and the conditions $FG^* = GF^*$ and $F^*(\nabla \bar{G} \nabla)^* = (\nabla \bar{G} \nabla) F$ amount to commutativity of the squares

$$\begin{tikzcd}
\bar{v}^{-\mathsf{T}} \arrow[r, "F^*"] \arrow[d, "G^*"'] & \bar{v} \arrow[d, "G"] \\
v^{-\mathsf{T}} \arrow[r, "F"'] & v
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
v^{-\mathsf{T}} \arrow[r, "F"] \arrow[d, "(\nabla \bar{G} \nabla)^*"'] & v \arrow[d, "\nabla \bar{G} \nabla"] \\
\bar{v}^{-\mathsf{T}} \arrow[r, "F^*"'] & \bar{v}.
\end{tikzcd}$$

If $Q = G \nabla \bar{G} \nabla$ is scalar, then both conditions evidently are equivalent.

2.6. The full dynamical quantum group $\mathrm{SU}_Q^{\mathrm{dyn}}(2)$**.** In [11], Koelink and Rosengren studied a dynamical quantum group $\mathcal{F}_R(\mathrm{SU}(2))$ that arises from a dynamical R-matrix via the generalized FRT-construction of Etingof and Varchenko. We first recall its definition, then show that this dynamical quantum group coincides with $A_{\mathrm{o}}^B(\nabla, F, G)$ for specific choice of B, Γ, ∇, F, G, and finally construct a refinement that includes several interesting limit cases.

We shall slightly reformulate the definition of $\mathcal{F}_R(\mathrm{SL}(2))$ and $\mathcal{F}_R(\mathrm{SU}(2))$ given in [11, §2.2] so that it fits better with our approach.

Fix $q \in (0,1)$. Let $\mathfrak{M}(\mathbb{C})$ be the algebra of meromorphic functions on the plane and let $\mathbb{Z}$ act on B such that $b \overset{k}{\mapsto} b_{(k)} := b(\,\cdot\, - k)$ for all $b \in B$, $k \in \mathbb{Z}$. Define $f \in \mathfrak{M}(\mathbb{C})$ by

$$f(\lambda) = q^{-1}\frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1} = \frac{q^{\lambda+2} - q^{-(\lambda+2)}}{q^{\lambda+1} - q^{-(\lambda-1)}} \quad \text{for all } \lambda \in \mathbb{C}. \tag{3}$$

Then the $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$-Hopf algebroid $\mathcal{F}_R(\mathrm{SL}(2))$ is the universal $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$-algebra with generators $\alpha, \beta, \gamma, \delta$ satisfying

$$\partial_\alpha = (1,1), \quad \partial_\beta = (1,-1), \quad \partial_\gamma = (-1,1), \quad \partial_\delta = (-1,-1), \tag{4}$$

$$\alpha\beta = s(f_{(1)})\beta\alpha, \quad \alpha\gamma = r(f)\gamma\alpha, \quad \beta\delta = r(f)\delta\beta, \quad \gamma\delta = s(f_{(1)})\delta\gamma, \tag{5}$$

$$\frac{r(f)}{s(f)}\delta\alpha - \frac{1}{s(f)}\beta\gamma = \alpha\delta - r(f)\gamma\beta = \frac{r(f_{(1)})}{s(f_{(1)})}\alpha\delta - r(f_{(1)})\beta\gamma = \delta\alpha - \frac{1}{s(f_{(1)})}\gamma\beta = 1, \tag{6}$$

and with comultiplication, counit and antipode given by

$$\begin{aligned} \Delta(\alpha) &= \alpha\tilde{\otimes}\alpha + \beta\tilde{\otimes}\gamma, \quad \Delta(\beta) = \alpha\tilde{\otimes}\beta + \beta\tilde{\otimes}\delta, \\ \Delta(\gamma) &= \gamma\tilde{\otimes}\alpha + \delta\tilde{\otimes}\gamma, \quad \Delta(\delta) = \gamma\tilde{\otimes}\beta + \delta\tilde{\otimes}\delta, \end{aligned} \tag{7}$$

$$\epsilon(\alpha) = \partial_\alpha^r = \partial_\alpha^s, \quad \epsilon(\beta) = \epsilon(\gamma) = 0, \quad \epsilon(\delta) = \partial_\delta^r = \partial_\delta^s, \tag{8}$$

$$S(\alpha) = \frac{r(f)}{s(f)}\delta, \quad S(\beta) = -\frac{1}{s(f)}\beta, \quad S(\gamma) = -r(f)\gamma, \quad S(\delta) = \alpha. \tag{9}$$

Equip $\mathfrak{M}(\mathbb{C})$ with the involution given by $b^*(\lambda) = \overline{b(\overline{\lambda})}$ for all $b \in \mathfrak{M}(\mathbb{C})$, $\lambda \in \mathbb{C}$. Then $\mathcal{F}_R(\mathrm{SL}(2))$ can be equipped with an involution such that

$$\alpha^* = \delta, \qquad \beta^* = -q\gamma, \qquad \gamma^* = -q^{-1}\beta, \qquad \delta^* = \alpha, \tag{10}$$

and one obtains a $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$-Hopf $*$-algebroid which is denoted by $\mathcal{F}_R(\mathrm{SU}(2))$ [11].

2.6.1. Proposition. *Let* $\nabla = \mathrm{diag}(1,-1)$, $F = \begin{pmatrix} 0 & -1 \\ f_{(1)}^{-1} & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$. *Then there exist isomorphisms of* $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$*-Hopf (*-)algebroids* $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F) \to \mathcal{F}_R(\mathrm{SL}(2))$ *and* $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F, G) \to \mathcal{F}_R(\mathrm{SU}(2))$ *whose extensions to matrices map* v *to* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Proof. First, note that the function $\lambda \mapsto q^\lambda$ and hence also f is self-adjoint, and that

$$\hat{F} := \nabla F \nabla = \begin{pmatrix} 0 & -1 \\ f^{-1} & 0 \end{pmatrix}, \quad \hat{G} := \nabla G \nabla = G, \quad FG^* = \begin{pmatrix} 1 & 0 \\ 0 & (qf_{(1)})^{-1} \end{pmatrix} = GF^*.$$

Therefore, $A := A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F)$ and $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F, G)$ are well-defined. Since $\nabla G \nabla \overline{G} = G^2 = q^{-1} \in M_2(\mathfrak{M}(\mathbb{C}))$, the latter algebra coincides with the former.

Write $v \in M_2(A)$ as $v = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ and write $(4)'$–$(10)'$ for the relations (4)–(10) with $\alpha', \beta', \gamma', \delta'$ instead of $\alpha, \beta, \gamma, \delta$. Then the relation $\partial_v = \nabla$ is equivalent to $(4)'$. The relation $v^{-\mathsf{T}} = r_2(\hat{F}^{-1}) v s_2(F)$ is equivalent to

$$v^{-1} = \left(\begin{pmatrix} 0 & r(f) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ s(f_{(1)}^{-1}) & 0 \end{pmatrix} \right)^{\mathsf{T}} = \begin{pmatrix} \frac{r(f)}{s(f)} \delta' & -\frac{1}{s(f)} \beta' \\ -r(f)\gamma' & \alpha' \end{pmatrix},$$

and multiplying out $v^{-1}v = 1 = vv^{-1}$ and using $(4)'$, we find that this relation is equivalent to $(5)'$ and $(6)'$. Hence, there exists an isomorphism of $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$-algebras $A \to \mathcal{F}_R(\mathrm{SL}(2))$ sending $\alpha', \beta', \gamma', \delta'$ to $\alpha, \beta, \gamma, \delta$. This isomorphism is compatible with the involution, comultiplication, counit and antipode because $(7)'$–$(10)'$ are equivalent to $\Delta_2(v) = v \tilde{\boxtimes} v$, $\epsilon_2(v) = \partial_v$, $S_2(v) = v^{-1}$ and

$$\bar{v} = r_2(\hat{G}^{-1}) v s_2(G) = \begin{pmatrix} 0 & q \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \delta' & -q\gamma' \\ -q^{-1}\beta' & \alpha' \end{pmatrix}. \blacksquare$$

