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VON NEUMANN MODELS AND THE OEUVRE OF JERZY ŁOŚ

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There is a line of great and famous Polish mathematicians, well recognized in the international mathematical community, as for instance Alfred Tarski, Stefan Banach, Hugo Steinhaus, Kazimierz Kuratowski, Wacław Sierpiński and indeed I would also add the name of Jerzy Maria Łoś to this list.

It's an honor for me to have the chance to speak here about the 'Von Neumann Models and the Oeuvre of Jerzy Łoś,' of course understanding this in the sense of 'the oeuvre related to von Neumann models.'

Jerzy Łoś has made decisive contributions to this theory as author but also as co–author, and he is the editor together with his wife M. W. Łoś and also A. Wieczorek of three volumes with collected contributions to Mathematical Economics and especially to the theory of von Neumann models.

Let me start to explain the von Neumann model of an expanding economy, as introduced in the paper of von Neumann (1937).

An economy with m production processes and n commodities is considered. Production in this model is defined by two (m, n)-technology matrices

$$A := (a_{ij})_{i=1,\dots,m,\ \ j=1,\dots,n}, \quad \text{and} \quad B := (b_{ij})_{i=1,\dots,m,\ \ j=1,\dots,n}.$$

The i-th production process (a_i, b_i) is represented by two row-vectors of A and B, the so-called input bundle and output bundle

$$a_i = (a_{i1}, \dots, a_{in}) \quad \text{and} \quad b_i = (b_{i1}, \dots, b_{in}),$$

respectively.

During a production period the input quantity $a_{ij} \in \mathbb{R}_+$ of commodity j is transformed by the i-th process into the output quantity $b_{ij} \in \mathbb{R}_+$ also of commodity j, if this

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process is run at intensity 1. If the i-th process is run at intensity $x_i \in \mathbb{R}_+$, the input and output quantities have to be multiplied by x_i.

If the processes (a_i, b_i) are run at intensities x_i, $i = 1, \ldots, m$, the total input and total output of the commodities $j = 1, \ldots, n$ are represented by the vectors

$$xA \quad \text{and} \quad xB, \tag{1}$$

respectively, x denoting the $(1, m)$-vector of intensities.

If $y_1, \ldots, y_n$ are the prices of the n commodities, then by a similar arithmetic the vectors of cost and revenues of the m processes are given by

$$Ay \quad \text{and} \quad By, \tag{2}$$

respectively, y denoting the $(n, 1)$-vector of prices.

Let $a \in \mathbb{R}_+$ be the growth rate; then the growth factor α is defined by

$$\alpha := 1 + a. \tag{3}$$

in a similar way a factor of interest β is introduced.

For a closed economy, i.e. an economy without foreign trade, the output xB of the past period has to be greater than or equal to the input of the following period, being understood as the input xA of the pre-period multiplied by the growth factor α, which yields the condition

$$xB \geq \alpha x A. \tag{4}$$

Without now entering into a detailed argumentation, the von Neumann model presents itself by the following conditions:

N1:	$xB \geq \alpha x A$	the output of the pre-period is not less than the input of the following period
N2:	$By \leq \alpha A y$	revenue is not higher than cost plus interest where the growth rate α coincides with the interest rate β
N3:	$xBy > 0$	the output vector weighted with prices is positive
N4:	$x \geq 0, x \neq 0$ $y \geq 0, y \neq 0$	intensities and prices are nonnegative

It is important, of course, to show that the conditions of a model do not exclude each other. Or in other words, one has to show that there exists a solution, fulfilling the various conditions of the model. Thereby it does not suffice—I refer to the 19th century economist Léon Walras of Lausanne—to construct a model with the same number of equations as variables.

A triple $(\bar{x}, \bar{y}, \bar{\alpha})$ fulfilling N1–N4 is called an equilibrium of the (standard) von Neumann model.

Condition N3 is not originated by von Neumann, cf. von Neumann (1937); it goes back to Kemeny, Morgenstern and Thompson (1956). In this sense one should correctly speak of the KMT-version of the von Neumann model; KMT denotes the group of authors Kemeny, Morgenstern and Thompson.

