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# BASE CHANGE FOR PICARD–VESSIOT CLOSURES

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*In honor of Michael F. Singer's* $60^{th}$*, the old half-way point ad me'ah v'esrim*

**Abstract.** The differential automorphism group, over $F$, $\Pi_1(F_1)$ of the Picard–Vessiot closure $F_1$ of a differential field $F$ is a proalgebraic group over the field $C_F$ of constants of $F$, which is assumed to be algebraically closed of characteristic zero, and its category of $C_F$ modules is equivalent to the category of differential modules over $F$. We show how this group and the category equivalence behave under a differential extension $E \supset F$, where $C_E$ is also algebraically closed.

**1. Introduction.** Let $F = F_0$ be a differential field with $C = C_F$ field of constants, and assume $C$ is algebraically closed of characteristic zero. As general references for differential fields and their Picard–Vessiot extensions we cite [6] and [1]. A *Picard–Vessiot closure* of $F$ is a differential field extension of $F$ in which every Picard–Vessiot extension of $F$ embeds and which is itself a union of Picard–Vessiot extensions of $F$. Such extensions exist [3, §3] and are unique up to isomorphism; we denote such an extension by $F_1$ and refer to it as the Picard–Vessiot closure of $F$. For example, if $F = \mathbb{C}$ is the complex numbers, then $\mathbb{C}_1 = \mathbb{C}(x, \{e^{\alpha x} | \alpha \in \mathbb{C} - \{0\}\})$, where $x' = 1$ and $(e^{\alpha x})' = \alpha e^{\alpha x}$. To explain the notation $F_1$, we note that in general, $F_1$ has proper Picard–Vessiot extensions: for example, $\mathbb{C}_1(\log x)$, where $(\log x)' = 1/x$. This leads to a whole hierarchy of extensions $F_0 \subseteq F_1 \subseteq \ldots$ where $F_{n+1} = (F_n)_1$ and then to a complete Picard–Vessiot closure $F_\infty = \cup_n F_n$ [5]. Although we use the notation for the hierarchy, our focus in this paper, however, is just the initial closure $F_1$.

We let $\Pi_1(F)$ be the group $\mathrm{Aut}_F^{\mathrm{diff}}(F_1)$ of differential automorphisms of $F_1$ over $F$, which is known to be a pro-affine algebraic group; briefly, a proalgebraic group [2]. There is a Galois correspondence between all differential fields intermediate between $F$ and $F_1$ and the Zariski closed subgroups of $\Pi_1(F)$. There is also a hierarchy (actually inverse

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system) of groups $\Pi_n(F) = \mathrm{Aut}_F^{\mathrm{diff}}(F_n)$, but these are not in general proalgebraic groups [5]. Although all intermediate fields are fixed fields of subgroups, there is not as yet a simple description of the corresponding subgroups [5].

The group $\Pi_1(F)$ figures in a categorical equivalence property for $F$ in the same way as the Galois group of the separable closure figures in a categorical equivalence property for ordinary fields; see [4]. Specifically, we define an $F \cdot D$ module $V$ to be finite dimensional $F$ vector space together with an additive group endomorphism $D_V$ such that $D_V(av) = a'v + aD_V(v)$. Morphisms of $F \cdot D$ modules are $F$-linear maps that preserve the corresponding endomorphisms. We let $\mathrm{Mod}_D(F)$ denote the category of $F \cdot D$ modules. We define a $\Pi_1(F)$ module $X$ to be a finite dimensional $C$ vector space on which $\Pi_1(F)$ acts such that the associated function $\Pi_1(F) \to GL(X)$ is a morphism of proalgebraic groups. Morphisms of $\Pi_1(F)$ modules are $C$-linear maps which are $\Pi_1(F)$ equivariant. We let $\mathrm{Mod}_C(\Pi_1(F))$ denote the category of $\Pi_1(F)$ modules. These two categories are equivalent [4, Theorem 3.7]:

$$\mathrm{Mod}_D(F) \to \mathrm{Mod}_C(\Pi_1(F)) \text{ by } V \mapsto \mathrm{Hom}_{F \cdot D}(V, F_1)$$

is an (anti)equivalence of tensored categories.

