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# ON LIPSCHITZ CONTINUITY OF THE SOLUTION MAP FOR TWO-DIMENSIONAL WAVE MAPS

PIERO D'ANCONA

*Università di Roma "La Sapienza"*
*Dipartimento di Matematica*
*Piazzale A. Moro 2, I-00185 Roma, Italy*
*E-mail: dancona@mat.uniroma1.it*

VLADIMIR GEORGIEV

*Dipartimento di Matematica*
*Università degli Studi di Pisa*
*Via F. Buonarroti 2, 56100 Pisa, Italy*
*E-mail: georgiev@dm.unipi.it*

**1. Introduction.** The purpose of this paper is to analyze the properties of the solution map

$$(u_0, u_1) \mapsto u(t,x)$$

to the Cauchy problem for the wave map equation

$$u_{tt} - \Delta u + (|u_t|^2 - |\nabla_x u|^2)u = 0$$

with initial data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

in the case when $x \in \mathbf{R}^2$ and the target is the unit sphere $\mathbf{S}^n$ (embedded in $\mathbf{R}^{n+1}$), $n \geq 2$. Thus

$$u : \mathbf{R}_t \times \mathbf{R}^2_x \to \mathbf{S}^n$$

and we have the additional constraint $|u| = 1$.

For this problem several results of global well-posedness are available under suitable smallness assumptions on the initial data (see [4], [12], [13], [7]). Moreover, the existence of a global weak solution in $H^1$ is known ([6], [14], [1]) and the existence and uniqueness

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of smooth solutions under suitable assumptions of symmetry is well-known for the case of geodesically convex two-dimensional targets (see [8], [2]). For the case of target $\mathbf{S}^2$ the existence of smooth classical solutions was recently proved by Struwe [10].

A basic question still open concerns the well posedness, even local in time, in the energy space, i.e., for data in $H^1 \times L^2$. This problem is strictly related to the properties of continuity and regularity of the solution map. Indeed, the classical definition of well-posedness implies in particular the continuity of this map; even in a modern sense, we may remark that the standard proofs of existence and uniqueness, which resort to some contraction method, have as a natural consequence the Lipschitz continuity of that map. To quantify this property, denote by $E(t, u)$ the energy of a solution $u$ at the time $t$:

$$E(t,u) = \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbf{R}^2)} + \|\nabla_x u(t,\cdot)\|^2_{L^2(\mathbf{R}^2)}.$$

Then we may say that the solution map is Lipschitz continuous if we may find a constant $C$ such that for any two solutions $u$, $v$ the following inequality holds:

$$E(t, u - v) \leq CE(0, u - v), \qquad \forall t \in [0, 1]. \tag{1}$$

Note that in this definition the existence of the solution map is not assumed.

The solution map is locally Lipschitz continuous if for any solution $u$ one can find positive constants $\delta, C$ such that for any solution $v$ with

$$E(0, u - v) \leq \delta$$

the inequality (1) holds.

Our goal here is to show, by a suitable counterexample, that the solution map *is not locally Lipschitz continuous*. More precisely, we prove the following:

THEOREM 1. *There exists a smooth solution* $u : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{S}^n$ *to the wave map equation, such that for any* $C > 0$, $\delta > 0$, *we can construct a smooth solution* $v : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{S}^n$ *to the wave map equation so that*

$$E(0, u - v) \leq \delta$$

*and the Lipschitz condition* (1) *is not satisfied at* $t = 1$.

We remark that the solutions used in the counterexample are radially symmetric, hence the symmetry assumption does not improve the regularity of the solution map.

