OPERATOR ALGEBRAS AND QUANTUM GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 98
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2012

ON WORONOWICZ'S APPROACH TO THE TOMITA-TAKESAKI THEORY

LÁSZLÓ ZSIDÓ

Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, 00133 Roma, Italy
E-mail: zsido@axp.mat.uniroma2.it

Abstract. The Tomita-Takesaki Theory is very complex and can be contemplated from different points of view. In the decade 1970–1980 several approaches to it appeared, each one seeking to attain more transparency. One of them was the paper of S. L. Woronowicz "Operator systems and their application to the Tomita-Takesaki theory" that appeared in 1979. Woronowicz's approach allows a particularly precise insight into the nature of the Tomita-Takesaki Theory and in this paper we present a brief, but fairly detailed version of his approach.

1. Introduction. The theory of M. Tomita of the standard form of general von Neumann algebras was a turning point in the theory of Operator Algebras and is up to this day one of the most important tools when working with von Neumann algebras. It became accessible in 1970 in the exposition of M. Takesaki [T], which contains so many fundamental contributions that the whole theory is usually referred to as the "Tomita-Takesaki Theory".

The starting point of the Tomita-Takesaki Theory in the case of a von Neumann algebra $\mathcal{M}$ having a bicyclic vector ξ_o is the following fundamental theorem:

The "projection" of the involution

$$\mathcal{M} \ni x \mapsto x^*$$

on the underlying Hilbert space, that is,

$$\mathcal{M}\xi_o \ni x\xi_o \mapsto x^*\xi_o,$$

is a closable antilinear operator and if S stands for its closure, $\Delta = S^*S$, and $S = J\Delta^{1/2}$

2010 *Mathematics Subject Classification*: Primary 46L10; Secondary 47B25, 46L40.
Key words and phrases: positive self-adjoint operator, analytic extension, polar decomposition, von Neumann algebra, Tomita-Takesaki theory.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc98-0-19

is the polar decomposition of S, then

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad t \in \mathbb{R}, \quad J\mathcal{M}J = \mathcal{M}'.$$

Thus we get the one-parameter automorphism group $\Delta^{it} \cdot \Delta^{-it}$, $t \in \mathbb{R}$, of $\mathcal{M}$ which plays a similar role in the study of the algebra $\mathcal{M}$ and of the positive linear form

$$\mathcal{M} \ni x \mapsto \omega_{\xi_o}(x) = (x\,\xi_o \,|\, \xi_o)$$

as the usual modular function of a localy compact group by working with functions on the group and with the Haar measure.

We notice that the Tomita-Takesaki Theory holds in a more general setting in which the positive form ω_{ξ_o} is replaced with a densely defined, not necessarily bounded positive form. However, the treatment of the general case can be reduced to the above one.

The treatment of the above fundamental theorem of the Tomita-Takesaki Theory is quite involved and can be contemplated from different points of view. In the decade 1970–1980 several approaches appeared to the theory, each one seeking to attain more transparency.

A common feature of the first three, of A. Van Daele ([V1]), U. Haagerup (see [B]) and L. Zsidó ([Z2]), is that they can be explained by using the notion of the "analytic generator" of one-parameter operator groups (see [Z1]).

A fourth aproach, of M. Rieffel - A. Van Daele [RV], is entirely done in terms of bounded linear operators.

The fifth aproach due to S. L. Woronowicz ([W3]) has a particular feature: first a general criterion is proved for a von Neumann algebra $\mathcal{M}$ and a non-singular, positive, self-adjoint operator Δ in order that $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$, $t \in \mathbb{R}$, holds true, and then it is verified that the operator Δ of the Tomita-Takesaki Theory satisfies the condition of the implementation criterion. The particular feature of Woronowicz's approach seems to reside in his implementation criterion, which could be useful also elsewhere. Indeed, the substantial part of the verification that the operator Δ of the Tomita-Takesaki Theory satisfies the condition of the criterion consists essentially in an application of S. Sakai's polar decomposition theorem for linear forms (see [S1]), completed with half of the proof of his Radon-Nikodym type theorem (see [S2]). Allowing a certain abuse in formulation, we may say that Sakai's work, coupled with Woronowicz's tailor-made implementation criterion, yields the fundamentals of the Tomita-Takesaki Theory.

It turns out that also the Woronowicz implementation criterion can be explained by using the "analytic generator" (and this leads to an unusual characterization of the analytic generators of automorphism groups [Z4]). The goal of this paper is to present a self-contained exposition of Woronowicz's criterion and its application to the fundamental theorem of the Tomita-Takesaki Theory.

2. Analytic extensions of groups of operators. We sketch here for further use some topics concerning analytic extensions of one-parameter groups of linear operators. In the exposition we follow works of I. Ciorănescu - L. Zsidó (see [CZ] and [Z3]) and U. Haagerup (see [H]).

The setting will be the following:

• H will denote a complex Hilbert space with inner product $(\,\cdot\,|\,\cdot\,)$, which is linear in the first variable and antilinear in the second variable.

• 1_H will denote the identity operator $H \to H$ and $1_{B(H)}$ will stand for the identity operator $B(H) \to B(H)$, but if $\lambda \in \mathbb{C}$ and there is no danger of confusion, we shall usually write simply λ instead of $\lambda 1_H$ or $\lambda 1_{B(H)}$.

• A denotes a non-singular, positive, self-adjoint linear operator in H.

• *wo* and *so* will denote the weak operator topology respectively the strong operator topology on the algebra $B(H)$ of all bounded linear operators on H.

• $\alpha_t^{(A)}$ stands for the $*$-automorphism of $B(H)$ implemented by $A^{it}, t \in \mathbb{R}$, that is,

$$\alpha_t^{(A)}(x) = \mathrm{Ad}\,(A^{it})(x) = A^{it} x\, A^{-it},\ x \in B(H).$$

• $\mathrm{conv}\,(M)$ will denote the convex hull of a subset M of some real vector space.

The *analytic extension in $z \in \mathbb{C}$ of the so-continuous one-parameter group*

$$\alpha^{(A)} : \mathbb{R} \ni t \mapsto \alpha_t^{(A)}$$

of $*$-automorphisms of $B(H)$ is the linear operator $\alpha_z^{(A)}$ in the Banach space $B(H)$ defined as follows:

$(x, y) \in B(H) \times B(H)$ belongs to the graph of $\alpha_z^{(A)}$, that is, x belongs to the domain $\mathcal{D}(\alpha_z^{(A)})$ of $\alpha_z^{(A)}$ and $y = \alpha_z^{(A)}(x)$

if and only if

there is a *wo*-continuous (or, equivalently, *so*-continuous) map F from the closed strip $\big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le |\Im z|,\ \Im\zeta \cdot \Im z \ge 0\big\}$ into $B(H)$, which is analytic in the interior of the strip and for which $F(t) = \alpha_t^{(A)}(x)$, $t \in \mathbb{R}$, as well as $F(z) = y$. We notice that for $B(H)$-valued maps *wo*-analyticity and analyticity in the norm topology are equivalent (see e.g. [HP], thm. 3.10.1).

Thus for $x \in \mathcal{D}(\alpha_z^{(A)})$ we have the *wo*-continuous mapping

$$\big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le |\Im z|,\ \Im\zeta \cdot \Im z \ge 0\big\} \ni \zeta \mapsto \alpha_\zeta^{(A)}(x) \in B(H),$$

which is analytic, and in particular continuous with respect to the operator norm, in the open strip $\big\{\zeta \in \mathbb{C}\,; 0 < |\Im\zeta| < |\Im z|,\ \Im\zeta \cdot \Im z > 0\big\}$.

The group property of $\alpha^{(A)}$ is preserved by analytic extension:

$$\begin{aligned} z_1, z_2 \in \mathbb{C},\ \Im z_1\, \Im z_2 \ge 0 \ &\Rightarrow\ \alpha_{z_1}^{(A)} \alpha_{z_2}^{(A)} = \alpha_{z_1+z_2}^{(A)}, \\ z \in \mathbb{C} \ &\Rightarrow\ \alpha_z^{(A)} \text{ injective and } \big(\alpha_z^{(A)}\big)^{-1} = \alpha_{-z}^{(A)}. \end{aligned} \tag{2.1}$$

In particular, if $x \in \mathcal{D}(\alpha_z^{(A)})$ then

$$\begin{aligned} &\sup\big\{\|\alpha_\zeta^{(A)}(x)\|\,;\ \zeta \in \mathbb{C}, |\Im\zeta| \le |\Im z|, \Im\zeta \cdot \Im z \ge 0\big\} \\ &\qquad = \sup\big\{\|\alpha_{i\beta}^{(A)}(x)\|\,;\ \beta \in \mathbb{R},\ |\beta| \le |\Im z|,\ \beta \cdot \Im z \ge 0\big\} < +\infty \end{aligned}$$

and by the maximum principle

$$\sup\big\{\|\alpha_\zeta^{(A)}(x)\|\,;\ \zeta \in \mathbb{C}, |\Im\zeta| \le |\Im z|, \Im\zeta \cdot \Im z \ge 0\big\} = \max\big(\|x\|, \|\alpha_z^{(A)}(x)\|\big). \tag{2.2}$$

We notice that (2.2) implies immediately that the graph of every $\alpha_z^{(A)}$ is norm-closed. Actually more is true: the graph of $\alpha_z^{(A)}$ is weak*-closed (see [CZ], Thm. 2.4 and [Z3], Thm. 1.1). By the Krein-Šmulian theorem this is equivalent to the following proposition, which actually can be proved also in a direct, elementary way:

PROPOSITION 2.1. *For any $z \in \mathbb{C}$, the closure of every bounded subset of the graph of $\alpha_z^{(A)}$ with respect to the product of the weak operator topologies is still contained in the graph of $\alpha_z^{(A)}$.*

We recall that if $F : \mathbb{R} \to B(H)$ is a mapping such that

– $\mathbb{R} \ni t \mapsto (F(t)\xi \,|\, \eta)$ is Lebesgue measurable for any $\xi, \eta \in H$ and
– $\mathbb{R} \ni t \mapsto \|F(t)\|$ is majorized by some $f \in L^1(\mathbb{R})$

then by the Riesz representation theorem there exists a uniquely defined $x_F \in B(H)$, the Lebesgue integral of F relative to the weak operator topology, satisfying

$$\int_{-\infty}^{+\infty} (F(t)\xi \,|\, \eta)\, dt = (x_F \xi \,|\, \eta), \qquad \xi, \eta \in H.$$

x_F is usually denoted by $wo\text{-} \int_{-\infty}^{+\infty} F(t)\, dt$ and satisfies $\|wo\text{-} \int_{-\infty}^{+\infty} F(t)\, dt\| \le \|f\|_1$.

In particular, for any $f \in L^1(\mathbb{R})$ we can define the linear operators

$$\alpha_f^{(A)} : B(H) \ni x \mapsto wo\text{-} \int_{-\infty}^{+\infty} f(t)\, \alpha_t^{(A)}(x)\, dt \in B(H)$$

for which we have $\|\alpha_f^{(A)}\| \le \|f\|_1$. It is easily seen that

$$\begin{aligned} &z \in \mathbb{C}, x \in \mathcal{D}(\alpha_z^{(A)}), f \in L^1(\mathbb{R}) \;\Rightarrow \\ &\alpha_f^{(A)}(x) \in \mathcal{D}(\alpha_z^{(A)}), \alpha_z^{(A)}\big(\alpha_f^{(A)}(x)\big) = \alpha_f^{(A)}\big(\alpha_z^{(A)}(x)\big). \end{aligned} \tag{2.3}$$

Indeed, if

$$F : \big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le |\Im z|,\, \Im\zeta \cdot \Im z \ge 0\big\} \to B(H)$$

is a bounded, *wo*-continuous mapping, which is analytic in the interior and for which $F(t) = \alpha_t^{(A)}(x), t \in \mathbb{R}$, then the mapping

$$F_f : \big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le |\Im z|,\, \Im\zeta \cdot \Im z \ge 0\big\} \ni \zeta \mapsto \int_{-\infty}^{+\infty} f(s)\, F(s+\zeta)\, ds \in B(H)$$

will be *wo*-continuous, analytic in the interior, and satisfying the conditions

$$F_f(t) = \alpha_t^{(A)}\big(\alpha_f^{(A)}(x)\big), t \in \mathbb{R}, \qquad F_f(z) = \alpha_f^{(A)}\big(\alpha_z^{(A)}(x)\big).$$

We say that $x \in B(H)$ is $\alpha^{(A)}$-*entire* if the orbit

$$\mathbb{R} \ni t \mapsto \alpha_t^{(A)}(x) \in B(H)$$

has a $B(H)$-valued entire extension, that is, x belongs to the domain of every $\alpha_z^{(A)}$. The set of all $\alpha^{(A)}$-entire elements of $B(H)$ is a vector space which is a *wo*-core of any $\alpha_z^{(A)}$, in particular it is *wo*-dense in $B(H)$:

PROPOSITION 2.2. *For any $z \in \mathbb{C}$ and $x \in \mathcal{D}(\alpha_z^{(A)})$, defining $f_n \in L^1(\mathbb{R})$, $n \ge 1$, by*

$$f_n(t) = \sqrt{\frac{n}{\pi}}\, e^{-n\,t^2},$$

every $\alpha^{(A)}_{f_n}(x) \in B(H)$ *is* $\alpha^{(A)}$*-entire and*

(i) $\sup_{\zeta\in K} \left\|\alpha^{(A)}_\zeta\big(\alpha^{(A)}_{f_n}(x)\big) - \alpha^{(A)}_\zeta(x)\right\| \to 0$ *for every compact subset* K *of the horizontal open strip* $\big\{\zeta\in\mathbb{C}\,;0<|\Im\zeta|<|\Im z|,\ \Im\zeta\cdot\Im z>0\big\}$;

(ii) $\sup_{\zeta\in K} \left|\big(\,\alpha^{(A)}_\zeta\big(\alpha^{(A)}_{f_n}(x)\big)\xi - \alpha^{(A)}_\zeta(x)\xi \,\big|\, \eta\big)\right| \to 0$ *for every* $\xi,\eta\in H$ *and every compact subset* K *of the horizontal closed strip* $\big\{\zeta\in\mathbb{C}\,;|\Im\zeta|\le|\Im z|,\ \Im\zeta\cdot\Im z\ge 0\big\}$;

(iii) $\sup\big\{\|\alpha^{(A)}_\zeta\big(\alpha^{(A)}_{f_n}(x)\big)\|\,;\,\zeta\in\mathbb{C},|\Im\zeta|\le|\Im z|,\Im\zeta\cdot\Im z\ge 0,n\ge 1\big\}$
$\le \max\big(\|x\|,\|\alpha^{(A)}_z(x)\|\big)$.

Proof. Direct computation shows that

$$F_n:\mathbb{C}\ni\zeta\mapsto wo\text{-}\int_{-\infty}^{+\infty}\sqrt{\frac{n}{\pi}}\,e^{-n(s-\zeta)^2}\,\alpha^{(A)}_s(x)\,ds\in B(H)$$

is an entire extension of $\mathbb{R}\ni t\mapsto\alpha^{(A)}_t\big(\alpha^{(A)}_{f_n}(x)\big)\in B(H)$ and so $\alpha^{(A)}_{f_n}(x)$ is $\alpha^{(A)}$-entire.

Since by (2.3)

$$\alpha^{(A)}_z\big(\alpha^{(A)}_{f_n}(x)\big)=\alpha^{(A)}_{f_n}\big(\alpha^{(A)}_z(x)\big),\qquad n\ge 1,$$

(iii) follows by using (2.2).

Let now K be a compact subset of $\big\{\zeta\in\mathbb{C}\,;0<|\Im\zeta|<|\Im z|,\ \Im\zeta\cdot\Im z>0\big\}$. A direct computation shows that, for any $\zeta\in K$ and $\delta>0$,

$$\begin{aligned}\left\|\alpha^{(A)}_\zeta\big(\alpha^{(A)}_{f_n}(x)\big)-\alpha^{(A)}_\zeta(x)\right\| &= \left\|wo\text{-}\int_{-\infty}^{+\infty}\sqrt{\frac{n}{\pi}}\,e^{-ns^2}\big(\alpha^{(A)}_{s+\zeta}(x)-\alpha^{(A)}_\zeta(x)\big)\,ds\right\|\\ &\le \sup_{|s|\le\delta}\left\|\alpha^{(A)}_{s+\zeta}(x)-\alpha^{(A)}_\zeta(x)\right\| + \frac{4}{\sqrt{\pi}}\max\big(\|x\|,\|\alpha^{(A)}_z(x)\|\big)\int_{\delta\sqrt{n}}^{+\infty}e^{-r^2}\,dr.\end{aligned}$$

Taking some $\delta_o>0$ and using the uniform continuity of $K+[-\delta_o,\delta_o]\ni\zeta\mapsto\alpha^{(A)}_\zeta(x)$ in the operator norm, the above estimate yields (i).

Finally, (ii) follows similarly as (i). ■

Let us call $x\in B(H)$ $\alpha^{(A)}$*-entire of exponential type* if x is $\alpha^{(A)}$-entire and there are constants (depending on x) such that

$$\|\alpha^{(A)}_z(x)\|\le c\,e^{\tau|z|},\qquad z\in\mathbb{C}.$$

The set of all $\alpha^{(A)}$-entire elements of exponential type of $B(H)$ is a linear subspace of the vector space of all $\alpha^{(A)}$-entire elements. We notice that it is the union of all Arveson spectral subspaces of the group $\alpha^{(A)}$ (see [Ar]), associated to the compact subsets of $\mathbb{R}$ (see [CZ], Corollary 5.7).

