

STABILITY FOR DISSIPATIVE MAGNETO-ELASTIC SYSTEMS

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Abstract. In this survey we first recall results on the asymptotic behavior of solutions in classical thermoelasticity. Then we report on recent results in linear magneto-thermo-elasticity and magneto-elasticity, respectively.

1. Introduction. The time-dependent system of elasticity is known to be of hyperbolic type, while that of heat conduction is parabolic. As third system we consider Maxwell's equations where displacement currents are neglected, thus resulting in a parabolic system for the magnetic field.

We are interested in the asymptotic behavior of solutions as time tends to infinity for different coupling of the systems as there are: thermo-elasticity, magneto-elasticity and magneto-thermo-elasticity. Due to the coupling of different types of differential equations with in part strongly different kind of behavior for example in bounded domains, the interesting question arises: what is the predominating part, and how is this expressed in terms of decay rates of the solutions?

We start with recalling well-known results in thermo-elasticity, then we report on recent results obtained in magneto-thermo-elasticity and in magneto-elasticity, respectively.

2. Typical features of thermo-elasticity. Let

$$u = u(t, x) \in \mathbb{R}^n, \quad n = 1, 2, 3, \quad \theta(t, x) \in \mathbb{R},$$

$t \geq 0$, $x \in \Omega \subset \mathbb{R}^n$, describe the displacement vector and the temperature difference, respectively, in a homogeneous, isotropic, linear medium. Then

$$\begin{aligned} u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \gamma \nabla \theta &= 0 & [F(\nabla^2 u, \dots)], \\ \theta_t - \kappa \Delta \theta + \gamma \operatorname{div} u_t &= 0 & [f(\nabla u_t, \nabla^2 \theta^2, \dots)] \end{aligned}$$

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plus initial and boundary conditions, e.g.

$$u = 0, \quad \theta = 0 \text{ on } \partial\Omega$$

([. . .] indicates a possible nonlinear structure).

Typical results. Let

$$E(t) := \frac{1}{2} \int_{\Omega} (|u_t|^2 + \mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + |\theta|^2)(t, x) \, dx$$

denote the energy at time t . Then we have:

$n = 1$:

- $\Omega = (0, 1)$: $E(t) \leq d_1 e^{-d_2 t} E(0)$, with positive constants d_1, d_2 .

Dissipation through heat conduction leads to overall exponential decay.

- $\Omega = \mathbb{R}$: $\|u_t(t), \theta(t)\|_{L^\infty} \leq C \cdot (1 + t)^{-1/2}$, $C > 0$ constant.

Behavior like heat conduction.

$n = 2, 3$:

• Reflecting rays imply slow decay: Whenever there exist reflecting rays that is a two-periodic orbit of the billiard in the domain, there is no uniform decay of the energy.

- $\Omega = B(0, 1)$ (unit ball), radial symmetry of the data: Energy decays exponentially.

- $E(t) \rightarrow 0 \Leftrightarrow \left(\left\{ \begin{array}{l} \Delta u + \lambda u = 0 \\ u|_{\partial\Omega} = 0 \\ \operatorname{div} u = 0 \end{array} \right\} \Rightarrow u = 0 \right) =: (*)$

(*) holds if $\Delta v + \lambda v = 0$, $v|_{\partial\Omega} = 0$ (scalar) has only simple eigenvalues.

- $\Omega = \mathbb{R}^3$: $U = U^P + U^S \in \overline{\nabla H^1} \oplus D_0$, where D_0 denotes divergence-free vector fields.

$U_{tt}^S - \mu \Delta U^S = 0$: no dissipation.

(U^S, θ) : like heat equation.

We remark that nonlinear systems in \mathbb{R}^1 are globally well-posed for small data, in \mathbb{R}^3 under radial symmetry in bounded domains (for small data), for $\Omega = \mathbb{R}^3$ if certain nonlinearities are excluded.

The results recalled in this section have been obtained over the last 30 years, among the contributors are Dafermos, Henry, Jiang, Kawashima, Kim, Koch, Lebeau, Lopes, Muñoz Rivera, Perissinotto, Shen, Shibata, Slemrod, Zheng, Zuazua and the author. A comprehensive survey with detailed references can be found in [4].