**2. Well-posedness of the Cauchy problem for semilinear wave equation.** The linear wave equation

$$\partial_t^2 u - \Delta u = 0 \tag{2}$$

with initial data

$$u(0, x) = u_0(x) \in \dot{H}^1(\mathbf{R}^n), \quad \partial_t u(0, x) = u_1(x) \in L^2(\mathbf{R}^n) \tag{3}$$

satisfies the energy estimate

$$\|\nabla_x u(t)\|_{L^2} + \|\partial_t u(t)\|_{L^2} \leq C\left(\|\nabla_x u_0\|_{L^2} + \|u_1\|_{L^2}\right) \tag{4}$$

provided the initial data $u_0, u_1$ belong to the Hilbert space

$$H = \dot{H}^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n). \tag{5}$$

Therefore for any $T>0$ we have a data-solution map $R$ defined in $H$ with values in $C(I;H)$, $I=[-T,T]$ so that $u(t,x)=R_0(u_0,u_1)$ is a solution to (2) in distribution sense in $(-T,T)\times\mathbf{R}^n$ and satisfies the initial conditions (3).

Moreover, $R_0$ is a bounded (hence continuous) linear operator

$$R_0:(u_0,u_1)\in H\to C(I;H).$$

A slight generalization of the above definition can be done by taking Banach space $X=X(I)\subseteq C(I;H)$ such that $R_0$ restricted to $H$ is a continuous linear operator

$$R_0:(u_0,u_1)\in H\to X.$$

Now we can consider the nonlinear Cauchy problem

$$\partial_t^2u-\Delta u=F(u),\ t\in[-T,T],\ x\in\mathbf{R}^n \tag{6}$$

with initial data (3). Here $F$ is a continuous map

$$F:u\in X\to F(u)\in Y \tag{7}$$

and $Y$ is a subset of the space of distributions $D'((-T,T)\times\mathbf{R}^n)$. The classical well-posedness usually is connected with the continuity of the mapping data-solution, $(u_0,u_1)\to u(t)$. More precisely, we shall say that the Cauchy problem (6) is well-posed in $H$ if there exists a Banach spaces $X=X(T)\subset C(I,H)$ and one can find a positive $r>0$ and a continuous operator

$$R:\{(u_0,u_1)\in H:\|(u_0,u_1)\|_H\leq r\}\to X=X(T),$$

so that $u(t)=R(u_0,u_1)(t)$ is a solution in distribution sense of (6) and satisfies the initial condition (3). The well-posedness of the Cauchy problem for the wave maps in $(t,x)\in\mathbf{R}\times\mathbf{R}$ is studied in [11].

In [3] even weaker regularity of $R$ is assumed, namely the uniform continuity of the mapping data-solution is studied for the case of Schrödinger type equations.

In the case when a standard contraction argument (see [9]) works one can show that the mapping

$$R:(u_0,u_1)\to u(t)$$

is locally Lipschitz continuous. More precisely, one can find a positive $r>0$ and $C$ so that

$$\|u-\tilde{u}\|_X\leq C\|(u_0,u_1)-(\tilde{u}_0,\tilde{u}_1)\|_H \tag{8}$$

for $(u_0,u_1)\in H$, $(\tilde{u}_0,\tilde{u}_1)\in H$, satisfying

$$\|(u_0,u_1)\|_H+\|(\tilde{u}_0,\tilde{u}_1)\|_H\leq r. \tag{9}$$

**3. Stereographic projection for wave maps.** The Cauchy problem for wave maps is the semilinear problem

$$(u_{tt}-\Delta u)+Q(\partial u)u=0, \tag{10}$$

with initial data

$$u(0,x)=u_0(x)\in\dot{H}^1(\mathbf{R}^2),\ \partial_tu(0,x)=u_1(x)\in L^2(\mathbf{R}^2), \tag{11}$$

where

$$Q(\partial u) = |\partial_t u|^2 - |\nabla_x u|^2. \tag{12}$$

Our first step is the reduction of the vector-valued wave equation (10) to a scalar one. For the purpose, we compose the wave map

$$u : (t, x) \in \mathbf{R} \times \mathbf{R}^2 \longrightarrow u = u(t, x) \in \mathbf{S}^2$$

with the stereographic projection

$$u = (u_1, u_2, u_3) \in \mathbf{S}^2 \longrightarrow z \in \mathbf{C} \cup \infty, \tag{13}$$

where

$$z = \frac{u_1 + iu_2}{1 + u_3}$$

and the south pole $S = (0, 0, -1)$ is mapped in $\infty$. The inverse map is

$$u_1 = \frac{2\,\mathrm{Re}\, z}{1 + |z|^2}\,, \;\; u_2 = \frac{2\,\mathrm{Im}\, z}{1 + |z|^2}\,, \;\; u_3 = \frac{1 - |z|^2}{1 + |z|^2}\,. \tag{14}$$

The metric induced by the projection is $(1 + |z|^2)^2 |dz|^2$.