$\alpha^{(A)}$-entire elements of exponential type of $B(H)$ can be produced as follows:

Let the support of $g\in C^2(\mathbb{R})$ be contained in $[-s_o,s_o]$ for some $s_o>0$ and let f denote the Fourier transform of g:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}g(s)e^{-its}\,ds=\frac{1}{2\pi}\int_{-s_o}^{s_o}g(s)e^{-its}\,ds,\qquad t\in\mathbb{R}.$$

Then f has an entire extension, which we continue to denote by f, given by the formula

$$f(z)=\frac{1}{2\pi}\int_{-s_o}^{s_o}g(s)e^{-izs}ds,\qquad z\in\mathbb{C}.$$

Since

$$(iz)^2 f(z) = \frac{1}{2\pi}\int_{-s_o}^{s_o} g''(s)e^{-izs}ds, \qquad z \in \mathbb{C},$$

we have the estimate

$$(1+|z|)^2|f(z)| \le \frac{(\|g\|_\infty + \|g''\|_\infty)s_o}{\pi} e^{|\Im z|}, \qquad z \in \mathbb{C}, \tag{2.4}$$

where $\|\cdot\|_\infty$ stands for the uniform norm. In particular, $f \in L^1(\mathbb{R})$ with

$$\|f\|_1 = \int_{-\infty}^{+\infty} |f(s)|\,ds \le \frac{(\|g\|_\infty + \|g''\|_\infty)s_o}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+s^2}\,ds = (\|g\|_\infty + \|g''\|_\infty)s_o \tag{2.5}$$

and we can apply the inversion formula for Fourier transforms obtaining

$$g(s) = \int_{-\infty}^{+\infty} f(t)\,e^{its}\,dt, \qquad s \in \mathbb{R}. \tag{2.6}$$

Now, for any $x \in B(H)$, $\alpha_f^{(A)}(x)$ is an $\alpha^{(A)}$-entire element of exponential type of $B(H)$. Indeed,

$$F : \mathbb{C} \ni \zeta \mapsto wo\text{-}\int_{-\infty}^{+\infty} f(s-\zeta)\,\alpha_s^{(A)}(x)\,ds \in B(H)$$

is an entire extension of $\mathbb{R} \ni t \mapsto \alpha_t^{(A)}\big(\alpha_f^{(A)}(x)\big) \in B(H)$ and, taking into account (2.4),

$$\begin{aligned}
\|F_n(\zeta)\| &\le \|x\|\frac{(\|g\|_\infty + \|g''\|_\infty)s_o}{\pi}e^{|\Im\zeta|}\int_{-\infty}^{+\infty}\frac{1}{1+|s-\zeta|^2}\,ds\\
&\le \|x\|\frac{(\|g\|_\infty + \|g''\|_\infty)s_o}{\pi}e^{|\Im\zeta|}\int_{-\infty}^{+\infty}\frac{1}{1+|s-\Re\zeta|^2}\,ds\\
&\le \|x\|\frac{(\|g\|_\infty + \|g''\|_\infty)s_o}{\pi}e^{|\Im\zeta|}\int_{-\infty}^{+\infty}\frac{1}{1+s^2}\,ds\\
&= (\|g\|_\infty + \|g''\|_\infty)s_o\|x\|e^{|\Im\zeta|}, \qquad \zeta \in \mathbb{C}.
\end{aligned}$$

The next proposition shows that already the vector space of $\alpha^{(A)}$-entire elements of exponential type of $B(H)$ is a *wo*-core of every $\alpha_z^{(A)}$:

PROPOSITION 2.3. *Let $z \in \mathbb{C}$ and $x \in \mathcal{D}(\alpha_z^{(A)})$ be arbitrary. Let further $g \in C^2(\mathbb{R})$ be the function defined by the formula*

$$g(s) := \begin{cases} (1-s^2)^3 & \text{for } |s| \le 1,\\ 0 & \text{for } |s| \ge 1.\end{cases}$$

Then $g(0) = 1$ and the support of g is contained in $[-1,1]$. Let finally f be the Fourier transform of g and define $f_n \in L^1(\mathbb{R})$, $n \ge 1$, by

$$f_n(t) = nf(nt).$$

Then every $\alpha_{f_n}^{(A)}(x) \in B(H)$ is $\alpha^{(A)}$-entire of exponential type and

(i) $\sup_{\zeta\in K}\big\|\,\alpha_\zeta^{(A)}\big(\alpha_{f_n}^{(A)}(x)\big) - \alpha_\zeta^{(A)}(x)\,\big\| \to 0$ *for every compact subset K of the horizontal open strip $\big\{\zeta \in \mathbb{C}\,; 0 < |\Im\zeta| < |\Im z|,\ \Im\zeta\cdot\Im z > 0\big\}$;*

(ii) $\sup_{\zeta\in K}\left|\left(\alpha_\zeta^{(A)}\left(\alpha_{f_n}^{(A)}(x)\right)\xi-\alpha_\zeta^{(A)}(x)\xi\,\middle|\,\eta\right)\right|\to 0$ *for every* $\xi,\eta\in H$ *and every compact subset* K *of the horizontal closed strip* $\left\{\zeta\in\mathbb{C}\,;|\Im\zeta|\le|\Im z|,\,\Im\zeta\cdot\Im z\ge 0\right\}$;

(iii) $\sup\left\{\left\|\alpha_\zeta^{(A)}\left(\alpha_{f_n}^{(A)}(x)\right)\right\|\,\zeta\in\mathbb{C},|\Im\zeta|\le|\Im z|,\Im\zeta\cdot\Im z\ge 0,\,n\ge 1\right\}$
$\le 7\max\left(\|x\|,\|\alpha_z^{(A)}(x)\|\right)$.

Proof. f_n is the Fourier transform of the function g_n defined by

$$g_n(s):=g\left(\frac{s}{n}\right),\qquad s\in\mathbb{R},$$

so by the discussion before the statement of Proposition 2.3 the elements $\alpha_{f_n}^{(A)}(x)$ of $B(H)$ are $\alpha^{(A)}$-entire of exponential type.

It is easy to verify that $\|g\|_\infty=1$ and $\|g''\|_\infty=6$, so by (2.5) we have $\|f\|_1\le 7$. Consequently

$$\|\alpha_{f_n}^{(A)}(x)\|\le\|f_n\|_1\|x\|=\|f\|_1\|x\|\le 7\|x\|.$$

Using (2.3) we get similarly

$$\left\|\alpha_z^{(A)}\left(\alpha_{f_n}^{(A)}(x)\right)\right\|=\left\|\alpha_{f_n}^{(A)}\left(\alpha_z^{(A)}(x)\right)\right\|\le\|f_n\|_1\|\alpha_z^{(A)}(x)\|\le 7\|\alpha_z^{(A)}(x)\|.$$

Now (iii) follows by (2.2).

Finally, taking into account that by (2.6)

$$\int_{-\infty}^{+\infty}f(t)\,dt=g(0)=1,$$

the proof of (i) and (ii) is similar to the proof of the analogous statements in Proposition 2.2. ∎

The multiplicativity and the $*$-map property of the mappings $\alpha_t^{(A)}, t\in\mathbb{R}$, yield for the analytic extensions $\alpha_z^{(A)}$:

$$\begin{aligned}z\in\mathbb{C},\,x,y\in\mathcal{D}(\alpha_z^{(A)})\ &\Rightarrow\ xy\in\mathcal{D}(\alpha_z^{(A)})\text{ and }\alpha_z^{(A)}(xy)=\alpha_z^{(A)}(x)\,\alpha_z^{(A)}(y),\\ z\in\mathbb{C},\,x\in\mathcal{D}(\alpha_z^{(A)})\ &\Rightarrow\ x^*\in\mathcal{D}(\alpha_{\overline{z}}^{(A)})\text{ and }\alpha_z^{(A)}(x)^*=\alpha_{\overline{z}}^{(A)}(x^*).\end{aligned}\tag{2.7}$$

In particular, the vector space of all $\alpha^{(A)}$-entire elements of $B(H)$ and the vector space of all $\alpha^{(A)}$-entire elements of exponential type of $B(H)$ are $*$-subalgebras of $B(H)$.

We notice that by (2.7) and by the closedness of the graph of $\alpha_z^{(A)}$ we have

$$z\in\mathbb{C},x\in\mathcal{D}(\alpha_z^{(A)})\ \Rightarrow\ \exp(x)\in\mathcal{D}(\alpha_z^{(A)}),\alpha_z^{(A)}\big(\exp(x)\big)=\exp\left(\alpha_z^{(A)}(x)\right).\tag{2.8}$$

Using the reflection principle it is easy to verify that for $x\in B(H)$ and $z\in\mathbb{C}$

$$x\in\mathcal{D}(\alpha_z^{(A)}),\alpha_z^{(A)}(x)\text{ self-adjoint}\ \Leftrightarrow\ x\in\mathcal{D}(\alpha_{2z}^{(A)}),\alpha_{2z}^{(A)}(x)=\alpha_{2\,\Re z}^{(A)}(x)^*\tag{2.9}$$

and in this case we have by (2.2) $\|\alpha_z^{(A)}(x)\|\le\|x\|$.

The next proposition is a "maximum principle for spectra":

Proposition 2.4. *For* $z_1\in\mathbb{C}$, $\Im z_1\ge 0$, $z_2\in\mathbb{C}$, $\Im z_2\le 0$ *and* $x\in\mathcal{D}(\alpha_{z_1}^{(A)})\cap\mathcal{D}(\alpha_{z_2}^{(A)})$, *denoting by* $\sigma(y)$ *the spectrum of* $y\in B(H)$, *we have:*

$$\sigma(x)\subset\operatorname{conv}\left(\sigma\left(\alpha_{z_1}^{(A)}(x)\right)\cup\sigma\left(\alpha_{z_2}^{(A)}(x)\right)\right).$$

Proof. Taking into account that $\operatorname{conv}\big(\sigma\big(\alpha^{(A)}_{z_1}(x)\big)\cup\sigma\big(\alpha^{(A)}_{z_2}(x)\big)\big)$ is a compact convex set in the complex plane, hence it is the intersection of all closed half-planes containing it, it is enough to prove that every closed half-plane Z, which contains the spectrum of $\alpha^{(A)}_{z_1}(x)$ and of $\alpha^{(A)}_{z_2}(x)$, contains also the spectrum of x. Clearly, we can consider only the case of the half-plane $Z=\{\zeta\in\mathbb{C}\,;\Re\zeta\le 0\}$, because the general case can be reduced to this one.

Let us denote $y=\exp\big(\alpha^{(A)}_{z_2}(x)\big)$. Since, by (2.1),

$$\alpha^{(A)}_{z_2}(x)\in\mathcal{D}(\alpha^{(A)}_{z_1}\alpha^{(A)}_{-z_2})=\mathcal{D}(\alpha^{(A)}_{z_1-z_2}),$$
$$\alpha^{(A)}_{z_1-z_2}\big(\alpha^{(A)}_{z_2}(x)\big)=\big(\alpha^{(A)}_{z_1}\alpha^{(A)}_{-z_2}\big)\big(\alpha^{(A)}_{z_2}(x)\big)=\alpha^{(A)}_{z_1}(x),$$

(2.8) implies that $y\in\mathcal{D}\big(\alpha^{(A)}_{z_1-z_2}\big)$ and $\alpha^{(A)}_{z_1-z_2}(y)=\exp\big(\alpha^{(A)}_{z_1}(x)\big)$. But the spectra of $\alpha^{(A)}_{z_2}(x)$ and $\alpha^{(A)}_{z_1}(x)$ being contained in the half-plane $\{\zeta\in\mathbb{C}\,;\Re\zeta\le 0\}$, by the spectral mapping theorem the spectra of $y=\exp\big(\alpha^{(A)}_{z_2}(x)\big)$ and $\alpha^{(A)}_{z_1-z_2}(y)=\exp\big(\alpha^{(A)}_{z_1}(x)\big)$ are contained in the closed unit disc, that is, the spectral radii $r(y)$ and $r\big(\alpha^{(A)}_{z_1-z_2}(y)\big)$ are ≤ 1.

Now, by (2.7) and (2.2), we have for every integer $k\ge 1$

$$\|\alpha^{(A)}_{-z_2}(y)^k\|=\|\alpha^{(A)}_{-z_2}(y^k)\|\le\max\big(\|y^k\|,\|\alpha^{(A)}_{z_1-z_2}(y^k)\|\big)=\max\big(\|y^k\|,\|\alpha^{(A)}_{z_1-z_2}(y)^k\|\big),$$
$$\|\alpha^{(A)}_{-z_2}(y)^k\|^{1/k}\le\max\big(\|y^k\|^{1/k},\|\alpha^{(A)}_{z_1-z_2}(y)^k\|^{1/k}\big)$$

and taking limits for $k\to\infty$ we obtain

$$r\big(\alpha^{(A)}_{-z_2}(y)\big)\le\max\big(r(y),\,r\big(\alpha^{(A)}_{z_1-z_2}(y)\big)\big)\le 1.$$

Since, by (2.8), $\alpha^{(A)}_{-z_2}(y)=\alpha^{(A)}_{-z_2}\big(\exp\big(\alpha^{(A)}_{z_2}(x)\big)=\exp\big(\alpha^{(A)}_{-z_2}\alpha^{(A)}_{z_2}(x)\big)=\exp(x)$ and by the spectral mapping theorem $\sigma\big(\exp(x)\big)=\exp\big(\sigma(x)\big)$, it follows that every $\lambda\in\sigma(x)$ satisfies the condition $|\exp(\lambda)|\le 1$, that is $\Re\lambda\le 0$. Thus $\sigma(x)\subset\{\zeta\in\mathbb{C}\,;\Re\zeta\le 0\}$. ■

The operators $\alpha^{(A)}_z, z\in\mathbb{C}$, can be described in terms of the powers A^{iz} (see [CZ], Thm. 6.2):

PROPOSITION 2.5. *For $z\in\mathbb{C}$ and $x\in B(H)$ the following conditions are equivalent:*

(i) $x\in\mathcal{D}(\alpha^{(A)}_z)$;
(ii) $A^{iz}xA^{-iz}$ *is defined and bounded on a core of* A^{-iz};
(iii) $A^{iz}xA^{-iz}$ *is defined and bounded on the whole domain of* A^{-iz}.

Moreover, if the above equivalent conditions are satisfied then

$$xA^{-iz}\subset A^{-iz}\alpha^{(A)}_z(x),$$

so $A^{iz}xA^{-iz}\subset\alpha^{(A)}_z(x)$ *and* $\alpha^{(A)}_z(x)$ *is equal to the closure* $\overline{A^{iz}xA^{-iz}}$.

The operator $\alpha^{(A)}_i$ (or, with a different choice, $\alpha^{(A)}_{-i}$) is called the *analytic generator* of the group $\alpha^{(A)}$.

It is easy to see that the point spectrum of $\alpha^{(A)}_i$ is contained in the positive half-line $[0,+\infty)$. However, the spectrum of $\alpha^{(A)}_i$ is equal to $\mathbb{C}$ unless A and A^{-1} are bounded (see A. Van Daele [V2] and G. A. Elliott - L. Zsidó [EZ]). Nevertheless, $\alpha^{(A)}_i$ always has densely defined resolvents at the points of $\mathbb{C}\setminus[0,+\infty)$ (see [CZ], Corollary 3.3):

THEOREM 2.6 (General resolvent formula). *If* $\lambda \in \mathbb{C} \setminus (-\infty, 0], \varepsilon > 0$ *and* $x \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)})$, *then* $x \in \mathcal{D}\big((\lambda 1_{B(H)} + \alpha_i^{(A)})^{-1}\big)$ *and*

$$\big(\lambda 1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) = \frac{1}{\lambda}\, x - \frac{1}{2\lambda} \int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta, \tag{2.10}$$

where the constant $0 < c < \min\{\varepsilon, 1\}$ *is arbitrary and* $\lambda^{i\zeta} = |\lambda|^{i\zeta} e^{-\theta\zeta}$ *provided that* $\lambda = |\lambda| e^{i\theta}$ *with* $-\pi < \theta < \pi$. *We underline that the integral on the right-hand side of the formula converges with respect to the operator norm.*

Proof. It is easy to see that the integral

$$\int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta$$

converges with respect to the operator norm for any $\lambda \in \mathbb{C} \setminus (-\infty, 0], \varepsilon > 0, x \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)})$ and $0 < c < \min\{\varepsilon, 1\}$. Let us denote

$$y = \frac{1}{\lambda}\, x - \frac{1}{2\lambda} \int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta.$$

If x is $\alpha^{(A)}$-entire then also y is $\alpha^{(A)}$-entire and, using the residue theorem, we obtain

$$\begin{aligned}
\alpha_i^{(A)}(y) &= \frac{1}{\lambda}\, \alpha_i^{(A)}(x) - \frac{1}{2\lambda} \int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_{\zeta+i}^{(A)}(x)\, d\zeta \\
&= \frac{1}{\lambda}\, \alpha_i^{(A)}(x) - \frac{1}{2\lambda} \int_{-\infty+i(c+1)}^{+\infty+i(c+1)} \frac{\lambda^{i(\zeta-i)}}{\sin(i\pi\big((\zeta-i)\big)}\, \alpha_\zeta^{(A)}(x)\, d\zeta \\
&= \frac{1}{\lambda}\, \alpha_i^{(A)}(x) + \frac{1}{2} \int_{-\infty+i(c+1)}^{+\infty+i(c+1)} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta \\
&= \frac{1}{\lambda}\, \alpha_i^{(A)}(x) + \frac{1}{2}\bigg(\int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta + 2\pi i \cdot \operatorname*{Res}_{\zeta=i} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\bigg) \\
&= \frac{1}{\lambda}\, \alpha_i^{(A)}(x) + (x - \lambda y) - \frac{1}{\lambda}\, \alpha_i^{(A)}(x) = x - \lambda y.
\end{aligned}$$

Thus

$$\big(\lambda + \alpha_i^{(A)}\big)(y) = x, \text{that is } \big(\lambda + \alpha_i^{(A)}\big)^{-1}(x) = y.$$

The case of arbitrary $x \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)})$ can be reduced to the $\alpha^{(A)}$-entire case by using Propositions 2.2 and 2.1. ∎

Theorem 2.6 implies above all that the group $\alpha^{(A)}$ is uniquely defined by its analytic generator (see [CZ], Thm. 4.4 and [H], Lemma 4.4):

COROLLARY 2.7 (Uniqueness). *If B is another non-singular, positive, self-adjoint linear operator in H then*

$$\alpha_t^{(A)} = \alpha_t^{(B)},\ t \in \mathbb{R} \ \Leftrightarrow\ \alpha_i^{(A)} \subset \alpha_i^{(B)}.$$

Proof. The implication $\Rightarrow$ is trivial.