We now refine the definition above as follows. The first idea is to replace the base $\mathfrak{M}(\mathbb{C})$ by the $\mathbb{Z}$-invariant subalgebra containing f and f^{-1}. This subalgebra can be described in terms of the functions $x(\lambda) = q^{\lambda}$, $y(\lambda) = q^{-\lambda}$ and $z = x - y$ as follows. Since $f = z_{(-2)}/z_{(-1)}$, this subalgebra is generated by all fractions $z_{(k)}/z_{(l)}$, where $k, l \in \mathbb{Z}$, and since $z_{(-1)} - qz = (q^{-1} - q)q^{-\lambda}$, also by all fractions $x/z_{(k)}$ and $y/z_{(k)}$, where $k \in \mathbb{Z}$. The second idea is to drop the relation $xy = 1$ to allow the limit cases $\lambda \to \pm\infty$, and regard x, y as canonical coordinates on $\mathbb{C}P^1$. Finally, we also regard q as a variable.

Let us now turn to the details. Denote by $R \subset \mathbb{C}(Q)$ the localization of $\mathbb{C}[Q]$ with respect to Q and the polynomials

$$S_k = (1 - Q^{2k})/(1 - Q^2) = 1 + Q^2 + \cdots + Q^{2(k-1)}, \quad \text{where } k \in \mathbb{N}.$$

Let $\mathbb{Z}$ act on the algebra $\mathbb{C}(Q, X, Y)$ of rational functions in Q, X, Y by

$$Q_{(k)} = Q, \qquad X_{(k)} = Q^{-k}X, \qquad Y_{(k)} = Q^k Y \qquad \text{for all } k \in \mathbb{Z},$$

where the lower index (k) denotes the action of k. Denote by $B \subset \mathbb{C}(Q, X, Y)$ the subalgebra generated by R and all elements

$$Z_{k,l} := (X - Y)_{(k)}/(X - Y)_{(l)}, \quad \text{where } k, l \in \mathbb{Z}.$$

We equip B with the induced action of $\mathbb{Z}$ and the involution given by $Q = Q^*$ and $Z_{k,l}^* = Z_{k,l}$ for all $k, l \in \mathbb{Z}$. Note that this involution is the one inherited from $\mathbb{C}(Q, X, Y)$ when $Q = Q^*$ and either $X^* = X$, $Y^* = Y$ or $X^* = -X$, $Y^* = -Y$. Finally, let

$$\nabla = (1, -1), \qquad F = \begin{pmatrix} 0 & -1 \\ Z_{0,-1} & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & -Q \\ 1 & 0 \end{pmatrix}.$$

Then $FG^* = G^*F$ and $G \nabla \bar{G} \nabla = G^2 = \mathrm{diag}(-Q, -Q)$.

2.6.2. Definition. We let $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) := A_{\mathrm{o}}^B(\nabla, F, G)$.

Thus, $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2))$ is generated by the entries $\alpha, \beta, \gamma, \delta$ of a 2×2-matrix v which satisfy the relations (4)–(10) with $Z_{-2,-1}$ and Q instead of f and q. This $(B, \mathbb{Z})$-Hopf $*$-algebroid aggregates several other interesting quantum groups and quantum groupoids which can be obtained by suitable base changes as follows.

Denote by $z \in \mathfrak{M}(\mathbb{C})$ the function $\lambda \mapsto q^\lambda - q^{-\lambda}$. Equip $\mathbb{C}(\lambda)$ with an involution such that $\lambda^* = \lambda$, and a $\mathbb{Z}$-action such that $\lambda_{(k)} = \lambda - k$. Let $\Omega = (0,1] \times [-\infty, \infty]$ and let $\mathbb{Z}$ act on $C(\Omega)$ by $g_{(k)}(q,t) = g(q, t-k)$ for all $g \in C(\Omega)$, $(q,t) \in \Omega$, $k \in \mathbb{Z}$.

2.6.3. Lemma. *There exist $\mathbb{Z}$-equivariant $*$-homomorphisms*

i) $\pi^q_{\mathfrak{M}(\mathbb{C})} \colon B \to \mathfrak{M}(\mathbb{C}),\quad Q \mapsto q,\qquad Z_{k,l} \mapsto \dfrac{z_{(k)}}{z_{(l)}},\qquad$ *for $q \in (0,1) \cup (1,\infty)$,*

ii) $\pi^1_{\mathfrak{M}(\mathbb{C})} \colon B \to \mathbb{C}(\lambda),\quad Q \mapsto 1,\qquad Z_{k,l} \mapsto \dfrac{\lambda - k}{\lambda - l}$,

iii) $\pi_{\pm\infty} \colon B \to R,\quad Q \mapsto Q,\qquad Z_{k,l} \mapsto \dfrac{Q^{\pm k}}{Q^{\pm l}} = Q^{\pm k \mp l}$,

iv) $\pi^q_{\pm\infty} \colon B \to \mathbb{C},\quad Q \mapsto q,\qquad Z_{k,l} \mapsto q^{\pm k \mp l},\qquad$ *for $q \in (0,\infty)$,*

v) $\pi_\Omega \colon B \to C(\Omega),\quad Q \mapsto ((q,t) \mapsto q),\quad Z_{k,l} \mapsto \left((q,t) \mapsto \begin{cases} \frac{q^{t-k}+q^{k-t}}{q^{t-l}+q^{l-t}}, & t \in \mathbb{R}, \\ q^{\pm k \mp l}, & t = \pm\infty \end{cases}\right)$.

Proof. i) Restrict the homomorphism $\pi \colon \mathbb{C}(Q, X, Y) \to \mathfrak{M}(\mathbb{C})$ given by $Q \mapsto q$, $X \mapsto (\lambda \mapsto q^\lambda)$, $Y \mapsto (\lambda \mapsto q^{-\lambda})$ to B.

ii) Use i) and the fact that for all $k, l \in \mathbb{Z}$ and $\lambda \in \mathbb{C} \setminus \{l\}$,

$$\lim_{q\to 1} \pi^q_{\mathfrak{M}(\mathbb{C})}(Z_{k,l})(\lambda) = \lim_{q \to 1} \frac{q^{\lambda-k} - q^{k-\lambda}}{q^{\lambda-l} - q^{l-\lambda}} = \frac{\lambda - k}{\lambda - l}.$$

iii) Define $\pi \colon \mathbb{C}[Q,X,Y] \to R$ by $Q \mapsto Q$, $X \mapsto 1$, $Y \mapsto 0$. Then π extends to the localization B of $\mathbb{C}[Q,X,Y]$, giving $\pi_{-\infty}$, because $\pi((X-Y)_{(k)}) = Q^{-k}$ is invertible for all $k \in \mathbb{Z}$. This homomorphism $\pi_{-\infty}$ evidently is involutive, and $\mathbb{Z}$-equivariant because $\pi_{-\infty}(Z_{k+j,l+j}) = \pi_{-\infty}(Z_{k,l})$ for all $j \in \mathbb{Z}$. Similarly, one obtains $\pi_{+\infty}$.

iv) Immediate from iii).

v) Define $\pi \colon \mathbb{C}[Q,X,Y] \to C((0,1] \times \mathbb{R})$ by $Q \mapsto ((q,t) \mapsto q)$, $X \mapsto ((q,t) \mapsto iq^t)$, $Y \mapsto ((q,t) \mapsto -iq^t)$. Since $\pi((X-Y)_{(k)}) = i(q^{t-k} + q^{k-t})$ is invertible for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$, this π extends to B. Moreover, each $\pi(Z_{k,l})$ extends to a continuous function on $C(\Omega)$ as desired, giving π_Ω. ∎

Note that $\pi^1_{+\infty} = \pi^1_{-\infty}$. Using this map, we obtain for each algebra C with an action by $\mathbb{Z}$ an $\mathbb{Z}$-equivariant homomorphism $\pi^1_C \colon B \to C$ sending Q and each $Z_{k,l}$ to 1_C.