The economic importance of such sectoral models results from the fact that they show the interdependences of the various production processes, or if you like, of the key industries. For the production of steel you need coal, and for the production of coal, steel is required. The analysis of Leontief of the American economy is justified by such considerations, cf. Leontief (1941).

Of course, you may argue that the available technologies cannot be comprehended by two technology matrices A and B of given dimension $m \times n$ and being constant over time. The available technologies are constantly altering and the development of the technological possibilities is even the engine of an economy.

Assuming the matrices A and B to be slightly perturbed so as to yield new matrices $\tilde{A}$ and $\tilde{B}$ a notion of stability may be introduced comparing the equilibrium solutions of (A, B) and $(\tilde{A}, \tilde{B})$. For a discussion of this ansatz we refer to Robinson (1974).

Von Neumann presented a proof of existence based on a generalization of the Brouwer fixed point theorem due to von Neumann, but named after Kakutani, wherein he had to make the assumption

$$A + B > 0, \tag{5}$$

later criticized and substituted by other pre-requirements in Kemeny, Morgenstern, Thompson (1956). This group of authors introduced the condition $xBy > 0$, which now became the critical point when proving the existence of an equilibrium solution.

The methods of analysis of Kemeny, Morgenstern and Thompson were game-theoretic; they could proceed in this way as meanwhile the notion of the matrix game was established.

To show the existence of an equilibrium fulfilling condition N3, Thompson's proof for instance makes use of the so-called Bohnenblust-Karlin-Shapley property. Kemeny, Morgenstern, Thompson (1956) is comparatively easy to read, which helped to make this theory popular among economists.

Of course, also the KMT-version of the von Neumann model is a model of a closed economy; i.e. there is neither an export nor an import of commodities.

In 1969 Morgenstern and Thompson (Morgenstern, Thompson 1969) presented their 'Open Expanding Economy Model' (OEEM), representing as well a sectoral model now of an open economy.

The notions of A, B, α, x, y are the same as before; w^e, w^i denote the export and import vectors, respectively, while z^p and z^l are the vectors of profits and losses, respectively.

Setting

$$M_\alpha := B - \alpha A \tag{6}$$

the OEEM is described as follows:

MT1: $xM_\alpha = w^e - w^i, \;\; w^e, w^i \geq 0, \;\; \langle w^e, w^{it} \rangle = 0,$
production plus imports = internal demands plus exports

MT2: $M_\alpha y = z^p - z^l, \;\; z^p, z^l \geq 0, \;\; \langle z^{pt}, z^l \rangle = 0$
value of outputs + losses of unprofitable sectors =
value of imports + profits of profitable sectors

MT3: $xBy > 0$ the value of output is positive

MT4: $x \in [t^m; t^M] =: T \subset {\mathbb{R}_+}^m,\ y \in [p^e; p^i] =: P \subset {\mathbb{R}_+}^n$
intensities and prices are bounded
($t^m > 0$: vector of minimal intensities, $t^M \geq t^m$, vector of maximal intensities)
($p^e > 0$: vector of export prices, $p^i \geq p^e$, vector of import prices)

MT5: $\langle w^e, p^e \rangle = \langle w^i, p^i \rangle$ balance of payment condition
$\langle z^p, t^M \rangle = \langle z^l, t^m \rangle$ balance of profit condition

$\langle \cdot, \cdot \rangle$ denotes the standard scalar product, while t indicates transposition.

The conditions MT1 and MT2 correspond to N1 and N2. MT1 and MT2 determine w^e, w^i and z^p, z^l as the positive and negative part of xM_α and $M_\alpha y$, respectively, with the consequence that an equilibrium of the OEEM has to be understood as a triple $(\bar{x}, \bar{y}, \bar{\alpha}), \bar{x}, \bar{y}$ being non-zero vectors, fulfilling MT1–MT5.

