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LEBESGUE TYPE POINTS IN STRONG $(\mathbf{C},\boldsymbol{\alpha})$ APPROXIMATION OF FOURIER SERIES

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Abstract. We present an estimation of the $H^{q,\alpha}_{k_0,k_r}f$ and $H^{\phi,\alpha}_{\lambda,u}f$ means as approximation versions of the Totik type generalization (see [5], [6]) of the result of G. H. Hardy, J. E. Littlewood. Some corollaries on the norm approximation are also given.

1. Introduction. Let L^p $(1<p<\infty)$ [resp. C] be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with p-th power [continuous] over $Q=[-\pi,\pi]$ and let $X=X^p$ where $X^p=L^p$ when $1<p<\infty$ or $X^p=C$ when $p=\infty$. Let us define the norm of $f\in X^p$ as

$$\|f\|_{X^p}=\|f(x)\|_{X^p}=\begin{cases}\left(\int_Q |f(x)|^p\,dx\right)^{1/p} & \text{when } 1<p<\infty,\\ \sup_{x\in Q}|f(x)| & \text{when } p=\infty.\end{cases}$$

Consider the trigonometric Fourier series

$$Sf(x)=\frac{a_o(f)}{2}+\sum_{k=0}^{\infty}\big(a_k(f)\cos kx+b_k(f)\sin kx\big)$$

and denote by S_kf and $\sigma_k^\alpha f$ the partial sums and the (C,α) means of Sf respectively. Let

$$H^{q,\alpha}_{k_0,k_r}f(x):=\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}|\sigma^\alpha_{k_\nu}f(x)-f(x)|^q\Big\}^{1/q},\quad q>0,\ \alpha>-1,$$

where $0<k_0<k_1<k_2<\ldots<k_r$, and

$$H^{\phi,\alpha}_{\lambda,u}f(x):=\sum_{\nu=0}^{\infty}\lambda_\nu(u)\phi(|\sigma^\alpha_\nu f(x)-f(x)|),\quad \alpha>-1,$$

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where (λ_ν) is a sequence of positive functions defined on the set $U \subseteq \mathbf{R}$ having at least one limit point u_0 (possibly $u_0 = \infty$) and a function $\phi : [0, \infty) \to \mathbf{R}$.

As a measure of approximation by the above quantities we use the pointwise characteristic

$$w_x f(\delta)_p := \Big\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \Big\}^{1/p}, \quad \text{where } \varphi_x(t) := f(x+t) + f(x-t) - 2f(x),$$

constructed on the base of definition of Lebesgue points (L^p-points) (cf. [1]).

We can observe that with $\widetilde{p} > p$ for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\| w_{\cdot} f(\delta)_p \|_{X^{\widetilde{p}}} \le \omega_{X^{\widetilde{p}}} f(\delta),$$

where $\omega_X f$ is the modulus of smoothness of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0 < |h| \le \delta} \| \varphi_{\cdot}(h) \|_X .$$

It is well-known that the $H^{q,\alpha}_{0,r} f(x)$-means tend to 0 (as $r \to \infty$ with $\alpha \in (-1/2, 0)$) at the L^p-points of $f \in L^p$ ($\frac{1}{1+\alpha} < p \le \infty$). In [1] this fact was proved by G. H. Hardy, J. E. Littlewood for $\alpha = 0$ as a generalization of the Fejér classical result on the convergence of the $(C, 1)$-means of Fourier series and in [3] these means were estimated for $\alpha < 0$ in terms of $w_x f(\delta)_p$. Here we present estimations of the $H^{q,\alpha}_{k_0,k_r} f(x)$-and$H^{\phi,\alpha}_{\lambda,u} f(x)$-means as approximation versions of the Totik type (see [5], [6]) generalization of the result of G. H. Hardy and J. E. Littlewood with the (C, α)-means $\sigma_k^\alpha f$ instead of Sf. We also give some corollaries on the norm approximation.

By K we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same at each occurrence.

2. Statement of results. Let us consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \le w_x(\delta_1) + w_x(\delta_2)$ for any $0 \le \delta_1 \le \delta_2 \le \delta_1 + \delta_2$ and let

$$L^p(w_x)^\star = \Big\{ f \in L^p : \Big[\frac{1}{\delta} \int_0^\delta |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p \, dt \Big]^{1/p} \le w_x(\gamma) \ \text{ and } w_x f(\delta)_p \le w_x(\delta) \Big\}.$$

In the same way let

$$L^p(\omega) = \{ f \in L^p : \omega_{L^p} f(\delta) \le \omega(\delta), \ \text{with a modulus of continuity } \omega \}.$$

We can now formulate our main results and begin with the following theorems:

THEOREM 1. *If $f \in L^p(w_x)^\star$ $(1 < p \le 2)$ and $0 \le k_0 < k_1 < k_2 < \ldots < k_r$ then, for $\alpha \in (-\frac{1}{q}, 0)$ where $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$H^{\alpha,q'}_{k_0,k_r} f(x) \le K \Big\{ \Big(\frac{r+1}{k_r+1} \Big)^\alpha w_x \Big(\frac{\pi}{k_0+1} \Big) + \frac{1}{k_0+1} \sum_{k=0}^{k_0} w_x \Big(\frac{\pi}{k+1} \Big) \Big\},$$

with $q' \in (0, q]$.