The lines through the origin are geodesics on $\mathbf{C}$. Hence, we can take a geodesic of type

$$\gamma : \mathrm{Im}\, z = h(\mathrm{Re}\, z), \;\; h(s) = As$$

in $\mathbf{C}$, where $A$ is a real constant. This geodesics generates a wave map $u = u_\gamma$ (see [7]). Indeed, taking

$$X(t, x) = \mathrm{Re}\, z(t, x),$$

from (14) we get

$$u_1 = \frac{2X}{1 + X^2 + h^2(X)}\,, \;\; u_2 = \frac{2h(X)}{1 + X^2 + h^2(X)}\,, \;\; u_3 = \frac{1 - X^2 - h^2(X)}{1 + X^2 + h^2(X)}\,. \tag{15}$$

Substitution of this ansatz into the wave map equation gives the following scalar equation

$$M(X)\Box X - L(X)Q(\partial X) = 0, \tag{16}$$

where

$$\begin{aligned} L(X) &= 4h(X)h'(X)(-3X^2 + h^2(X) + 1) \\ &\qquad - (1 - (h'(X))^2)(2X^3 - 6Xh^2(X) - 2X), \\ M(X) &= -X^4 + (1 + h^2(X))^2 - 2X^3 h(X)h'(X) \\ &\qquad - 2X(1 + h^2(X))h(X)h'(X). \end{aligned} \tag{17}$$

To verify that the wave map equation is reduced to this scalar equation, we start with the relation

$$\partial_{x_j} \frac{2X}{1 + X^2 + h^2(X)} = 2\partial_{x_j} X \left( \frac{1 - X^2 + h^2(X) - 2Xh(X)h'(X)}{(1 + X^2 + h^2(X))^2} \right), \tag{18}$$

$$\partial_{x_j} \frac{2h(X)}{1 + X^2 + h^2(X)} = 2\partial_{x_j} X \left( \frac{-2Xh(X) + (1 + X^2 - h^2(X))h'(X)}{(1 + X^2 + h^2(X))^2} \right) \tag{19}$$

and

$$\partial_{x_j} \frac{1 - X^2 - h^2(X)}{1 + X^2 + h^2(X)} = -4\partial_{x_j} X \left( \frac{X + h(X))h'(X)}{(1 + X^2 + h^2(X))^2} \right). \tag{20}$$

The above three relations imply

$$Q(\partial u) = 4Q(\partial X)\frac{1+(h'(X))^2}{(1+X^2+h^2(X))^2}\,. \tag{21}$$

For the second derivative (note that $h''=0$) we have

$$\partial_{x_jx_j}\frac{2X}{1+X^2+h^2(X)} = \frac{2(\partial_{x_jx_j}X)M_1(X)-4(\partial_{x_j}X)^2L_1(X)}{(1+X^2+h^2(X))^3}\,, \tag{22}$$

where

$$\begin{aligned} L_1(X) &= X(1+X^2+h^2(X))(1+(h'(X))^2) \\ &\qquad + 2(X+h(X)h'(X))(1-X^2+h^2(X)-2Xh(X)h'(X)), \\ M_1 &= (1-X^2+h^2(X)-2Xh(X)h'(X))(1+X^2+h^2(X)). \end{aligned} \tag{23}$$

These relations imply

$$\Box\left(\frac{2X}{1+X^2+h^2(X)}\right) = \frac{2(\Box X)M_1(X)-4Q(\partial X)L_1(X)}{(1+X^2+h^2(X))^3}, \tag{24}$$

so combining this identity and (21), we obtain (16).