Let us now assume that $\alpha_i^{(A)} \subset \alpha_i^{(B)}$. Then

$$\big(e^s + \alpha_i^{(A)}\big)^{-1} \subset \big(e^s + \alpha_i^{(B)}\big)^{-1}, \qquad s \in \mathbb{R}. \tag{2.11}$$

Since the the set of all $\alpha^{(A)}$-entire elements of $B(H)$ is $\bigcap_{n\in\mathbb{Z}}\mathcal{D}((\alpha_i^{(A)})^n)$, every $\alpha^{(A)}$-entire x is also $\alpha^{(B)}$-entire. Taking into account (2.11), Theorem 2.6 yields for every $\alpha^{(A)}$-entire x

$$\int_{-\infty+i\frac{1}{2}}^{+\infty+i\frac{1}{2}} \frac{e^{is\zeta}}{\sin(i\pi\zeta)}\,\alpha_\zeta^{(A)}(x)\,d\zeta = \int_{-\infty+i\frac{1}{2}}^{+\infty+i\frac{1}{2}} \frac{e^{is\zeta}}{\sin(i\pi\zeta)}\,\alpha_\zeta^{(B)}(x)\,d\zeta, \qquad s\in\mathbb{R},$$

that is,

$$\int_{-\infty}^{+\infty} \frac{e^{ist}}{\cosh(\pi t)}\,\alpha_{t+\frac{1}{2}}^{(A)}(x)\,dt = \int_{-\infty}^{+\infty} \frac{e^{ist}}{\cosh(\pi t)}\,\alpha_{t+\frac{1}{2}}^{(B)}(x)\,dt, \qquad s\in\mathbb{R}.$$

By the injectivity of the Fourier transformation it follows

$$\alpha_{t+\frac{1}{2}}^{(A)}(x) = \alpha_{t+\frac{1}{2}}^{(B)}(x), \qquad t\in\mathbb{R}$$

and therefore the entire mappings

$$\mathbb{C}\ni\zeta\mapsto\alpha_\zeta^{(A)}(x) \text{ and } \mathbb{C}\ni\zeta\mapsto\alpha_\zeta^{(B)}(x)$$

coincide. In particular, $\alpha_t^{(A)}(x)=\alpha_t^{(B)}(x)$, $t\in\mathbb{R}$.

Now the *wo*-density of the $\alpha^{(A)}$-entire elements of $B(H)$ (Proposition 2.2) implies that $\alpha_t^{(A)}(x)=\alpha_t^{(B)}(x)$, $t\in\mathbb{R}$, holds for every $x\in B(H)$. ∎

REMARK. We notice that (2.10) can be formulated also as follows:

$$\alpha_i^{(A)}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x) = \frac{1}{2}\int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\,\alpha_\zeta^{(A)}(x)\,d\zeta,$$

that is,

$$\alpha_i^{(A)}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda^{-z}\Big(-\frac{\pi}{\sin(\pi z)}\,\alpha_{iz}^{(A)}(x)\Big)dz.$$

Thus, for every $x\in\mathcal{D}(\alpha_i^{(A)})$,

$$\Phi:[0,+\infty)\ni\lambda\mapsto\alpha_i^{(A)}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)$$

is the inverse Mellin transform of

$$\varphi:\big\{z\in\mathbb{C}\,;\,0<\Re z<1\big\}\ni z\mapsto-\frac{\pi}{\sin(\pi z)}\,\alpha_{iz}^{(A)}(x)$$

and, since the conditions of [D], Kap. 6, §8, Satz 3 are satisfied, it follows that φ is the Mellin transform of Φ:

$$-\frac{\pi}{\sin(\pi z)}\,\alpha_{iz}^{(A)}(x) = \int_0^{+\infty}\lambda^{z-1}\alpha_i^{(A)}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)\,d\lambda.$$

Consequently, for $\zeta\in\mathbb{C}, 0<\Im\zeta<1$,

$$\alpha_\zeta^{(A)}(x) = \frac{\sin(i\pi\zeta)}{\pi}\int_0^{+\infty}\lambda^{-i\zeta-1}\alpha_i^{(A)}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)\,d\lambda \tag{2.12}$$

is the $(-i\zeta)^{\text{th}}$ "Balakrishnan power" (see [Ba]) of the analytic generator $\alpha_i^{(A)}$ (hence the $(i\zeta)^{\text{th}}$ power of $\alpha_{-i}^{(A)}=\big(\alpha_i^{(A)}\big)^{-1}$). (2.12) is a variant for $\alpha^{(A)}$ of the Stone Representation Theorem for strongly continuous one-parameter unitary groups: If $U_t, t\in\mathbb{R}$, is a strongly continuous one-parameter group of unitary operators then the analytic generator $B=U_i$

is a non-singular, positive, self-adjoint linear operator and we have $U_t = B^{-it}$ or, with $U_{-i} = B^{-1}$, $U_t = \left(B^{-1}\right)^{it}$.

Though the resolvents $\left(\lambda 1_{B(H)} + \alpha_i^{(A)}\right)^{-1}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, of the analytic generator of $\alpha^{(A)}$ are in general not everywhere defined, they have the same domain and the resolvent equation holds:

Corollary 2.8 (Resolvent equation). *For every λ_1, $\lambda_2 \in \mathbb{C} \setminus (-\infty, 0]$ we have*

$$\mathcal{D}\big((\lambda_1 1_{B(H)} + \alpha_i^{(A)})^{-1}\big) = \mathcal{D}\big((\lambda_2 1_{B(H)} + \alpha_i^{(A)})^{-1}\big) \supset \bigcup_{\varepsilon>0} \mathcal{D}(\alpha_{i\varepsilon}^{(A)}) \cup \bigcup_{\varepsilon>0} \mathcal{D}(\alpha_{-i\varepsilon}^{(A)}) \tag{2.13}$$

and the resolvent equation holds:

$$\begin{aligned} &\left(\lambda_1 1_{B(H)} + \alpha_i^{(A)}\right)^{-1} - \left(\lambda_2 1_{B(H)} + \alpha_i^{(A)}\right)^{-1} \\ &= (\lambda_2 - \lambda_1) \left(\lambda_1 1_{B(H)} + \alpha_i^{(A)}\right)^{-1} \left(\lambda_2 1_{B(H)} + \alpha_i^{(A)}\right)^{-1} \\ &= (\lambda_2 - \lambda_1) \left(\lambda_2 1_{B(H)} + \alpha_i^{(A)}\right)^{-1} \left(\lambda_1 1_{B(H)} + \alpha_i^{(A)}\right)^{-1}. \end{aligned} \tag{2.14}$$

Proof. By Theorem 2.6 we have for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$

$$\mathcal{D}\big((\lambda + \alpha_i^{(A)})^{-1}\big) \supset \bigcup_{\varepsilon>0} \mathcal{D}(\alpha_{i\varepsilon}^{(A)}).$$

On the other hand, if $0 < \varepsilon \leq 1$ and $x \in \mathcal{D}(\alpha_{-i\varepsilon}^{(A)})$ then $\alpha_{-i\varepsilon}^{(A)}(x) \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)})$ so we can consider

$$y = (\lambda + \alpha_i^{(A)})^{-1} \alpha_{-i\varepsilon}^{(A)}(x).$$

Since

$$\lambda y + \alpha_i^{(A)}(y) = \alpha_{-i\varepsilon}^{(A)}(x) \;\Rightarrow\; \alpha_i^{(A)}(y) = \alpha_{-i\varepsilon}^{(A)}(x) - \lambda y \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)}),$$

by (2.1) it follows that $y \in \mathcal{D}(\alpha_{i(1+\varepsilon)}^{(A)})$ and

$$\lambda \alpha_{i\varepsilon}^{(A)}(y) + \alpha_i^{(A)}\big(\alpha_{i\varepsilon}^{(A)}(y)\big) = \alpha_{i\varepsilon}^{(A)}\big(\alpha_{-i\varepsilon}^{(A)}(x)\big) = x,$$

that is, $x \in \mathcal{D}\big((\lambda + \alpha_i^{(A)})^{-1}\big)$.

Consequently, also

$$\mathcal{D}\big((\lambda + \alpha_i^{(A)})^{-1}\big) \supset \bigcup_{0<\varepsilon<1} \mathcal{D}(\alpha_{-i\varepsilon}^{(A)}) = \bigcup_{\varepsilon>0} \mathcal{D}(\alpha_{-i\varepsilon}^{(A)}).$$

Now let λ_1, $\lambda_2 \in \mathbb{C} \setminus (-\infty, 0]$ be arbitrary.

For every $x \in \mathcal{D}\big((\lambda_2 + \alpha_i^{(A)})^{-1}\big)$, by Theorem 2.6 $(\lambda_2 + \alpha_i^{(A)})^{-1}(x) \in \mathcal{D}(\alpha_i^{(A)})$ belongs to $\mathcal{D}\big((\lambda_1 + \alpha_i^{(A)})^{-1}\big)$, so we can consider

$$z = (\lambda_2 - \lambda_1)\left(\lambda_1 + \alpha_i^{(A)}\right)^{-1}\left(\lambda_2 + \alpha_i^{(A)}\right)^{-1}(x) + \left(\lambda_2 + \alpha_i^{(A)}\right)^{-1} \in \mathcal{D}(\alpha_i^{(A)}).$$

Since

$$\begin{aligned} \alpha_i^{(A)}(z) &= (\lambda_2 - \lambda_1) \underbrace{\alpha_i^{(A)} \left(\lambda_1 + \alpha_i^{(A)}\right)^{-1}}_{= 1_H - \lambda_1 (\lambda_1 + \alpha_i^{(A)})^{-1}} \left(\lambda_2 + \alpha_i^{(A)}\right)^{-1}(x) + \underbrace{\alpha_i^{(A)} \left(\lambda_2 + \alpha_i^{(A)}\right)^{-1}}_{= 1_H - \lambda_2 (\lambda_2 + \alpha_i^{(A)})^{-1}}(x) \\ &= x - \lambda_1 z, \end{aligned}$$

that is, $\lambda_1 z + \alpha_i^{(A)}(z) = x$, we have $x \in \mathcal{D}\big((\lambda_1 + \alpha_i^{(A)})^{-1}\big)$ and $(\lambda_1 + \alpha_i^{(A)})^{-1}(x) = z$.

Thus we have verified that

$$\mathcal{D}\big((\lambda_2+\alpha_i^{(A)})^{-1}\big)\subset\mathcal{D}\big((\lambda_1+\alpha_i^{(A)})^{-1}\big)$$

and the first equality in (2.14) holds true.

Interchanging λ_1 with λ_2, it also follows that

$$\mathcal{D}\big((\lambda_2+\alpha_i^{(A)})^{-1}\big)\supset\mathcal{D}\big((\lambda_1+\alpha_i^{(A)})^{-1}\big)$$

together with the validity of the second equality in (2.14). ∎

Furthermore, $\big(\lambda+\alpha_i^{(A)}\big)^{-1}(x)$ depends analytically on λ for every x in the common domain of the resolvents of $\alpha_i^{(A)}$:

Corollary 2.9 (Analyticity of the resolvent). *If $x\in\mathcal{D}\big((\lambda 1_{B(H)}+\alpha_i^{(A)})^{-1}\big)$ then*

$$\mathbb{C}\setminus(-\infty,0]\ni\lambda\mapsto\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)\in B(H)$$

is an analytic map and we have for any integer $n\geq 0$ and any $\lambda\in\mathbb{C}\setminus(-\infty,0]$:

$$\frac{d^n}{d\lambda^n}\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)=(-1)^n\,n!\,\big(\lambda 1_{B(H)}+\alpha_i^{(A)}\big)^{-n-1}(x). \tag{2.15}$$

Proof. Let $x\in\mathcal{D}\big((\lambda+\alpha_i^{(A)})^{-1}\big)$ be arbitrary. By the resolvent equation (2.14) and by Theorem 2.6 we have for every $\lambda\in\mathbb{C}\setminus(-\infty,0]$:

$$\begin{aligned}
&(\lambda+\alpha_i^{(A)})^{-1}(x)\\
&\qquad=(1+\alpha_i^{(A)})^{-1}(x)+(1-\lambda)\,(\lambda+\alpha_i^{(A)})^{-1}\big((1+\alpha_i^{(A)})^{-1}(x)\big)\\
&\qquad=\frac{1}{\lambda}\,(1+\alpha_i^{(A)})^{-1}(x)-\frac{1-\lambda}{2\,\lambda}\int_{-\infty+\frac{i}{2}}^{+\infty+\frac{i}{2}}\frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\,\alpha_\zeta^{(A)}\big((1+\alpha_i^{(A)})^{-1}(x)\big)d\zeta.
\end{aligned}$$

Using the above equality it is easily seen that

$$R_x:\mathbb{C}\setminus(-\infty,0]\ni\lambda\mapsto\big(\lambda+\alpha_i^{(A)}\big)^{-1}(x)\in B(H)$$

is an analytic map.

For the proof of (2.15) we use induction with respect to n.

(2.15) clearly holds for $n=0$ and let us now assume that, for some $n\geq 1$,

$$R_x^{(k)}(\lambda)=(-1)^k\,k!\,\big(\lambda+\alpha_i^{(A)}\big)^{-k-1}(x)$$

holds for any $0\leq k\leq n-1$, $x\in\mathcal{D}\big((1+\alpha_i^{(A)})^{-1}\big)$ and $\lambda\in\mathbb{C}\setminus(-\infty,0]$. We have to prove that then

$$\frac{d}{d\lambda}\big(\lambda+\alpha_i^{(A)}\big)^{-n}(x)=-n\big(\lambda+\alpha_i^{(A)}\big)^{-n-1}(x) \tag{2.16}$$

for any $x\in\mathcal{D}\big((1+\alpha_i^{(A)})^{-1}\big)$ and $\lambda\in\mathbb{C}\setminus(-\infty,0]$.

For let $x \in \mathcal{D}\big((1+\alpha_i^{(A)})^{-1}\big)$ and $\lambda, \lambda' \in \mathbb{C} \setminus (-\infty, 0], \lambda \neq \lambda'$ be arbitrary. Using (2.1) we get:

$$\begin{aligned} &\frac{1}{\lambda'-\lambda}\left((\lambda'+\alpha_i^{(A)})^{-n}(x)-(\lambda+\alpha_i^{(A)})^{-n}(x)\right) \\ &= \frac{1}{\lambda'-\lambda}\sum_{k=0}^{n-1}(\lambda'+\alpha_i^{(A)})^{-(n-k-1)}\big((\lambda'+\alpha_i^{(A)})^{-1}-(\lambda+\alpha_i^{(A)})^{-1}\big)(\lambda+\alpha_i^{(A)})^{-k}(x) \\ &= -\sum_{k=0}^{n-1}(\lambda'+\alpha_i^{(A)})^{-(n-k)}(\lambda+\alpha_i^{(A)})^{-k-1}(x). \end{aligned}$$

Consequently

$$\begin{aligned} &\left\|\frac{1}{\lambda'-\lambda}\left((\lambda'+\alpha_i^{(A)})^{-n}(x)-(\lambda+\alpha_i^{(A)})^{-n}(x)\right)+n\big(\lambda+\alpha_i^{(A)}\big)^{-n-1}(x)\right\| \\ &\qquad\le \sum_{k=0}^{n-1}\left\|(\lambda'+\alpha_i^{(A)})^{-(n-k)}(\lambda+\alpha_i^{(A)})^{-k-1}(x)-(\lambda+\alpha_i^{(A)})^{-(n-k)}(\lambda+\alpha_i^{(A)})^{-k-1}(x)\right\| \end{aligned}$$

and to verify (2.16) it is enough to show that the mapping

$$\mathbb{C}\setminus(-\infty,0] \ni \lambda' \mapsto (\lambda'+\alpha_i^{(A)})^{-(n-k)}(\lambda+\alpha_i^{(A)})^{-k-1}(x)$$

is operator norm-continuous for every $0 \le k \le n-1$. But by the induction assumption the $(n-k-1)^{\text{th}}$ (operator norm continuous) derivative of the analytic mapping

$$\mathbb{C}\setminus(-\infty,0] \ni \lambda' \mapsto \frac{(-1)^{n-k-1}}{(n-k-1)!}(\lambda'+\alpha_i^{(A)})^{-1}(\lambda+\alpha_i^{(A)})^{-k-1}(x)$$

is exactly the above map. ∎

Now we can describe invariance for the group $\alpha^{(A)}$ in terms of its analytic generator $\alpha_i^{(A)}$:

THEOREM 2.10 (General invariance theorem). *Let $X \subset B(H)$ be an operator norm-closed linear subspace, and $x \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)}), \varepsilon > 0$. Then the following statements are equivalent:*

(i) $\quad \big(1_H+\alpha_i^{(A)}\big)^{-k}(x) \in X, \quad k \ge 0\,,$

(ii) $\quad \big(\lambda\, 1_H+\alpha_i^{(A)}\big)^{-1}(x) \in X, \quad \lambda \in \mathbb{C}\setminus(-\infty,0]\,,$

(iii) $\quad \alpha_\zeta^{(A)}(x) \in X, \quad \zeta \in \mathbb{C}, 0 < \Im\zeta < \varepsilon\,.$

If the closed balls of X are wo-closed (that is, X is weak-closed) then the above conditions are equivalent also to*

(iv) $\quad \alpha_t^{(A)}(x) \in X, \quad t \in \mathbb{R}\,.$

Proof. By Corollaries 2.8 and 2.9

$$\mathbb{C}\setminus(-\infty,0] \ni \lambda \mapsto \big(\lambda+\alpha_i^{(A)}\big)^{-1}(x) \in B(H)$$

is an analytic map and (2.15) holds for every integer $n \ge 0$ and every $\lambda \in \mathbb{C}\setminus(-\infty,0]$. Hence (ii)⇒(i) follows immediately. Conversely, (i) implies that for each bounded linear functional φ on $B(H)$, which vanishes on X, the derivatives of any order of the analytic function

$$\mathbb{C}\setminus(-\infty,0] \ni \lambda \mapsto \langle\big(\lambda+\alpha_i^{(A)}\big)^{-1}(x), \varphi\rangle$$

vanish at $\lambda = 1$. Therefore the above analytic function vanishes identically and by the Hahn-Banach theorem (ii) follows.

