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A NEW APPROACH TO STUDY HYPERBOLIC-PARABOLIC COUPLED SYSTEMS

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1. Introduction. In many physical problems, like in thermoelasticity, viscous fluid etc., people often meet hyperbolic-parabolic coupled systems. There is a rich literature devoted to the existence, regularity and long time asymptotics of solutions to initial (-boundary) value problems for linear and nonlinear hyperbolic-parabolic coupled systems, mostly relying on the framework of semigroup theory or of abstract evolutional equations, cf. [11, 15, 21] and references therein. To describe properties of solutions precisely, it would be very interesting and useful if one could decouple the hyperbolic and parabolic operators in the coupled systems. In this note, we shall survey our recent works in developing a general procedure to weakly decouple the hyperbolic-parabolic systems as well as the applications. The decoupling argument is inspired by Taylor's work [18], in which an idea was introduced to decouple the different characteristic fields for studying the reflection of singularities in hyperbolic problems. In Section 3, by using the decoupling argument, several interesting applications will be given ranging from the propagation of singularities either in nonlinear thermoelasticity or in viscous fluids, to find that there is a cone of dependence for the propagation of singularities in nonlinear coupled systems, which shows that the hyperbolic operators in coupled systems play a dominant role in determining the regularity of solutions.

2. A general idea for decoupling hyperbolic-parabolic coupled systems. For any given $\Omega \subseteq \mathbb{R}^n$, denote by $\Psi^k(\Omega)$ the set of pseudodifferential operators of order k with symbols in $S^k(\Omega \times \mathbb{R}^n_\xi)$ ([3, 10]). Consider the following linear hyperbolic-parabolic

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coupled system in $\mathbb{R}_t \times \mathbb{R}_x^n$:

$$
(2.1)\qquad \begin{cases} \partial_t u + \sum_{j=1}^n A_j(t,x)\partial_{x_j} u + \sum_{j=1}^n B_j(t,x)\partial_{x_j} v + A_0(t,x)u + B_0(t,x)v = f(t,x) \\ \partial_t v - \sum_{j,k=1}^n a_{jk}(t,x)\partial^2_{x_j x_k} v + \sum_{j=1}^n C_j(t,x)\partial_{x_j} u + \sum_{j=1}^n d_j(t,x)\partial_{x_j} v \\ \qquad\qquad + C_0(t,x)u + d_0(t,x)v = g(t,x) \end{cases}
$$

where all coefficients are supposed to be smooth for simplicity, $u = (u_1, \dots, u_N)^T(t,x)$, $v(t,x)$ is scalar, $\{A_j(t,x)\}_{j=0}^n$ are $N \times N$ matrices, $\{B_j(t,x)\}_{j=0}^n$ are $N \times 1$ vectors, $\{C_j(t,x)\}_{j=0}^n$ are $1 \times N$ vectors, $\{d_j(t,x)\}_{j=0}^n$ are scalar, and $(a_{jk}(t,x))_{n\times n}$ is positively definite, which means there is $a_0 > 0$ such that

$$
\sum_{j,k=1}^n a_{jk}(t,x)\xi_j\xi_k \geq a_0|\xi|^2
$$

for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. We are going to seek for a transformation of (u,v) in order to decouple the system (2.1). Let $\Lambda_a = (1 - \sum_{j,k=1}^n a_{jk}\partial^2_{x_j x_k})^{\frac{1}{2}}$, then from (2.1) we know that $U = (u,v)^T$ satisfies the system

$$
(2.2)\qquad \partial_t U + H_2(t,x,D_x)U + H_1(t,x,D_x)U + H_0(t,x,D_x)U = F(t,x)
$$

where $F = (f,g)^t$, $H_2(t,x,D_x) = \mathrm{diag}\big[\mathbf{0}_{N\times N}, \Lambda_a^2\big] \in \Psi^2(\mathbb{R}^n)$ and $H_1 \in \Psi^1(\mathbb{R}^n)$, $H_0 \in \Psi^0(\mathbb{R}^n)$ are given obviously by (2.1). As in [18], let $U^{(1)} = (I + K_1)U$ where $K_1(t,x,D_x) \in \Psi^{-1}(\mathbb{R}^n)$ will be determined later, then from (2.2) we have

$$
(2.3)\qquad \begin{aligned} &\partial_t U^{(1)} + H_2 U^{(1)} + H_1 U^{(1)} + [K_1, H_2]U \\ &\qquad + \big([K_1, H_1] - \partial_t K_1 + (I + K_1)H_0\big)U = (I + K_1)F. \end{aligned}
$$