In the special case $h(X)=AX$, where $A$ is a real constant, we obtain

$$\begin{aligned} L(X) &= 4A^2X(-3X^2+A^2X^2+1)-2X(1-A^2)(X^2-3A^2X^2-1) \\ &= 2X(1+A^2)(1-X^2(1+A^2)), \\ M(X) &= -X^4+(1+A^2X^2)^2-2A^2X^4-2A^2X^2(1+A^2X^2) \\ &= (1-X^2(1+A^2))(1+X^2(1+A^2)). \end{aligned} \tag{25}$$

The equation (16) suggests us to take $X$ so that the equation

$$\Box X + f(X)Q(\partial X) = 0 \tag{26}$$

be satisfied. Here

$$f(X) = -\frac{2X(1+A^2)}{1+X^2(1+A^2)}\,. \tag{27}$$

It is clear that (26) implies (16). This scalar nonlinear wave equation can be transformed into linear wave equation (see [5]) by the aid of the transform

$$Y = G(X) \equiv \int_0^X \mathrm{e}^{F(s)}ds, \quad F(s) = \int_0^s f(\sigma)\,d\sigma.$$

So using (27), we find $F(s) = -\ln(1+B^2s^2)$, where $B=\sqrt{1+A^2}$ and

$$Y = B^{-1}\arctan(BX).$$

In conclusion, given any solution of the linear wave equation

$$\Box Y = 0 \tag{28}$$

the function

$$X = B^{-1}\tan(BY) \tag{29}$$

is a solution of the scalar nonlinear wave equation (26) and from (15) we see that the function $u = u_A(t,x)$ defined by

$$u_1 = \frac{\sin(2BY)}{B}, \; u_2 = \frac{A\sin(2BY)}{B}, \; u_3 = \cos(2BY), \qquad B = \sqrt{1+A^2} \tag{30}$$

is a wave map. The special solutions of (28) we shall use have the form

$$Y(t,x) = \mathrm{Re} \int_{\mathbf{R}^3} \sin(t|\xi|) \mathrm{e}^{(ix\xi)} \varphi(\xi) \frac{d\xi}{|\xi|}. \tag{31}$$

With this choice we have

$$Y(0,x) = 0, \; \|\partial_t Y(t,\cdot)\|_{L^2} + \|\nabla_x Y(t,\cdot)\|_{L^2} \le C\|\varphi\|_{L^2}$$

for any $t \ge 0$. These relations and (30) imply

$$\begin{aligned} &\|\partial_t u_A(0,\cdot)\|_{L^2} + \|\nabla_x u_A(0,\cdot)\|_{L^2} \le C\|\varphi\|_{L^2}, \\ &C^{-1}|A_1 - A_2| \, \|\varphi\|_{L^2} \\ &\qquad \le \big\|\partial_t\big(u_{A_1}(0,\cdot) - u_{A_2}(0,\cdot)\big)\big\|_{L^2} + \big\|\nabla_x\big(u_{A_1}(0,\cdot) - u_{A_2}(0,\cdot)\big)\big\|_{L^2} \\ &\qquad\qquad \le C|A_1 - A_2| \, \|\varphi\|_{L^2} \end{aligned} \tag{32}$$

with some constant $C$ independent of $\varphi, A, A_1, A_2$. Indeed, we have

$$\partial_t u_A(0) = \partial_t Y(0,x)(2,2A,0), \; \nabla_x u_A(0) = \nabla_x Y(0,x)(2,2A,0).$$

**4. The solution map for wave maps is not Lipschitz continuous.** Take two real numbers $A, \tilde{A}$ such that

$$0 \le A < \tilde{A},$$

$\tilde{A}$ is close enough to $A$, and consider the wave maps $u_A$ and $u_{\tilde{A}}$ constructed in (30). If the solution map is Lipschitz continuous, then the estimate (32) implies that

$$\|\partial_t(u_A(t,\cdot) - u_{\tilde{A}}(t,\cdot))\|_{L^2} + \|\nabla_x(u_A(t,\cdot) - u_{\tilde{A}}(t,\cdot))\|_{L^2} \le C|A - \tilde{A}|\|\varphi\|_{L^2}.$$