3. Magneto-thermo-elasticity. We consider the system of magneto-thermo-elasticity for the displacement vector u , the temperature difference θ and the magnetic field h :

$$u_{tt} - Eu - \alpha[\nabla \times h] \times \vec{H} + \gamma \nabla \theta = 0,$$

$$h_t - \Delta h - \beta \nabla \times [u_t \times \vec{H}] = 0,$$

$$\theta_t - \kappa \Delta \theta + \gamma \operatorname{div} u_t = 0,$$

$$Eu = [(C_{ijkl} u_{,i}^k)_{,j}]_{i=1,2,3},$$

in the homogeneous isotropic case under consideration we have

$$Eu = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u,$$

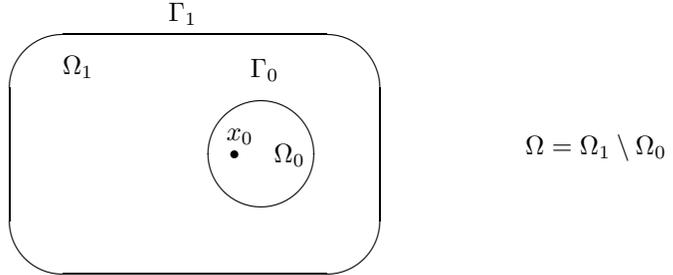
and $\vec{H} = (0, 0, H)'$. Additionally we have the initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad h(0, x) = h_0(x), \quad \theta(0, x) = \theta_0(x).$$

First we shall consider the Cauchy problem, then the bounded domain case for which we shall assume the following setup.

$\Omega \subset \mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Omega$ and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 \neq \emptyset$. Assume that there is $x_0 \in \Omega$ such that

$$\begin{aligned} \Gamma_0 &= \{x \in \Gamma : (x - x_0) \cdot \nu(x) \leq 0\}, \\ \Gamma_1 &= \{x \in \Gamma : (x - x_0) \cdot \nu(x) \geq a > 0\}. \end{aligned}$$



Boundary conditions for h and θ :

$$\nu \times (\nabla \times h) = 0, \quad \nu \cdot h = 0, \quad \theta = 0 \quad \text{on } \Gamma$$

and a memory type boundary condition for u :

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma_0, \quad u + r * \partial_\nu u = 0 \quad \text{on } \Gamma_1. \\ \partial_\nu u &:= (C_{ijkl} u_{,i}^k \nu^j)_{i=1,2,3} - \alpha H h^3 \nu + \alpha H \nu^3 h, \\ (r * f)(t) &:= \int_0^t r(t-s) f(s) ds, \\ \partial_\nu u &= -\tau u_t - \tau g * u_t, \quad \tau > 0. \end{aligned}$$

First we shall assume that g essentially decays exponentially, i.e. for $t \geq 0$

$$\left. \begin{aligned} 0 &< g(t) \leq c_0 e^{-g_0 t}, \\ -c_1 g(t) &\leq g'(t) \leq -c_2 g(t), \\ -c_3 g'(t) &\leq g''(t) \leq -c_4 g'(t) \end{aligned} \right\}$$

with positive constants c_j . Example:

$$\tilde{g}(t) = c_0 e^{-g_0 t}.$$

Second, we shall consider polynomially decaying kernels satisfying for $t \geq 0$

$$\left. \begin{aligned} 0 < g(t) &\leq b_0(1+t)^{-p}, \\ -b_1g(t)^{(p+1)/p} &\leq g'(t) \leq -b_2g(t)^{(p+1)/p}, \\ -b_3|g'(t)|^{(p+2)/(p+1)} &\leq g''(t) \leq -b_4|g'(t)|^{(p+2)/(p+1)} \end{aligned} \right\}$$

with positive constants b_j . Example:

$$\bar{g}(t) = b_0(1+t)^{-p}.$$

3.1. The Cauchy problem

$$\begin{aligned} u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \alpha H\nabla h^3 - \alpha H \frac{\partial h}{\partial x_3} + \gamma\nabla\theta &= 0, \\ h_t - \Delta h - \beta H \frac{\partial u_t}{\partial x_3} + \beta H(0, 0, \operatorname{div} u_t)' &= 0, \\ \theta_t - \kappa\Delta\theta + \gamma \operatorname{div} u_t &= 0. \end{aligned}$$