Temporarily, we assume that $I + K_1$ is invertible, which will be realized by slightly modifying K_1 later. Then, by using $(I + K_1)^{-1} = I - K_1(I + K_1)^{-1}$, the system (2.3) can be rewritten as the following one for $U^{(1)}$,

$$
(2.4)\qquad \partial_t U^{(1)} + H_2 U^{(1)} + \big([K_1, H_2] + H_1\big)U^{(1)} + H_0^{(1)} U^{(1)} = F^{(1)},
$$

where $F^{(1)} = (I + K_1)F$, and

$$
H_0^{(1)} = \big([K_1, H_1] + (I + K_1)H_0 - \partial_t K_1 - [K_1, H_2]K_1\big)(I + K_1)^{-1} \in \Psi^0(\mathbb{R}^n).
$$

Noting that the eigenvalues of the symbol of $H_2(t,x,D_x)$ are zero and $\langle\xi\rangle_a^2 := 1 + \sum_{j,k=1}^n a_{jk}\xi_j\xi_k$ with the multiplicity of zero being N, as in [18], if we define $K_1(t,x,\xi)$ as

$$
K_1(t,x,\xi) = \begin{pmatrix} \mathbf{0}_{N\times N} & K_{12} \\ K_{21} & 0 \end{pmatrix}
$$

with K_{12} of dimension $N \times 1$, and set the matrix

$$
K_1(t,x,\xi)H_2(t,x,\xi) - H_2(t,x,\xi)K_1(t,x,\xi) + H_1(t,x,\xi)
$$

to be the block form

$$
\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}
$$

with J_1 of dimension $N \times N$, then we can easily obtain

$$K_1(t,x,\xi) = i\langle\xi\rangle_a^{-2} \begin{pmatrix} \mathbf{0}_{N\times N} & -\sum_{j=1}^n B_j(t,x)\xi_j \\ \sum_{j=1}^n C_j(t,x)\xi_j & 0 \end{pmatrix} \in S^{-1}(\mathbb{R}^n)$$

and

$$J_1 = i\sum_{j=1}^n A_j(t,x)\xi_j, \quad J_2 = i\sum_{j=1}^n d_j(t,x)\xi_j.$$

Obviously, there is $M > 0$ such that when $|\xi| \geq M$, $I + K_1(t,x,\xi)$ is invertible. Hence, we can find a smooth symbol

$$\tilde{K}_1(t,x,\xi) = \begin{cases} K_1(t,x,\xi), & \text{when } |\xi| \geq M \\ 0, & \text{when } |\xi| \ll 1, \end{cases}$$

such that $I + \tilde{K}_1(t,x,\xi)$ is invertible for any $\xi \in \mathbb{R}^n$, and

$$U^{(1)} = \big(I + \tilde{K}_1(t,x,D_x)\big)U$$

satisfies

$$\partial_t U^{(1)} + H_2(t,x,D_x)U^{(1)} + H_1^{(1)}(t,x,D_x)U^{(1)} + H_0^{(1)}(t,x,D_x)U^{(1)} = F^{(1)}, \tag{2.5}$$

where $H_1^{(1)}(t,x,D_x) = \operatorname{diag}\big[\sum_{j=1}^n A_j(t,x)\partial_{x_j}, \sum_{j=1}^n d_j(t,x)\partial_{x_j}\big]$, and $H_0^{(1)}(t,x,D_x)$ belongs to $\Psi^0(\mathbb{R}^n)$. Next, if we define $U^{(2)} = \big(I + K_2(t,x,D_x)\big)U^{(1)}$ with $K_2 \in \Psi^{-2}(\mathbb{R}^n)$, then by the same procedure as above we can find K_2 such that $U^{(2)}$ satisfies a system with the same second and first order terms as in (2.5), its zero-th order term is decoupled, and $I + K_2(t,x,\xi)$ is invertible. In this way, we can decouple further terms by setting $U^{(j)} = \big(I + K_j(t,x,D_x)\big)U^{(j-1)}$ with $K_j \in \Psi^{-j}(\mathbb{R}^n)$ for any $j \geq 3$. For any fixed integer $m \in \mathbb{N}$, if we define $K(t,x,D_x) \in \Psi^{-1}(\mathbb{R}^n)$ by

$$I + K = (I + K_{m+1}) \cdot \cdots \cdot (I + K_2)(I + \tilde{K}_1)$$

then $V = \big(I + K(t,x,D_x)\big)U$ satisfies the following decoupled system modulo Ψ^{-m}:

$$\partial_t V + H_2(t,x,D_x)V + H_1^{(1)}(t,x,D_x)V + H_0(t,x,D_x)V + R_m(t,x,D_x)V = \mathcal{F}, \tag{2.6}$$

where $R_m \in \Psi^{-m}$, and $H_0(t,x,D_x) = \operatorname{diag}[H_0^{(1)}, H_0^{(2)}] \in \Psi^0(\mathbb{R}^n)$ with $H_0^{(1)}$ being $N \times N$ and $H_0^{(2)}$ scalar.