This paper explores the change in the groups, and the corresponding change in the category equivalence, when the field $F$ is extended. Assume $E \supset F$ is a differential field extension, with $C_E$ algebraically closed. A corresponding morphism $\Pi_1(E) \to \Pi_1(F)$, such as would be present with the Galois groups of separable closures of ordinary fields, cannot exist: the domain is proalgebraic over $C_E$ and the range is proalgebraic over the subfield $C_F$. On the other hand, there is an obvious natural transformation $\mathrm{Mod}_D(F) \to \mathrm{Mod}_D(E)$ given by $V \mapsto E \otimes_F V$. By the category equivalence, this implies a natural transformation $\mathrm{Mod}_C(\Pi_1(F)) \to \mathrm{Mod}_{C_E}(\Pi_1(E))$. We will show that this latter is realized on the level of the underlying groups, namely via the proalgebraic group $\Pi$ over $C_E$ given by base change of $\Pi_1(F)$ to $C_E$. We show, using Galois theory, that this group is a quotient of $\Pi_1(E)$. Then we show that the natural transformation $\mathrm{Mod}_C(\Pi_1(F)) \to \mathrm{Mod}_{C_E}(\Pi_1(E))$ is given by the composition of $\mathrm{Mod}_C(\Pi_1(F)) \to \mathrm{Mod}_{C_E}(\Pi)$ by $X \mapsto C_E \otimes_C X$ with $\mathrm{Mod}_{C_E}(\Pi) \to \mathrm{Mod}_{C_E}(\Pi_1(E))$ coming from $\Pi_1(E) \to \Pi$.

**2. Base change and categorical equivalence.** Our first result, which mimics a similar fact for simple rings, shows that differential simplicity is preserved by certain base changes.

LEMMA 2.1. *Let $R$ be a simple differential $F$ algebra and $E \supset F$ a differential field extension generated by constants (i.e., $E = C_E \cdot F$). Assume that every constant of $R$ is in $C_F$. (This is always the case if $R$ is an affine $F$ algebra.) Then $R \otimes_F E$ is a simple differential $E$ algebra.*

*Proof.* Let $S$ be the ring compositum $C_E \cdot F$, so $S = \{\sum c_i f_i | c_i \in C_E, f_i \in F\}$. Note that $E$ is the quotient field of $S$, so that nonzero ideals of $R \otimes_F E$ have nonzero intersection with $R \otimes_F S$. Let $I \supseteq R \otimes_F S$ be a nonzero ideal, and suppose $\sum_1^n a_i \otimes b_i \in I$ with 1) all $b_i \in C_E$ and 2) $n$ minimal. (Note that this implies that $b_1, \dots b_n$ are linearly independent over $F$, and that no $a_i$ is zero.)

Let $J = \{\alpha \in R | \exists a_i(\alpha) \in R$ such that $\alpha \otimes b_1 + \sum_2^n a_i(\alpha) \otimes b_i \in I\}$.

We claim that $J$ is an ideal of $R$: for if $z = \alpha \otimes b_1 + \sum_2^n a_i(\alpha) \otimes b_i \in I$ and $w = \beta \otimes b_1 + \sum_2^n a_i(\beta) \otimes b_i \in I$ then $z + w = (\alpha + \beta) \otimes b_1 + \sum_2^n (a_i(\alpha) + a_i(\beta)) \otimes b_i \in I$ so that $\alpha + \beta \in J$. And if $r \in R$, then $rz = r\alpha \otimes b_1 + \sum_2^n ra_i(\alpha) \otimes b_i \in I$ which implies that $r\alpha \in J$. Finally, since the $b_i$ are constants, and $D(z) = D(\alpha) \otimes b_1 + \sum_2^n D(a_i(\alpha)) \otimes b_i$ is in $I$, we have. $D(\alpha) \in J$. Thus $J$ is a differential ideal of $R$, nonzero since $a_1 \in J$.