Using Fourier transform

$$v := \hat{u}, \quad w := \hat{h}, \quad \psi := \hat{\theta}.$$

we get

$$\begin{aligned} v_{tt} + \mu|\eta|^2v + (\mu + \lambda)(\eta v)\eta - i\alpha H\eta w^3 + i\alpha H\eta_3w - i\gamma\psi\eta &= 0, \\ w_t + |\eta|^2w + i\beta H\eta_3v_t - \beta H(0, 0, v_t\eta)' &= 0, \\ \psi_t + \kappa|\eta|^2\psi - i\gamma v_t\eta &= 0. \end{aligned}$$

Next,

$$\frac{d}{dt}\mathcal{E}_1(t) = -\frac{\alpha}{\beta}|\eta|^2|w|^2 - \kappa|\eta|^2|\psi|^2$$

where

$$\mathcal{E}_1(t) := \frac{1}{2}\left\{|v_t|^2 + \mu|\eta|^2|v|^2 + (\mu + \lambda)(\eta v)^2 + \frac{\alpha}{\beta}|w|^2 + |\psi|^2\right\}(t).$$

Let

$$\mathcal{A} \equiv \mathcal{A}(\eta) := |\eta|^2\left(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\right).$$

THEOREM 1. *If the initial data satisfy*

$$\int_{\mathbb{R}^n} \left[\frac{(1 + |\eta|^2)\mathcal{A}^2}{|\eta|^2} \right]^{m+1} \mathcal{E}_1(0, \eta) \, d\eta < \infty,$$

$m \in \mathbb{N}_0$, then the energy term

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left\{ |u_t|^2 + \mu|\nabla u|^2 + (\mu + \lambda)|\operatorname{div} u|^2 + \frac{\alpha}{\beta}|h|^2 + |\theta|^2 \right\}(t, x) \, dx$$

associated to a solution (u, h, θ) to the Cauchy problem decays polynomially:

$$E(t) = \mathcal{O}(t^{-m}) \quad \text{as } t \rightarrow \infty.$$

3.2. The bounded domain case. Assuming $\operatorname{div} h_0 = 0$, we have

$$\operatorname{div} h(t, \cdot) = 0 \quad \text{for all } t \geq 0.$$

Let

$$F(t) := \frac{1}{2} \int_{\Omega} \left(|u_t|^2 + C_{ijkl} u_{,i}^k u_{,j}^l + \frac{\alpha}{\beta} |h|^2 + |\theta|^2 \right) (t, x) \, dx \\ - \frac{\tau}{2} \int_{\Gamma_1} (g' \square u)(t, z) \, dz + \frac{\tau}{2} g(t) \int_{\Gamma_1} |u|^2(t, z) \, dz,$$

where

$$(f \square \varphi)(t) := \int_0^t f(t-s) |\varphi(t) - \varphi(s)|^2 \, ds.$$

THEOREM 2. *Let g be an exponentially decaying resolvent kernel. Then the energy F decays exponentially, i.e.*

$$\exists d_0, d_1 > 0 \quad \forall t \geq 0 \quad F(t) \leq d_0 e^{-d_1 t} F(0).$$

THEOREM 3. *Let g be a polynomially decaying resolvent kernel. Then the energy F decays polynomially, i.e.*

$$\exists d_2 > 0 \quad \forall t \geq 0 \quad F(t) \leq \frac{d_2}{(1+t)^p} F(0).$$

The results in this section have been obtained in collaboration with J. E. Muñoz Rivera [5]. For previous or related work see [1, 2, 3, 7, 8, 9] and further references in [5].