Remark 2.1.

(1) Obviously, the above procedure also works for the case that the second equation in (2.1) is replaced by a parabolic system.

(2) If (2.1) is a quasilinear coupled system, we can decouple it by using paradifferential operators (see Section 3.3).

3. Applications

3.1. *Finite speeds for the propagation of singularities in coupled systems.* As the first application of our decoupling idea, we shall show that hyperbolic-parabolic coupled systems still have finite speeds for the propagation of singularities as for the case of pure hyperbolic equations.

THEOREM 3.1. *Consider the Cauchy problem for the semilinear version of hyperbolic-parabolic coupled system* (2.1):

$$\left\{\begin{array}{l}\partial_t u+\sum_{j=1}^n A_j(t,x)\partial_{x_j}u+\sum_{j=1}^n B_j(t,x)\partial_{x_j}v=f(u,v)\\ \partial_t v-\sum_{j,k=1}^n a_{jk}(t,x)\partial^2_{x_jx_k}v+\sum_{j=1}^n C_j(t,x)\partial_{x_j}u+\sum_{j=1}^n d_j(t,x)\partial_{x_j}v=g(u,v)\\ t=0:\ u=u_0(x),\ v=v_0(x)\end{array}\right. \tag{3.1}$$

with all notation being the same as in (2.1). *For any open* $\omega\subseteq\mathbb{R}^n$, *let* $\Omega\subseteq[0,+\infty)\times\mathbb{R}^n$ *be the determinacy domain of* ω *with respect to the hyperbolic operator*

$$\partial_t+\sum_{j=1}^n A_j(t,x)\partial_{x_j}.$$

If the initial data satisfy

$$u_0\in H^s(\mathbb{R}^n)\cap C^\infty(\omega),\quad v_0\in H^{s+1}(\mathbb{R}^n)\cap C^\infty(\omega) \tag{3.2}$$

for a fixed $s>n/2$, *then there is* $T>0$ *such that*

$$(u,v)\in C^\infty\big(\Omega\cap\{0\le t<T\}\big). \tag{3.3}$$

PROOF. By employing the decoupling technique for the problem (3.1), it follows that $V(t,x)=(V_1,V_2)^T=(I+K(t,x,D_x))(u,v)^T$ satisfies the Cauchy problem

$$\left\{\begin{array}{l}\partial_t V_1+\sum_{j=1}^n A_j(t,x)\partial_{x_j}V_1=\mathcal{F}_1(V_1,V_2)\\ \partial_t V_2-\sum_{j,k=1}^n a_{jk}(t,x)\partial^2_{x_jx_k}V_2+\sum_{j=1}^n d_j(t,x)\partial_{x_j}V_2=\mathcal{F}_2(V_1,V_2)\\ t=0:\ V_1=V_1^{(0)}(x),\ V_2=V_2^{(0)}(x)\end{array}\right. \tag{3.4}$$

with $(V_1^{(0)}(x),V_2^{(0)}(x))^T=(I+K(0,x,D_x))(u_0,v_0)^T$. Under assumption (3.2), we have

$$V_1^{(0)}\in H^s(\mathbb{R}^n)\cap C^\infty(\omega),\quad V_2^{(0)}\in H^{s+1}(\mathbb{R}^n)\cap C^\infty(\omega). \tag{3.5}$$

Applying the classical theory of hyperbolic and parabolic equations to (3.4), we obtain the following estimate for the norm of (V_1,V_2) in $C([0,T],H^s(\mathbb{R}^n))\times C([0,T],H^{s+1}(\mathbb{R}^n))$:

$$\begin{aligned}&\|V_1(t,\cdot)\|^2_{H^s}+\|V_2(t,\cdot)\|^2_{H^{s+1}}\\ &\quad\le C_0\Big(\big\|V_1^{(0)}\big\|^2_{H^s}+\big\|V_2^{(0)}\big\|^2_{H^{s+1}}+\int_0^t\big(\|\mathcal{F}_1(V)(t_1,\cdot)\|^2_{H^s}+\|\mathcal{F}_2(V)(t_1,\cdot)\|^2_{H^s}\big)\,dt_1\Big)\end{aligned}$$

with a constant $C_0>0$, which easily implies the existence and uniqueness of the solution $V=(V_1,V_2)^T\in C\big([0,T],H^s(\mathbb{R}^n)\big)\times C\big([0,T],H^{s+1}(\mathbb{R}^n)\big)$ to (3.4) for certain $T>0$ by using a classical argument. From the regularity of parabolic equations, we also have

$$V_2\in L^2\big(0,T;H^{s+2}(\mathbb{R}^n)\big)\cap H^1\big(0,T;H^s(\mathbb{R}^n)\big). \tag{3.6}$$