Hence, using Lemma 2, we can immediately obtain

THEOREM 2. *If* $f \in L^p(\omega)$ $(1 < p \le 2)$ *and* $0 \le k_0 < k_1 < k_2 < \ldots < k_r$*, then, for* $\alpha \in (-\frac{1}{q}, 0)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, we have*

$$\left\| H^{\alpha,q'}_{k_0,k_r} f(\cdot) \right\|_{L^p} \le K \Big\{ \Big(\frac{r+1}{k_r+1} \Big)^{\alpha} \omega\Big(\frac{\pi}{k_0+1} \Big) + \frac{1}{k_0+1} \sum_{k=0}^{k_0} \omega\Big(\frac{\pi}{k+1} \Big) \Big\},$$

where $q' \in (0, q]$.

Writing for $\alpha \in (-\frac{1}{2}, 0)$

$\Phi^{\alpha} = \{ \phi : \phi(0) = 0,\ \phi \nearrow,\ \phi(2u) \le K_0 \phi(u)$ for $u \in (0, +\infty),$ with $K_0 < 2^{-1-1/\alpha} \}$

and, as in the Leindler monograph [2, p. 15],

$$\Lambda_s(N_m) = \Bigg\{ (\lambda_\nu) : \Bigg(\frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} (\lambda_\nu)^s \Bigg)^{1/s} \le K \Bigg(\frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \Bigg) \text{ for } s \ge 1, \text{ and } N_m < N_{m+1},\ N_0 = 0,\ N_{-1} = -1 \Bigg\},$$

we can formulate the next theorem on the base of the previous one and Lemma 3.

THEOREM 3. *If* $f \in L^p(w_x)^\star$ $(1 < p \le 2)$, $(\lambda_\nu) \in \Lambda_s(N_m)$, $\alpha \in (-1 + \frac{1}{p}, 0)$ *and* $\phi \in \Phi^{\alpha}$*, then*

$$H^{\phi,\alpha}_{\lambda,u} f(x) \le K \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu(u) \phi\Bigg(\frac{1}{N_{m-2}+1} \sum_{k=0}^{N_{m-2}} w_x\Big(\frac{\pi}{k+1} \Big) \Bigg),$$

for any $u \in U$.

By the same proof using Lemmas 2 and 4 we can obtain also the following theorem and corollary:

THEOREM 4. *If* $f \in L^p(\omega)$ $(1 < p \le 2)$*, for* $(\lambda_\nu) \in \Lambda_s(N_m)$, $\alpha \in (-1 + \frac{1}{p}, 0)$ *and* $\phi \in \Phi^{\alpha}$*, then*

$$\left\| H^{\phi,\alpha}_{\lambda,u} f(\cdot) \right\|_{L^p} \le K \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu(u) \phi\Bigg(\frac{1}{N_{m-2}+1} \sum_{k=0}^{N_{m-2}} \omega\Big(\frac{\pi}{k+1} \Big) \Bigg),$$

for any $u \in U$.

COROLLARY 1. *If we additionally assume that* $\lim_{u \to u_0} \lambda_\nu(u) = 0$ *for all* ν *and that* $\sum_{\nu}^{\infty} \lambda_\nu(u)$ *converges, then*

$$\lim_{u \to u_0} H^{\phi,\alpha}_{\lambda,u} f(x) = 0$$

at every L^p*-point* x *of the function* f *and*

$$\lim_{u \to u_0} \left\| H^{\phi,\alpha}_{\lambda,u} f(\cdot) \right\|_{L^p} = 0.$$

3. Auxiliary results. With the notation

$$\Phi_x f(\delta, \gamma) := \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} \varphi_x(t)\, dt, \quad w_x f(\delta, \gamma)_p := \Big[\frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} |\varphi_x(t)|^p\, dt \Big]^{1/p}$$

we will need

LEMMA 1. *If* $f \in L^p(w_x)^\star$ $(p \geq 1)$, *then*

$$|\Phi_x f(\delta,\gamma)| \leq w_x f(\delta,\gamma)_p \leq w_x(\delta) + w_x(\gamma)$$

for any positive γ, δ.

Proof. The first inequality is evident and we only prove the second one here.

If $f \in L^p(w_x)^\star$, then

$$\left[\frac{1}{\delta}\int_0^\delta |\varphi_x(t)|^p\,dt\right]^{1/p} \leq w_x(\delta)$$

whence

$$\begin{aligned}&\left[\frac{1}{\delta}\int_\gamma^{\gamma+\delta} |\varphi_x(t)|^p\,dt\right]^{1/p} - w_x(\delta)\\ &\leq \left[\frac{1}{\delta}\int_0^\delta |\varphi_x(t+\gamma)|^p\,dt\right]^{1/p} - \left[\frac{1}{\delta}\int_0^\delta |\varphi_x(t)|^p\,dt\right]^{1/p}\\ &\leq \left[\frac{1}{\delta}\int_0^\delta |\varphi_x(t+\gamma)-\varphi_x(t)|^p\,dt\right]^{1/p}.\end{aligned}$$

By the assumption

$$\left[\frac{1}{\delta}\int_\gamma^{\gamma+\delta} |\varphi_x(t)|^p\,dt\right]^{1/p} - w_x(\delta) \leq w_x(\gamma)$$

and our relation follows. ∎

Analogously, we can give inequalities for norms.