2.6.4. Proposition. *There exist isomorphisms of Hopf $*$-algebroids as follows:*

i) $(\pi^q_{\mathfrak{M}(\mathbb{C})})_* \mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) \cong \mathcal{F}_R(\mathrm{SU}(2))$ *for each $q \in (0,1) \cup (1,\infty)$;*

ii) $(\pi^q_{-\infty})_* \mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))$ *for each $q \in (0,\infty)$;*

iii) $(\pi^q_{\infty})_* \mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))^{\mathsf{op}}$ *for each $q \in (0,\infty)$;*

iv) $(\pi^1_{\mathbb{C}[X]})_* \mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2))$ *is isomorphic to the $(\mathbb{C}[X], \mathbb{Z})$-Hopf $*$-algebroid in Example 2.4.14.*

Proof. i) This is immediate from the definitions and Proposition 2.6.1.

ii), iii) Let $\pi^{\pm} = \pi^q_{\pm\infty}$. Then $(\pi^{\pm})_*\mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2))$ is generated by the entries $\alpha', \beta', \gamma', \delta$ of a matrix v' such that $\beta' = -q\gamma'^*$ and $\delta' = \alpha'^*$. Moreover, $v'^{-\mathsf{T}} = \pi^{\pm}_2(F)^{-1}v'\pi^{\pm}_2(F)$ and $\bar{v}' = \pi^{\pm}_2(G)^{-1}v'\pi^{\pm}_2(G)$, where

$$\pi^-_2(F) = \begin{pmatrix} 0 & -1 \\ \pi(Z_{0,-1}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix} = \pi^{\pm}_2(G), \qquad \pi^+_2(F) = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

In the case of π_-, we find that v' is unitary, and obtain the usual presentation of $\mathcal{O}(\mathrm{SU}_q(2))$. Multiplying out the relation $v'^{-\mathsf{T}} = \pi^+_2(F)^{-1}v'\pi^+_2(F)$, one easily verifies the assertion on π^+.

iv) Immediate from the relations $(\pi^1_{\mathbb{C}[X]})_2(F) = (\pi^1_{\mathbb{C}[X]})_2(G) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. ∎

We expect most of the results of [11] to carry over from $\mathcal{F}_R(\mathrm{SU}(2))$ to $\mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2))$.

3. The level of universal C^*-algebras. Throughout this section, we shall only work with unital C^*-algebras. We assume all $*$-homomorphisms to be unital, and B to be a commutative, unital C^*-algebra equipped with a left action of a discrete group Γ. Given a subset X of a normed space V, we denote by $\overline{X} \subseteq v$ its closure and by $[X] \subseteq V$ the closed linear span of X.

3.1. The maximal cotensor product of C^*-algebras with respect to $C^*(\Gamma)$. This subsection reviews the cotensor product of C^*-algebras with respect to the Hopf C^*-algebra $C^*(\Gamma)$ and develops the main properties that will be needed in §3.2. The material presented here is certainly well known to the experts, but we didn't find a suitable reference.

We first recall a few preliminaries.

Let A be a $*$-algebra. A *representation* of A is a $*$-homomorphism into a C^*-algebra. Such a representation π is *universal* if every other representation of A factorizes uniquely through π. A universal representation exists if and only if for each $a \in A$,

$$|a| := \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-homomorphism of } A \text{ into some } C^*\text{-algebra}\} < \infty.$$

Indeed, if $|a|$ is finite for all $a \in A$, then the separated completion of A with respect to $|-|$ carries a natural structure of a C^*-algebra, which is denoted by $C^*(A)$ and called the enveloping C^*-algebra of A, and the natural representation $A \to C^*(A)$ is universal.

The maximal tensor product of C^*-algebras A and C is the enveloping C^*-algebra of the algebraic tensor product $A \otimes C$, and will be denoted by $A \hat{\otimes} C$.

The full group C^*-algebra $C^*(\Gamma)$ of Γ is the enveloping C^*-algebra of the group algebra $\mathbb{C}\Gamma$. We denote by $\Delta_\Gamma \colon C^*(\Gamma) \to C^*(\Gamma)\hat{\otimes}C^*(\Gamma)$ the *comultiplication*, given by $\gamma \mapsto \gamma \otimes \gamma$ for all $\gamma \in \Gamma$, and by $\epsilon_\Gamma \colon C^*(\Gamma) \to \mathbb{C}$ the *counit*, given by $\gamma \mapsto 1$ for all $\gamma \in \Gamma$. Clearly, $(\epsilon_\Gamma \hat{\otimes} \mathrm{id})\Delta_\Gamma = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon_\Gamma)\Delta_\Gamma$.

A completely positive (contractive) map, or brielfy c.p.(c.)-map, from a C^*-algebra A to a C^*-algebra C is a linear map $\phi \colon A \to C$ such that $\phi_n \colon M_n(A) \to M_n(C)$ is positive (and $\|\phi_n\| \leq 1$) for all $n \in \mathbb{N}$.

3.1.1. Definition. A $(\mathbb{C},\Gamma)$-*C^*-algebra* is a unital C^*-algebra A with injective unital $*$-homomorphisms $\delta_A\colon A \to C^*(\Gamma)\hat{\otimes}A$ and $\bar{\delta}_A\colon A \to A\hat{\otimes}C^*(\Gamma)$ such that $(\mathrm{id}\hat{\otimes}\delta_A)\circ\delta_A = (\Delta_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A$, $(\bar{\delta}_A\hat{\otimes}\mathrm{id})\circ\bar{\delta}_A = (\mathrm{id}\hat{\otimes}\Delta_\Gamma)\circ\bar{\delta}_A$ and $(\delta_A\hat{\otimes}\mathrm{id})\circ\bar{\delta}_A = (\mathrm{id}\hat{\otimes}\bar{\delta}_A)\circ\delta_A$. A *morphism* of $(\mathbb{C},\Gamma)$-C^*-algebras A and C is a unital $*$-homomorphism $\pi\colon A \to C$ satisfying $\delta_C\circ\pi = (\mathrm{id}\hat{\otimes}\pi)\circ\delta_A$ and $\bar{\delta}_C\circ\pi = (\pi\hat{\otimes}\mathrm{id})\circ\bar{\delta}_A$. We denote by $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ the category of all $(\mathbb{C},\Gamma)$-C^*-algebras. Replacing $*$-homomorphisms by c.p.-maps, we define c.p.-maps of $(\mathbb{C},\Gamma)$-C^*-algebras and the category $\mathbf{C^*\text{-}Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$.

3.1.2. Remark. Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra. Then $(\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A = \mathrm{id}_A$ because

$$\delta_A(\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A = (\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})\circ(\mathrm{id}\hat{\otimes}\delta_A)\circ\delta_A = ((\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\Delta_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A = \delta_A,$$

and likewise $(\mathrm{id}\hat{\otimes}\epsilon_\Gamma)\circ\bar{\delta}_A = \mathrm{id}_A$.

Let A and C be $(\mathbb{C},\Gamma)$-C^*-algebras. Then the maximal tensor product $A\hat{\otimes}C$ is a $(\mathbb{C},\Gamma)$-C^*-algebra with respect to $\delta_A\hat{\otimes}\mathrm{id}$ and $\mathrm{id}\hat{\otimes}\bar{\delta}_C$, and the assignments $(A,C)\mapsto A\hat{\otimes}C$ and $(\phi,\psi)\mapsto\phi\hat{\otimes}\psi$ define a product $-\hat{\otimes}-$ on $\mathbf{C^*\text{-}Alg}^{(\mathrm{c.p.})}_{(\mathbb{C},\Gamma)}$ that is associative in the obvious sense. Unless Γ is trivial, this product cannot be unital because it forgets $\bar{\delta}_A$ and δ_C.