Using (3.6) and applying the theory of hyperbolic equations (cf. [1, 4]) to problem (3.4) for V_1, we obtain

$$V_1\in C\big([0,T],H^{s+2}_{\rm loc}(\omega_t)\big)\cap C^1\big([0,T],H^{s+1}_{\rm loc}(\omega_t)\big) \tag{3.7}$$

where $\omega_t = \Omega \cap \{t = \text{const.}\}$. Denoting by $C_0^\infty(\overline{\Omega}_T)$ the set of smooth functions infinitely order vanishing on $\partial\Omega_T \setminus \{t = 0, T\}$, and letting $\chi \in C_0^\infty(\overline{\Omega}_T)$, from (3.4) we know that χV_2 satisfies

$$\begin{cases} \Big(\partial_t - \sum_{j,k=1}^n a_{jk}(t,x)\partial^2_{x_j x_k} + \sum_{j=1}^n d_j(t,x)\partial_{x_j}\Big)(\chi V_2) = \mathcal{F}_{2,\chi}(V_1, V_2) \\ (\chi V_2)(0,x) = \chi(0,x)V_2^{(0)}(x), \end{cases} \tag{3.8}$$

where $\mathcal{F}_{2,\chi}(V) = \chi\mathcal{F}_2(V) + [L,\chi]V_2$ belongs to $L^2(0,T; H^{s+1}(\mathbb{R}^n))$ by using (3.6), (3.7). Here, L denotes the parabolic operator appearing on the left-hand side of (3.8). Employing the classical theory of parabolic equations for (3.8), we deduce that

$$\chi V_2 \in L^2\big(0,T; H^{s+3}(\mathbb{R}^n)\big) \cap H^1(0,T; H^{s+1}(\mathbb{R}^n))$$

which implies

$$V_2 \in L^2\big(0,T; H^{s+3}_{\text{loc}}(\omega_t)\big) \cap H^1(0,T; H^{s+1}_{\text{loc}}(\omega_t)) \tag{3.9}$$

from the arbitrariness of χ. Thus, we can continue this process and eventually obtain the conclusion. ∎

Remark 3.1. It is not difficult to see that the above result holds as well for the case that the nonlinear function f on the right hand side of (3.1) also depends on the first order derivatives of v in space variables.

3.2. *Microlocal analysis in semilinear thermoelastic systems.* A typical example of (3.1) is the system of thermoelasticity ([11, 15]). Let us study the following semilinear problem of thermoelasticity in three space variables $x = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$\begin{cases} u_{tt} - (2\mu + \lambda)\nabla \operatorname{div} u + \mu \operatorname{rot}(\operatorname{rot} u) + \gamma_1 \nabla\theta = f(u, u_t, \nabla u, \nabla\theta, \theta) \\ \theta_t - \beta^2\Delta\theta + \gamma_2 \operatorname{div} u_t = g(u, u_t, \nabla u, \theta), \end{cases} \tag{3.10}$$

where all coefficients $(\mu, \lambda, \gamma_1, \gamma_2, \beta)$ are smooth functions of (t,x) with μ, $\lambda + \mu$ and β being positive, ∇u ($\nabla\theta$ resp.) denotes the gradient of u (θ resp.) with respect to the space variables (x_1, x_2, x_3), and nonlinear functions f and g are smooth in their arguments with $f(0) = g(0) = 0$. Here u represents the displacement, and $\theta = T_a - T_0$ is the temperature difference. The first result on the propagation of weak singularities in thermoelasticity was given in [16, 17]. Here, by employing the decoupling idea of Section 2, we can obtain rather deeper results on the regularity of solutions to (3.10).

Theorem 3.2.

(1) *For a fixed point $(x_0, \xi_0) \in T^*(\mathbb{R}^3) \setminus \{0\}$, suppose that $(u_0, \theta_0) \in H^s(\mathbb{R}^3) \cap H^r_{ml}(x_0, \xi_0)$, $u_1 \in H^{s-1}(\mathbb{R}^3) \cap H^{r-1}_{ml}(x_0, \xi_0)$ for any fixed $\frac{5}{2} < s \le r < 2s - \frac{5}{2}$. Let $\gamma(t) = \{(t, x(t); \tau(t), \xi(t)) \in T^*(\mathbb{R}^4) \setminus \{0\} : 0 \le t < T_0\}$ be a null bicharacteristic of $\partial_t^2 - (2\mu + \lambda)\Delta$ or $\partial_t^2 - \mu\Delta$ with $x(0) = x_0$ and $\xi(0) = \xi_0$. Then the problem* (3.10) *has a unique solution*:

$$\begin{cases} u \in C\big([0,T], H^s \cap H^r_{ml}(x(t), \xi(t))\big) \cap C^1\big([0,T], H^{s-1} \cap H^{r-1}_{ml}(x(t), \xi(t))\big) \\ \theta \in C\big([0,T], H^s \cap H^r_{ml}(x(t), \xi(t))\big), \end{cases} \tag{3.11}$$

where the notation $u_0 \in H^r_{ml}(x_0,\xi_0)$ means that there are a cutoff function $\varphi \in C_0^\infty(\mathbb{R}^3)$ with $\varphi(x_0)=1$, and a cone K in $\mathbb{R}^3\setminus\{0\}$ about the direction ξ_0 such that $(1+|\xi|^2)^{r/2}\times \chi_K(\xi)\widehat{\varphi u_0}(\xi)\in L^2(\mathbb{R}^3)$ ([1, 10]).

(2) Let $\{t=g_1(x)\}$ and $\{t=g_2(x)\}$ be the forward characteristic cones issuing from the origin for the operators $\partial_t^2-(2\mu+\lambda)\Delta$ and $\partial_t^2-\mu\Delta$ respectively, and denote by $\mathcal{C}_+$ the set $\{t>g_1(x)\}\setminus\{t=g_2(x)\}$. If $(u_0,\theta_0)\in H^s(\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3\setminus\{0\})$ and $u_1\in H^{s-1}(\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3\setminus\{0\})$ for $s>\frac{5}{2}$, then the local solution (u,θ) obtained above satisfies

$$\begin{cases} u\in C\big([0,T],H^{2s-5/2-\epsilon}_{\mathrm{loc}}(\mathcal{C}_+)\big)\cap C^1\big([0,T],H^{2s-7/2-\epsilon}_{\mathrm{loc}}(\mathcal{C}_+)\big)\\ \theta\in C([0,T],H^{2s-5/2-\epsilon}_{\mathrm{loc}}(\mathcal{C}_+))\end{cases}\tag{3.12}$$

for any $\epsilon>0$.

Sketch of the Proof. (1) As usual (see, e.g., [11, 15]), by taking the decomposition

$$u=u^p+u^s$$

with u^p being the potential part, $\operatorname{rot} u^p=0$, and u^s the solenoidal part, $\operatorname{div} u^s=0$, the system (3.10) is transformed into the following one for the unknown (u^p,u^s,θ):

$$\begin{cases} u^p_{tt}-\alpha^2\Delta u^p+\gamma_1\nabla\theta=f^p(u^p,u^p_t,\nabla u^p,u^s,u^s_t,\nabla u^s,\nabla\theta,\theta)\\ u^s_{tt}-a^2\Delta u^s=f^s(u^p,u^p_t,\nabla u^p,u^s,u^s_t,\nabla u^s,\nabla\theta,\theta)\\ \theta_t-\beta^2\Delta\theta+\gamma_2\operatorname{div}u^p_t=g(u^p,u^p_t,\nabla u^p,u^s,u^s_t,\nabla u^s,\theta)\end{cases}\tag{3.13}$$

with smooth functions f^p and f^s, where $\alpha=\sqrt{2\mu+\lambda}$ and $a=\sqrt{\mu}$ satisfying $\alpha>a$, and $\Delta=\sum_{j=1}^3\partial^2_{x_j}$. By setting

$$\begin{cases} u^p_+=(\partial_t+i\alpha\Lambda)u^p, \quad u^p_-=(\partial_t-i\alpha\Lambda)u^p\\ u^s_+=(\partial_t+ia\Lambda)u^s, \quad u^s_-=(\partial_t-ia\Lambda)u^s\end{cases}\tag{3.14}$$

with $\Lambda=(1-\Delta)^{1/2}$, and using an argument similar to Section 2, we conclude that there is $K(t,x,D_x)\in\Psi^{-1}(\mathbb{R}^3)$ such that

$$V=(V_1,V_2)^T:=\big(I+K(t,x,D_x)\big)(u^p_+,u^p_-,u^s_+,u^s_-,\theta)^T,$$

with V_1 denoting the first twelve components of V, satisfies the problem

$$\begin{cases} \partial_tV_1+A_1(t,x,D_x)V_1=B_0^{(1)}\cdot F_1(B_0^{(2)}\cdot V_1,B_1\cdot V_2)\\ \partial_tV_2-\beta^2\Delta V_2=B_0^{(3)}\cdot F_2(B_0^{(4)}\cdot V)\\ V_1(0,\cdot)\in H^{s-1}(R^3)\cap H^{r-1}_{ml}(x_0,\xi_0), \quad V_2(0,\cdot)\in H^s(R^3)\cap H^r_{ml}(x_0,\xi_0),\end{cases}\tag{3.15}$$