LEMMA 2. *If* $f \in L^p(\omega)$ $(p \geq 1)$, *then*

$$\|\Phi_. f(\delta,\gamma)\|_{L^p} \leq \|w_. f(\delta,\gamma)_p\|_{L^p} \leq 2\omega(\delta) + 2\omega(\gamma)$$

and

$$\left\|\left[\frac{1}{\delta}\int_0^\delta |\varphi_.(t)-\varphi_.(t\pm\gamma)|^p\,dt\right]^{1/p}\right\|_{L^p} \leq 2\omega(\gamma)$$

for any positive γ, δ.

Proof. If $f \in L^p(\omega)$, then, by the monotonicity of the norm as a functional and by the above lemma,

$$\|\Phi_. f(\delta,\gamma)\|_{L^p} \leq \left\|w_. f(\delta,\gamma)_p\right\|_{L^p},$$

and consequently

$$\begin{aligned}\left\|w_. f(\delta,\gamma)_p\right\|_{L^p} &= \left\{\int_{-\pi}^{\pi}\left[\frac{1}{\delta}\int_\gamma^{\gamma+\delta}|\varphi_x(t)|^p\,dt\right]dx\right\}^{1/p} = \left\{\frac{1}{\delta}\int_\gamma^{\gamma+\delta}\left[\int_{-\pi}^{\pi}|\varphi_x(t)|^p\,dx\right]dt\right\}^{1/p}\\ &\leq \left\{\frac{1}{\delta}\int_\gamma^{\gamma+\delta}[2\omega_{L^p} f(t)]^p\,dt\right\}^{1/p} \leq 2\omega_{L^p} f(\delta+\gamma),\end{aligned}$$

whence our first result follows.

For the next one we will change the order of integration. Then

$$\Big\{\frac{1}{\delta}\int_0^\delta\Big[\int_{-\pi}^{\pi}|\varphi_x(t)-\varphi_x(t\pm\gamma)|^p\,dx\Big]\,dt\Big\}^{1/p}$$
$$\le\Big\{\frac{1}{\delta}\int_0^\delta\Big[\int_{-\pi}^{\pi}(|f(x+t)-f(x+t\pm\gamma)|+|f(x-t)-f(x-t\mp\gamma)|)^p\,dx\Big]\,dt\Big\}^{1/p}$$
$$\le\Big\{\frac{1}{\delta}\int_0^\delta[2\omega_{L^p}f(\gamma)]^p\,dt\Big\}^{1/p}=2\omega_{L^p}f(\gamma)$$

and thus our proof is complete. ■

In the proof of our last two theorems we will use the next fundamental lemmas.

LEMMA 3. *If* $f\in L^p(w_x)^\star$ $(1<p\le 2)$, $\phi\in\Phi^\alpha$ $(-1+\frac{1}{p}<\alpha<0)$ *and* $\lambda_\nu(m)=\frac{1}{N_m+1}$ *for* $\nu=N_{m-2}+1,N_{m-2}+2,\ldots,N_m$, *and* $\lambda_\nu(m)=0$ *otherwise, then*

$$H^{\phi,\alpha}_{\lambda,m}f(x)\le K\phi\bigg(\frac{1}{N_{m-2}+1}\sum_{k=0}^{N_{m-2}}w_x\Big(\frac{\pi}{k+1}\Big)\bigg),$$

where $m=0,1,2,\ldots.$

Proof. If $w_x(\delta)\equiv 0$ then f is a constant function and our inequality is true. Thus we can assume that $w_x(\delta)>0$ for $\delta>0$.

We use the following notation with integers $\mu>0$ and ν:

$$\Omega_x(N_{m-2})=\frac{1}{N_{m-2}+1}\sum_{k=0}^{N_{m-2}}w_x\Big(\frac{\pi}{k+1}\Big),$$
$$\Delta_\mu=\{\nu\in[N_{m-2}+1,N_m]:|\sigma^\alpha_\nu f(x)-f(x)|\ge\mu\Omega_x(N_{m-2})\},$$
$$\Gamma_\mu=\{\nu\in[N_{m-2}+1,N_m]:(\mu-1)\Omega_x(N_{m-2})\le|\sigma^\alpha_\nu f(x)-f(x)|\le\mu\Omega_x(N_{m-2})\},$$
$$\Theta=\{\mu\in\mathbb{N}:\Gamma_\mu\ne\varnothing\}.$$

Then

$$H^{\phi,\alpha}_{\lambda,m}f(x)\le\frac{1}{N_m+1}\sum_{\mu\in\Theta}\sum_{\nu\in\Gamma_\mu}\phi(|\sigma^\alpha_\nu f(x)-f(x)|)\le\frac{1}{N_m+1}\sum_{\mu\in\Theta}\sum_{\nu\in\Gamma_\mu}\phi(\mu\Omega_x(N_{m-2}))$$
$$=\frac{1}{N_m+1}\sum_{\mu\in\Theta}|\Gamma_\mu|\phi(\mu\Omega_x(N_{m-2}))\le\frac{1}{N_m+1}\sum_{\mu\in\Theta}|\Delta_{\mu-1}|\phi(\mu\Omega_x(N_{m-2})).$$