In the von Neumann model, KMT version, the fulfillment of condition N3 was the crucial point in proving the existence of an equilibrium solution; but the pre-assumptions of Morgenstern, Thompson (1969) make the fulfillment of MT3 a matter of course.

To show the existence of an equilibrium of the OEEM Morgenstern, Thompson use a family of pairs of linear programs dual to each other and parametrized by $\alpha, 0 \leq \alpha < \infty$.

From this for instance it follows that there exists an $\bar{\alpha} \in \mathbb{R}_+$ such that

$$\bar{x} M_{\bar{\alpha}} y \geq 0 \geq x M_{\bar{\alpha}} \bar{y}, \qquad x \in T,\ y \in P. \tag{7}$$

I became acquainted with Prof. Łoś in February 1972; he organized a symposium of Mathematical Methods in Economics with a broad international spectrum of participants.

Before going to Warsaw myself, I was sent the paper Łoś (1971): 'A simple proof ...,' which I studied carefully—it dealt with von Neumann models in linear spaces ordered by means of cones. It was indeed an interesting paper—one thing however, needs to be mentioned: The title 'Simple proof ... ' seems to be a slight understatement.

Moreover a paper circulated among the participants of this Warsaw Symposium—the so-called Aarhus paper, Łoś (1967). It was a well elaborated, typewritten manuscript of a lecture Prof. Łoś had given at Aarhus, which was very interesting to read. It was something like a toolbox already containing main ideas of the further developments of the oeuvre of J. Łoś. This indeed is my impression today, when studying the papers of Jerzy Łoś as well as those of his pupils. I will come back to this point.

When coming to Warsaw, of course, I had more or less known the published literature about von Neumann models; new to me was the fundamental approach of Prof. Łoś.

Almost at the beginning of the symposium Prof. Łoś developed in a seminar talk his view of the matter, presenting the following diagram:

input, output mapping

$$\underset{\text{space of intensities}}{X} \xrightarrow{\ \cdot\, A,\ \cdot B\ } \underset{\text{space of commodities}}{W}$$

cost, revenue mapping

$$\underset{\text{space of values}}{Z} \xleftarrow{\ A\,\cdot, B\,\cdot\ } \underset{\text{space of prices}}{Y}$$

Fig. 1

The m-dimensional linear space X (intensities) will be transformed by the linear mappings $\cdot A$, $\cdot B$ into the n-dimensional linear space W (commodities).

The n-dimensional space Y (prices), i.e. the dual space to W (commodities), will be mapped by the mappings $A\,\cdot, B\,\cdot$, dual to $\cdot\, A, \cdot\, B$, into the space Z (values), i.e. the dual space to X (intensities).

This diagram allows to recognize the mathematical possibilities and challenges of a further development of the theory of von Neumann models—essentially realized by Jerzy Łoś.

If we accept the linearity of the mappings in the diagram from above there are two points which merit to be discussed and which need to be discussed regarding development of the von Neumann theory of an expanding economy in the status of the early seventies. These are

- The introduction and the role of cone induced orders in the respective linear spaces, and
- Overcoming the (traditional) duality thinking,

and both are main foci in the oeuvre of Jerzy Łoś.

Of course, this is a mathematical challenge, which, of course, has to be based on economic ideas. That Łoś was a brillant mathematician need never be debated; more astonishing—at least to me—was his careful economic reasoning.

As a further focus of J. Łoś in cooperation with his wife, M. W. Łoś, might be seen the analysis of classical economic results within von Neumann models, as e.g. the relationship between the real–wage rate and the interest rate and/or between the level of consumption and the rate of growth, cf. J. Łoś, M. W. Łoś (1974) or the study of the paradoxes of capital theory, cf. J. Łoś, M. W. Łoś (1976), treating and generalizing Samuelson's Non–substitution theorem.