where $A_1(t,x,D_x)=\operatorname{diag}[-i\alpha\Lambda I,i\alpha\Lambda I,-ia\Lambda I,ia\Lambda I]$ with $I=I_{3\times3}$ being the 3×3 unit matrix, $B_0^{(k)}(t,x,D_x)\in\Psi^0(\mathbb{R}^3)$, $B_1(t,x,D_x)\in\Psi^1(\mathbb{R}^3)$, and $F_k(\cdot)$ are smooth in their arguments. By using the classical theory of hyperbolic problems and parabolic problems, we obtain a unique solution to (3.15):

$$\begin{cases} V_1\in C([0,T],H^{s-1}(\mathbb{R}^3))\\ V_2\in L^2(0,T;H^{s+1}(\mathbb{R}^3))\cap H^1(0,T;H^{s-1}(\mathbb{R}^3)).\end{cases}\tag{3.16}$$

Using (3.16) and the theory of hyperbolic equations ([1, 4]), we deduce that

$$V_1\in C\big([0,T],H^{s-1}(\mathbb{R}^3)\cap H^{\min(r-1,s)}_{ml}(x(t),\xi(t))\big).$$

If $s \le r \le s+1$, we conclude that

$$V_1 \in C\big([0,T], H^{s-1}(\mathbb{R}^3) \cap H^{r-1}_{ml}(x(t),\xi(t))\big), \tag{3.17}$$

which implies

$$u \in C\big([0,T], H^s \cap H^r_{ml}(x(t),\xi(t))\big) \cap C^1\big([0,T], H^{s-1} \cap H^{r-1}_{ml}(x(t),\xi(t))\big).$$

Using (3.17) and the theory of parabolic problems gives

$$V_2 \in L^2\big(0,T; H^{s+1} \cap H^{r+1}_{ml}(x(t),\xi(t))\big) \cap H^1\big(0,T; H^{s-1} \cap H^{r-1}_{ml}(x(t),\xi(t))\big)$$

which implies

$$\theta \in C\big([0,T], H^s \cap H^r_{ml}(x(t),\xi(t))\big).$$

When $r > s+1$, we can continue this process, and conclude (3.11) eventually.

(2) For any $(t_0, x(t_0)) \in \mathcal{C}_+$, and $\xi(t_0) \in \mathbb{R}^3 \setminus 0$, let $\tau(t_0) \in \mathbb{R}$ be such that $P_0 = (t_0, x(t_0), \tau(t_0), \xi(t_0))$ is a characteristic point for the hyperbolic operators $L_1 = \partial_t^2 - (2\mu+\lambda)\Delta$ or $L_2 = \partial_t^2 - \mu\Delta$. Denote by $\gamma(t) = \{(t, x(t), \tau(t), \xi(t)\} \subset T^*(\mathbb{R}^4) \setminus 0$ a null bicharacteristic of L_1 or L_2 passing through P_0. Obviously, the projection in (t,x)-space of $\gamma(t)$ intersects with $\{t=0\}$ at $x \neq 0$, where (u,θ) is smooth by using Theorem 3.1. Thus, by applying Theorem 3.2(1) we obtain

$$\begin{cases} u \in C\big([0,T], H^{2s-5/2-\epsilon}_{ml}(x(t),\xi(t))\big) \cap C^1\big([0,T], H^{2s-7/2-\epsilon}_{ml}(x(t),\xi(t))\big) \\ \theta \in C\big([0,T], H^{2s-5/2-\epsilon}_{ml}(x(t),\xi(t))\big) \end{cases}$$

for any $\epsilon > 0$, which is equivalent to the assertion (3.12) by using the arbitrariness of $(x(t_0), \xi(t_0))$, because the above result holds obviously for the case that $\gamma = \{(t, x(t), \tau(t), \xi(t))\}$ is a bicharacteristic of L_1 or L_2 while $(t_0, x(t_0), \tau(t_0), \xi(t_0))$ is not a characteristic point. ∎