Using Theorem 1 with $q'=1$ and $k_\nu\in\Delta_{\mu-1}$ we can estimate

$$(\mu+1)\Omega_x(N_{m-2})\le\frac{1}{|\Delta_{\mu-1}|}\sum_{\nu\in\Delta_{\mu-1}}|\sigma^\alpha_\nu f(x)-f(x)|$$
$$\le K\Big(\frac{|\Delta_{\mu-1}|^\alpha}{(N_m+1)^\alpha}+1\Big)\Omega_x(N_{m-2})\le 2K\Big(\frac{|\Delta_{\mu-1}|}{N_m+1}\Big)^\alpha\Omega_x(N_{m-2}),$$

whence $|\Delta_{\mu-1}| \le (N_m+1)(\frac{\mu-1}{2K})^{1/\alpha}$ and therefore

$$H^{\phi,\alpha}_{\lambda,m} f(x) \le \frac{1}{N_m+1}\sum_{\mu\in\Theta} N_m\Big(\frac{\mu-1}{K}\Big)^{1/\alpha}\phi(\mu\Omega_x(N_{m-2}))$$
$$\le K\sum_{\mu\in\Theta}\Big(\frac{\mu-1}{K}\Big)^{1/\alpha}\phi(\mu\Omega_x(N_{m-2})).$$

Since $\phi\in\Phi^\alpha$ we have

$$\begin{aligned}
H^{\phi,\alpha}_{\lambda,m} f(x) &\le K\Big\{\phi(\Omega_x(N_{m-2})) + \sum_{n=0}^{\infty}\sum_{\mu=2^n+1}^{2^{n+1}}\Big(\frac{\mu-1}{K}\Big)^{1/\alpha}\phi(\mu\Omega_x(N_{m-2}))\Big\}\\
&\le K\Big\{\phi(\Omega_x(N_{m-2})) + \sum_{n=0}^{\infty}\sum_{\mu=2^n+1}^{2^{n+1}}\Big(\frac{2^n}{K}\Big)^{1/\alpha}\phi(2^{n+1}\Omega_x(N_{m-2}))\Big\}\\
&\le K\phi(\Omega_x(N_{m-2})) + K\sum_{n=0}^{\infty}2^n 2^{n/\alpha}K_0\phi(2^n\Omega_x(N_{m-2}))\\
&\le K\phi(\Omega_x(N_{m-2})) + K\sum_{n=0}^{\infty}2^{n(1+1/\alpha)}K_0^{n+1}\phi(\Omega_x(N_{m-2}))\\
&\le K_0K\phi(\Omega_x(N_{m-2}))
\end{aligned}$$

and our proof is complete. ∎

LEMMA 4. *If $f\in L^p(\omega)$ $(1<p\le 2)$, $\phi\in\Phi^\alpha$ $(-1+\frac{1}{p}<\alpha<0)$ and $\lambda_\nu(m)=\frac{1}{N_m+1}$ for $\nu=N_{m-2}+1, N_{m-2}+2,\ldots,N_m$ and $\lambda_\nu(m)=0$ otherwise, then*

$$\big\|H^{\phi,\alpha}_{\lambda,m} f(\cdot)\big\|_{L^p} \le K\phi\Big(\frac{1}{N_{m-2}+1}\sum_{k=0}^{N_{m-2}}\omega\Big(\frac{\pi}{k+1}\Big)\Big),$$

where $m=0,1,2,\ldots$.

Proof. The proof is similar to the above one, the only difference is that we have to use Theorem 2 instead of Theorem 1. ∎

4. Proofs of the results.

We only prove Theorems 1 and 3.

4.1. Proof of Theorem 1.

Let as usual

$$H^{q,\alpha}_{k_0k_r} f(x) = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_0^\pi \varphi_x(t)K^\alpha_{k_\nu}(t)\,dt\Big|^q\Big\}^{1/q},$$

where $K^\alpha_{k_\nu}(t)=\frac{1}{A^\alpha_{k_\nu}}\sum_{k=0}^{k_\nu}A^{\alpha-1}_{k_\nu-k}\frac{\sin((2k+1)t/2)}{2\sin t/2}$. By the known notation and estimates ([7, Vol. I, pp. 94–95])

$$\begin{aligned}
K^\alpha_k(t) &= \frac{1}{A^\alpha_k}\frac{\sin[(k+(1+\alpha)/2)t-\pi\alpha/2]}{(2\sin t/2)^{1+\alpha}} + \frac{\alpha}{k+1}\frac{M}{(2\sin t/2)^2}\\
&= K^\alpha_{k,1}(t)+K^\alpha_{k,2}(t) \quad\text{with } |M|\le 1,
\end{aligned}$$
$$|K^\alpha_k(t)|\le 2k,\qquad |K^\alpha_k(t)|\le Kk^{-\alpha}t^{-\alpha-1}$$