It follows that $1 \in J$ and hence that $I$ contains a element $u = 1 \otimes b_1 + \sum_2^n c_i \otimes b_i$. The linear independence of $b_1, \ldots, b_n$ over $F$ implies that $u \neq 0$. Now $D(u) = \sum_2^n D(c_i) \otimes b_i$ is also in $I$, and by minimality of $n$ must be 0. Again, since the $b_i$ are linearly independent over $F$, $D(c_2) = \cdots = D(c_n) = 0$. By assumption, the $c_i$ are constants of $F$, and in particular belong to $F$, so $u = 1 \otimes x$ where $x = b_1 + \sum_2^n c_i b_i$. Thus any nonzero differential ideal of $R \otimes_F S$ contains a nonzero element of the form $1 \otimes x$. Such elements are invertible in $R \otimes_F E$, which implies that this latter is simple. ∎

Because Picard–Vessiot closures are unions of Picard–Vessiot extensions, and these latter are quotient fields of differentially simple rings, we can use the lemma to get embeddings of Picard–Vessiot closures.

Corollary 2.2. *Let $E \supseteq F$ be a differential field extension with $C_E$ algebraically closed. Then there is an embedding $F_1 \to E_1$ compatible with the inclusion.*

*Proof.* Let $R$ be the subring of $F_1$ consisting of all elements satisfying a monic linear homogeneous differential equation over $F$. $R$ is a simple differential ring all of whose constants lie in $F$, being a direct limit of such [3]. First consider the field $FC_E$. By Lemma 2.1, $R \otimes_F FC_E$ is a simple $FC_E$ algebra, as well as a direct limit of affine such, and generated over $FC_E$ by solutions of differential equations. It follows that its quotient field $Q(R \otimes_F FC_E)$ is an infinite Picard–Vessiot extension of $FC_E$, which contains the quotient field $F_1$ of $R$, and which in turn is contained in $(FC_E)_1$. Thus it suffices to show that this latter embeds in $E_1$; in other words, we may assume $F \subseteq E$ is a no new constants extension. Now let $L$ be a monic linear homogeneous differential operator over $F$ and let $V_L$ be a full set of solutions of $L = 0$ in $E_1$. $V_L$ is a finite dimensional $C$ vector space, and since its elements satisfy $L = 0$ the $F$ space $\sum FD^i V_L$ is finite dimensional over $F$. It is clear that $F < V_L >$ is a Picard–Vessiot extension of $F$ for $L$ inside $E_1$. The compositum of all these for all $L$ is then a, hence by uniqueness the, Picard–Vessiot closure of $F$, contained in $E_1$. ∎

Now using the corollary we can show how the categorical equivalence works with base change.

Corollary 2.3. *Let $E \supseteq F$ be a differential field extension with $C_E$ algebraically closed. Let $V$ be an $F \cdot D$ module. Then*

$$C_E \otimes_C Hom_{F \cdot D}(V, F_1) = Hom_{E \cdot D}(E \otimes_F V, E_1).$$

*Proof.* By Corollary 2.2 we may assume $F_1 \subseteq E_1$. Then

$$\mathrm{Hom}_{E \cdot D}(E \otimes_F V, E_1) = \mathrm{Hom}_{F \cdot D}(V, E_1).$$

Consider any $F \cdot D$ homomorphism $f : V \to E_1$. Any element $y$ of $V$ satisfies some differential equation $L$ over $F$, say of order $n$. We have $f(y)$ in the solutions of $L=0$ inside $E_1$. The equation $L=0$ has a solution space $V_L \subset F_1$ which has a basis $x_1,\dots,x_n$ linearly independent over the constants $C_F$, and hence with nonzero Wronskian. This Wronskian is nonzero as an element of $E_1$, from which it follows that $x_1,\dots,x_n$ are linearly independent over $C_E$ as well. Thus $L^{-1}(0) \subset E_1$ is $C_E V_L$, and the multiplication map $C_E \otimes_C V_L \to C_E V_L$ is an isomorphism. Moreover, $f(y) \in C_E V_L$. Let $\mathcal{L}$ denote the set of all such equations $L=0$ which arise from the elements of $V$, and let $V_{\mathcal{L}}$ denote the sum of the $V_L$ for $L \in \mathcal{L}$. Then $f(V) \subset C_E V_{\mathcal{L}}$. Moreover, the Wronskian argument above implies that a $C$ basis for $V_{\mathcal{L}}$ is a $C_E$ basis for $C_E V_{\mathcal{L}}$, so that the multiplication map is an isomorphism. It follows that $\mathrm{Hom}_{F\cdot D}(V,E_1) = \mathrm{Hom}_{F\cdot D}(V, C_E \otimes_C V_{\mathcal{L}})$ and this latter is $C_E \otimes_C \mathrm{Hom}_{F\cdot D}(V, V_{\mathcal{L}})$. Now $V_{\mathcal{L}}$ is also an $F$ subspace of $F_1$ (it is the $\Pi_1(F)$ module generated by the sum of the images of $V$ in $F_1$) and as such finite dimensional ($\mathrm{Hom}_{F\cdot D}(V,F_1)$ has a finite $C$ basis [4, Proposition 3.6]). Since $C_E V_{\mathcal{L}} = C_E \otimes_C V_{\mathcal{L}}$, we then have $\mathrm{Hom}_{F\cdot D}(V, C_E \otimes_C V_{\mathcal{L}}) = C_e \otimes_C \mathrm{Hom}_{F\cdot D}(V, V_{\mathcal{L}}) = C_E \otimes_C \mathrm{Hom}_{F\cdot D}(V, F_1)$, which completes the proof. ∎