For the proof of (ii)$\Leftrightarrow$(iii) we notice that, choosing some $0 < c < \min\{\varepsilon, 1\}$, by the Hahn-Banach theorem (ii) is equivalent to the validity of

$$\langle \big(\lambda + \alpha_i^{(A)}\big)^{-1}(x), \varphi\rangle = 0, \qquad \lambda \in \mathbb{C} \setminus (-\infty, 0] \tag{*}$$

for every bounded linear functional φ on $B(H)$ vanishing on X, while (iii) is equivalent to the validity of

$$\langle \alpha_\zeta^{(A)}(x), \varphi\rangle = 0, \qquad \zeta \in \mathbb{R} + ic \tag{**}$$

for the same functionals φ. Thus (ii)$\Leftrightarrow$(iii) follows if we show (*)$\Leftrightarrow$(**) for any bounded linear functional φ on $B(H)$.

(*)$\Leftarrow$(**) is an immediate consequence of formula (2.10) in Theorem 2.6.

Conversely, assuming that (*) is verified, by formula (2.10)

$$\int_{-\infty}^{+\infty} \frac{\lambda^{it-c}}{\sin(i\pi t - \pi c)} \langle \alpha_{t+ic}^{(A)}(x), \varphi\rangle \, dt = \int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)} \langle \alpha_\zeta^{(A)}(x), \varphi\rangle \, d\zeta = 2\langle x, \varphi\rangle$$

holds for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, so we have

$$\int_{-\infty}^{+\infty} \frac{e^{ist}}{\sin(i\pi t - \pi c)} \langle \alpha_{t+ic}^{(A)}(x), \varphi\rangle \, dt = 2\langle x, \varphi\rangle e^{sc}, \qquad s \in \mathbb{R}.$$

Since the left-hand side is a bounded function of s, we must have $\langle x, \varphi\rangle = 0$. But then

$$\int_{-\infty}^{+\infty} \frac{e^{ist}}{\sin(i\pi t - \pi c)} \langle \alpha_{t+ic}^{(A)}(x), \varphi\rangle \, dt = 0, \qquad s \in \mathbb{R}$$

and by the injectivity of the Fourier transformation we conclude that (**) holds true.

If the closed balls of X are wo-closed then (iii)$\Rightarrow$(iv) is a consequence of Proposition 2.2 while the converse implication follows also easily by using (2.2) and the Hahn-Banach theorem. ■

When working with the resolvents of the analytic generator $\alpha_i^{(A)}$, the following computation rules can be useful:

PROPOSITION 2.11.

$$\mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\alpha_i^{(A)}\big) = \mathcal{D}\big(\alpha_i^{(A)}\big) \text{ and } (1_H + \alpha_i^{(A)})^{-1}\alpha_i^{(A)} \subset \alpha_i^{(A)}(1_H + \alpha_i^{(A)})^{-1},$$

$$x \in \mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\big) \;\Rightarrow\; \begin{cases} x^* \in \mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\big) \text{ and} \\ \big(1_H + \alpha_i^{(A)}\big)^{-1}(x^*) = \big(\alpha_i^{(A)}\,(1_H + \alpha_i^{(A)})^{-1}(x)\big)^*. \end{cases}$$

Proof. For the proof of the first statement let $x \in \mathcal{D}\big(\alpha_i^{(A)}\big)$ be arbitrary. By Theorem 2.6 we have $x \in \mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\big)$ and with $y = \big((1_H + \alpha_i^{(A)})^{-1}(x)$ we obtain successively

$$\begin{aligned} x &= y + \alpha_i^{(A)}(y), \\ \alpha_i^{(A)}(y) &= x - y = (1_H + \alpha_i^{(A)})^{-1}(1_H + \alpha_i^{(A)})(x) - (1_H + \alpha_i^{(A)})^{-1}(x) \\ &= (1_H + \alpha_i^{(A)})^{-1}\alpha_i^{(A)}(x). \end{aligned}$$

For the proof of the second statement let now $x \in \mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\big)$ be arbitrary and put $y = (1_H + \alpha_i^{(A)})^{-1}(x)$. Then $x = y + \alpha_i^{(A)}(y)$ and by (2.7) we deduce successively

$$x^* = y^* + \alpha_{-i}^{(A)}(y^*) = (1_H + \alpha_i^{(A)})\big(\alpha_{-i}^{(A)}(y^*)\big),$$
$$x^* \in \mathcal{D}\big((1_H + \alpha_i^{(A)})^{-1}\big) \text{ and } (1_H + \alpha_i^{(A)})^{-1}(x^*) = \alpha_{-i}^{(A)}(y^*) = \alpha_i^{(A)}(y)^*. \ \blacksquare$$

3. Polar decomposition relative to a positive, self-adjoint operator. In all this section A will denote a non-singular, positive, self-adjoint linear operator in a complex Hilbert space H.

If $x \in B(H)$ and the linear operator xA is closable then $\overline{xA}$ is densely defined and closed, so we can consider its polar decomposition $\overline{xA} = v|\overline{xA}|$. Then

$$\mathcal{D}(\,\overline{xA}\,) = \mathcal{D}(|\,\overline{xA}\,|), \quad |\,\overline{xA}\,| = \overline{|\,\overline{xA}\,|\,|\,\mathcal{D}(A)},$$
$$v^*v \text{ is the orthogonal projection onto } \overline{|\,\overline{xA}\,|\big(\mathcal{D}(|\,\overline{xA}\,|)\big)} = H \ominus \operatorname{Ker}(\,\overline{xA}\,),$$
$$v\,v^* \text{ is the orthogonal projection onto } \overline{\overline{xA}\big(\mathcal{D}(\,\overline{xA}\,)\big)} = \overline{xA\big(\mathcal{D}(A)\big)} = \overline{x(H)}.$$

Let us call the partial isometry v the *phase of x relative to A* and denote it by $\mathrm{phase}_A(x)$.

The goal of this section is to point out a basic connection between phase relative to A and the analytic extensions of the group $\alpha^{(A)}$ discussed in the preceding section. What we are really doing is a reconsideration of Section 4 of the paper of S. L. Woronowicz [W3] in the language of the analytic generator.

If $x \in B(H)$ is such that xA is closable then the usually unbounded, positive, self-adjoint linear operator $|\,\overline{xA}\,|$, which is a complicated function of x, can be expressed in a simple way with the help of a certain $a \in B(H)$:

PROPOSITION 3.1. *Let $x \in B(H)$ be such that xA is closable. Then there exists a unique $a \in B(H)$ such that $aA \subset |\,\overline{xA}\,|$, and hence $\overline{aA} = |\,\overline{xA}\,|$. Moreover,*

$$x = \mathrm{phase}_A(x)a, \quad a = \mathrm{phase}_A(x)^*x$$

and

$$x \text{ invertible } \Rightarrow \begin{cases} \mathrm{phase}_A(x) \text{ is unitary,} \\ a \text{ is invertible,} \\ aA \text{ and } a^{-1}A \text{ are self-adjoint and positive.} \end{cases}$$

Proof. Let us denote, for convenience, $v = \mathrm{phase}_A(x)$. Since

$$\big\|\,|\,\overline{xA}\,|\,\xi\big\| = \big\|\,\overline{xA}\,\xi\big\| = \big\|\,xA\,\xi\big\| \le \|x\|\,\|A\,\xi\|, \qquad \xi \in \mathcal{D}(A)$$

and $A\big(\mathcal{D}(A)\big)$ is dense in H, there exists a uniquely defined $a \in B(H)$ with $aA\xi = |\,\overline{xA}\,|\,\xi$ for all $\xi \in \mathcal{D}(A)$, that is, with $aA \subset |\,\overline{xA}\,|$. Moreover, taking again into account the density of $A\big(\mathcal{D}(A)\big)$ in H,

$$xA = v\,|\overline{xA}|\,|\,\mathcal{D}(A) = vaA, \quad aA = |\overline{xA}|\,|\,\mathcal{D}(A) = v^*xA$$

imply

$$x = va, \quad a = v^*x.$$

Let us now assume that x is invertible. Then xA is closed and $\operatorname{Ker}(xA) = \{0\}$, hence $H \ominus \operatorname{Ker}(\,\overline{xA}\,) = H$ and $v^*v = 1_H$ follows. On the other hand, $x(H) = H$ implies

$vv^* = 1_H$, so v is unitary and $a = v^*x$ is invertible. Consequently $aA = \overline{aA} = |\,\overline{xA}\,|$ is self-adjoint and positive, that is,

$$aA = (aA)^* = Aa^* \geq 0. \tag{3.1}$$

(3.1) implies $a^{-1}A = A(a^*)^{-1} = A(a^{-1})^* = (a^{-1}A)^*$, so also $a^{-1}A$ is self-adjoint. Again by (3.1), $a^{-1}A = A(a^*)^{-1} = a^{-1}(aA)(a^{-1})^* \geq 0$. ■

The decomposition $x = \mathrm{phase}_A(x)a$ established in Proposition 3.1 will be called the *polar decomposition of x relative to A*. Denoting a by $|x|_A$, the polar decomposition of x relative to A takes the form

$$x = \mathrm{phase}_A(x)|x|_A.$$

It is easy to see that, for $x, v, a \in B(H)$,

$$\left.\begin{array}{l} xA \text{ is closable} \\ |x|_A = a \\ \mathrm{phase}_A(x) = v \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x = v\,a \\ aA \text{ is positive and essentially self-adjoint} \\ v^*v \text{ is the orthogonal projection onto } \overline{a(H)}. \end{array}\right. \tag{3.2}$$

Let us now describe with the help of $\alpha^{(A)}$ those $a \in B(H)$ for which aA is symmetric respectively positive:

Proposition 3.2. *For $a \in B(H)$ we have:*

$$\begin{aligned} aA \textit{ symmetric} &\Leftrightarrow a \in \mathcal{D}(\alpha^{(A)}_{i/2}), \alpha^{(A)}_{i/2}(a) \textit{ self-adjoint} \Leftrightarrow a \in \mathcal{D}(\alpha^{(A)}_{i}), \alpha^{(A)}_{i}(a) = a^*, \\ &\Rightarrow a \in \mathcal{D}\big((1_H + \alpha^{(A)}_i)^{-1}\big), \alpha^{(A)}_i(1_H + \alpha^{(A)}_i)^{-1}(a) \textit{ self-adjoint}, \\ aA \geq 0 &\Leftrightarrow a \in \mathcal{D}(\alpha^{(A)}_{i/2}), \alpha^{(A)}_{i/2}(a) \geq 0. \end{aligned}$$

Proof. Clearly

$$aA \text{ symmetric, that is, } aA \subset (aA)^* = A\,a^* \Leftrightarrow \mathcal{D}(A^{-1}a\,A) = \mathcal{D}(A) \text{ and } A^{-1}a\,A \subset a^*$$

and so, by Proposition 2.5, aA is symmetric if and only if $a \in \mathcal{D}(\alpha^{(A)}_i)$ and $\alpha^{(A)}_i(a) = a^*$. In this case, according to Proposition 2.11, we have $a \in \mathcal{D}\big((1_{B(H)} + \alpha^{(A)}_i)^{-1}\big)$ and

$$\begin{aligned} \alpha^{(A)}_i\,(1_{B(H)} + \alpha^{(A)}_i)^{-1}(a) &= (1_{B(H)} + \alpha^{(A)}_i)^{-1}\alpha^{(A)}_i(a) = (1_{B(H)} + \alpha^{(A)}_i)^{-1}(a^*) \\ &= \big(\alpha^{(A)}_i\,(1_{B(H)} + \alpha^{(A)}_i)^{-1}(a)\big)^*, \end{aligned}$$

that is, $\alpha^{(A)}_i\,(1_{B(H)} + \alpha^{(A)}_i)^{-1}(a)$ is self-adjoint.

Moreover, by (2.9) also the equivalence

$$a \in \mathcal{D}(\alpha^{(A)}_i), \alpha^{(A)}_i(a) = a^* \Leftrightarrow a \in \mathcal{D}(\alpha^{(A)}_{i/2}), \alpha^{(A)}_{i/2}(a) = \alpha^{(A)}_{i/2}(a)^*$$

holds true.

Let us now assume that $aA \geq 0$. By the above part of the proof we have $a \in \mathcal{D}(\alpha^{(A)}_{i/2})$ and Proposition 2.5 yields

$$\mathcal{D}\big(A^{-1/2}(aA)\,A^{-1/2}\big) = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(A^{-1/2}), \qquad A^{-1/2}(aA)\,A^{-1/2} \subset \alpha^{(A)}_{i/2}(a).$$

Consequently

$$\big(\alpha^{(A)}_{i/2}(a)\,\xi \,|\, \xi\big) = \big((aA)\,A^{-1/2}\,\xi \,|\, A^{-1/2}\,\xi\big) \geq 0, \qquad \xi \in \mathcal{D}(A^{1/2}) \cap \mathcal{D}(A^{-1/2})$$

and, taking into account the density of $\mathcal{D}(A^{1/2})\cap\mathcal{D}(A^{-1/2})$ in H, the positivity of $\alpha^{(A)}_{i/2}(a)$ follows.

Conversely, if $a \in \mathcal{D}(\alpha^{(A)}_{i/2})$ and $\alpha^{(A)}_{i/2}(a) \geq 0$ then, by Proposition 2.5, we have

$$\mathcal{D}\big(A^{-1/2}a\,A^{1/2}\big) = \mathcal{D}(A^{1/2}) \text{ and } A^{-1/2}a\,A^{1/2} \subset \alpha^{(A)}_{i/2}(a),$$

so

$$\begin{aligned}\big(aA\,\xi\,\big|\,\xi\big) &= \big(A^{1/2}(A^{-1/2}a\,A^{1/2})\,A^{1/2}\,\xi\,\big|\,\xi\big) = \big(\alpha^{(A)}_{i/2}(a)\,A^{1/2}\,\xi\,\big|\,A^{1/2}\,\xi\big)\\ &\geq 0, \qquad \xi \in \mathcal{D}(A).\end{aligned}$$

In other words $aA \geq 0$. ∎

Next we describe those invertible operators $a \in B(H)$ for which both aA and $a^{-1}A$ are positive:

PROPOSITION 3.3. *For an invertible $a \in B(H)$ the following are equivalent:*

(i) $aA \geq 0$ *and* $a^{-1}A \geq 0$*;*
(ii) aA *is self-adjoint and positive;*
(iii) aA *and* $a^{-1}A$ *are both self-adjoint and positive;*
(iv) $a,\, a^{-1} \in \mathcal{D}(\alpha^{(A)}_{i/2})$ *and* $\alpha^{(A)}_{i/2}(a), \alpha^{(A)}_{i/2}(a^{-1}) \geq 0$.

Proof. The equivalence (ii)⇔(iii) is an immediate consequence of Proposition 3.1, (iii)⇒(i) is obvious and (i)⇒(iv) follows by Proposition 3.2. Therefore only (iv)⇒(iii) remains to be proved.

Let $aA = w\,|aA|$ be the polar decomposition of the closed operator aA. According to Proposition 3.1 there exists an invertible $b \in B(H)$ for which

$$bA = |aA|,\ a = w\,b,\ b^{-1}A \text{ is self-adjoint and positive.}$$

We shall prove that $w = 1_H$: this will imply the self-adjointness and the positivity of $aA = bA = |aA|$ and of $a^{-1}A = b^{-1}A$.