Dividing by $|A - \tilde{A}|$ and taking the limit $\tilde{A} \to A$, we get

$$\|\partial_t \partial_A u_A(t,\cdot)\|_{L^2} + \|\nabla_x \partial_A u_A(t,\cdot)\|_{L^2} \le C\|\varphi\|_{L^2}.$$

From (30) we obtain

$$\begin{aligned} \partial_A u_1 &= \frac{2A}{B^2} Y\cos(2BY) - \frac{A}{B^3}\sin(2BY), \\ \partial_A u_2 &= \frac{2A^2}{B^2} Y\cos(2BY) + \frac{1}{B^3}\sin(2BY), \\ \partial_A u_3 &= -\frac{2A}{B} Y\sin(2BY). \end{aligned}$$

Taking the time derivative, we find the following pointwise estimate

$$|\partial_t \partial_A u_A(t,x)| \ge C_0(A)|Y(t,x)| \, |\partial_t Y(t,x)| - C_1(A)|\partial_t Y(t,x)|,$$

where $C_0(A) > 0$ provided $A > 0$. For space derivatives we have an analogous estimate

$$|\nabla_x \partial_A u_A(t,x)| \ge C_0(A)|Y(t,x)| \, |\nabla_x Y(t,x)| - C_1(A)|\nabla_x Y(t,x)|.$$

Therefore, we take $A > 0$, say $A = 1$, fix it and then the assumption that the solution map is Lipschitz continuous implies that

$$\begin{aligned} C_0(A)\|Y(t,\cdot)\partial_t Y(t,\cdot)\|_{L^2} + C_0(A)\|Y(t,\cdot)\nabla_x Y(t,\cdot)\|_{L^2} \\ - C_1(A)\|\partial_t Y(t,\cdot)\|_{L^2} - C_1(A)\|\nabla_x Y(t,\cdot)\|_{L^2} \\ \leq \|\partial_t \partial_A u_A(t,\cdot)\|_{L^2} + \|\nabla_x \partial_A u_A(t,\cdot)\|_{L^2} \leq C\|\varphi\|_{L^2}. \end{aligned}$$

From (31) we have the classical energy estimate

$$\|\partial_t Y(t,\cdot)\|_{L^2} + \|\nabla_x Y(t,\cdot)\|_{L^2} \leq C\|\varphi\|_{L^2}$$

and we arrive at

$$\|Y(t,\cdot)\partial_t Y(t,\cdot)\|_{L^2} + \|Y(t,\cdot)\nabla_x Y(t,\cdot)\|_{L^2} \leq C\|\varphi\|_{L^2}. \tag{33}$$

Recall that this estimate is valid locally, i.e. only for $\|\varphi\|_{L^2} \leq r$ according to (9). It is a standard argument that shows that the estimate (33) with $\|\varphi\|_{L^2} \leq r$ implies the scale invariant estimate

$$\|Y(t,\cdot)\partial_t Y(t,\cdot)\|_{L^2} + \|Y(t,\cdot)\nabla_x Y(t,\cdot)\|_{L^2} \leq C\|\varphi\|^2_{L^2} \tag{34}$$

without any upper bound on $\|\varphi\|_{L^2}$. It is clear also that if the estimate (34) is valid for real valued functions $Y$, then the same estimate is valid for complex valued functions $Y$ so we can take

$$Y(t,x) = \int_{\mathbf{R}^3} \sin(t|\xi|)\mathrm{e}^{ix\xi}\varphi(\xi)\,\frac{d\xi}{|\xi|}\,. \tag{35}$$

In the remaining part of this section we shall show that the estimate (34) will lead to a contradiction.