With respect to the restrictions of $\delta_A\hat{\otimes}\mathrm{id}$ and $\mathrm{id}\hat{\otimes}\bar{\delta}_C$, the subspace

$$A\overset{\Gamma}{\hat{\otimes}}C := \{x\in A\hat{\otimes}C : (\bar{\delta}_A\hat{\otimes}\mathrm{id})(x) = (\mathrm{id}\hat{\otimes}\delta_C)(x)\}\subseteq A\hat{\otimes}C$$

evidently is a $(\mathbb{C},\Gamma)$-C^*-algebra again. Moreover, given morphisms of $(\mathbb{C},\Gamma)$-C^*-algebras $\phi\colon A\to C$ and $\psi\colon D\to E$, the product $\phi\hat{\otimes}\psi$ restricts to a morphism $\phi\overset{\Gamma}{\hat{\otimes}}\psi\colon A\overset{\Gamma}{\hat{\otimes}}D\to C\overset{\Gamma}{\hat{\otimes}}E$. We thus obtain a second product $-\overset{\Gamma}{\hat{\otimes}}-$ on $\mathbf{C^*\text{-}Alg}^{(\mathrm{c.p.})}_{(\mathbb{C},\Gamma)}$ that is associative in the natural sense, and unital in the following sense.

Regard $C^*(\Gamma)$ as a $(\mathbb{C},\Gamma)$-C^*-algebra with respect to Δ_Γ. Then for each $(\mathbb{C},\Gamma)$-C^*-algebra A, the maps δ_A and $\bar{\delta}_A$ are isomorphisms of $(\mathbb{C},\Gamma)$-C^*-algebras

$$\delta_A\colon A\xrightarrow{\cong}C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A, \qquad \bar{\delta}_A\colon A\xrightarrow{\cong}A\overset{\Gamma}{\hat{\otimes}}C^*(\Gamma).$$

Indeed, they evidently are morphisms, and surjective because

$$x = (\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})((\Delta_\Gamma\hat{\otimes}\mathrm{id})(x)) = (\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})((\mathrm{id}\hat{\otimes}\delta_A)(x)) = \delta_A((\epsilon_\Gamma\hat{\otimes}\mathrm{id})(x))$$

for each $x\in C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A$ and likewise $y=\bar{\delta}_A((\mathrm{id}\hat{\otimes}\epsilon_\Gamma)(y))$ for each $y\in A\overset{\Gamma}{\hat{\otimes}}C^*(\Gamma)$.

We next construct a natural transformation $p\colon(-\hat{\otimes}-)\to(-\overset{\Gamma}{\hat{\otimes}}-)$ which will be needed to prove associativity of the product of (B,Γ)-C^*-algebras in §3.2. The construction is based on ideas taken from [1, §7], and carries over from $C^*(\Gamma)$ to any Hopf C^*-algebra H equipped with a Haar mean $H\hat{\otimes}H\to H$; see also [13].

3.1.3. Lemma. *There exists a unique state ν on $C^*(\Gamma)\hat{\otimes}C^*(\Gamma)$ such that $\nu(\gamma\otimes\gamma')=\delta_{\gamma,\gamma'}1$ for all $\gamma,\gamma'\in\Gamma$. Moreover, $\nu\circ\Delta_\Gamma=\epsilon_\Gamma$ and $(\mathrm{id}\hat{\otimes}\nu)\circ(\Delta_\Gamma\hat{\otimes}\mathrm{id})=(\nu\hat{\otimes}\mathrm{id})\circ(\mathrm{id}\hat{\otimes}\Delta_\Gamma)$.*

Proof. This follows from [13, Theorem 0.1], but let us include the short direct proof. Uniqueness is clear. To construct ν, denote by $(\epsilon_\gamma)_{\gamma\in\Gamma}$ the canonical orthonormal basis of $l^2(\Gamma)$, by $\lambda, \rho\colon C^*(\Gamma) \to \mathcal{L}(l^2(\Gamma))$ the representations given by $\lambda(\gamma)\epsilon_{\gamma'} = \epsilon_{\gamma\gamma'}$ and $\rho(\gamma)\epsilon_{\gamma'} = \epsilon_{\gamma'\gamma^{-1}}$ for all $\gamma, \gamma' \in \Gamma$, and by $\lambda \times \rho\colon C^*(\Gamma)\hat{\otimes}C^*(\Gamma) \to \mathcal{L}(l^2(\Gamma))$ the representation given by $x \otimes y \mapsto \lambda(x)\rho(y)$. Then $\nu := \langle \epsilon_e | (\lambda \times \rho)(-)\epsilon_e \rangle$ satisfies $\nu(\gamma \otimes \gamma') = \delta_{\gamma,\gamma'} 1$ for all $\gamma, \gamma' \in \Gamma$. The remaining equations follow easily. ∎

3.1.4. Lemma. *i) For every $(\mathbb{C}, \Gamma)$-C^*-algebra A, the maps*

$$\bar{p}_A := (\mathrm{id}\,\hat{\otimes}\nu)(\bar{\delta}_A\hat{\otimes}\,\mathrm{id})\colon A\hat{\otimes}C^*(\Gamma) \to A, \quad p_A := (\nu\hat{\otimes}\,\mathrm{id})(\mathrm{id}\,\hat{\otimes}\delta_A)\colon C^*(\Gamma)\hat{\otimes}A \to A$$

are morphisms in $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$ and satisfy $p_A \circ \delta_A$ and $\bar{p}_A \circ \bar{\delta}_A = \mathrm{id}$.

ii) The families $(p_A)_A$ and $(\bar{p}_A)_A$ are natural transformations from $-\hat{\otimes}C^*(\Gamma)$ and $C^*(\Gamma)\hat{\otimes}-$, respectively, to id, regarded as functors on $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$.

Proof. i) The map p_A is a morphism in $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$ because $\bar{\delta}_A \circ p_A = (p_A\hat{\otimes}\,\mathrm{id}) \circ \bar{\delta}_A$ and

$$\begin{aligned}
\delta_A \circ p_A &= (\nu\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\delta_A)(\mathrm{id}\,\hat{\otimes}\delta_A) \\
&= (\nu\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\Delta_\Gamma\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_A) \\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\Delta_\Gamma\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_A) \\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\delta_A) \circ (\Delta_\Gamma\hat{\otimes}\,\mathrm{id}) = (\mathrm{id}\,\hat{\otimes}p_A) \circ (\Delta_\Gamma\hat{\otimes}\,\mathrm{id}).
\end{aligned}$$

Moreover, $p_A \circ \delta_A = (\nu\hat{\otimes}\,\mathrm{id})(\mathrm{id}\,\hat{\otimes}\delta_A)\delta_A = (\nu\Delta_\Gamma\hat{\otimes}\,\mathrm{id})\delta_A = (\epsilon_\Gamma\hat{\otimes}\,\mathrm{id})\delta_A = \mathrm{id}$ and similarly $\bar{p}_A \circ \bar{\delta}_A = \mathrm{id}$.

ii) This follows from the fact that $(\delta_A)_A$ and $(\bar{\delta}_A)_A$ are natural transformations. ∎

3.1.5. Proposition. *i) Let A, C be $(\mathbb{C}, \Gamma)$-C^*-algebras. Then the map*

$$p_{A,C} := (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A\hat{\otimes}\delta_C)\colon A\hat{\otimes}C \to A\hat{\otimes}C$$

is equal to $(\mathrm{id}\,\hat{\otimes}p_C) \circ (\bar{\delta}_A\hat{\otimes}\,\mathrm{id})$ and $(\bar{p}_A\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_C)$, a morphism in $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$, and a conditional expectation onto $A \overset{\Gamma}{\hat{\otimes}} C \subseteq A\hat{\otimes}C$ in the sense that $p_{A,C}(xyz) = xp_{A,C}(y)z$ for all $x, z \in A \overset{\Gamma}{\hat{\otimes}} C$ and $y \in A\hat{\otimes}C$.

ii) The family $(p_{A,C})_{A,C}$ is a natural transformation from $-\hat{\otimes}-$ to $-\overset{\Gamma}{\hat{\otimes}}-$, regarded as functors on $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)} \times \mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$.