$$\text{and } |K_{k,1}^{\alpha}(t)| \le K k^{-\alpha} t^{-\alpha-1},\ |K_{k,2}^{\alpha}(t)| \le K(k+1)^{-1} t^{-2}$$
$$\text{for } \alpha \in (-1,1),\ t \in [-\pi,\pi],$$

we can write

$$H_{k_0 k_r}^{q,\alpha} f(x) \le A_r + B_r + C_r + D_r,$$

where

$$A_r = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_0^{2\delta_\nu} \varphi_x(t) K_{k_\nu}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q},$$
$$B_r = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q},$$
$$C_r = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi} \varphi_x(t) K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q},$$
$$D_r = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi} \varphi_x(t) K_{k_\nu,2}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q},$$

with $\delta_\nu = \delta_{\nu,\alpha} = \frac{\pi}{k_\nu+(1+\alpha)/2}$, and $\gamma_r = \frac{\pi}{r+(1+\alpha)/2}$ $(\nu = 0,1,2,\ldots,r)$.

Since $|K_{k_\nu}^{\alpha}(t)| \le 2k_\nu$, for $\nu = 0,1,2,\ldots,r$, we have

$$A_r \le \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{2k_\nu}{\pi}\int_0^{2\delta_\nu} |\varphi_x(t)|\,dt\Big]^q\Big\}^{1/q} \le 4\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}[w_x f(\delta_\nu)_1]^q\Big\}^{1/q} \le 4 w_x(\delta_0).$$

The terms B_r, C_r and D_r will be estimated by the Totik method [6].

At the beginning we divide the sum in the term B_r into two parts with the index ν_0 $(0 \le \nu_0 \le r)$ which is defined as follows: $\nu_0 = 0$ in the case $r \le k_0$ and $\nu_0 = \nu > 0$ if $k_{\nu-1} < r \le k_\nu$:

$$\begin{aligned}
B_r &= \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\le \Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q} + \Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\le \Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\frac{k_\nu+1}{\pi}\int_{2\gamma_r}^{2\delta_\nu} \Big|\varphi_x(t)\Big|\,dt\Big]^q\Big\}^{1/q} + \Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\quad + \Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r} \varphi_x(t) K_{k_\nu,2}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\le 2\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}[w_x f(2\delta_\nu)_1]^q\Big\}^{1/q} + B_{r,1} + B_{r,2} \le 4 w_x(\delta_0) + B_{r,1} + B_{r,2}\,.
\end{aligned}$$

Next, the term $B_{r,1}$ is divided into three parts, namely

$$\begin{aligned}
B_{r,1} &= \Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}\varphi_x(t)K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&= \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\Big(\int_{2\delta_\nu}^{2\gamma_r}+\int_{\delta_\nu}^{2\gamma_r-\delta_\nu}+\int_{2\gamma_r-\delta_\nu}^{2\gamma_r}-\int_{\delta_\nu}^{2\delta_\nu}\Big)\varphi_x(t)K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\le B_{r,1}^1+B_{r,1}^2+B_{r,1}^3.
\end{aligned}$$

By a simple calculation

$$\begin{aligned}
B_{r,1}^1 &= \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\Big(\int_{2\delta_\nu}^{2\gamma_r}+\int_{\delta_\nu}^{2\gamma_r-\delta_\nu}\Big)\varphi_x(t)K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&= \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}(\varphi_x(t)K_{k_\nu,1}^{\alpha}(t)+\varphi_x(t-\delta_\nu)K_{k_\nu,1}^{\alpha}(t-\delta_\nu))\,dt\Big|^q\Big\}^{1/q}\\
&\le \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}(\varphi_x(t)-\varphi_x(t-\delta_\nu))K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\quad+\frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}\varphi_x(t-\delta_\nu)(K_{k_\nu,1}^{\alpha}(t)+K_{k_\nu,1}^{\alpha}(t-\delta_\nu))\,dt\Big|^q\Big\}^{1/q}.
\end{aligned}$$

Using the inequality

$$\Big|\Big(2\sin\frac{t}{2}\Big)^{-1-\alpha}-\Big(2\sin\frac{t-\delta}{2}\Big)^{-1-\alpha}\Big|\le K\delta t^{-2-\alpha}\ \text{for}\ t\in[\delta,\pi]$$