Since the unitary operator w is equal to 1_H if and only if its spectrum $\sigma(w)$ is contained in $[0, +\infty)$, we have to prove that $\sigma(w) \subset [0, +\infty)$.

We shall use the formulas

$$w = ab^{-1}, \quad w^* = w^{-1} = ba^{-1}.$$

By Proposition 3.2 we have $b, b^{-1} \in \mathcal{D}(\alpha^{(A)}_{i/2})$ and $\alpha^{(A)}_{i/2}(b), \alpha^{(A)}_{i/2}(b^{-1}) \geq 0$, so (2.7) yields

$$w \in \mathcal{D}(\alpha^{(A)}_{i/2}) \text{ and } \alpha^{(A)}_{i/2}(w\,) = \alpha^{(A)}_{i/2}(a)\,\alpha^{(A)}_{i/2}(b^{-1}) \text{ with } \alpha^{(A)}_{i/2}(a),\ \alpha^{(A)}_{i/2}(b^{-1}) \geq 0, \tag{3.3}$$

$$w^* \in \mathcal{D}(\alpha^{(A)}_{i/2}) \text{ and } \alpha^{(A)}_{i/2}(w^*) = \alpha^{(A)}_{i/2}(b)\,\alpha^{(A)}_{i/2}(a^{-1}) \text{ with } \alpha^{(A)}_{i/2}(b),\ \alpha^{(A)}_{i/2}(a^{-1}) \geq 0. \tag{3.4}$$

Using the second implication in (2.7) we can transcribe (3.4) as follows:

$$w \in \mathcal{D}(\alpha^{(A)}_{-i/2}) \text{ and } \alpha^{(A)}_{-i/2}(w) = \alpha^{(A)}_{i/2}(a^{-1})\,\alpha^{(A)}_{i/2}(b) \text{ with } \alpha^{(A)}_{i/2}(a^{-1}), \alpha^{(A)}_{i/2}(b) \geq 0. \tag{3.5}$$

Let us recall that

$$\sigma(pq) \subset [0, +\infty), \qquad 0 \leq p,\, q \in B(H). \tag{3.6}$$

Indeed, with $x := p^{1/2}$ and $y := p^{1/2}q$ we have $xy = p\,q$ and $yx = p^{1/2}q\,p^{1/2} \geq 0$, so the classical equality $\sigma(x\,y)\cup\{0\} = \sigma(y\,x)\cup\{0\}$ yields $\sigma(pq) \subset \sigma(p^{1/2}q\,p^{1/2})\cup\{0\} \subset [0, +\infty)$.

Now by (3.3), (3.5) and (3.6) we have

$$w \in \mathcal{D}(\alpha^{(A)}_{i/2}) \cap \mathcal{D}(\alpha^{(A)}_{-i/2}), \quad \sigma\big(\alpha^{(A)}_{i/2}(w)\big) \cup \sigma\big(\alpha^{(A)}_{-i/2}(w)\big) \subset [0,+\infty)$$

and using Proposition 2.4 we deduce the desired positivity $\sigma(w) \subset [0,+\infty)$. ■

After this preparation we can prove that the resolvent $(1_H + \alpha^{(A)}_i)^{-1}$ can be expressed in terms of the phase relative to A (cf. [W3], Proposition 1.1 and Lemma 4.1):

THEOREM 3.4 (Woronowicz's resolvent formula). *If $x \in B(H)$ and xA is symmetric. then $x \in \mathcal{D}(\alpha^{(A)}_i)$ and*

$$\big(1_H + \alpha^{(A)}_i\big)^{-1}(x) = x - wo\text{-}\lim_{0\neq\varepsilon\to 0} \frac{1}{2\varepsilon i}\big(\mathrm{phase}_A(1_H + i\varepsilon x) - 1_H\big). \tag{3.7}$$

Proof. Since xA is symmetric, by Proposition 3.2 we have

$$x \in \mathcal{D}(\alpha^{(A)}_i), \alpha^{(A)}_i(x) = x^*. \tag{3.8}$$

Let us denote

$$x_\varepsilon = \frac{1}{2\varepsilon i}\big(\mathrm{phase}_A(1_H + i\varepsilon x) - 1_H\big), \quad \varepsilon \in \mathbb{R}\setminus\{0\}.$$

First step: proof that $\|x_\varepsilon\| \le \|x\|/2$ for $0 < |\varepsilon| < \|x\|^{-1}$. Let us first verify that

$$\big\{\big((1_H + i\varepsilon x)A\,\xi \,\big|\, \xi\big)\,; \xi \in \mathcal{D}(A)\big\} \subset \big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le |\varepsilon|\,\|x\|\,\Re\zeta\big\}. \tag{3.9}$$

For let $\xi \in \mathcal{D}(A)$ be arbitrary. By Proposition 3.2

$$x \in \mathcal{D}(\alpha^{(A)}_{i/2}) \text{ and } \alpha^{(A)}_{i/2}(x) \text{ is self-adjoint}$$

and, using Proposition 2.5, we obtain successively:

$$\mathcal{D}(A^{-1/2}x\,A^{1/2}) = \mathcal{D}(A^{1/2}) \text{ and } A^{-1/2}xA^{1/2} \subset \alpha^{(A)}_{i/2}(x),$$

$$xA \subset A^{1/2}(A^{-1/2}x\,A^{1/2})A^{1/2} \subset A^{1/2}\alpha^{(A)}_{i/2}(x)A^{1/2},$$

$$\big((1_H + i\varepsilon x)A\,\xi \,\big|\, \xi\big) = \|A^{1/2}\xi\|^2 + i\varepsilon\big(xA\,\xi \,\big|\, \xi\big) = \underbrace{\|A^{1/2}\xi\|^2}_{\ge 0} + i\varepsilon\,\underbrace{\big(\alpha^{(A)}_{i/2}(x)A^{1/2}\xi \,\big|\, A^{1/2}\xi\big)}_{\in\mathbb{R}}.$$

Consequently the complex number $\zeta = \big((1_H + i\varepsilon x)\,A\,\xi \,\big|\, \xi\big)$ satisfies the condition

$$|\Im\zeta| \le |\varepsilon|\,\|\alpha^{(A)}_{i/2}(x)\|\,\|A^{1/2}\xi\|^2 = |\varepsilon|\,\|\alpha^{(A)}_{i/2}(x)\|\,\Re\zeta.$$

But by (3.8) and by (2.2) we have $\|\alpha^{(A)}_{i/2}(x)\| \le \|x\|$, so ζ satisfies $|\Im\zeta| \le |\varepsilon|\,\|x\|\,\Re\zeta$.

(3.9) means that $(1_H + i\varepsilon x)A$ is a *sectorial operator* (see [Kt], V.3.10 and [W2]) whose sector

$$\mathcal{S}\big((1_H + i\varepsilon x)\,A\big) := \big\{\big((1_H + i\varepsilon x)\,A\,\xi \,\big|\, \xi\big)\,;\, \xi \in \mathcal{D}(A)\big\}$$

is contained in the angle $\big\{\zeta \in \mathbb{C}\,; |\Im\zeta| \le \varepsilon\,\|x\|\,\Re\zeta\big\}$. Actually $(1_H + i\varepsilon x)A$ is an *m-sectorial* (= maximal sectorial) *operator*: since

$$1_H + (1_H + i\varepsilon x)\,A = \big(1_H + i\varepsilon xA\,(1_H + A)^{-1}\big)(1_H + A)$$

and $\|\varepsilon xA\,(1_H + A)^{-1}\| \le |\varepsilon|\,\|x\| < 1$, -1 does not belong to the spectrum of $(1_H + i\varepsilon x)A$.

According to Woronowicz's result from [W2] it follows now that the spectrum of the unitary operator $\mathrm{phase}_A(1_H + i\varepsilon x)$ occurring in the polar decomposition of $(1_H + i\varepsilon x)A$ is contained in the sector $\mathcal{S}\big((1_H+i\varepsilon x)\,A\big)$ and hence in the angle $\big\{\zeta\in\mathbb{C}\,;|\Im\zeta|\le|\varepsilon|\,\|x\|\,\Re\zeta\big\}$. In other words

$$\sigma\big(\mathrm{phase}_A(1_H+i\varepsilon x)\big)\subset\big\{\zeta\in\mathbb{C}\,;\,|\zeta|=1,\,|\Im\zeta|\le|\varepsilon|\,\|x\|\,\Re\zeta\big\}. \tag{3.10}$$

Using (3.10) it is easy to verify that the spectral radius $r(x_\varepsilon)$ of the operator x_ε is $\le\|x\|/2$. But x_ε is normal and therefore $\|x_\varepsilon\|=r(x_\varepsilon)$.

Second step: $\lim_{0\neq\varepsilon\to0}\|x_\varepsilon-x_\varepsilon^*\|=0$. For any $0<|\varepsilon|<\|x\|^{-1}$, since $\mathrm{phase}_A(1_H+i\varepsilon x)=1_H+2\varepsilon i x_\varepsilon$ is a unitary operator, we have

$$\begin{gathered}1_H=(1_H+2\varepsilon i x_\varepsilon)^*(1_H+2\varepsilon i x_\varepsilon)=1_H+2\varepsilon i(x_\varepsilon-x_\varepsilon^*)+4\varepsilon^2x_\varepsilon^*x_\varepsilon,\\ x_\varepsilon-x_\varepsilon^*=2\varepsilon i x_\varepsilon^*x_\varepsilon.\end{gathered}$$

Using the estimate obtained in the first step of the proof for $\|x_\varepsilon\|$ we conclude:

$$\|\,x_\varepsilon-x_\varepsilon^*\|=\|2\varepsilon i\,x_\varepsilon^*x_\varepsilon\|\le\frac{|\varepsilon|\,\|x\|^2}{2}\xrightarrow{\varepsilon\to0}0.$$

Third step: if y is a wo-limit point at 0 of the function $\mathbb{R}\setminus\{0\}\ni\varepsilon\mapsto x_\varepsilon\in B(H)$ then $x-y=\big(1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)$. First of all we notice that by the second step of the proof we have $y=y^*$.

Let $\varepsilon\in\mathbb{R}\setminus\{0\}$ be arbitrary. By the definition of x_ε and of $\mathrm{phase}_A(1_H+i\varepsilon x)$ we have

$$(1_H+2\varepsilon i x_\varepsilon)^*(1_H+i\varepsilon x)\,A=\big(\mathrm{phase}_A(1_H+i\varepsilon x)\big)^*(1_H+i\varepsilon x)\,A=|(1_H+i\varepsilon x)\,A|\ge0$$

and it follows that for every $\xi\in\mathcal{D}(A)$

$$\begin{aligned}0&\le\big((1_H+i\varepsilon x)\,A\xi\,\big|\,(1_H+2\varepsilon i x_\varepsilon)\,\xi\big)=\big(A\xi+i\varepsilon xA\xi\,\big|\,\xi+2\varepsilon i x_\varepsilon\xi\big)\\ &=\underbrace{\big(A\xi\,|\,\xi\big)}_{\in\mathbb{R}}+i\varepsilon\underbrace{\big(xA\xi\,|\,\xi\big)}_{\in\mathbb{R}}-2\varepsilon i\big(A\xi\,|\,x_\varepsilon\xi\big)+2\varepsilon^2\big(xA\xi\,|\,x_\varepsilon\xi\big),\\ 0&=\big(xA\xi\,|\,\xi\big)-2\,\Re\,\big(A\xi\,|\,x_\varepsilon\xi\big)+2\varepsilon\,\Im\,\big(xA\xi\,|\,x_\varepsilon\xi\big).\end{aligned}$$

For $\varepsilon\to0$ the above equality yields

$$\begin{aligned}\big(xA\xi\,|\,\xi\big)&=2\,\Re\,\big(A\xi\,|\,y\xi\big)=\big(A\xi\,|\,y\xi\big)+\big(y\xi\,|\,A\xi\big)=\big(y^*A\xi\,|\,\xi\big)+\big(y\xi\,|\,A\xi\big)\\ &=\big(yA\xi\,|\,\xi\big)+\big(y\xi\,|\,A\xi\big),\end{aligned}$$

that is, $\big((x-y)\,A\xi\,|\,\xi\big)=\big(y\xi\,|\,A\xi\big)$.

Using the polarization identity we obtain

$$\big((x-y)\,A\xi\,|\,\eta\big)=\big(y\xi\,|\,A\eta\big),\qquad \xi,\eta\in\mathcal{D}(A)$$

and it follows successively that

$$\begin{gathered}\xi\in\mathcal{D}(A)\;\Rightarrow\;y\xi\in\mathcal{D}(A^*)=\mathcal{D}(A)\text{ and }(x-y)A\xi=A^*y\,\xi=Ay\,\xi,\\ \mathcal{D}\big(A^{-1}(x-y)A\big)=\mathcal{D}(A)\text{ and }A^{-1}(x-y)\,A\subset y.\end{gathered}$$

But by Proposition 2.5 the last assertion is equivalent to

$$x-y\in\mathcal{D}(\alpha_i^{(A)})\text{ and }\alpha_i^{(A)}(x-y)=y.$$

Therefore $\big(1_{B(H)}+\alpha_i^{(A)}\big)(x-y)=x-y+y=x$, that is, $x-y=\big(1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)$.

Fourth and last step: end of the proof. Since the closed balls in $B(H)$ are *wo*-compact, the bounded function

$$(-\|x\|^{-1},0)\cup(0,\|x\|^{-1})\ni\varepsilon\mapsto x_\varepsilon\in B(H) \tag{3.11}$$

has *wo*-limit points in 0. But by the third step of the proof all these limit points are equal to $x-\left(1_{B(H)}+\alpha_i^{(A)}\right)^{-1}(x)$. Consequently the limit $wo\text{-}\lim_{0\neq\varepsilon\to 0}x_\varepsilon$ exists and is equal to $x-\left(1_{B(H)}+\alpha_i^{(A)}\right)^{-1}(x)$. ∎

Now we shall prove the Woronowicz implementation criterion which characterizes the situation in which the group $\alpha^{(A)}$ leaves invariant a von Neumann algebra $\mathcal{M}\subset B(H)$ in terms of the phase relative to A (see [W1], Thm. 3.1 and [W3], Proposition 4.3):

THEOREM 3.5 (Woronowicz's invariance criterion). *Let $\mathcal{M}\subset B(H)$ be a von Neumann algebra. Then the following statements are equivalent:*

(i) $\alpha_t^{(A)}(\mathcal{M})=\mathcal{M}$ *for every* $t\in\mathbb{R}$;
(ii) $x\in\mathcal{M}$, xA *closable* $\Rightarrow$ $\mathrm{phase}_A(x)\in\mathcal{M}$;
(iii) $x\in\mathcal{M}$, x *invertible* $\Rightarrow$ $\mathrm{phase}_A(x)\in\mathcal{M}$.

Proof. First we prove that (i)⇒(ii).

For this purpose let us assume the validity of (i) and let $x\in\mathcal{M}$ be such that xA is closable. We have to prove that if the polar decomposition of $\overline{xA}$ is $\overline{xA}=v|\overline{xA}|$ then $v\in\mathcal{M}$. By the second commutant theorem of John von Neumann this is equivalent to the commutation of v with every element of the commutant $\mathcal{M}'$ of $\mathcal{M}$. But by (i) we have

$$\alpha_t^{(A)}(\mathcal{M}')=\mathcal{M}' \text{ for every } t\in\mathbb{R}$$

and Proposition 2.3 yields the *wo*-density in $\mathcal{M}'$ of the $\alpha^{(A)}$-entire elements of $\mathcal{M}'$ of exponential type. Consequently it is enough to prove that v commutes with every $\alpha^{(A)}$-entire element of exponential type of $\mathcal{M}'$.

Let us recall that by Proposition 2.5

$$x'A\subset A\,\alpha_i^{(A)}(x'),\qquad x'\in B(H)\ \alpha^{(A)}\text{-entire}. \tag{3.12}$$

Next we verify that

$$x'\,\overline{xA}\subset\overline{xA}\,\alpha_i^{(A)}(x'),\qquad x'\in\mathcal{M}'\ \alpha^{(A)}\text{-entire}. \tag{3.13}$$

For every $\xi\in\mathcal{D}(\,\overline{xA}\,)$ there exists a sequence $\left(\xi_k\right)_{k\geq 1}$ in $\mathcal{D}(A)$ such that

$$\xi_k\to\xi \ \text{ and } \ xA\,\xi_k\to\overline{xA}\,\xi.$$

Using the permutability of $x\in\mathcal{M}$ and $x'\in\mathcal{M}'$ as well as (3.12) we deduce for every k

$$x'\,xA\,\xi_k=x\,x'A\,\xi_k=xA\,\alpha_i^{(A)}(x')\,\xi_k.$$

Thus $\alpha_i^{(A)}(x')\,\xi_k\to\alpha_i^{(A)}(x')\,\xi$ and $xA\,\alpha_i^{(A)}(x')\,\xi_k=x'\,xA\,\xi_k\to x'\,\overline{xA}\,\xi$. Since xA is closable, we get

$$\alpha_i^{(A)}(x')\,\xi\in\mathcal{D}(\,\overline{xA}\,) \ \text{ and } \ \overline{xA}\,\alpha_i^{(A)}(x')\,\xi=x'\,\overline{xA}\,\xi.$$

Now we verify that

$$x'\,|\,\overline{xA}\,|^2\subset|\,\overline{xA}\,|^2\alpha_{2i}^{(A)}(x'),\qquad x'\in\mathcal{M}'\ \alpha^{(A)}\text{-entire}. \tag{3.14}$$

Indeed, by (3.12), the commutation of $x^* \in \mathcal{M}$ and $\alpha_i^{(A)}(x') \in \mathcal{M}'$, and (3.13) (applied with x' replaced by $\alpha_i^{(A)}(x')$) we have

$$\begin{aligned} x' \,|\,\overline{xA}\,|^2 &= x'\,Ax^*\overline{xA} \subset A\,\alpha_i^{(A)}(x')\,x^*\overline{xA} = A\,x^*\alpha_i^{(A)}(x')\,\overline{xA} \subset A\,x^*\overline{xA}\,\alpha_{2i}^{(A)}(x') \\ &= |\,\overline{xA}\,|^2\alpha_{2i}^{(A)}(x'). \end{aligned}$$

A first consequence of (3.14) is that $v^*v \in \mathcal{M}$. For the proof let us recall that v^*v is the orthogonal projection onto $\overline{|\,\overline{xA}\,|\big(\mathcal{D}(|\,\overline{xA}\,|)\big)}$. Since this subspace is equal to the closure of $|\,\overline{xA}\,|^2\big(\mathcal{D}(|\,\overline{xA}\,|^2)\big)$ and by (3.14) every element of the $*$-algebra of all $\alpha^{(A)}$-entire elements of $\mathcal{M}'$ leaves $|\,\overline{xA}\,|^2\big(\mathcal{D}(|\,\overline{xA}\,|^2)\big)$ invariant, it follows that v^*v commutes with every $\alpha^{(A)}$-entire element of $\mathcal{M}'$ and thus $v^*v \in \mathcal{M}'' = \mathcal{M}$.