Proof. i) The equality follows immediately from the definitions and implies that $p_{A,C}$ is a morphism as claimed. Next, $p_{A,C}(A\hat{\otimes}C) \subseteq A \overset{\Gamma}{\hat{\otimes}} C$ because

$$\begin{aligned}
(\bar{\delta}_A\hat{\otimes}\,\mathrm{id}) \circ p_{A,C} &= (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A\hat{\otimes}\delta_C) \\
&= (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\Delta_\Gamma\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A\hat{\otimes}\delta_C) \\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\Delta_\Gamma\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A\hat{\otimes}\delta_C) = (\mathrm{id}\,\hat{\otimes}\delta_C) \circ p_{A,C}.
\end{aligned}$$

On the other hand, $p_{A,C}(x) = (\bar{p}_A\hat{\otimes}\,\mathrm{id})((\mathrm{id}\,\hat{\otimes}\delta_C)(x)) = (\bar{p}_A\hat{\otimes}\,\mathrm{id})((\delta_A\hat{\otimes}\,\mathrm{id})(x)) = x$ for all $x \in A \overset{\Gamma}{\hat{\otimes}} C$. Thus, $p_{A,C}$ is a completely positive projection from $A\hat{\otimes}C$ onto $A \overset{\Gamma}{\hat{\otimes}} C$ and hence a conditional expectation (see, e.g., [5, Proposition 1.5.7]).

ii) Straightforward. ∎

Denote by $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)} \subseteq \mathbf{Alg}^*_{(C,\Gamma)}$ the full subcategory formed by all $(\mathbb{C},\Gamma)$-$*$-algebras that have an enveloping C^*-algebra. We shall need an adjoint pair of functors

$$*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)} \underset{(-)_{*,*}}{\overset{C^*(-)}{\rightleftarrows}} \mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}. \tag{11}$$

The functor $C^*(-)$ is defined as follows. Let $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$. Using the universal property of $C^*(A)$, we obtain unique $*$-homomorphisms $\delta_{C^*(A)} \colon C^*(A) \to C^*(\Gamma)\hat{\otimes}C^*(A)$ and $\bar{\delta}_{C^*(A)} \colon C^*(A) \to C^*(A)\hat{\otimes}C^*(\Gamma)$ such that $\delta_{C^*(A)}(a) = \gamma \otimes a$ and $\bar{\delta}_{C^*(A)}(a) = a \otimes \gamma'$ for all $a \in A_{\gamma,\gamma'}, \gamma, \gamma' \in A$, and with respect to these $*$-homomorphisms, $C^*(A)$ becomes a $(\mathbb{C},\Gamma)$-C^*-algebra. Moreover, every morphism $\pi \colon A \to C$ in $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ extends uniquely to a $*$-homomorphism $C^*(\pi) \colon C^*(A) \to C^*(C)$ which is a morphism in $\mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}$.

The functor $(-)_{*,*}$ is defined as follows. Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra and let

$$A_{\gamma,\gamma'} := \{a \in A : \delta(a) = \gamma \otimes a, \bar{\delta}(a) = a \otimes \gamma'\} \subseteq A \quad \text{for all } \gamma, \gamma' \in \Gamma.$$

Then the sum $A_{*,*} := \sum_{\gamma,\gamma'} A_{\gamma,\gamma'} \subseteq A$ is a $(\mathbb{C},\Gamma)$-$*$-algebra, and, every morphism $\pi \colon A \to \mathbb{C}$ of $(\mathbb{C},\Gamma)$-C^*-algebras restricts to a morphism $\pi_{*,*} \colon A_{*,*} \to C_{*,*}$ of $(\mathbb{C},\Gamma)$-$*$-algebras. We thus obtain a functor $(-)_{*,*} \colon \mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)} \to \mathbf{Alg}^*_{(C,\Gamma)}$.

3.1.6. LEMMA. *$(-)_{*,*}$ takes values in $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$.*

Proof. Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra. Then for every $*$-representation π of $A_{*,*}$, the restriction to the C^*-subalgebra $A_{e,e}$ is contractive and thus $\|\pi(a)\|^2 = \|\pi(a^*a)\| \leq \|a^*a\| = \|a\|^2$ for all $a \in A_{\gamma,\gamma'}$ $\gamma, \gamma' \in \Gamma$. Since such elements a span $A_{*,*}$, we can conclude $|a'| < \infty$ for all $a' \in A_{*,*}$. ∎

For every $(\mathbb{C},\Gamma)$-C^*-algebra A, the morphisms p_A and $\bar{p}_A$ yield a morphism

$$P_A := \bar{p}_A \circ (p_A \hat{\otimes} \mathrm{id}) = p_A \circ (\mathrm{id} \hat{\otimes} \bar{p}_A) \colon C^*(\Gamma)\hat{\otimes}A\hat{\otimes}C^*(\Gamma) \to A \quad \text{in} \quad \mathbf{C}^*\text{-}\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}.$$

3.1.7. LEMMA. *i) Let $A \in \mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}$. Then for all $\gamma, \gamma', \beta, \beta' \in \Gamma$,*

$$P_A(\gamma \otimes A \otimes \gamma') = A_{\gamma,\gamma'}, \quad P_A(\beta \otimes A_{\gamma,\gamma'} \otimes \beta') = \delta_{\beta,\gamma}\delta_{\beta',\gamma'}A_{\gamma,\gamma'}, \quad A = \overline{A_{*,*}}.$$

ii) Let $A \in *\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$. Then $C^*(A)_{\gamma,\gamma'} = \overline{A_{\gamma,\gamma'}}$ for all $\gamma, \gamma' \in \Gamma$.

Proof. We only prove i); assertion ii) follows similarly. First, $P_A(\gamma \otimes A \otimes \gamma') \subseteq A_{\gamma,\gamma'}$ because P_A is a morphism in $\mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}$ and $\Delta_\Gamma(\gamma'') = \gamma'' \otimes \gamma''$ for $\gamma'' = \gamma, \gamma'$. This inclusion, the relation $C^*(\Gamma)\hat{\otimes}A\hat{\otimes}C^*(\Gamma) = \overline{\sum_{\gamma,\gamma'} \gamma \otimes A \otimes \gamma'}$ and continuity and surjectivity of P_A imply $\overline{A_{*,*}} = A$. The equation $P_A(\beta \otimes A_{\gamma,\gamma'} \otimes \beta') = \delta_{\beta,\gamma}\delta_{\beta',\gamma'}A_{\gamma,\gamma'}$ follows from the definitions and implies that the inclusion $P_A(\gamma \otimes A \otimes \gamma') \subseteq A_{\gamma,\gamma'}$ is an equality. ∎

For every A in $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and C in $\mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}$, we get canonical morphisms $\eta_A \colon A \to C^*(A)_{*,*}$ in $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and $\epsilon_C \colon C^*(C_{*,*}) \to C$ in $\mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}$.