and integrating by parts we obtain

$$\begin{aligned}
B_{r,1}^1 \le{}& \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi A_{k_\nu}^{\alpha}}\int_{2\delta_\nu}^{2\gamma_r}\frac{d}{dt}\Big[\int_0^t(\varphi_x(u)-\varphi_x(u-\delta_\nu))\\
&\cdot\sin\Big(\Big(k_\nu+\frac{1+\alpha}{2}\Big)u-\frac{\alpha\pi}{2}\Big)\,du\Big]\frac{dt}{(2\sin t/2)^{1+\alpha}}\Big|^q\Big\}^{1/q}\\
&+\frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi A_{k_\nu}^{\alpha}}\int_{2\delta_\nu}^{2\gamma_r}\varphi_x(t-\delta_\nu)\Big(\frac{1}{(2\sin t/2)^{1+\alpha}}\\
&-\frac{1}{(2\sin(t-\delta_\nu)/2)^{1+\alpha}}\Big)\sin\Big(\Big(k_\nu+\frac{1+\alpha}{2}\Big)t-\frac{\alpha\pi}{2}\Big)\,dt\Big|^q\Big\}^{1/q}\\
\le{}& \frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi A_{k_\nu}^{\alpha}}\Big[\int_0^t(\varphi_x(u)-\varphi_x(u-\delta_\nu))\\
&\cdot\sin\Big(\Big(k_\nu+\frac{1+\alpha}{2}\Big)u-\frac{\alpha\pi}{2}\Big)\,du\frac{1}{(2\sin t/2)^{1+\alpha}}\Big]_{t=2\delta_\nu}^{t=2\gamma_r}\\
&+\frac{1}{\pi A_{k_\nu}^{\alpha}}\int_{2\delta_\nu}^{2\gamma_r}\Big[\int_0^t(\varphi_x(u)-\varphi_x(u-\delta_\nu))\\
&\cdot\sin\Big(\Big(k_\nu+\frac{1+\alpha}{2}\Big)u-\frac{\alpha\pi}{2}\Big)\,du\Big]\frac{(-1-\alpha)\cos t/2}{(2\sin t/2)^{2+\alpha}}\Big|^q\Big\}
\end{aligned}$$

$$+ K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi A_{k_\nu}^{\alpha}}\int_{2\delta_\nu}^{2\gamma_r}\Big|\varphi_x(t-\delta_\nu)\Big|\delta_\nu t^{-2-\alpha}\,dt\Big|^q\Big\}^{1/q}.$$

By the assumption

$$\begin{aligned}
B_{r,1}^1 &\le \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{1}{\gamma_r^{1+\alpha}}\int_0^{2\gamma_r}|\varphi_x(u)-\varphi_x(u-\delta_\nu)|\,du \\
&\quad + \frac{1}{(\delta_\nu)^{1+\alpha}}\int_0^{2\delta_\nu}|\varphi_x(u)-\varphi_x(u-\delta_\nu)|\,du \\
&\quad + \int_{2\delta_r}^{2\gamma_r}\Big(\int_0^t|\varphi_x(u)-\varphi_x(u-\delta_\nu)|\,du\Big)\frac{dt}{t^{2+\alpha}}\Big]^q\Big\}^{1/q} \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\int_{\delta_\nu}^{2\gamma_r-\delta_\nu}\frac{|\varphi_x(t)|}{t^{2+\alpha}}\,dt\Big]^q\Big\}^{1/q} \\
&\le \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[[\gamma_r^{-\alpha}+(\delta_\nu)^{-\alpha}]w_x(\delta_\nu)+w_x(\delta_\nu)\int_{2\delta_r}^{2\gamma_r}\frac{dt}{t^{1+\alpha}}\Big]^q\Big\}^{1/q} \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\int_{\delta_\nu}^{2\gamma_r}\frac{|\varphi_x(t)|}{t^{2+\alpha}}\,dt\Big]^q\Big\}^{1/q}
\end{aligned}$$

and since the function $w_x(t)/t$ is almost nonincreasing,

$$\begin{aligned}
B_{r,1}^1 &\le \frac{K}{(k_r+1)^\alpha}\gamma^{-\alpha}w_x(\delta_0) + \frac{K}{(k_r+1)^\alpha}w_x(\delta_0)\Big[\frac{t^{-\alpha}}{-\alpha}\Big]_{2\delta_0}^{2\gamma_r} \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\int_{\delta_\nu}^{2\gamma_r}\frac{1}{t^{2+\alpha}}\frac{d}{dt}\Big(\int_0^t\Big|\varphi_x(u)\Big|\,du\Big)\,dt\Big]^q\Big\}^{1/q} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0) + \frac{K}{(k_r+1)^\alpha}w_x(\delta_0)\Big[\frac{(2\gamma_r)^{-\alpha}}{-\alpha}\Big] \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Bigg\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Bigg[\delta_\nu\Big[\frac{1}{t^{2+\alpha}}\int_0^t|\varphi_x(u)|\,du\Big]_{t=\delta_r}^{2\gamma_r} \\
&\quad + \delta_\nu\int_{\delta_\nu}^{2\gamma_r}\frac{1}{t^{3+\alpha}}\Big(\int_0^t|\varphi_x(u)|\,du\Big)\,dt\Bigg]^q\Bigg\}^{1/q} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0) \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\frac{1}{(2\gamma)^{1+\alpha}}w_x(2\gamma)+\delta_\nu\int_{\delta_\nu}^{2\gamma_r}\frac{w_x(t)}{t^{2+\alpha}}\Big]^q\Big\}^{1/q} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0) \\
&\quad + \frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\frac{1}{(2\gamma)^{\alpha}}\frac{w_x(\delta_\nu)}{\delta_\nu}+\delta_\nu\frac{w_x(\delta_\nu)}{\delta_\nu}\int_{\delta_\nu}^{2\gamma_r}\frac{1}{t^{1+\alpha}}\Big]^q\Big\}^{1/q} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0)
\end{aligned}$$

$$+\frac{K}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{w_x(\delta_\nu)}{(2\gamma)^\alpha}+w_x(\delta_\nu)\frac{(2\gamma)^{-\alpha}}{-\alpha}\Big]^q\Big\}^{1/q}$$
$$\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0).$$