As is usual in economic models, the order defined by components is tacitly meant; this is for instance the case in the von Neumann model as also in the OEEM:

$$x = (x_1, \ldots, x_m) \in X,\, x \geq 0 :\Longleftrightarrow x_i \geq 0, \quad i \in \mathbb{N}_m. \tag{8}$$

From functional analysis we know the method to introduce an order by means of cones, which is applied here with respect to our finite dimensional linear spaces X,Z,Y,W:

Let $X^+ \subset X$ be a pointed closed and convex cone of X, then a homogeneous, additive, reflexive and transitive relation (order) $\geq_{X^+}$ may be introduced for $x^1, x^2 \in X$ by:

$$x^1 \geq_{X^+} x^2 \,:\iff\, x^1 - x^2 \in X^+; \tag{9}$$

we speak of X^+ as of the order cone of X.

The dual space Z of X has then to be ordered by the dual cone $(X^+)^* \subset Z$ of X^+ defined as

$$(X^+)^* := \{z \in Z : \langle x, z\rangle \geq 0, \text{ for all } x \in X^+\} =: Z^+. \tag{10}$$

We denote Z^+ as the order cone of Z.

The order $\geq_{Z^+}$ is defined in accordance with (9).

In the same way we introduce orders $\geq_{Y^+}$ and $\geq_{W^+}$ based on the order cones Y^+ and W^+ dual to each other, being subsets of Y and W, respectively.

The linear spaces X,Z,Y and W ordered by the cones X^+, Z^+, Y^+, W^+ are represented by the pairs (X, X^+),(Z, Z^+),(Y, Y^+) and (W, W^+), respectively.

The concept of von Neumann models in linear spaces ordered by means of cones was introduced in Łoś (1971) 'A simple proof ...', in—let me say—an abstract way, i.e. without discussing the economic meaning of a von Neumann model generalized in this way. The proof for the existence of an equilibrium to N1–N4, where the orders have to be changed accordingly, uses a theorem of alternative, i.e. a so-called Farkas Lemma, formulated for finite dimensional linear spaces ordered by means of cones. For orders defined by components this lemma is proved in Gale (1960), Theorem 2.8.

At that time, parallel to the von Neumann activities of J. Łoś, several authors started to study for instance the theory of linear programming and related the theory of linear equations and equalities in linear spaces ordered by means of cones. The proof of the main theorem about the existence of optimal solutions to a pair of linear programs dual to each other is normally based on a lemma of the Farkas type.

For an immediate reference we specify the linear spaces $(X, X^+), (Z, Z^+), (Y, Y^+)$ and (W, W^+) determining the order cones X^+, Z^+, Y^+ and W^+:

The order cones X^+ and Y^+ are generated by $T \subset X$ and $P \subset Y$, respectively, as introduced in the OEEM in the following way:

$$X^+ := \{x \in X : x = \lambda \cdot t,\ \ \lambda \in \mathbb{R}_+,\ \ t \in T\} \tag{11}$$

and

$$Y^+ := \{y \in Y : y = \lambda \cdot p,\ \ \lambda \in \mathbb{R}_+,\ \ p \in P\}, \tag{12}$$

respectively, while the cones W^+ of W and Z^+ of Z are the dual cones $(Y^+)^*$ and $(X^+)^*$ of Y^+ and X^+, respectively.

The following lemma allows an economic interpretation of the abstract von Neumann model, cf. Moeschlin (1974b)/(1977).

LEMMA 1. *The equilibrium* $(\bar{x}, \bar{y}, \bar{\alpha})$ *of the OEEM is an equilibrium of a von Neumann model in the linear spaces* $(X, X^+), (Z, Z^+), (Y, Y^+)$ *and* (W, W^+) *and an equilibrium* $(\bar{x}, \bar{y}, \bar{\alpha})$ *of a von Neumann model in the linear spaces* $(X, X^+), (Z', Z^+), (Y, Y^+)$ *and* (W, W^+) *is up to a multiplication of* $\bar{x}$ *and* $\bar{y}$ *by positive scalars an equilibrium of the OEEM.*

I'm going to sketch the proof of the $\Longrightarrow$ part of the above statement. To this end let $(\bar{x}, \bar{y}, \bar{\alpha})$ be an equilibrium solution of the OEEM.