3.1.8. PROPOSITION. *The functors $C^*(-)$ and $(-)_{*,*}$ are adjoint, where the unit and counit of the adjunction are the families $(\eta_A)_A$ and $(\epsilon_C)_C$, respectively. Furthermore, $(-)_{*,*}$ is faithful.*

Proof. Let $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$ and $C \in \mathbf{C}^*\text{-}\mathbf{Alg}_{(B,\Gamma)}$. Since the representation $A \to C^*(A)$ has dense image and is universal, the assignment $(C^*(A) \xrightarrow{\pi} C) \mapsto (A \xrightarrow{\eta_A} C^*(A)_{*,*} \xrightarrow{\pi_{*,*}} C_{*,*})$

yields a bijective correspondence between morphisms $C^*(A) \to C$ and morphisms $A \to C_{*,*}$. The functor $(-)_{*,*}$ is faithful because $A_{*,*} \subseteq A$ is dense. ∎

3.1.9. Remark. Similar arguments as in the proof of Lemma 3.1.7 show that for all $A, C \in \mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$, $D, E \in *\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and all $\gamma, \gamma'' \in \Gamma$,

$$(A \overset{\Gamma}{\hat{\otimes}} C)_{\gamma,\gamma''} = \overline{\sum_{\gamma'} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''}}, \qquad (C^*(D) \overset{\Gamma}{\hat{\otimes}} C^*(E))_{\gamma,\gamma'} = \overline{\sum_{\gamma'} D_{\gamma,\gamma'} \otimes E_{\gamma',\gamma''}}.$$

A *short exact sequence* of $(\mathbb{C},\Gamma)$-C^*-algebra is a sequence of morphisms $J \xrightarrow{\iota} A \xrightarrow{\pi} C$ in $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ such that $\ker \iota = 0$, $\iota(J) = \ker \pi$ and $\pi(A) = C$. A functor on $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ is *exact* if it maps short exact sequences to short exact sequences.

3.1.10. Proposition. *For every $(\mathbb{C},\Gamma)$-C^*-algebra D, the functors $- \overset{\Gamma}{\hat{\otimes}} D$ and $D \overset{\Gamma}{\hat{\otimes}} -$ on $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ are exact.*

Proof. If $J \xrightarrow{\iota} A \xrightarrow{\pi} C$ is a short exact sequence in $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$, then $J \hat{\otimes} D \xrightarrow{\iota \hat{\otimes} \mathrm{id}} A \hat{\otimes} D \xrightarrow{\pi \hat{\otimes} \mathrm{id}} C \hat{\otimes} D$ is exact (see, e.g., [5, Proposition 3.7]), whence $\ker(\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id}) = 0$ and

$$\begin{aligned} \ker(\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id}) = p_{A,D}(\ker(\pi \hat{\otimes} \mathrm{id})) &= p_{A,D}((\iota \hat{\otimes} \mathrm{id})(J \hat{\otimes} D)) \\ &= (\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(p_{J,D}(J \hat{\otimes} D)) = (\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(J \overset{\Gamma}{\hat{\otimes}} D), \\ (\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(A \overset{\Gamma}{\hat{\otimes}} D) &= (\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(p_{A,D}(A \hat{\otimes} D)) \\ &= p_{C,D}((\pi \hat{\otimes} \mathrm{id})(A \hat{\otimes} D)) = p_{C,D}(C \hat{\otimes} D) = C \overset{\Gamma}{\hat{\otimes}} D. \end{aligned}$$ ∎

3.2. The monoidal category of (B,Γ)-C^*-algebras. We now define an analogue of (B,Γ)-$*$-algebras on the level of universal C^*-algebras, and construct a monoidal product which is unital and associative.

3.2.1. Definition. A (B,Γ)-*C^*-algebra* is a $(\mathbb{C},\Gamma)$-C^*-algebra A equipped with unital $*$-homomorphisms $r_A, s_A \colon B \to A_{e,e}$ such that $A_{*,*}$ is a (B,Γ)-$*$-algebra with respect to the map $r_A \times s_A \colon B \otimes B \to A_{e,e}$, $b \otimes b' \mapsto r_A(b)s_A(b')$. A *morphism* of (B,Γ)-C^*-algebras is a $B \otimes B$-linear morphism of $(\mathbb{C},\Gamma)$-C^*-algebras. We denote by $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$ the category of all (B,Γ)-C^*-algebras. Replacing $*$-homomorphisms by c.p.-maps, we define c.p.-maps of (B,Γ)-C^*-algebras and the category $\mathbf{C^*\text{-}Alg}^{\mathrm{c.p.}}_{(B,\Gamma)}$.

Denote by $*\text{-}\mathbf{Alg}^0_{(B,\Gamma)} \subseteq *\text{-}\mathbf{Alg}_{(B,\Gamma)}$ the full subcategory formed by all (B,Γ)-$*$-algebras that have an enveloping C^*-algebra. This category is related to $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$ as follows. If $C \in \mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$, then $C_{*,*} \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$ by Lemma 3.1.6. Conversely, if $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$, then $C^*(A)$ carries a natural structure of a (B,Γ)-C^*-algebra. The canonical maps $\eta_A \colon A \to C^*(A)_{*,*}$ and $\epsilon_C \colon C^*(C_{*,*}) \to C$ are morphisms in $*\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$ and $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$, respectively, and Proposition 3.1.8 therefore implies:

3.2.2. COROLLARY. *The assignments $A \mapsto C^*(A), \pi \mapsto C^*(\pi)$ and $A \mapsto A_{*,*}, \pi \mapsto \pi_{*,*}$ form a pair of adjoint functors*

$$*\text{-}\mathbf{Alg}^0_{(B,\Gamma)} \underset{(-)_{*,*},}{\overset{C^*(-)}{\rightleftarrows}} \mathbf{C}^*\text{-}\mathbf{Alg}_{(B,\Gamma)}$$

where the unit and counit of the adjunction are the families $(\eta_A)_A$ and $(\epsilon_C)_C$, respectively. Furthermore, $(-)_{*,*}$ is faithful.

Let A and C be (B,Γ)-C^*-algebras. Then the $(\mathbb{C},\Gamma)$-C^*-algebra $A \overset{\Gamma}{\hat{\otimes}} C$ is a (B,Γ)-C^*-algebra with respect to the $*$-homomorphisms $r\colon b \mapsto r_A(b)\hat{\otimes}1$ and $s\colon b' \mapsto 1\hat{\otimes}s_C(b')$, and the assignments $(A,C) \mapsto A \overset{\Gamma}{\hat{\otimes}} C$ and $(\phi,\psi) \mapsto \phi \overset{\Gamma}{\hat{\otimes}} \psi$ define a product $-\overset{\Gamma}{\hat{\otimes}}-$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(B,\Gamma)}$ that is associative in the obvious sense. Using the map

$$t_{A,C}\colon B \to A\overset{\Gamma}{\hat{\otimes}}C, \quad b \mapsto s_A(b)\hat{\otimes}1 - 1\hat{\otimes}r_C(b),$$

we define an ideal $(t_{A,C}(B)) \subseteq A\overset{\Gamma}{\hat{\otimes}}C$. Since $t_{A,C}(B) \subseteq (A\overset{\Gamma}{\hat{\otimes}}C)_{e,e}$, the quotient

$$A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}C := (A\overset{\Gamma}{\hat{\otimes}}C)/(t_{A,C}(B)).$$

inherits the (B,Γ)-C^*-algebra structure of $A\overset{\Gamma}{\hat{\otimes}}C$. For every pair of morphisms $\phi\colon A \to C$ and $\psi\colon D \to E$ in $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(B,\Gamma)}$, the morphism $\phi\overset{\Gamma}{\hat{\otimes}}\psi$ maps $t_{A,D}(B)$ to $t_{C,E}(B)$ and thus factorizes to a morphism $\phi\underset{B}{\overset{\Gamma}{\hat{\otimes}}}\psi\colon A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D \to C\underset{B}{\overset{\Gamma}{\hat{\otimes}}}E$. We thus obtain a product $-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(B,\Gamma)}$, and the canonical quotient map $q_{A,C}\colon A\overset{\Gamma}{\hat{\otimes}}C \to A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}C$ yields a natural transformation $q = (q_{A,C})_{A,C}$ from $-\overset{\Gamma}{\hat{\otimes}}-$ to $-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-$.