3.3. *Propagation of singularities in viscous fluids.* In this subsection, we shall briefly explain how the decoupling idea of Section 2 can be applied to study the compressible Navier-Stokes equations with heat conduction by using paradifferential equations. It will be seen that this argument also works for general nonlinear hyperbolic-parabolic coupled equations. Denote by ρ, $u = (u_1, \dots, u_n)^T$, θ, $p = p(\rho,\theta)$ and $e = e(\rho,\theta)$ the fluid density, velocity, temperature, pressure and internal energy satisfying $e'_\theta(\rho,\theta)$, $k > 0$ is the thermal conduction coefficient, and λ, μ are the Lamè constants satisfying $\lambda+\mu > 0$ and $\mu > 0$. The motion of the compressible viscous fluids is governed by the following Navier-Stokes equations:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & t > 0,\ x \in \mathbb{R}^n \\ \partial_t(\rho u_i) + \operatorname{div}(\rho u u_i) + \partial_i p = \mu\Delta u_i + (\lambda+\mu)\partial_i(\operatorname{div} u), & i = 1,\dots,n \\ \partial_t(\rho e) + \operatorname{div}(\rho u e) + p \operatorname{div} u - k\Delta\theta = \frac{\mu}{2}(\partial_i u_j + \partial_j u_i)^2 + \lambda(\operatorname{div} u)^2. \end{cases} \tag{3.18}$$

We are going to study equations (3.18) with the initial data

$$(\rho, u, \theta)\big|_{t=0} = (\rho_0(x), u_0(x), \theta_0(x)). \tag{3.19}$$

First, we recall a classical result from [12, 13] as follows: For any given $s > \frac{n}{2} + 2$, if $(\rho_0, u_0, \theta_0) \in H^s(\mathbb{R}^n)$, then there is a unique local solution to (3.18), (3.19) with the properties:

$$\begin{cases} \rho \in C\big([0,T), H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T), H^{s-1}(\mathbb{R}^n)\big) \\ u, \theta \in C\big([0,T), H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T), H^{s-2}(\mathbb{R}^n)\big) \\ \nabla u, \nabla\theta \in L^2(0,T; H^s(\mathbb{R}^n)). \end{cases} \tag{3.20}$$

In order to decouple the system (3.18), we should use the paradifferential operators developed by Bony (cf. [2, 5]). Given a $R \gg 1$, and $0 < \epsilon_1 < \epsilon_2 \ll 1$, let $\chi(\theta,\eta) = \phi(\theta,\eta)s(\eta)$ with

$$s(\eta) = \begin{cases} 0, & |\eta| \le R \\ 1, & |\eta| \ge 2R, \end{cases} \qquad \phi(\theta,\eta) = \begin{cases} 1, & |\theta| \le \epsilon_1|\eta| \\ 0, & |\theta| \ge \epsilon_2|\eta|. \end{cases}$$

The paraproduct operator T_a is defined by

$$T_a u(x) = \mathcal{F}^{-1}\Big(\int \chi(\xi-\eta,\eta)\hat{a}(\xi-\eta)\hat{u}(\eta)\,d\eta\Big)$$

for any $a, u \in S'(\mathbb{R}^n)$. With (ρ, u, θ) being given in (3.20), the equations (3.18) can be paralinearized as follows:

$$\begin{cases} \partial_t\rho + T_\rho \operatorname{div} u + (T_u\cdot\nabla)\rho + T_{\operatorname{div} u}\rho + \displaystyle\sum_{j=1}^n T_{\partial_{x_j}\rho} u_j = r_\rho \\ \partial_t u - T_{\rho^{-1}}Lu + (T_u\cdot\nabla)u + T_{\rho^{-1}p'_\rho}\nabla\rho + T_{\rho^{-1}p'_\theta}\nabla\theta + T_a U = r_u \\ \partial_t\theta - T_{k/(\rho e'_\theta)}\Delta\theta + (T_u\cdot\nabla)\theta + T_e\partial u + T_b U = r_\theta, \end{cases} \tag{3.21}$$

where $Lu = \mu\Delta u + (\lambda+\mu)\nabla(\operatorname{div} u)$ is an elliptic system, $U = (\rho, u, \theta)^t$, $a, b \in C([0,T), H^{s-2}(\mathbb{R}^n))$ smoothly depend on $(U, \nabla U, \nabla^2 U)$, $T_u\cdot\nabla = \sum_{j=1}^n T_{u_j}\partial_j$,

$$T_e\partial u := T_{(p-\rho^2 e'_\rho - 2\lambda \operatorname{div} u)/(\rho e'_\theta)} \operatorname{div} u - 2\sum_{j,k=1}^n T_{\mu(\partial_j u_k + \partial_k u_j)/(\rho e'_\theta)}\partial_j u_k$$

and

$$r_\rho \in C\big([0,T), H^{2s-n/2-1}(\mathbb{R}^n)\big), \qquad r_u, r_\theta \in C\big([0,T), H^{2s-n/2-2}(\mathbb{R}^n)\big).$$