In fact, the estimate (34) will imply

$$\left|\int \Psi(x)Y(t,x)\overline{\partial_t Y(t,x)}\,dx\right| \leq C\|\Psi\|_{L^2}\|\varphi\|^2_{L^2} \tag{36}$$

for any $\Psi \in L^2$. Using the Plancherel identity and (35) we see that this inequality yields

$$\left|\int\int \widehat{\Psi}(\xi-\eta)\cos(t|\xi|)\sin(t|\eta|)\varphi(\xi)\overline{\varphi(\eta)}\,d\xi\,\frac{d\eta}{|\eta|}\right| \leq C\|\Psi\|_{L^2}\|\varphi\|^2_{L^2}. \tag{37}$$

Given any even integer $M > 2$, we set (compare with [5])

$$\varphi_M(\xi) = H(A_M)\frac{1}{|\xi|\ln^{5/8}|\xi|}\,, \tag{38}$$

where

$$A_M = \left\{\xi \in \mathbf{R}^2 : 2 \leq |\xi| \leq M, \mathrm{dist}(|\xi|, 8\mathbf{Z}+1) < 1/2\right\} \tag{39}$$

and $H(A)$ denotes the characteristic function of the set $A$. The condition $\mathrm{dist}(|\xi|, 8\mathbf{Z}+1) < 1/2$ is needed to assure the inequality

$$\varphi_M(\xi)\sin(t_0|\xi|) \geq C\varphi_M(\xi) \geq 0,\ \ \varphi_M(\xi)\cos(t_0|\xi|) \geq C\varphi_M(\xi) \geq 0 \tag{40}$$

with $C > 0$ and $t_0 = \pi/4$. For $\Psi$ we take

$$\widehat{\Psi_M}(\xi) = H(2 \leq |\xi| \leq M)\frac{1}{|\xi|\ln^{9/16}|\xi|}\,. \tag{41}$$

For any $M > 3$ we have the estimates

$$\|\varphi_M\|_{L^2(\mathbf{R}^2)} \le C, \ \|\Psi_M\|_{L^2(\mathbf{R}^2)} \le C \tag{42}$$

with some constant $C$ independent of $M > 3$. Further, we take $N \in 8\mathbf{Z}+1$ and $M \in 16\mathbf{Z}+1$ so that

$$3 < N < \frac{M}{2}.$$

Using the non-negativity property (40), we find

$$\int\int \widehat{\Psi_M}(\xi-\eta)\cos(\pi|\xi|/4)\sin(\pi|\eta|/4)\varphi_M(\xi)\overline{\varphi_M(\eta)}\,d\xi\,\frac{d\eta}{|\eta|}$$
$$\ge C\int_{3<|\eta|<N}\int_{|\xi|>2N}\widehat{\Psi_M}(\xi-\eta)\varphi_M(\xi)\varphi_M(\eta)\,d\xi\,\frac{d\eta}{|\eta|}.$$

For $|\xi| \ge 3, |\eta| \ge 3$ and $|\eta| < |\xi|/2$ we have

$$|\xi-\eta| \sim |\xi|, \ \ln|\xi-\eta| \sim \ln|\xi|.$$

So the estimate (37) and the definition (41) of $\widehat{\Psi_M}$ lead to the estimate

$$\int_{3<|\eta|<N}\int_{|\xi|>2N}\frac{\varphi_M(\xi)}{|\xi|\ln^{9/16}|\xi|}\varphi_M(\eta)\,d\xi\,\frac{d\eta}{|\eta|} \le C \tag{43}$$

with some constant $C > 0$ independent of $M, N$. Now the definition (38) of $\varphi_M$ implies

$$\int_{3<|\eta|<N}\varphi_M(\eta)\,\frac{d\eta}{|\eta|} \sim \sum_{2\le j\le N,\, j\in 8\mathbf{Z}+1}\frac{1}{j\ln^{5/8} j} \sim \ln^{3/8} N.$$

In a similar way

$$\int_{|\xi|>2N}\frac{\varphi_M(\xi)}{|\xi|\ln^{9/16}|\xi|}\,d\xi \sim \sum_{2N\le j\le M,\, j\in 8\mathbf{Z}+1}\frac{1}{j\ln^{9/16+5/8} j} \sim \frac{1}{\ln^{3/16} N}$$

provided $M \ge N^2$. Consequently, the estimate (43) will lead to

$$\ln^{3/16} N \le C$$

with $C > 0$ independent of $N$. This estimate is an obvious contradiction. This concludes the proof of Theorem 1.

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