3.2.3. REMARKS. i) For all (B,Γ)-C^*-algebras A, C, we have $[t_{A,C}(B)(A\overset{\Gamma}{\hat{\otimes}}C)] = (t_{A,C}(B)) = [(A\overset{\Gamma}{\hat{\otimes}}C)t_{A,C}(B)]$. Indeed, a short calculation shows that for all $\gamma,\gamma',\gamma'' \in \Gamma$, $a \in A_{\gamma,\gamma'}$, $c \in C_{\gamma',\gamma''}$, $b \in B$, $(a\otimes c)t_{A,C}(b) = t_{A,C}(\gamma'(b))(a\otimes c)$, and now the assertion follows from Remark 3.1.9.

ii) For every (B,Γ)-C^*-algebra D, the functors $-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D$ and $D\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-$ on $\mathbf{C}^*\text{-}\mathbf{Alg}_{(B,\Gamma)}$ preserve surjections because the functors $-\overset{\Gamma}{\hat{\otimes}}D$ and $D\overset{\Gamma}{\hat{\otimes}}-$ do so by Proposition 3.1.10.

We show that the full crossed product $B\hat{\rtimes}\Gamma := C^*(B\rtimes\Gamma)$ is the unit for the product $-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-$. Denote by $\iota_\Gamma\colon C^*(\Gamma) \to B\hat{\rtimes}\Gamma$ the natural inclusion.

3.2.4. PROPOSITION. *i) For each (B,Γ)-C^*-algebra A, the $*$-homomorphisms*

$$L_A\colon A \xrightarrow{\delta_A} C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A \xrightarrow{\iota_\Gamma\overset{\Gamma}{\hat{\otimes}}\mathrm{id}} (B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A \xrightarrow{q_{B\hat{\rtimes}\Gamma,A}} (B\hat{\rtimes}\Gamma)\underset{B}{\overset{\Gamma}{\hat{\otimes}}}A$$

and

$$R_A\colon A \xrightarrow{\bar{\delta}_A} A \overset{\Gamma}{\hat{\otimes}} C^*(\Gamma) \xrightarrow{\mathrm{id} \overset{\Gamma}{\hat{\otimes}} \iota_\Gamma} A \overset{\Gamma}{\hat{\otimes}} (B \hat{\rtimes} \Gamma) \xrightarrow{q_{A, B\hat{\rtimes}\Gamma}} A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (B \hat{\rtimes} \Gamma),$$

are isomorphisms of (B,Γ)-C^**-algebras.*

ii) The families $R = (R_A)_A$ *and* $L = (L_A)_A$ *form natural isomorphism from* id *to* $((B \hat{\rtimes} \Gamma) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -)$ *and* $(- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (B \hat{\rtimes} \Gamma))$*, respectively, regarded as functors on* $\mathbf{C^*\text{-}Alg}^{(\mathrm{c.p.})}_{(B,\Gamma)}$.

Proof. One easily checks that each L_A is a morphism of (B,Γ)-C^*-algebras and that $L = (L_A)_A$ is a natural transformation. We show that L_A is an isomorphism for every (B,Γ)-C^*-algebra A. The assertions concerning $R = (R_A)_A$ then follow similarly.

To prove that L_A is surjective, we only need to show that $(t_{B\hat{\rtimes}\Gamma,A}(B)) + C^*(\Gamma) \overset{\Gamma}{\hat{\otimes}} A$ is dense in $(B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A$. But by Remark 3.1.9, elements of the form

$$b\gamma \otimes a = t_{B\hat{\rtimes}\Gamma,A}(b)(\gamma \otimes a) + \gamma \otimes r_A(b)a, \quad \text{where } b \in B, a \in A_{\gamma,\gamma'}, \gamma, \gamma' \in \Gamma,$$

are linearly dense in $(B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A$.

To prove that L_A is injective, we only need to show that the intersection

$$J := (\iota_\Gamma(C^*(\Gamma)) \overset{\Gamma}{\hat{\otimes}} A) \cap (t_{B\hat{\rtimes}\Gamma,A}(B)) \subseteq (B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A$$

equals 0. Since $J = \overline{J_{*,*}}$ by Lemma 3.1.7, it suffices to show that $J_{\gamma,\gamma'} = 0$ for all $\gamma, \gamma' \in \Gamma$. Note that $J_{\gamma,\gamma'} = [\gamma \otimes A_{\gamma,\gamma'}] \cap [(B\gamma \otimes A_{\gamma,\gamma'}) t_{B\hat{\rtimes}\Gamma,A}(B)]$. For each $\gamma, \gamma' \in \Gamma$, define a linear map $R_{\gamma,\gamma'}\colon B\gamma \otimes A_{\gamma,\gamma'} \to A_{\gamma,\gamma'}$ by $b\gamma \otimes a \mapsto r(b)a$. Then $R_{e,e}$ extends to a $*$-homomorphism on the C^*-subalgebra $B \hat{\otimes} A_{e,e} \subseteq (B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A$, and each $R_{\gamma,\gamma'}$ extends to a bounded linear map on $[B\gamma \otimes A_{\gamma,\gamma'}] \subseteq (B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A$ because

$$\|R_{\gamma,\gamma'}(z)\|^2 = \|R_{\gamma,\gamma'}(z) R_{\gamma,\gamma'}(z)^*\| = \|R_{e,e}(zz^*)\| \leq \|zz^*\| = \|z\|^2$$

for all $z \in B\gamma \otimes A_{\gamma,\gamma'}$. Now, $R_{\gamma,\gamma'}(z t_{B\hat{\rtimes}\Gamma,A}(b)) = 0$ for all $z \in [B\gamma \otimes A_{\gamma,\gamma'}]$ and $b \in B$, and $R_{\gamma,\gamma'}(\gamma \otimes a) = a$ for all $a \in A_{\gamma,\gamma'}$. Consequently, $J_{\gamma,\gamma'} = 0$. ∎

We now show that the product $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$ is associative. Let A, C, D be (B,Γ)-C^*-algebras, denote by $a_{A,C,D}\colon (A \overset{\Gamma}{\hat{\otimes}} C) \overset{\Gamma}{\hat{\otimes}} D \to A \overset{\Gamma}{\hat{\otimes}} (C \overset{\Gamma}{\hat{\otimes}} D)$ the canonical isomorphism and let

$$\Phi_{A,C,D} := q_{\left(A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C\right), D} \circ (q_{A,C} \overset{\Gamma}{\hat{\otimes}} \mathrm{id})\colon (A \overset{\Gamma}{\hat{\otimes}} C) \overset{\Gamma}{\hat{\otimes}} D \to (A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D,$$

$$\Psi_{A,C,D} := q_{A, \left(C \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D\right)} \circ (\mathrm{id} \overset{\Gamma}{\hat{\otimes}} q_{C,D})\colon A \overset{\Gamma}{\hat{\otimes}} (C \overset{\Gamma}{\hat{\otimes}} D) \to A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (C \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D).$$

3.2.5. Lemma. *i)* $\ker \Phi_{A,C,D}$ *and* $\ker \Psi_{A,C,D}$ *are generated as ideals by* $t_{A,C}(B) \otimes 1_D + t_{\left(A \overset{\Gamma}{\hat{\otimes}} C, D\right)}(B)$ *and* $1_A \otimes t_{C,D}(B) + t_{A, \left(C \overset{\Gamma}{\hat{\otimes}} D\right)}(B)$*, respectively.*

ii) *There exists a unique isomorphism of* (B,Γ)*-*C^**-algebras* $\tilde{a}_{A,C,D}\colon (A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D \to A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (C \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D)$ *such that* $\tilde{a}_{A,C,D} \circ \Phi_{A,C,D} = \Psi_{A,C,D} \circ a_{A,C,D}$.