By (7) we have

$$\bar{x} M_{\bar{\alpha}} y \geq 0 \quad \text{for all } y \in P \quad \text{or equivalently for all } y \in Y^+, \tag{13}$$

$$0 \geq x M_{\bar{\alpha}} \bar{y} \quad \text{for all } x \in T \quad \text{or equivalently for all } x \in X^+. \tag{14}$$

Based on the definition of W^+ and Z^+ as dual cone of Y^+ and X^+, respectively, (13) and (14) may be written as

$$\bar{x} M_{\bar{\alpha}} \geq_{W^+} 0 \tag{15}$$

and

$$M_{\bar{\alpha}}\bar{y} \leq_{Z^+} 0, \tag{16}$$

respectively.

$\bar{x} \in T$ and $\bar{y} \in P$ clearly implies $\bar{x} \in X^+$ and $\bar{y} \in Y^+$ and therefore we get

$$\bar{x} \geq_{X^+} 0 \quad \text{and} \quad \bar{y} \geq_{Y^+} 0, \tag{17}$$

respectively. ■

The reformulation of the OEEM as a von Neumann model in linear spaces ordered by some special cones shows that the theory of von Neumann models in such spaces is well suited to represent the balanced trade of an economy exchanging with the outside world some commodities for other commodities at given prices.

But the von Neumann models in linear spaces ordered by means of cones open even other possibilities: for instance they would also give the framework to represent economic-ecological problems.

Decisive for Lemma 1 is the fact that we have a saddle–point situation, but it can easily be generalized to situations such as we have for bimatrix games; see Ballarini, Moeschlin (1976).

Reformulating the OEEM as von Neumann model in linear spaces ordered by special cones allows to give a proof of the existence of an equilibrium of the OEEM under relaxed assumptions compared with those of Morgenstern, Thompson (1969), cf. Moeschlin (1974a) and Moeschlin (1977).

A proof of the existence of an equilibrium to the OEEM under relaxed assumptions had already be given before in Mardoń (1974) by other methods.

In Łoś (1974) (The existence of equilibrium in an open expanding economy), trade activities at given prices are treated within the framework of the standard von Neumann model (the orders being defined by components), with the alteration that the input and output matrices are enlarged, having a typical structure with even negative entries.

Łoś formulated—in an abstract way—the question of how to prove the existence of an equilibrium solution to such a model in a problem session just at the beginning of the Symposium of Mathematical Models in Economics which motivated me to give a proof for the existence of an equilibrium, cf. Moeschlin (1973), based on Moeschlin (1974a).

The first who considered von Neumann models with non–nonnegative input and output matrices was Mrs. Longina Mardoń (Mardoń (1974)); she formulated the OEEM as a standard von Neumann model, i.e. as she says: in a closed form, with extended input and output matrices with also negative entries. The proof for the existence of an equilibrium under relaxed assumption compared with Morgenstern, Thompsom (1969) is based on a theorem of Tucker, cf. Tucker (1956).

Summarizing the situation L. Mardoń had—based on extended input and output matrices with even negative entries—reformulated the OEEM in the form of the standard von Neumann model, i.e. the orders being defined by components, while in Moeschlin (1974b) and (1977) the same was done, constructing a von Neumann model in linear spaces ordered by special cones.

The question now was posed: how is a von Neumann model of an open economy of the Mardoń–type related to an open von Neumann model represented in linear spaces ordered by means of cones, as done by Moeschlin?

The answer was given in Łoś (1976b), where he shows for an abstract presentation of the Mardoń model that the latter may—by decomposition—be related to the cone model determined by Moeschlin, in such a way that they have the same equilibria.

The theory developed by Łoś is indeed sophisticated.

The discussion of open von Neumann models suggests studying the economic effects of a cooperation of two economies. In J. Łoś, M. W. Łoś (1977) the weak equilibrium of a von Neumann model (not necessarily fulfilling condition N3) is related to the optimal solutions of corresponding linear programs, which allows together with a partition of A and B in corresponding submatrices to show that a cooperation of two economies allows to maximize the value of demand of both economies at common equilibrium prices. The connection of two economies by a market is also addressed in J. Łoś, M. W. Łoś (1979).