Next, by the evident estimate $\left|K^\alpha_{k_\nu,1}(t)\right|\le Kk_\nu^{-\alpha}t^{-\alpha-1}$,

$$B^2_{r,1}=\frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r-\delta_\nu}^{2\gamma_r}\varphi_x(t)K^\alpha_{k_\nu,1}(t)\,dt\Big|^q\Big\}^{1/q}$$
$$\le K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{(k_\nu+1)^\alpha}\int_{2\gamma_r-\delta_\nu}^{2\gamma_r}\frac{|\varphi_x(t)|}{t^{1+\alpha}}\,dt\Big|^q\Big\}^{1/q}$$
$$\le K\frac{1}{(k_r+1)^\alpha\gamma_r^{1+\alpha}}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\int_{2\gamma_r-\delta_\nu}^{2\gamma_r}|\varphi_x(t)|\,dt\Big]^q\Big\}^{1/q}$$
$$\le\frac{K\gamma_r^{-1-\alpha}}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\int_0^{2\gamma_r}|\varphi_x(t)|\,dt-\int_{\delta_\nu}^{2\gamma_r}|\varphi_x(t-\delta_\nu)|\,dt\Big|^q\Big\}^{1/q}$$
$$\le\frac{K\gamma_r^{-1-\alpha}}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\int_0^{\delta_\nu}|\varphi_x(t)|\,dt+\int_{\delta_\nu}^{2\gamma_r}\big||\varphi_x(t)|-|\varphi_x(t-\delta_\nu)|\big|\,dt\Big]^q\Big\}^{1/q}$$
$$\le\frac{K\gamma_r^{-\alpha}}{(k_r+1)^\alpha}\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{1}{2\gamma_r}\int_0^{\delta_\nu}|\varphi_x(t)|\,dt+\frac{1}{2\gamma_r}\int_{\delta_\nu}^{2\gamma_r}|\varphi_x(t)-\varphi_x(t-\delta_\nu)|\,dt\Big]^q\Big\}^{1/q}$$
$$\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{\delta_\nu}{\gamma_r}w_x(\delta_\nu)+w_x(\delta_\nu)\Big]^q\Big\}^{1/q}$$
$$\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}[w_x(\delta_\nu)]^q\Big\}^{1/q}\le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_0)$$

and

$$B^3_{r,1}=\frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{\delta_\nu}^{2\delta_\nu}\varphi_x(t)K^\alpha_{k_\nu,1}(t)\,dt\Big|^q\Big\}^{1/q}$$
$$\le\frac{1}{2}\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{k_\nu+1}{\pi}\int_{\delta_r}^{2\delta_r}|\varphi_x(t)|\,dt\Big|^q\Big\}^{1/q}$$
$$\le\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}[w_x(2\delta_r)]^q\Big\}^{1/q}\le 2w_x(\delta_0).$$

For an estimate of $B_{r,2}$ we observe that $|K^\alpha_{k_\nu,2}(t)|\le K(k_\nu+1)^{-1}t^{-2}$ and therefore

$$B_{r,2}=K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_\nu}^{2\gamma_r}\varphi_x(t)K^\alpha_{k_\nu,2}(t)\,dt\Big|^q\Big\}^{1/q}$$
$$\le K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{1}{k_\nu+1}\int_{2\delta_\nu}^{2\gamma_r}\frac{|\varphi_x(t)|}{t^2}\,dt\Big]^q\Big\}^{1/q}$$

$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\delta_\nu\int_{\delta_\nu}^{\pi}\frac{|\varphi_x(t)|}{t^2}\,dt\Big]^q\Big\}^{1/q}$$

$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\frac{1}{k_\nu+1}\sum_{k=0}^{k_\nu}w_x\Big(\frac{\pi}{k+1}\Big)\Big]^q\Big\}^{1/q}$$

$$\leq K\frac{1}{k_0+1}\sum_{k=0}^{k_0}w_x\Big(\frac{\pi}{k+1}\Big).$$

We divide the term C_r into two parts using the notation

$$\Phi_x f(\delta_{0,1},t) := \frac{1}{\delta_{0,1}}\int_t^{t+\delta_{0,1}}\varphi_x(u)\,du \text{ with } \delta_0=\delta_{0,1}=\frac{\pi}{k_0+1}:$$

$$C_r = \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi}\varphi_x(t)K_{k_\nu,1}^{\alpha}(t)\,dt\Big|^q\Big\}^{1/q}$$
$$\leq \frac{K\gamma_r^{1/q}}{A_{k_r}^{\alpha}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi}[\varphi_x(t)-\Phi_x f(\delta_{0,1},t)]\frac{\sin[(k_\nu+(1+\alpha)/2)t-\pi\alpha/2]}{(2\sin t/2)^{1+\alpha}}\,dt\Big|^q\Big\}^{1/q}$$
$$+\frac{K\gamma_r^{1/q}}{A_{k_r}^{\alpha}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi}\Phi_x f(\delta_{0,1},t)\frac{\sin[(k_\nu+(1+\alpha)/2)t-\pi\alpha/2]}{(2\sin t/2)^{1+\alpha}}\,dt\Big|^q\Big\}^{1/q}$$
$$= C_{r,1}+C_{r,2}.$$