4. Magneto-elasticity. Let $\Omega \subset \mathbb{R}^2$ be bounded. The system of magneto-elasticity for the displacement $u = (u^1, u^2, 0) = \underline{u}(t, x_1, x_2)$ and the magnetic field $h = (h^1, h^2, 0)$ to be considered is the following (here $\vec{H} = (H, 0, 0)'$):

$$u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u - \alpha (\nabla \times h) \times \vec{H} = 0, \\ h_t - \Delta h - \beta \nabla \times (u_t \times \vec{H}) = 0, \\ u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad h(t=0) = h_1, \\ u = 0, \quad \nu \cdot h = 0, \quad \nu \times (\nabla \times h) = 0 \quad \text{on } \partial\Omega.$$

Here, ν denotes again the exterior normal and we have $\operatorname{div} h = 0$. Admissible domains are those that are homeomorphic to the unit ball and for which one of the following three conditions I–III is satisfied:

I. Ω is the union of finitely many rectangles with axes parallel to the x_1 - and x_2 -axes, respectively, see Figure 1.

II. Ω satisfies $\nu_1 \nu_2 = 0$ in the first quadrant (where $x_1 \geq 0$ and $x_2 \geq 0$) and in the third quadrant (where $x_1 \leq 0$ and $x_2 \leq 0$). In the second and fourth quadrants Ω satisfies $x\nu \geq \delta_0 > 0$, for some δ_0 , see Figure 2.

III. Ω satisfies $\nu_1 \nu_2 = 0$ in the second and fourth quadrants. In the first and third quadrants Ω satisfies $x\nu \geq \delta_0 > 0$, for some δ_0 , see Figure 3.

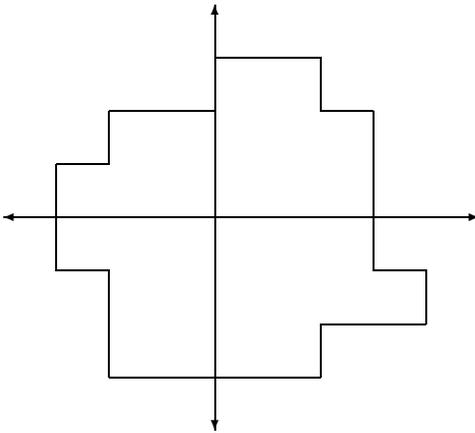


Figure 1. Type I

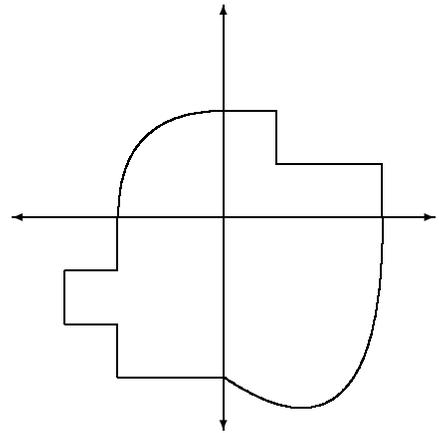


Figure 2. Type II

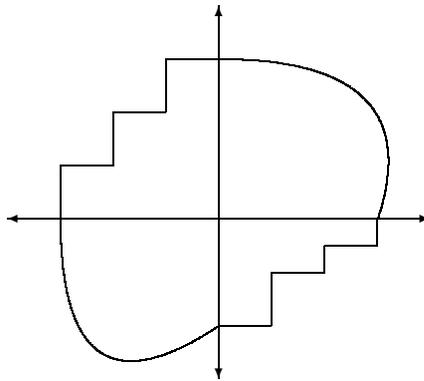


Figure 3. Type III

Let

$$E(t) \equiv E(t; u, h) := \frac{1}{2} \int_{\Omega} |u_t|^2 + \mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2 + \frac{\alpha}{\beta} |h|^2 dx,$$

$$E_j(t) := E(t; \partial_t^j u, \partial_t^j h), \quad j \geq 0.$$

Then a polynomial decay of the energy is described by

THEOREM 4. *There exists $d > 0$ such that for all $t \geq 0$ we have $E(t) \leq d \sum_{j=0}^7 E_j(0) \cdot \frac{1}{t}$.*

(Use $\omega := \nabla \times h = \partial_1 h^2 - \partial_2 h^1$, $\omega|_{\partial\Omega} = 0$.)

The results in this section have been obtained in collaboration with J. E. Muñoz Rivera [6]. For previous or related work see [1, 7] and further references in [6].

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