The homomorphism spaces of Corollary 2.3 are values of category equivalences

$$\mathrm{Mod}_D(F) \to \mathrm{Mod}_{C_F}(\Pi_1(F)) \text{ by } W \mapsto \mathrm{Hom}_{F\cdot D}(W, F_1)$$

$$\mathrm{Mod}_D(E) \to \mathrm{Mod}_{C_E}(\Pi_1(E)) \text{ by } V \mapsto \mathrm{Hom}_{F\cdot D}(V, E_1).$$

[4, Theorem 3.7]. Corollary 2.3 shows that the base change functor $W \mapsto E \otimes_F W$ from $\mathrm{Mod}_D(F) \to \mathrm{Mod}_D(E)$ carries the one equivalence to the other. To see this on the level over modules over groups, we need look at the effect of base change on the groups.

Thus we now record the map on absolute differential Galois groups arising from the inclusion $F_1 \subseteq E_1$. As $\Pi_1(E)$ is a proalgebraic group over $C_E$ and $\Pi_1(F)$ a group over $C_F$, we have to deal with change of scalars. For that, we return to the notation used in the proof of Corollary 2.2 and in particular to the infinite Picard–Vessiot intermediate field $Q(R \otimes_F FC_E)$ between $FC_E$ and $(FC_E)_1$. Let $\Pi$ denote the corresponding quotient group of $\Pi_1(FC_E)$.

Lemma 2.4.

$$\Pi = \Pi_1(F) \otimes_{C_F} C_E.$$

*Proof.* Let $\overline{(\cdot)}$ denote algebraic closure. We continue to use the notation of the proof of Corollary 2.2. By [1, Theorem 5.12] we have

$$\overline{F} \otimes_F R = \overline{F} \otimes_C C[\Pi_1(F)]$$

and

$$\overline{FC_E} \otimes_{FC_E} (R \otimes_F FC_E) = \overline{FC_E} \otimes_{C_E} C_E[\Pi].$$

The left hand side of the second equation above simplifies, by “canceling” the tensors over $FC_E$, to $\overline{FC_E} \otimes_F R$, then we insert a tensor over $\overline{F}$ to obtain $\overline{FC_E} \otimes_{\overline{F}} (\overline{F} \otimes_F R)$ which by the first equation above reduces to $\overline{FC_E} \otimes_C C[\Pi_1(F)]$; then we insert a tensor over $C_E$ to obtain $\overline{FC_E} \otimes_{C_E} (C_E \otimes_C C[\Pi_1(F)])$.