On the other hand, iterating (3.14) we obtain

$$x'\,|\,\overline{xA}\,|^{2n} \subset |\,\overline{xA}\,|^{2n}\alpha_{2ni}^{(A)}(x'), \qquad x' \in \mathcal{M}'\ \alpha^{(A)}\text{-entire},\ n \geq 0 \text{ integer.} \tag{3.15}$$

Let e_k, $k \geq 1$, be the spectral projection of $|\,\overline{xA}\,|$ associated to the interval $[k^{-1}, k]$. Then the restriction of $|\,\overline{xA}\,|\,e_k$ to $e_k(H)$ is an invertible, bounded, positive operator, all of whose complex powers are defined by functional calculus. Let $\big(|\,\overline{xA}\,|\,e_k\big)^z$, $z \in \mathbb{C}$, denote the element of $B(H)$ which on the subspace $e_k(H)$ is equal to the above mentioned z^{th} power, while it vanishes on $H \ominus e_k(H)$.

Let x' be an arbitrary $\alpha^{(A)}$-entire element of exponential type of $\mathcal{M}'$. Since the function

$$F_{k,k'} : \mathbb{C} \ni z \mapsto e_{k'}\,x'\,\big(|\,\overline{xA}\,|\,e_k\big)^z - \big(|\,\overline{xA}\,|\,e_{k'}\big)^z\alpha_z^{(A)}(x')\,e_k \in B(H), \qquad k,\,k' \geq 1$$

is entire and of exponential type (that is, for suitable constants τ, $c \geq 0$, the inequality $\|F(z)\| \leq c\,e^{\tau|z|}$ holds for every $z \in \mathbb{C}$), bounded on the imaginary axis, and by (3.15) it vanishes at every even natural number, according to a classical theorem of F. Carlson (see e.g. [PS], III. Problem 328) it must vanish identically. Thus, for $z = 1$ we have

$$e_{k'}x'\,|\,\overline{xA}\,|\,e_k = |\,\overline{xA}\,|\,e_{k'}\,\alpha_i^{(A)}(x')\,e_k, \qquad k,\,k' \geq 1, \tag{3.16}$$

We show that (3.16) implies

$$x'\,|\,\overline{xA}\,| \subset |\,\overline{xA}\,|\alpha_i^{(A)}(x'). \tag{3.17}$$

For let $\xi \in \mathcal{D}(|\,\overline{xA}\,|)$ be arbitrary. Since $e_k\,\xi \to v^*v\,\xi$, $|\,\overline{xA}\,|\,e_k\,\xi \to |\,\overline{xA}\,|\,\xi$ and the operators $v^*v \in \mathcal{M}$ and $\alpha_i^{(A)}(x') \in \mathcal{M}'$ are commuting, using (3.16) we deduce

$$\begin{aligned} e_{k'}x'\,|\,\overline{xA}\,|\,\xi &= |\,\overline{xA}\,|\,e_{k'}\,\alpha_i^{(A)}(x')\,v^*v\,\xi = |\,\overline{xA}\,|\,e_{k'}\,v^*v\,\alpha_i^{(A)}(x')\,\xi \\ &= |\,\overline{xA}\,|\,e_{k'}\,\alpha_i^{(A)}(x')\,\xi, \qquad k' \geq 1. \end{aligned}$$

Taking into account that by (3.13) $\alpha_i^{(A)}(x')\,\xi \in \mathcal{D}(\,\overline{xA}\,) = \mathcal{D}(|\,\overline{xA}\,|)$, the former equality can be written also as

$$e_{k'}x'\,|\,\overline{xA}\,|\,\xi = e_{k'}|\,\overline{xA}\,|\,\alpha_i^{(A)}(x')\,\xi, \qquad k' \geq 1.$$

Now, since $e_{k'} \overset{so}{\to} v^*v$, v^*v commutes with x' and $v^*v\,|\,\overline{xA}\,| = |\,\overline{xA}\,|$, we obtain

$$x'\,|\,\overline{xA}\,|\,\xi = x'\,v^*v\,|\,\overline{xA}\,|\,\xi = v^*v\,x'\,|\,\overline{xA}\,|\,\xi = v^*v\,|\,\overline{xA}\,|\,\alpha_i^{(A)}(x')\,\xi = |\,\overline{xA}\,|\,\alpha_i^{(A)}(x')\,\xi,$$

which is exactly (3.17).

We conclude that for any $\alpha^{(A)}$-entire element of exponential type $x' \in \mathcal{M}'$ both (3.13) and (3.17) hold. Therefore we have, for every $\xi \in \mathcal{D}(|\,\overline{xA}\,|)$,

$$x'\,v\,|\,\overline{xA}\,|\,\xi = x'\,\overline{xA}\,\xi \overset{(3.13)}{=} \overline{xA}\,\alpha_i^{(A)}(x')\,\xi = v\,|\,\overline{xA}\,|\,\alpha_i^{(A)}(x')\,\xi \overset{(3.17)}{=} v\,x'\,|\,\overline{xA}\,|\,\xi.$$

In other words $x'v$ and $v\,x'$ are equal on the subspace $\overline{|\,\overline{xA}\,|\big(\mathcal{D}(|\,\overline{xA}\,|)\big)} = v^*vH$, that is, $x'v\,v^*v = v\,x'v^*v$. But v is a partial isometry and v^*v commutes with x', hence we deduce that $x'v = x'v\,v^*v = v\,x'v^*v = v\,v^*v\,x' = v\,x'$ and the proof of (i)$\Rightarrow$(ii) is complete.

(ii)$\Rightarrow$(iii) being trivial, it remains to prove (iii)$\Rightarrow$(i).

Thus let us assume that (iii) holds and let us consider

$$X = \big\{x \in \mathcal{M}\,;\, xA \ \text{ symmetric}\big\} + i\,\big\{x \in \mathcal{M}\,;\, xA \ \text{ symmetric}\big\}.$$

Then X is a linear subspace of $\mathcal{M}$ and, by Proposition 3.2, also of $\mathcal{D}(\alpha_i^{(A)})$. We next show that X generates the von Neumann algebra $\mathcal{M}$.

Since the von Neumann algebra $\mathcal{M}$ is generated by all invertible, positive operators belonging to it, it is enough to prove that any such operator a belongs to the von Neumann algebra generated by X. Let

$$a^{1/2} = \text{phase}_A(a^{1/2})|a^{1/2}|_A$$

be the polar decomposition of $a^{1/2}$ relative to A. By (iii) we have $\text{phase}_A(a^{1/2}) \in \mathcal{M}$, so $|a^{1/2}|_A \in \mathcal{M}$. Since $|a^{1/2}|_A A$ is self-adjoint, $|a^{1/2}|_A$ belongs to X and so

$$a = (a^{1/2})^* a^{1/2} = |a^{1/2}|_A^*|a^{1/2}|_A$$

belongs to the von Neumann algebra generated by X.

Taking into account that X generates the von Neumann algebra $\mathcal{M}$, by Theorem 2.10 (i) follows once we prove that

$$\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(X) \subset X. \tag{3.18}$$

But if $x \in \mathcal{M}$ and xA is symmetric then, according to (iii) and Theorem 3.4, we have $\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) \in \mathcal{M}$. On the other hand,

$$\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) = \frac{1}{2}\,x + \bigg(\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) - \frac{1}{2}\,x\bigg),$$

where $\frac{1}{2}\,xA$ is symmetric. Showing that $i\big(\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) - \frac{1}{2}\,x\big)A$ is symmetric it will follow that $\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) \in X$ and so (3.18) will be proved.

By Proposition 3.2 $i\big(\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) - \frac{1}{2}\,x\big)A$ is symmetric exactly when

$$\alpha_i^{(A)}\bigg(\big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x) - \frac{1}{2}\,x\bigg) = \bigg(\frac{1}{2}\,x - \big(1_{B(H)} + \alpha_i^{(A)}\big)^{-1}(x)\bigg)^*.$$

But, taking into account that by Proposition 3.2 $\alpha_i^{(A)}(x) = x^*$, this implies by using

Proposition 2.11:

$$\begin{aligned}\alpha_i^{(A)}\Big(\big(1_{B(H)}+\alpha_i^{(A)}\big)^{-1}(x)-\frac{1}{2}\,x\Big) &= \big(1_{B(H)}+\alpha_i^{(A)}\big)^{-1}\big(\alpha_i^{(A)}(x)\big)-\frac{1}{2}\,\alpha_i^{(A)}(x)\\ &= \big(1_{B(H)}+\alpha_i^{(A)}\big)^{-1}\big(x^*\big)-\frac{1}{2}\,x^*\\ &= \Big(\alpha_i^{(A)}\,(1_{B(H)}+\alpha_i^{(A)})^{-1}(x)-\frac{1}{2}\,x\Big)^*\\ &= \Big(\frac{1}{2}\,x-(1_{B(H)}+\alpha_i^{(A)})^{-1}(x)\Big)^* . \ \blacksquare\end{aligned}$$

4. The foundation of the Tomita-Takesaki Theory. Let H, K be complex Hilbert spaces. We recall that the *conjugate Hilbert space* $\overline{K}$ of K has the same additive group structure as K, but

$$\lambda\,\eta \ \text{ in } \overline{K}=\overline{\lambda}\,\eta \text{ in } K, \qquad \lambda\in\mathbb{C},\ \eta\in\overline{K},$$
$$(\eta_1\,|\,\eta_2) \text{ in } \overline{K}=(\eta_2\,|\,\eta_1) \text{ in } K, \quad \eta_1,\ \eta_2\in\overline{K}.$$

Any antilinear operator $T : H \supset \mathcal{D}(T)\to K$ can be considered a linear operator taking values in $\overline{K}$ and as such we can apply to it the results of the theory of linear operators. Thus, assuming that T is densely defined, $\eta\in K$ belongs to the domain of the adjoint T^* when the linear functional $\mathcal{D}(T)\ni\xi\mapsto\big(\eta\,|\,T(\xi)\big)$ is bounded and then

$$\big(\xi\,|\,T^*(\eta)\big)=\big(\eta\,|\,T(\xi)\big), \quad \xi\in\mathcal{D}(T).$$

Furthermore, if T is densely defined and closable, then

$$(\overline{T})^*=T^* : K\supset\mathcal{D}(T^*)\to H$$

is a densely defined closed antilinear operator, $T^*\overline{T}$ is a positive, self-adjoint linear operator and we can consider the polar decomposition of $\overline{T}$:

$$\overline{T}=V\,(T^*\overline{T})^{1/2},$$

where $V : H\to K$ is an antilinear operator which carries isometrically the closure of $(T^*\overline{T})^{1/2}\big(\mathcal{D}(\overline{T})\big)$ onto the closure of $T\big(\mathcal{D}(\overline{T})\big)$ and vanishes on the orthogonal complement of $(T^*\overline{T})^{1/2}\big(\mathcal{D}(\overline{T})\big)$.

In all this section H will denote a non-zero complex Hilbert space, and $\mathcal{M}\subset B(H)$ a von Neumann algebra endowed with a *bicyclic vector* ξ_o, that is, a vector $\xi_o\in H$ for which $\overline{\mathcal{M}\,\xi_o}=H$ (cyclicity relative to $\mathcal{M}$) and $\overline{\mathcal{M}'\,\xi_o}=H$ (cyclicity relative to $\mathcal{M}'$). We recall that $\overline{\mathcal{M}'\,\xi_o}=H$ is equivalent to

$$x\in\mathcal{M},\quad x\,\xi_o=0\ \Rightarrow\ x=0 \tag{4.1}$$

(ξ_o separating for $\mathcal{M}$) and similarly for $\overline{\mathcal{M}'\,\xi_o}=H$.

(4.1) means that the linear map $\mathcal{M}\ni x\mapsto x\,\xi_o\in H$ is injective. Thus the $*$-operation on $\mathcal{M}$ can be transported over H: the antilinear operator

$$S_o : \mathcal{M}\,\xi_o\ni x\,\xi_o\mapsto x^*\xi_o\in\mathcal{M}\,\xi_o\subset H$$

is well-defined. If ξ_o is a *trace vector* relative to $\mathcal{M}$, that is, the linear functional

$$\mathcal{M}\ni x\mapsto\omega_{\xi_o}(x):=(x\,\xi_o\,|\,\xi_o)$$

satisfies the condition $\omega_{\xi_o}(x^*x) = \omega_{\xi_o}(x\,x^*)$, $x \in \mathcal{M}$, then S_o is isometrical. In general S_o is not even bounded, but it is always closable:

Indeed, if $(x_n)_{n\geq 1}$ is a sequence in $\mathcal{M}$ such that $x_n\,\xi_o \to 0$ and $x_n^*\,\xi_o \to \zeta$, then

$$(\zeta\,|\,x'\,\xi_o) = \lim_{n\to\infty} (x_n^*\,\xi_o\,|\,x'\,\xi_o) = \lim_{n\to\infty} \big((x')^*\xi_o\,|\,x_n\,\xi_o) = 0, \qquad x' \in \mathcal{M}'$$

and therefore $\zeta = 0$.

Thus $S = \overline{S_o}$ is an invertible, densely defined, closed antilinear operator with $S^{-1} = S$, hence also $S^* = S_o^*$ is an invertible, densely defined, closed antilinear operator satisfying $(S^*)^{-1} = S^*$. $\Delta = S^*S$ will be a non-singular, positive, self-adjoint linear operator, called the *modular operator* associated to the pair $(\mathcal{M}, \xi_o)$. Finally, if $S = J\,\Delta^{1/2}$ is the polar decomposition of S then $J : H \to H$ will be an invertible antilinear isometry, called the *modular conjugation* associated to $(\mathcal{M}, \xi_o)$. The unicity of the polar decomposition $S = S^{-1} = \Delta^{-1/2}J^{-1} = J^*(J\,\Delta^{-1/2}J^*)$ yields:

$$\begin{aligned} &J = J^* = J^{-1} \text{ and } \Delta^{1/2} = J\,\Delta^{-1/2}J^* = J\,\Delta^{-1/2}J, \\ &S = J\,\Delta^{1/2} = \Delta^{-1/2}J,\ S^* = J\,\Delta^{-1/2} = \Delta^{1/2}J. \end{aligned} \tag{4.2}$$

We recall that a linear operator $L : H \supset \mathcal{D}(T) \to H$ is called *affiliated* to $\mathcal{M}$ if $u'\,L\,(u')^* = L$ for all unitaries $u' \in \mathcal{M}'$. By the second commutant theorem of John von Neumann every bounded linear operator $H \to H$ affiliated to $\mathcal{M}$ belongs to $\mathcal{M}$. Clearly, the adjoint and the operators involved in the polar decomposition of a linear operator affiliated to $\mathcal{M}$ are still affiliated to $\mathcal{M}$.

According to the definition of S, $\mathcal{M}\,\xi_o$ is a core for it, on which S is given by the formula $S(x\,\xi_o) = x^*\xi_o, x \in \mathcal{M}$. In the next lemma a similar statement is proved for S^*:

LEMMA 4.1. *$\eta \in H$ belongs to the domain of S^* if and only if there exists a densely defined, closed linear operator R affiliated to $\mathcal{M}'$ such that*

$$\xi_o \in \mathcal{D}(R) \cap \mathcal{D}(R^*) \ \textit{ and } \ \eta = R\,\xi_o$$

and then $S^*\eta = R^*\xi_o$. It follows that $\mathcal{M}'\xi_o$ is a core for S^* on which S^* is given by the formula $S^*(x'\,\xi_o) = (x')^*\xi_o, y \in \mathcal{M}'$.

Proof. Let us first prove that, for $\eta \in \mathcal{D}(S^*)$, the linear operator $\mathcal{M}\,\xi_o \ni x\,\xi_o \mapsto x\,\eta \in H$ is closable and its closure R_η is affiliated to $\mathcal{M}'$.