Concerning now the overcoming of the duality concept in the theory of linear economic models, Łoś has contributed two papers.

In a first paper, Łoś (1974), Łoś introduces labour and consumption and wages—as for instance earlier done by Malinvaud (1959), Morishima (1960), Morgenstern, Thompson (1967), Morishima (1971) and others—into the von Neumann model, which now yields the so-called three-matrix von Neumann model $(A_1, A_2; B)$, with A_1 and A_2 acting as input matrix and cost matrix, respectively, and B as both the output and the revenue matrix.

An equilibrium of this model is a quadruple $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$, $0 \leq \bar{x} \in \mathbb{R}_+{}^m$ and $0 \leq \bar{y} \in \mathbb{R}_+{}^n$ being two non–zero vectors and $0 < \bar{\lambda}, \bar{\mu} \in \mathbb{R}_+$, such that

$$\bar{\lambda} \cdot \bar{x} A_1 \geq \bar{x} B, \tag{18}$$

$$B\bar{p} \leq \bar{\mu} \cdot A_2 \bar{p}, \tag{19}$$

$$\bar{\lambda} \cdot \bar{x} A_1 \bar{p} \;=\; \bar{\mu} \cdot \bar{x} A_2 \bar{p} \;=\; \bar{x} B \bar{p} > 0, \tag{20}$$

where the orders $\geq$ are defined by components.

The proof of the existence of an equilibrium solution given by Łoś is based on the original fixed point theorem by von Neumann.

In Ballarini, Moeschlin (1976) the OEEM as treated in Moeschlin (1974b) and Moeschlin (1977) is generalized to a corresponding three-matrix model (including now consumption and labour) where the proof of existence is based—according to Łoś (1974) with orders now defined by components—on the fixed point theorem of von Neumann.

Finally in Łoś (1976) the three-matrix model is extended to a four-matrix model $(A_1, B_1; A_2, B_2)$, A_1,A_2,B_1,B_2 acting as input, cost, output and revenue matrix, respectively, but now formulated in linear spaces ordered by means of cones.

In Łoś (1976) the four-matrix model with extended equilibrium conditions according to the three-matrix model is related to a bimatrix game having the same equilibrium as the four-matrix von Neumann model showing that the growth rate and the rate of interest enjoy an optimality property. The proof of existence of an equilibrium solution corresponds to the one in Łoś (1974).

I have already mentioned the Aarhus paper, i.e. Łoś (1967), in which—according to my impression of today—various ideas of the further oeuvre of J. Łoś are formulated. Although I do not possess this Aarhus paper any more and I have to rely on my memory, I'm sure that the bimatrix game just mentioned is anticipated there as a zero–sum game within linear spaces with orders defined by components, as the special game theoretic approach of J. Łoś to von Neumann models.

The same holds true for the lemma of the Farkas type, which plays a special role in the proofs of J. Łoś.

Another concept of J. Łoś (1967), which I didn't find again in his literature, but which is taken up in Sosnowska (1976), is the theory of the perfect aggregate, where macro data remain invariant against the aggregation of processes.

The oeuvre of J. Łoś is characterized by a multitude of methods and concepts; for instance that of von Neumann models in linear spaces ordered by means of cones. His results as well as those of his school are deep-lying, for instance also those of Bromek, Kaniewska, Łoś (1976), where the existence of equilibrium of the various types of von Neumann model are examined. There exist indeed types of von Neumann models—not mentioned here—where it could be shown that the (pretended) equilibrium does not exist. J. Łoś has dealt with almost any question that can be posed in connection with von Neumann models as—for example—also with that of the cooperation of two economies.

With his systematic research work J. Łoś has left behind a well developed theory of von Neumann models.

This theory is and will remain strongly connected with the name of J. Łoś.

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