Proof. i) By Proposition 3.1.10, $\ker(q_{A,C} \overset{\Gamma}{\hat{\otimes}} \mathrm{id}_D) = (\ker q_{A,C}) \overset{\Gamma}{\hat{\otimes}} D = (t_{A,C}(B)) \overset{\Gamma}{\hat{\otimes}} D$, and $\ker q_{\left(A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C\right),D}$ is generated as an ideal by $(q_{A,C} \overset{\Gamma}{\hat{\otimes}} \mathrm{id}_D)\big(t_{\left(A \overset{\Gamma}{\hat{\otimes}} C, D\right)}(B)\big)$. The assertion on $\Phi_{A,C,D}$ follows, and the assertion concerning $\Psi_{A,C,D}$ follows similarly.

ii) Using i), one easily verifies that $a_{A,C,D}(\ker \Phi_{A,C,D}) = \ker \Psi_{A,C,D}$. We thus get an isomorphism $\tilde{a}_{A,C,D}$ of C^*-algebras which is easily seen to be an isomorphism of (B,Γ)-C^*-algebras. ∎

3.2.6. Proposition. *The family* $(\tilde{a}_{A,C,D})_{A,C,D}$ *is a natural isomorphism from* $(- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$ *to* $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -)$.

Proof. By Lemma 3.2.5, we only need to check naturality which is straightforward. ∎

3.3. Free dynamical quantum groups on the level of universal C^*-algebras.

Given the monoidal structure on the category of all (B,Γ)-C^*-algebras, the definitions in §2.1–§2.4 carry over as follows:

3.3.1. Definition. A *compact* (B,Γ)*-Hopf* C^**-algebroid* is a (B,Γ)-C^*-algebra A with a morphism $\Delta\colon A \to A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} A$ satisfying

i) $(\Delta \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \mathrm{id}) \circ \Delta = (\mathrm{id} \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \Delta) \circ \Delta$ (*coassociativity*),

ii) $[\Delta(A)(1 \otimes A_{e,*})] = A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} A = [(A_{*,e} \otimes 1)\Delta(A)]$, where $A_{e,*} = [\sum_\gamma A_{e,\gamma}] \subseteq A$ and $A_{*,e} = [\sum_\gamma A_{\gamma,e}] \subseteq A$ (*cancellation*).

A *counit* for a compact (B,Γ)-Hopf C^*-algebroid (A,Δ) is a morphism $\epsilon\colon A \to B \hat{\rtimes} \Gamma$ of (B,Γ)-C^*-algebras satisfying $(\epsilon \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \mathrm{id}) \circ \Delta = \mathrm{id}_A = (\mathrm{id} \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \epsilon) \circ \Delta$. A *morphism* of compact (B,Γ)-Hopf C^*-algebroids (A,Δ_A) and (C,Δ_C) is a morphism $\pi\colon A \to C$ satisfying $\Delta_C \circ \pi = (\pi \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \pi) \circ \Delta_A$. We denote the category of all compact (B,Γ)-Hopf C^*-algebroids by $\mathbf{C^*\text{-}Hopf}_{(B,\Gamma)}$.

Denote by $\mathbf{Hopf}^0_{(B,\Gamma)}$ the full subcategory of $\mathbf{Hopf}^*_{(B,\Gamma)}$ formed by all (B,Γ)-Hopf $*$-algebroids (A,Δ,ϵ,S) where $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$.

3.3.2. Proposition. *Let* $(A,\Delta,\epsilon,S) \in \mathbf{Hopf}^*_{(B,\Gamma)}$. *Then* Δ *extends to a morphism of* (B,Γ)*-*C^**-algebras* $\Delta_{C^*(A)}\colon C^*(A) \to C^*(A) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C^*(A)$ *such that* $(C^*(A), \Delta_{C^*(A)})$ *is a compact* (B,Γ)*-Hopf* C^**-algebroid with counit* $C^*(\epsilon)\colon C^*(A) \to C^*(B \rtimes \Gamma) = B \hat{\rtimes} \Gamma$.

Proof. The composition of Δ with the canonical map $A\tilde{\otimes}A \to C^*(A) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C^*(A)$ extends to a morphism $\Delta_{C^*(A)}$ by the universal property of $C^*(A)$. Coassociativity of Δ and density of A in $C^*(A)$ imply coassociativity of $\Delta_{C^*(A)}$, and cancellation follows from Remark 2.1.9 ii). ■

The assignments $(A, \Delta, \epsilon, S) \mapsto (C^*(A), \Delta_{C^*(A)})$ and $\pi \mapsto C^*(\pi)$ evidently form a functor $\mathbf{Hopf}^0_{(B,\Gamma)} \to \mathbf{C^*}\text{-}\mathbf{Hopf}_{(B,\Gamma)}$.

We now apply this functor to the free unitary and free orthogonal dynamical quantum groups $A^B_{\mathrm{u}}(\nabla, F)$ and $A^B_{\mathrm{o}}(\nabla, F, G)$ introduced in Definition 2.4.4, Theorem 2.4.5 and Definition 2.4.11, Theorem 2.4.12, respectively.

Let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\nabla = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in M_n(B \rtimes \Gamma)$.

Assume that $F \in \mathrm{GL}_n(B)$ be ∇-even in the sense that $\nabla F \nabla^{-1} \in M_n(B)$. Then the (B,Γ)-Hopf $*$-algebroid $A^B_{\mathrm{u}}(\nabla, F)$ is generated by a copy of $B \otimes B$ and entries of a unitary matrix $v \in M_n(A^B_{\mathrm{u}}(\nabla, F))$ and therefore has an enveloping C^*-algebra. Applying the functor $C^*(-)$ and unraveling the definitions, we find:

3.3.3. Corollary. *$C^*(A^B_{\mathrm{u}}(\nabla, F))$ is the universal C^*-algebra generated by a inclusion $r \times s$ of $B \otimes B$ and by the entries of a unitary $n \times n$-matrix v subject to the relations*

i) $v_{ij} r(b) = r(\gamma_i(b)) v_{ij}$ and $v_{ij} s(b) = s(\gamma_j(b)) v_{ij}$ for all i, j and $b \in B$,
ii) $v^{-\mathsf{T}} = \bar{v}$ is invertible and $r_n(\nabla F \nabla^{-1}) \bar{v}^{-\mathsf{T}} = v s_n(F)$.

It has the structure of a compact (B,Γ)-Hopf C^*-algebroid with counit, where for all i, j,

$$\delta(v_{ij}) = \gamma_i \otimes v_{ij}, \quad \bar{\delta}(v_{ij}) = v_{ij} \otimes \gamma_j, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}, \quad \epsilon(v_{ij}) = \delta_{i,j}. \tag{12}$$

Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd in the sense that $\nabla F \nabla, \nabla G \nabla \in M_n(B)$, and assume that $GF^* = FG^*$. If $G^{-1}F = \lambda \bar{H} H^{\mathsf{T}}$ for some $\lambda \in \mathbb{C}$ and some ∇-even $H \in \mathrm{GL}_n(B)$, then $A^B_{\mathrm{o}}(\nabla, F, G)$ is generated by a copy of $B \otimes B$ and entries of a unitary matrix $u \in M_n(A^B_{\mathrm{o}}(\nabla, F, G))$ by Remark 2.4.13 iii), and therefore has an enveloping C^*-algebra.

3.3.4. Corollary. *$C^*(A^B_{\mathrm{o}}(\nabla, F, G))$ is is the universal C^*-algebra generated by a inclusion $r \times s$ of $B \otimes B$ and by the entries of an invertible $n \times n$-matrix v subject to the relations*

i) $v_{ij} r(b) = r(\gamma_i(b)) v_{ij}$ and $v_{ij} s(b) = s(\gamma_j(b)) v_{ij}$ for all i, j and $b \in B$,
ii) $r_n(\nabla F \nabla) v^{-\mathsf{T}} = v s_n(F)$ and $r_n(\nabla G \nabla) \bar{v} = v s_n(G)$.

It carries the structure of a compact (B,Γ)-Hopf C^*-algebroid with counit such that (12) *holds.*

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