Indeed, if $(x_n)_{n\geq 1}$ is a sequence in $\mathcal{M}$ such that $x_n\,\xi_o \to 0$ and $x_n\,\eta \to \zeta$, then

$$(\zeta\,|\,x\,\xi_o) = \lim_{n\to\infty} (x_n\,\eta\,|\,x\,\xi_o) = \lim_{n\to\infty} (\eta\,|\,S\,x^*x_n\,\xi_o) = \lim_{n\to\infty} (x_n\,\xi_o\,|\,x\,S^*\eta) = 0$$

for all $x \in \mathcal{M}$ and therefore $\zeta = 0$. Moreover, for every unitary $u \in \mathcal{M}$ we have

$$u\,R_\eta\,u^*(x\,\xi_o) = u\,u^*x\,\eta = x\,\eta = R_\eta\,(x\,\xi_o), \qquad x \in \mathcal{M}.$$

For every $\eta \in \mathcal{D}(S^*)$, since $S^*\eta = (S^*)^{-1}\eta \in \mathcal{D}(S^*)$, also $R_{S^*\eta}$ is defined. Let us verify that

$$R_{S^*\eta} \subset R_\eta^*.$$

Indeed, we have for every $x \in \mathcal{M}$

$$\begin{aligned} \big(R_\eta(y\,\xi_o)\,|\,x\,\xi_o\big) &= \big(y\,\eta\,|\,x\,\xi_o\big) = \big(\eta\,|\,S\,x^*y\,\xi_o\big) = \big(y\,\xi_o\,|\,x\,S^*\eta\big) \\ &= \big(y\,\xi_o\,|\,R_{S^*\eta}(x\,\xi_o)\big), \qquad y \in \mathcal{M} \end{aligned}$$

and therefore $x\,\xi_o \in \mathcal{D}(R_\eta^*)$ and $R_\eta^*(x\,\xi_o) = R_{S^*\eta}(x\,\xi_o)$.

By the above, for any $\eta \in \mathcal{D}(S^*)$ we have the densely defined, closed linear operator R_η, affiliated to $\mathcal{M}'$, such that

$$\xi_o \in \mathcal{D}(R_\eta) \cap \mathcal{D}(R_\eta^*) \text{ and } R_\eta(\xi_o) = \eta,\ R_\eta^*(\xi_o) = R_{S^*\eta}(\xi_o) = S^*\eta.$$

Conversely, let us now assume that for $\eta \in H$ there is a densely defined, closed linear operator R, affiliated to $\mathcal{M}'$, for which $\xi_o \in \mathcal{D}(R) \cap \mathcal{D}(R^*)$ and $\eta = R\,\xi_o$. Since

$$\big(\eta \,|\, S(u\,\xi_o)\big) = \big(R\,\xi_o \,|\, u^*\xi_o\big) = \big(u\,R\,\xi_o \,|\, \xi_o\big) = \big(R\,u\,\xi_o \,|\, \xi_o\big) = \big(u\,\xi_o \,|\, R^*\xi_o\big)$$

holds for every unitary $u \in \mathcal{M}$ and $\mathcal{M}$ is the linear span of all unitaries contained in $\mathcal{M}$, the equality $\big(\eta \,|\, S(x\,\xi_o)\big) = \big(x\,\xi_o \,|\, R^*\xi_o\big)$ holds for every $x \in \mathcal{M}$. In other words, $\eta \in \mathcal{D}(S^*)$ and $S^*\eta = R^*\xi_o$.

In particular, for every $x' \in \mathcal{M}'$ we have $x'\xi_o \in \mathcal{D}(S^*)$ and $S^*(x'\xi_o) = (x')^*\xi_o$.

It remains to show that $\mathcal{M}'\xi_o$ is a core for S^*. This means by the above that if R is a densely defined, closed linear operator R, affiliated to $\mathcal{M}'$, for which $\xi_o \in \mathcal{D}(R) \cap \mathcal{D}(R^*)$, then $(R\,\xi_o,\, R^*\xi_o)$ belongs to the closure of $\big\{(x'\xi_o,\, (x')^*\xi_o)\,;\, x' \in \mathcal{M}'\big\} \subset H \times H$.

Let $R = v'|R|$ be the polar decomposition of R and let us denote by e_n' the spectral projection of the positive, self-adjoint operator $|R|$, associated to the interval $[0, n]$. Then $v' \in \mathcal{M}'$ and $|R|$ is affiliated to $\mathcal{M}'$, so $0 \le |R|\,e_n' \in \mathcal{M}'$ for every $n \ge 1$.

Put $x_n' = v'(|R|\,e_n') \in \mathcal{M}', n \ge 1$. By $\xi_o \in \mathcal{D}(R) = \mathcal{D}(|R|)$ we have $|R|\,e_n'\,\xi_o \to |R|\,\xi_o$, so

$$x_n'\xi_o \to v'|R|\,\xi_o = R\,\xi_o.$$

On the other hand, $\xi_o \in \mathcal{D}(R^*)$ and $R^* = |R|\,(v')^*$ yield $(v')^*\xi_o \in \mathcal{D}(|R|)$, so we have also

$$(x_n')^*\xi_o = (|R|\,e_n')\,(v')^*\xi_o \to |R|\,(v')^*\xi_o = R^*\xi_o. \ \blacksquare$$

Since $S\,\xi_o = \xi_o$ and, by Lemma 4.1, $S^*\xi_o = \xi_o$, we have also $\Delta\,\xi_o = \xi_o$ and it follows that

$$\Delta^z\xi_o = \xi_o\ ,\ z \in \mathbb{C}, \qquad J\,\xi_o = \xi_o. \tag{4.3}$$

We notice that all the above topics originate in the original work of M. Tomita and M. Takesaki ([T]).

Let ω_{ξ_o} denote the linear functional $B(H) \ni x \mapsto (x\,\xi_o \,|\, \xi_o)$. In the next lemma we characterize in terms of the modular operator Δ those $a \in \mathcal{M}$ for which the linear functional $\mathcal{M} \ni x \mapsto \omega_{\xi_o}(xa)$ is positive. Its proof is based on S. Sakai's method to prove his Radon-Nikodym type theorem for normal positive linear functionals (see [S2]).

LEMMA 4.2. *For $a \in \mathcal{M}$ the following conditions are equivalent:*

(i) *the linear functional $\mathcal{M} \ni x \mapsto \omega_{\xi_o}(xa)$ is positive;*
(ii) *there exists a positive $a' \in \mathcal{M}'$ with $a\,\xi_o = a'\xi_o$;*
(iii) *$a \in \mathcal{D}\big(\alpha_{i/2}^{(\Delta)}\big)$ and $\alpha_{i/2}^{(\Delta)}(a) \ge 0$.*

Moreover, if the above conditions are satisfied then, with $r(a)$ denoting the spectral radius of a, we have $a' = J\,\alpha_{i/2}^{(\Delta)}(a)\,J \le r(a)1_H$.

Proof. First we prove that (i)⇒(ii) and $a' \le r(a)1_H$.

The positivity of the linear functional $\varphi : \mathcal{M} \ni x \mapsto \omega_{\xi_o}(xa)$ implies its self-adjointness, that is, the validity of $\varphi(x^*) = \overline{\varphi(x)}, x \in \mathcal{M}$. Consequently

$$\omega_{\xi_o}(a^*x) = (a^*x\,\xi_o\,|\,\xi_o) = (\xi_o\,|\,x^*a\,\xi_o) = \overline{\varphi(x^*)} = \varphi(x) = \omega_{\xi_o}(xa), \qquad x \in \mathcal{M}. \tag{4.4}$$

Using now the Schwarz inequality repeatedly, we deduce for every $0 \le x \in \mathcal{M}$

$$\begin{aligned}
\varphi(x) &= \omega_{\xi_o}(x^{1/2}x^{1/2}a)\\
&\le \omega_{\xi_o}(x)^{1/2}\omega_{\xi_o}(a^*x\,a)^{1/2} \overset{(4.4)}{=} \omega_{\xi_o}(x)^{1/2}\omega_{\xi_o}(x\,a^2)^{1/2} = \omega_{\xi_o}(x)^{1/2}\omega_{\xi_o}(x^{1/2}x^{1/2}a^2)^{1/2}\\
&\le \omega_{\xi_o}(x)^{1/2+1/2^2}\omega_{\xi_o}\big((a^2)^*x\,a^2\big)^{1/2^2} \overset{(4.4)}{=} \omega_{\xi_o}(x)^{1/2+1/2^2}\omega_{\xi_o}\big(x\,a^{2^2}\big)^{1/2^2}\\
&\qquad\qquad\qquad\qquad\qquad\cdots\cdots\cdots\\
&\le \omega_{\xi_o}(x)^{1/2+1/2^2+\,\dots\,+1/2^n}\omega_{\xi_o}\big(x\,a^{2^n}\big)^{1/2^n}\\
&\le \omega_{\xi_o}(x)^{1/2+1/2^2+\,\dots\,+1/2^n}\,\|\,\omega_{\xi_o}\|^{1/2^n}\,\|x\|^{1/2^n}\,\|\,a^{2^n}\,\|^{1/2^n} \to \omega_{\xi_o}(x)\,r(a).
\end{aligned}$$

Thus we have proved

$$0 \le \varphi(x) \le r(a)\,\omega_{\xi_o}(x) = r(a)\,(x\xi_o\,|\,\xi_o), \qquad 0 \le x \in \mathcal{M}. \tag{4.5}$$

Using again the Schwarz inequality, (4.5) yields

$$|\varphi(y^*x)| \le \varphi(y^*y)^{1/2}\varphi(x^*x)^{1/2} \le r(a)\,\|x\,\xi_o\|\,\|y\,\xi_o\|, \qquad x,\, y \in \mathcal{M},$$

so

$$(\mathcal{M}\,\xi_o) \times (\mathcal{M}\,\xi_o) \ni (x\,\xi_o,\, y\,\xi_o) \mapsto \theta(x\,\xi_o,\, y\,\xi_o) := \varphi(y^*x) \in \mathbb{C} \tag{4.6}$$

is a well defined bounded sesquilinear form θ and by the cyclicity of ξ_o relative to $\mathcal{M}$ it can be extended to a bounded sesquilinear form $H \times H \to \mathbb{C}$, still denoted by θ, for which

$$0 \le \theta(\xi,\, \xi) \le r(a)\,(\xi\,|\,\xi), \qquad \xi \in H.$$

Applying now the Riesz representation theorem, we deduce the existence of a unique $a' \in B(H)$ satisfying the condition

$$\theta(\xi,\, \eta) = (a'\xi\,|\,\eta), \qquad \xi,\, \eta \in H. \tag{4.7}$$

Moreover, $0 \le a' \le r(a)\,1_H$ and a' is affiliated to $\mathcal{M}'$, so $a' \in \mathcal{M}'$:

Indeed, if $u \in \mathcal{M}$ is unitary then we have for every $x, y \in \mathcal{M}$

$$\begin{aligned}
\big((u\,a'u^*)\,x\,\xi_o\,\big|y\,\xi_o\big) &= \big(a'u^*x\,\xi_o\,\big|\,u^*y\,\xi_o\big) \overset{(4.7)}{=} \theta(u^*x\,\xi_o,\, u^*y\,\xi_o) \overset{(4.6)}{=} \varphi\big((u^*y)^*u^*x\big)\\
&= \varphi(y^*x) \overset{(4.6)}{=} \theta(x\,\xi_o,\, y\,\xi_o) \overset{(4.7)}{=} (a'x\,\xi_o\,|\,y\,\xi_o)
\end{aligned}$$

and by the cyclicity of ξ_o relative to $\mathcal{M}$ we conclude that $u\,a'u^* = a'$.

Finally, taking into account the cyclicity of ξ_o relative to $\mathcal{M}$, the equality $a\,\xi_o = a'\xi_o$ follows from

$$(a\,\xi_o\,|\,x\,\xi_o) = \varphi(x^*) \overset{(4.6)}{=} \theta(\xi_o,\, x\,\xi_o) \overset{(4.7)}{=} (a'\xi_o\,|\,x\,\xi_o), \qquad x \in \mathcal{M}.$$

Next we prove that (ii)⇒(iii) and $a' = J\,\alpha^{(\Delta)}_{i/2}(a)\,J$.

(ii) implies

$$a\,\Delta^{1/2}J = x\,S^* \subset S^*a' = \Delta^{1/2}J\,a'. \tag{4.8}$$

Indeed, since $\mathcal{M}'\xi_o$ is by Lemma 4.1 a core for S^*, it is enough to verify that

$$x\, S^* x'\xi_o = S^* a' x'\xi_o \text{ for every } x' \in \mathcal{M}'.$$

But we have $x\, S^* x'\xi_o = x\,(x')^*\xi_o = (x')^* x\,\xi_o = (x')^* a'\xi_o = S^* a' x'\xi_o$.

Let us finally prove the implication (iii)⇒(i).

Let $x \in \mathcal{M}$ be arbitrary. By (2.7) we have

$$a^* \in \mathcal{D}\big(\alpha^{(\Delta)}_{-i/2}\big) \text{ and } \alpha^{(\Delta)}_{-i/2}(a^*) = \alpha^{(\Delta)}_{i/2}(a)^* = \alpha^{(\Delta)}_{i/2}(a),$$

so by Proposition 2.5 $a^*\Delta^{-1/2} \subset \Delta^{-1/2}\alpha^{(\Delta)}_{i/2}(a)$. Consequently

$$x\, a\, \xi_o = S\, a^* S\, x\,\xi_o = S\, a^*\Delta^{-1/2} J\, x\,\xi_o = S\,\Delta^{-1/2}\alpha^{(\Delta)}_{i/2}(a)\, J\, x\,\xi_o = J\,\alpha^{(\Delta)}_{i/2}(a)\, J\, x\,\xi_o$$

and we conclude that

$$\omega_{\xi_o}(x^* x\, a) = (x\, a\,\xi_o \,|\, x\,\xi_o) = \big(J\underbrace{\alpha^{(\Delta)}_{i/2}(a)}_{\geq 0} J\, x\,\xi_o \,|\, x\,\xi_o\big) \geq 0. \ \blacksquare$$

For any $x \in \mathcal{M}$ we can apply to the *wo*-continuous linear functional $\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}$

$$\mathcal{M} \ni y \mapsto \omega_{\xi_o}(yx)$$

S. Sakai's polar decomposition theorem (see [S1]): there exists a partial isometry $v \in \mathcal{M}$ and a normal positive linear functional $\big|\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}\big|$ on $\mathcal{M}$ such that

$$\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M} = \big|\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}\big|(\,\cdot\, v), \quad \big|\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}\big| = \omega_{\xi_o}(\,\cdot\, v^* x)\,|\,\mathcal{M}.$$

Thus we can make $\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}$ positive by multiplying x on the left with the partial isometry v^*:

$$\omega_{\xi_o}(\,\cdot\, v^* x)\,|\,\mathcal{M} = \big|\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}\big| \geq 0.$$

Now we shall discuss the connection between the polar decomposition of the functional $\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M}$ and the polar decomposition of x relative to the modular operator Δ. The proof of the next lemma is based on S. Sakai's method to prove his polar decomposition theorem for normal linear functionals.

Lemma 4.3. *For every $x \in \mathcal{M}$ there exists a uniquely defined partial isometry $u \in \mathcal{M}$ such that*

$$\omega_{\xi_o}(\cdot\, ux)\,|\,\mathcal{M} \geq 0 \textit{ and } u^*u \textit{ is the orthogonal projection onto } \overline{x(H)}. \tag{4.9}$$

Moreover, $x\Delta$ is closable, $|x|_\Delta = ux$ and $\mathrm{phase}_\Delta(x) = u^*$.

Proof. The existence and uniqueness of the partial isometry u is essentially a rephrasing of the polar decomposition of the linear functional $\mathcal{M} \ni y \mapsto \omega_{\xi_o}(yx)$. However we prefer to give a direct proof. Since the case $x = 0$ is trivial, we shall consider only the case $x \neq 0$.

Let φ denote the *wo*-continuous linear functional $\omega_{\xi_o}(\cdot\, x)\,|\,\mathcal{M}$. Since the closed unit ball $\mathcal{M}_1$ of $\mathcal{M}$ is *wo*-compact, the *wo*-compact, convex set

$$\mathcal{K} = \big\{y \in \mathcal{M}\,;\ \|y\| \leq 1,\ \varphi(y) = \|\varphi\|\big\}$$

is not empty, so by the Krein-Milman theorem it has an extreme point u. Since $\mathcal{K}$ is an extremal subset of $\mathcal{M}_1$, u is also an extreme point of $\mathcal{M}_1$ and therefore, by a classical theorem of R. V. Kadison, it is a partial isometry (see [Kd]).

Let us define $\psi = \omega_{\xi_o}(\,\cdot\, u_o x)\,|\,\mathcal{M} = \varphi(\,\cdot\, u_o)$. Since

$$\psi(1_H) = \varphi(u_o) = \|\varphi\| \geq \|\psi\| \geq \psi(1_H),$$

ψ is a positive linear functional. We are going to prove

$$\omega_{\xi_o}(yx) = \varphi(y) = \psi(y\, u_o^*) = \varphi(y\, u_o^* u_o) = \omega_{\xi_o}(y\, u_o^* u_o x), \qquad y \in \mathcal{M}. \tag{4.10}$$

Indeed, let us assume that there exists some $y \in \mathcal{M}$ with $\varepsilon := \varphi\big(y\,(1_H - u_o^* u_o)\big) > 0$. Then we have for every natural number n on the one hand

$$\varphi\big(n\, u_o + y\,(1_H - u_o^* u_o)\big) = n\,\|\varphi\| + \varepsilon,$$

and on the other hand

$$\begin{aligned}\varphi\big(n\, u_o + y\,(1_H - u_o^* u_o)\big) &\leq \|\varphi\|\,\|n\, u_o + y\,(1_H - u_o^* u_o)\| \\ &= \|\varphi\|\,\big\|\big(n\, u_o + y\,(1_H - u_o^* u_o)\big)\big(n\, u_o^* + (1_H - u_o^* u_o)\, y^*\big)\big\|^{1/2} \\ &= \|\varphi\|\,\big\|n^2 u_o\, u_o^* + y\,(1_H - u_o^* u_o)\, y^*\big\|^{1/2} \leq \|\varphi\|\,\big(n^2 + \|y\|^2\big)^{1/2}.\end{aligned}$$

It follows that $n\,\|\varphi\| + \varepsilon \leq \|\varphi\|\,\big(n^2 + \|y\|^2\big)^{1/2}$, but this is false for large n values.