In a way similar to that in Section 2, we can find

$$K = \begin{pmatrix} 0 & -A_1^{(1)}A_2^{-1} \\ A_2^{-1}A_1^{(2)} & 0 \end{pmatrix}$$

with

$$A_2 = -\begin{pmatrix} T_{\rho^{-1}}L & 0 \\ 0 & T_{k/(\rho e'_\theta)}\Delta \end{pmatrix}, \quad A_1^{(1)} = (T_\rho \nabla^T, 0), \quad A_1^{(2)} = \begin{pmatrix} T_{\rho^{-1}p'_\rho}\nabla \\ 0 \end{pmatrix},$$

such that $V = (V_1, V_2)^t = (I+K)U$, with V_1 being scalar and V_2 being the last $(n+1)$ components of V, satisfies the weakly coupled system

$$\begin{cases} \partial_t V_1 + (u\cdot\nabla)V_1 + A_0 V + A_{-1}U = r_1 \\ \partial_t V_2 - \begin{pmatrix} T_{\rho^{-1}}L & 0 \\ 0 & T_{k/(\rho e'_\theta)}\Delta \end{pmatrix} V_2 + B_1 V_2 + B_0 V + B_{-1}U = r_2, \end{cases} \tag{3.22}$$

where

$$\begin{cases} r_1 \in C\big([0,T), H^{2s-n/2-1}(\mathbb{R}^n)\big), \quad r_2 \in C\big([0,T), H^{2s-n/2-2}(\mathbb{R}^n)\big) \\ B_1 \in C([0,T), \mathrm{Op}(\Sigma^1_{s-n/2-1})) \end{cases}$$

with $\mathrm{Op}(\Sigma^1_\sigma)(\mathbb{R}^n)$ being the set of paradifferential operators of order one in x-variables (cf. [2, 5]), and

$$\begin{cases} A_0, B_0 : C\big([0,T), H^q(\mathbb{R}^n)\big) \longrightarrow C\big([0,T), H^q(\mathbb{R}^n)\big) \\ A_{-1}, B_{-1} : C\big([0,T), H^q(\mathbb{R}^n)\big) \longrightarrow C\big([0,T), H^{q+1}(\mathbb{R}^n)\big) \end{cases}$$

are bounded for any $q \in \mathbb{R}$.

Remark 3.2. To see what V_2, the part of $U = (\rho, u, \theta)^T$ having smoothing effect, is, let us consider the equations of isothermal fluids, i.e. we do not need the equation for the balance of energy in (3.18). Then, by direct computation, we deduce that

$$\mu\Delta V_2 + (\lambda+\mu)\nabla(\operatorname{div} V_2) = \nabla F - \mu \operatorname{rot}(\operatorname{rot} u) + \text{“}\ldots\text{”},$$

where the dots represent some terms smoother than the first two terms on the right-hand side, and

$$F = (\lambda + 2\mu)\operatorname{div} u - p(\rho)$$

is the *effective viscous flux*, which was throughly studied by D. Hoff et al. ([8, 9]).

Similarly to Theorem 3.2(2), by using the theory of hyperbolic equations and a classical bootstrap argument for the equations (3.22), we can establish the following result:

Theorem 3.3. *For any fixed* $\frac{n}{2} + 2 < s \le r \le 2s - \frac{n}{2}$, *suppose that*

$$(\rho_0, u_0, \theta_0) \in H^s(\mathbb{R}^n) \cap H^r_{ml}(x_0, \xi_0)$$

with a fixed $(x_0, \xi_0) \in T^*(\mathbb{R}^n) \setminus 0$. *Let* $u \in C\big([0,T), H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T), H^{s-2}(\mathbb{R}^n)\big)$ *be the velocity component of the unique local solution of* (3.18), *and* $\gamma(t) = (t, x(t); \tau(t), \xi(t))$ $(0 \le t < T)$ *be a null bicharacteristic of* $\partial_t + u \cdot \nabla$ *with* $x(0) = x_0$ *and* $\xi(0) = \xi_0$. *Then the unique solution* (ρ, u, θ) *of* (3.18) *satisfies*

$$(\rho, u, \theta) \in C\big([0,T), H^s \cap H^r_{ml}(x(t), \xi(t))\big).$$

Remark 3.3.

(1) The detail of the proof for Theorem 3.2 can be found in [6, 19], and the detail of Theorem 3.3 can be found in [7]. Moreover, in [7], we have also obtained a result on the propagation of microlocal Hölder regularity for the Navier-Stokes equations (3.18).

(2) Recently, in [9], D. Hoff obtained an interesting result on the propagation of discontinuities in compressible Navier-Stokes equations of isothermal fluids in two space dimensions.

(3) Some other interesting applications of the decoupling idea can be developed as well. In [20], we have partially applied this idea to study the $L^p{-}L^q$ decay rate of solutions to Cauchy problems for linear thermoelastic systems with time-dependent coefficients. In [14], we use this decoupling idea to study the global existence of smooth solutions to the Navier-Stokes equations.

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