Returning to the second equation above, we conclude that

$$\overline{FC_E} \otimes_{C_E} (C_E \otimes_C C[\Pi_1(F)]) = \overline{FC_E} \otimes_{C_E} C_E[\Pi],$$

which implies the conclusion of the lemma. ■

The inclusion $FC_E \subseteq E$ induces a map of absolute Galois groups, which we now record:

COROLLARY 2.5. *Let $F \supseteq E$ be a no new constants extension. Then the embedding $F_1 \to E_1$ of Picard–Vessiot closures induces a morphism $\Pi_1(E) \to \Pi_1(F)$.*

*Proof.* The embedding comes from Corollary 2.2. We assume it is an inclusion, and continue to use the notation of that Corollary. Let $\sigma \in \Pi_1(E)$. For any monic homogeneous linear differential operator $L$ over $F$, $\sigma(V_L) = V_L$, where $V_L$ is the set of solutions of $L = 0$ in $E_1$. It follows that $\sigma$ preserves $F < V_L >$ and hence the compositum of all such over of all $L$, which we have identified as $F_1$. Thus the restriction map $\sigma \mapsto \sigma|F_1$ is the desired morphism. ■

Combining Lemma 2.4 with Corollary 2.5 we obtain a homomorphism regardless of constant extensions:

PROPOSITION 2.6. *Let $E \supseteq F$ be an extension of differential fields with $C_E$ algebraically closed. Then there is an induced morphism*

$$\Pi_1(E) \to \Pi_1(F)_{C_E}$$

*of absolute Galois groups, where*

$$C_E[\Pi_1(F)_{C_E}] = C_E \otimes_C C[\Pi_1(F)].$$

*Proof.* By Corollary 2.5 we have a morphism $\Pi_1(E) \to \Pi_1(FC_E)$ coming from restriction. We have a morphism $\Pi_1(FC_E) \to \Pi$, also arising from restriction, and then the description of $\Pi$ in Lemma 2.4 concludes the proof. ■

The morphism of absolute Galois groups in Proposition 2.6 determines a functor $\mathrm{Mod}_{C_F}(\Pi_1(F)) \to \mathrm{Mod}_{C_E}(\Pi_1(E))$: if $V$ is a $\Pi_1(F)$-module, then $C_E \otimes_C V$ is a $\Pi_1(F)_{C_E}$-module, and then the morphism $\Pi_1(E) \to \Pi_1(F)_{C_E}$ of $C_E$ groups means $C_E \otimes_C V$ can be considered as a module over $\Pi_1(E)$.

We combine this functor of group categories with the previously defined functor of differential module categories and obtain the following commutative diagram:

THEOREM 2.7. *Let $E \supseteq F$ be an extension of differential fields with $C_E$ algebraically closed. Then there is a commutative diagram:*

$$\begin{tikzcd} Mod_D(F) \arrow[r, "U_F"] \arrow[d, "\Phi"] & Mod_{C_F}(\Pi_1(F)) \arrow[d, "\Psi"] \\ Mod_D(E) \arrow[r, "U_E"] & Mod_{C_E}(\Pi_1(E)) \end{tikzcd}$$

*where $U_F(V) = Hom_{F.D}(W, F_1)$, $U_E(W) = Hom_{E.D}(W, E_1)$, $\Phi(V) = E \otimes_F V$ and $\Psi(X) = C_E \otimes_C X$ as a $\Pi_1(E)$ module.*

*Proof.* For an $F \cdot D$ module $V$, we have $\Psi(U_F(V)) = C_E \otimes \mathrm{Hom}_{F \cdot D}(V, F_1)$ (as a $\Pi_1(E)$ module) and $U_E(\Phi(V) = \mathrm{Hom}_{E \cdot D}(E \otimes_F V, E_1)$. Corollary 2.3 shows the spaces are isomorphic. ∎

## References


[1] A. Magid, *Lectures on Differential Galois Theory*, University Lecture Series 7, American Mathematical Society, Providence RI, 1997 (second printing with corrections).

[2] A. Magid, *The Picard–Vessiot antiderivative closure*, J. Algebra 244 (2001), 1–18.

[3] A. Magid, *The Picard–Vessiot closure in differential Galois theory*, in: Differential Galois Theory, Banach Center Publications Publications 58, Polish Academy of Sciences, Warszawa, 2002, 151–164.

[4] A. Magid, *Universal covers and category theory in polynomial and differential Galois theory*, Fields Institute Communications 43 (2004), 373–383.

[5] A. Magid, *The Complete Picard–Vessiot closure* in: L. Guo and W. Sit (eds.), Differential Algebra and Related Topics, World Scientific, Hackensack NJ, to appear.

[6] M. van der Put and M. Singer, *Galois Theory of Linear Differential Equations*, Springer, New York, 2003.