(4.10) implies immediately

$$(x\,\xi_o\,|\,y\xi_o) = \omega_{\xi_o}(y^* x) = \omega_{\xi_o}(y^* u_o^* u_o x) = (u_o^* u_o x\,\xi_o\,|\,y\,\xi_o), \qquad y \in \mathcal{M}.$$

Using the cyclicity property $\overline{\mathcal{M}\,\xi_o} = H$ we first deduce $x\,\xi_o = u_o^* u_o x\,\xi_o$, and using then the separating property (4.1) we get $x = u_o^* u_o x$. Thus the orthogonal projection $u_o^* u_o$ majorizes the orthogonal projection e onto $\overline{x(H)}$. Accordingly $u = u_o e \in \mathcal{M}$ is a partial isometry with $u^* u = e$ and $ux = u_o x$. Thus $\omega_{\xi_o}(\cdot\, ux)\,|\,\mathcal{M} = \omega_{\xi_o}(\cdot\, u_o x)\,|\,\mathcal{M} = \psi \geq 0$ and with this the existence of u is proved.

To prove the uniqueness, let $v \in \mathcal{M}$ be another partial isometry satisfying

$$\omega_{\xi_o}(\cdot\, vx)\,|\,\mathcal{M} \geq 0 \text{ and } v^* v \text{ is the orthogonal projection onto } \overline{x(H)}.$$

Using the Schwarz inequality for the functional $\omega_{\xi_o}(\,\cdot\, vx)\,|\,\mathcal{M}$ we get

$$\begin{aligned}\big|\,\omega_{\xi_o}\big((1_H - v\,v^*)\,u\,x\big)\big| &= \big|\,\omega_{\xi_o}\big((1_H - v\,v^*)\,u\,v^* v\,x\big)\big| \\ &\leq \omega_{\xi_o}\big((1_H - v\,v^*)\,v\,x\big)^{1/2}\omega_{\xi_o}\big(v\,u^* u\,v^* v\,x\big)^{1/2} = 0.\end{aligned}$$

Now a second application of the Schwarz inequality, but this time for the functional $\omega_{\xi_o}(\,\cdot\, ux)\,|\,\mathcal{M}$, yields further for every $y \in \mathcal{M}$

$$\begin{aligned}\big|\,\big((1_H - v\,v^*)\,u\,x\,\xi_o\,\big|\,y\,\xi_o\big)\big| &= \big|\,\omega_{\xi_o}\big(y^*(1_H - v\,v^*)\,u\,x\big)\big| \\ &\leq \omega_{\xi_o}\big(y^* yux\big)^{1/2}\omega_{\xi_o}\big((1_H - vv^*)\,ux\big)^{1/2} = 0\,.\end{aligned}$$

By the cyclicity property $\overline{\mathcal{M}\,\xi_o} = H$ and by the separating property (4.1) it follows that $(1_H - vv^*)\,u = 0$. Thus

$$uu^* = vv^* uu^* = (vv^* uu^*)^* = uu^* vv^* \leq vv^*.$$

Since in the above reasoning we can interchange u with v, also the converse inequality holds and therefore $uu^* = vv^*$. It follows that vu^* is a unitary element of $uu^* \mathcal{M}\, uu^*$:

$$(vu^*)^* vu^* = uv^* vu^* = uu^* uu^* = uu^* \text{ and } vu^*(vu^*)^* = vu^* uv^* = vv^* vv^* = vv^* = uu^*.$$

Let $vu^* = a + ib$ with $a, b \in \mathcal{M}$ self-adjoint. Clearly, $\|a\|, \|b\| \leq 1$ and $a, b \in \mathcal{M}$. Moreover, since vu^* is a normal operator, a and b commute. Now

$$\underbrace{\omega_{\xi_o}(aux)}_{\in \mathbb{R}} + i\, \underbrace{\omega_{\xi_o}(bux)}_{\in \mathbb{R}} = \omega_{\xi_o}(vu^*ux) = \omega_{\xi_o}(vx) = \|\omega_{\xi_o}(\cdot\, vx)\|$$

implies that $\omega_{\xi_o}(bux) = 0$ and so

$$\omega_{\xi_o}(aux) = \|\omega_{\xi_o}(\cdot\, vx)\| \geq \|\omega_{\xi_o}(\cdot\, uv^*v\, x)\| = \|\omega_{\xi_o}(\cdot\, ux)\|.$$

Consequently $\omega_{\xi_o}(aux) = \|\omega_{\xi_o}(\cdot\, ux)\| = \omega_{\xi_o}(ux)$, that is, $\omega_{\xi_o}\big((uu^* - a)\, ux\big) = 0$. Using the Schwarz inequality for the functional $\omega_{\xi_o}(\cdot\, ux) \,|\, \mathcal{M}$, we get for every $y \in \mathcal{M}$

$$\begin{aligned} \big|\, \big((uu^* - a)ux\, \xi_o \,\big|\, y\, \xi_o\big) \big| &= \big|\, \omega_{\xi_o}\big(y^*(uu^* - a)\, ux\big) \big| \\ &= \big|\, \omega_{\xi_o}\big(y^*(uu^* - a)^{1/2}(uu^* - a)^{1/2}ux\big) \big| \\ &\leq \omega_{\xi_o}\big(y^*(uu^* - a)\, yux\big)^{1/2} \omega_{\xi_o}\big((uu^* - a)\, ux\big)^{1/2} = 0. \end{aligned}$$

Hence, by the cyclicity property $\overline{\mathcal{M}\, \xi_o} = H$ and by the separating property (4.1), we have $(uu^* - a)u\, x = 0$. Since u vanishes on the orthogonal complement of the range of x, it follows that $(uu^* - a)\, u = 0$, which implies $uu^* = uu^*u\, u^* = a\, uu^* = a$. Thus $a \geq 0$ and

$$1 = \|vu^*\|^2 = \|a + ib\|^2 = \|a^2 + b^2\| = \|uu^* + b^2\| = 1 + \|b\|^2,$$

so $b = 0$.

We conclude that $vu^* = uu^*$ and thus $v = vv^*v = vu^*u = uu^*u = u$. This proves the uniqueness of u.

To finish the proof, we have to verify that

$$x\Delta \text{ is closable, } |x|_\Delta = ux \text{ and } \mathrm{phase}_\Delta(x) = u^*.$$

By (3.2) this means:

$$\begin{aligned} &x = u^*(ux), \\ &ux\Delta \ \text{ is positive and essentially self-adjoint,} \\ &uu^* \text{ is the orthogonal projection onto } \overline{ux(H)}. \end{aligned}$$

Since u^*u is the orthogonal projection onto $\overline{x(H)}$, we have $x = u^*(ux)$ and uu^* is the orthogonal projection onto $\overline{ux(H)}$. Thus it remains to prove only that $ux\Delta$ is positive and essentially self-adjoint.

Since $\omega_{\xi_o}(\cdot\, ux) \,|\, \mathcal{M} \geq 0$, by Lemma 4.2 we have $ux \in \mathcal{D}\big(\alpha_{i/2}^{(\Delta)}\big)$ and $\alpha_{i/2}^{(\Delta)}(ux) \geq 0$. Using now Proposition 3.2, we deduce that $ux\, \Delta \geq 0$. In particular $ux\, \Delta$ is closable and we can consider the polar decomposition $ux = \mathrm{phase}_\Delta(ux)\, |ux|_\Delta$ relative to Δ. For convenience, we shall use the notations

$$b = |ux|_\Delta \text{ and } w = \mathrm{phase}_\Delta(ux)$$

By Proposition 3.2 and Lemma 4.2 we have $\omega_{\xi_o}(\,\cdot\, w^*ux) \,|\, \mathcal{M} = \omega_{\xi_o}(\,\cdot\, b) \,|\, \mathcal{M} \geq 0$. On the other hand, since w^*w is the orthogonal projection onto $\overline{b(H)}$, ww^* will be the orthogonal projection onto $w\big(\overline{b(H)}\big) = \overline{wb(H)} = \overline{ux(H)}$. Thus we have:

$$\omega_{\xi_o}(\,\cdot\, w^*ux) \,|\, \mathcal{M} \geq 0 \text{ and } ww^* \text{ is the orthogonal projection onto } \overline{ux(H)}. \tag{4.11}$$

We recall that $\omega_{\xi_o}(\,\cdot\, ux)\,|\,\mathcal{M} \geq 0$, so, denoting by f the orthogonal projection onto $\overline{ux(H)}$, we have also

$$\omega_{\xi_o}(\,\cdot\, fux)\,|\,\mathcal{M} \geq 0 \text{ and } f^*f = f \text{ is the orthogonal projection onto } \overline{ux(H)}. \tag{4.12}$$

Taking into account (4.11) and (4.12), by the above proved uniqueness result we infer that $w^* = f$. Consequently $b = w^*ux = ux$ and so $ux\Delta = b\Delta$ is essentially self-adjoint. ∎

REMARK. Perhaps the following two operator theoretical properties of the modular operator Δ, which are immediate consequences of Lemma 4.3, deserve attention:

$$x \in \mathcal{M} \;\Rightarrow\; x\,\Delta \text{ is closable,} \tag{4.13}$$

$$x \in \mathcal{M},\, x\,\Delta \geq 0 \;\Rightarrow\; x\,\Delta \text{ is essentially self-adjoint.} \tag{4.14}$$

(4.13) is already part of the statement of Lemma 4.3.

To verify (4.14), let $x \in \mathcal{M}$ be such that $x\,\Delta \geq 0$. According to Proposition 3.2 and Lemma 4.2 we have $\omega_{\xi_o}(\,\cdot\, x)\,|\,\mathcal{M} \geq 0$, so Lemma 4.3 yields $x = |x|_\Delta$. Consequently $x\Delta = |x|_\Delta\Delta$ is essentially self-adjoint.

Let us finally prove the fundamental theorem of the Tomita-Takesaki Theory (in the case of a bicyclic vector):

THEOREM 4.4 (The fundamental theorem of the Tomita-Takesaki Theory). *The following two conditions are satisfied:*

(i) $\alpha_t^{(\Delta)}(\mathcal{M}) = \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$, $t \in \mathbb{R}$;
(ii) $J\mathcal{M}J = \mathcal{M}'$.

Proof. By Lemma 4.3

$$x \in \mathcal{M} \;\Rightarrow\; xA \text{ is closable and } \mathrm{phase}_\Delta(x) \in \mathcal{M}.$$

By Theorem 3.5 the statement of (i) follows.

According to (i) and Proposition 2.2 the $\alpha^{(\Delta)}$-entire elements of $\mathcal{M}$ are *wo*-dense in $\mathcal{M}$, so for $J\mathcal{M}J \subset \mathcal{M}'$ it is enough to prove that $JxJ \in \mathcal{M}'$ for every $x \in \mathcal{M} \cap \mathcal{D}(\alpha_{i/2}^{(\Delta)})$.

Let thus $x \in \mathcal{M} \cap \mathcal{D}(\alpha_{i/2}^{(\Delta)})$ be arbitrary and let us consider the *wo*-continuous mapping

$$F : \big\{\zeta \in \mathbb{C}\,;\, 0 \leq \Im\zeta \leq 1/2\big\} \ni \zeta \mapsto \alpha_\zeta^{(A)}(x) \in B(H),$$

which is analytic in the open strip $\big\{\zeta \in \mathbb{C}\,;\, 0 < \Im\zeta < 1/2\big\}$. Since $F(t) \in \mathcal{M}$ for all $t \in \mathbb{R}$, and $\mathcal{M}$ is *wo*-closed, using the Hahn-Banach theorem it is easily seen that $F(\zeta) \in \mathcal{M}$ for all ζ in the domain of F. In particular, $\alpha_{i/2}^{(\Delta)}(x) \in \mathcal{M}$. Furthermore, by Proposition 2.5 we have $x\,\Delta^{1/2} \subset \Delta^{1/2}\,\alpha_{i/2}^{(\Delta)}(x)$. Consequently for every unitary $u \in \mathcal{M}$ and every $y \in \mathcal{M}$ we have:

$$\begin{aligned}
uJxJu^*y\,\xi_o &= uJxJSy^*u\xi_o = uJx\,\Delta^{1/2}y^*u\xi_o = uJ\Delta^{1/2}\alpha_{i/2}^{(\Delta)}(x)y^*u\xi_o \\
&= uS\alpha_{i/2}^{(\Delta)}(x)y^*u\xi_o = uu^*y\alpha_{i/2}^{(\Delta)}(x)^*\xi_o = S\alpha_{i/2}^{(\Delta)}(x)y^*\xi_o \\
&= J\Delta^{1/2}\alpha_{i/2}^{(\Delta)}(x)y^*\xi_o = Jx\,\Delta^{1/2}y^*\xi_o = JxJSy^*\xi_o \\
&= JxJy\,\xi_o.
\end{aligned}$$

Therefore JxJ is affiliated to $\mathcal{M}'$ and being bounded, it belongs to $\mathcal{M}'$.

Observing now that, by Lemma 4.1 and by (4.2), the modular operator and the modular conjugation associated to $(\mathcal{M}', \xi_o)$ are Δ^{-1} respectively J, we can apply the above proved inclusion $J\mathcal{M}J \subset \mathcal{M}'$ with $\mathcal{M}$ and $\mathcal{M}'$ interchanged. We get $J\mathcal{M}'J \subset \mathcal{M}$, that is, $\mathcal{M}' \subset J\mathcal{M}J$, and with this also the proof of (ii) is finished. ∎

Acknowledgments. This research was partially supported by MIUR, INdAM and EU.

References

[Ar] W. Arveson, *On groups of automorphisms of operator algebras*, J. Funct. Anal. 15 (1974), 217–243.

[Ba] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math. 10 (1960), 419–437.

[B] G. Benfatto, C. D'Antoni, F. Nicolò and G. C. Rossi, *Introduzione alla teoria delle algebre di von Neumann e teorema di Tomita-Takesaki*, Da seminari tenuti dal Prof. S. Doplicher negli AA 1973-74 e 1974-75, Quaderni del CNR, GNAFA e GNFM.

[CZ] I. Ciorănescu and L. Zsidó, *Analytic generators for one-parameter groups*, Tôhoku Math. J. 28 (1976), 327–362.

[D] G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Dover Publications, New York, 1943.

[EZ] G. A. Elliott and L. Zsidó, *Almost uniformly continuous automorphism groups of operator algebras*, J. Operator Theory 8 (1982), 227–277.

[H] U. Haagerup, *Operator valued weights in von Neumann algebras, I*, J. Funct. Anal. 32 (1979), 175–206.

[HP] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloquium Publ. 31, Providence, RI, 1957.

[Kd] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. 54 (1951), 325–338.

[Kt] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[PS] G. Pólya and G. Szegő, *Problems and Theorems in Analysis I*, Springer, Berlin, 1972.

[RV] M. Rieffel and A. van Daele, *A bounded operator approach to Tomita-Takesaki theory*, Pacific J. Math. 69 (1977), 187–221.

[S1] S. Sakai, *On linear functionals of W^*-algebras*, Proc. Japan Acad. 34 (1958), 571–574.

[S2] S. Sakai, *A Radon-Nikodym theorem in W^*-algebras*, Bull. Amer. Math. Soc. 71 (1965), 149–151.

[T] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Math. 128, Springer, Berlin, 1970.

[V1] A. van Daele, *A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras*, J. Funct. Anal. 15 (1974), 378–393.

[V2] A. van Daele, *On the spectrum of the analytic generator*, Math. Scand. 37 (1975), 307–318.

[W1] S. L. Woronowicz, *On the purification of factor states*, Comm. Math. Phys. 28 (1972), 221–235.

[W2] S. L. Woronowicz, *A remark on the polar decomposition of m-sectorial operators*, Letters on Math. Phys. 1 (1977), 429–433.

[W3] S. L. Woronowicz, *Operator systems and their application to the Tomita-Takesaki theory*, J. Operator Theory 2 (1979), 169–209.

[Z1] L. Zsidó, *Analytic generators and the foundation of the Tomita-Takesaki theory of Hilbert algebras*, in: Proc. International School of Math. Phys. (Camerino, 1974), 182–267.

[Z2] L. Zsidó, *A proof of Tomita's fundamental theorem in the theory of standard von Neumann algebras*, Rev. Roumaine Math. Pures Appl. 20 (1975), 609–619.

[Z3] L. Zsidó, *Spectral and ergodic properties of the analytic generator*, J. Approximation Theory 20 (1977), 77–138.

[Z4] L. Zsidó, *The characterization of the analytic generator of $*$-automorphism groups*, Proc. Symposia in Pure Math. 38 (1982), 381–383.