Then, by the Hausdorff-Young inequality,

$$C_{r,1}\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{\pi}\big|[\Phi_x f(\delta_{0,1},t)-\varphi_x(t)](2\sin t/2)^{-1-\alpha}\big|^p\,dt\Big\}^{1/p}$$
$$\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{\pi}\big|[\Phi_x f(\delta_{0,1},t)-\varphi_x(t)]t^{-1-\alpha}\big|^p\,dt\Big\}^{1/p}.$$

Applying the generalized Minkowski inequality, we get

$$C_{r,1}\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{\pi}\Big|\frac{1}{t^{1+\alpha}\delta_{0,1}}\int_t^{t+\delta_{0,1}}[\varphi_x(u)-\varphi_x(t)]\,du\Big|^p\,dt\Big\}^{1/p}$$
$$=K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{\pi}\Big[\frac{1}{t^{1+\alpha}\delta_{0,1}}\int_0^{\delta_{0,1}}|\varphi_x(u+t)-\varphi_x(t)|\,du\Big]^p\,dt\Big\}^{1/p}$$
$$\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}}\frac{1}{\delta_{0,1}}\int_0^{\delta_{0,1}}\Big[\int_{2\gamma}^{\pi}\frac{|\varphi_x(u+t)-\varphi_x(t)|^p}{t^{(1+\alpha)p}}\,dt\Big]^{1/p}du$$
$$\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}\delta_{0,1}}\int_0^{\delta_{0,1}}\Big[\int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p}}\frac{d}{dt}\Big(\int_0^t|\varphi_x(u+v)-\varphi_x(v)|^p\,dv\Big)\,dt\Big]^{1/p}du.$$

Next, the partial integration gives

$$C_{r,1}\leq K\frac{\gamma^{1/q}}{A_{k_r}^{\alpha}\delta_{0,1}}\int_0^{\delta_{0,1}}\Big[\int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p}}\frac{d}{dt}\Big(\int_0^t|\varphi_x(u+v)-\varphi_x(v)|^p\,dv\Big)\,dt\Big]^{1/p}du$$

$$= K\frac{\gamma^{1/q}}{A^{\alpha}_{k_r}\delta_{0,1}}\int_0^{\delta_{0,1}}\Big\{\Big[\frac{1}{t^{(1+\alpha)p}}\int_0^t|\varphi_x(u+v)-\varphi_x(v)|^p\,dv\Big]^{\pi}_{t=2\gamma}$$
$$+(1+\alpha)p\int_{2\gamma}^{\pi}\Big[\frac{1}{t^{(1+\alpha)p+1}}\int_0^t|\varphi_x(u+v)-\varphi_x(v)|^p\,dv\Big]\,dt\Big\}^{1/p}\,du.$$

Since $f\in L^p(w_x)^\star$ and $\alpha q>-1$,

$$C_{r,1}\le K\frac{\gamma^{1/q}}{A^{\alpha}_{k_r}\delta_{0,1}}\int_0^{\delta_{0,1}}\Big\{\Big[\frac{1}{\pi^{(1+\alpha)p}}\int_0^{\pi}|\varphi_x(u+v)-\varphi_x(v)|^p\,dv\Big]$$
$$+(1+\alpha)p\int_{2\gamma}^{\pi}\Big[\frac{1}{t^{1+\alpha}}w_x(u)\Big]^p\,dt\Big\}^{1/p}\,du$$
$$\le K\frac{\gamma^{1/q}}{A^{\alpha}_{k_r}\delta_{0,1}}\int_0^{\delta_{0,1}}\Big\{[w_x(u)]^p+[w_x(u)]^p\int_{2\gamma}^{\pi}\frac{dt}{t^{(1+\alpha)p}}\Big\}^{1/p}\,du$$
$$\le K\frac{\gamma^{1/q}w_x(\delta_{0,1})}{A^{\alpha}_{k_r}}\gamma^{(1-(1+\alpha)p)/p}\le K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha}w_x(\delta_{0,1}).$$

Let us now estimate $C_{r,2}$. First we suppose that $r\le k_0$. Then, by the partial integration,

$$C_{r,2}=\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\frac{d}{dt}\big(\sin((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)\big)\,dt\Big|^q\Big\}^{1/q}$$
$$=\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}}\Big\{\sum_{\nu=0}^{r}\Big|\Big[\frac{\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\frac{\cos((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)}{k_\nu+(1+\alpha)/2}\Big]^{\pi}_{2\gamma}$$
$$+\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{d}{dt}\Big(\frac{\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)\frac{\cos[(k_\nu+(1+\alpha)/2)t-\pi\alpha/2]}{k_\nu+(1+\alpha)/2}\,dt\Big|^q\Big\}^{1/q}$$
$$\le\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{\Phi_xf(\delta_{0,1},2\gamma)}{(2\sin\gamma)^{1+\alpha}}\frac{\cos((k_\nu+(1+\alpha)/2)2\gamma-\pi\alpha/2)}{k_\nu+(1+\alpha)/2}\Big|^q\Big\}^{1/q}$$
$$+\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}(k_0+(1+\alpha)/2)}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{d}{dt}\Big(\frac{\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)$$
$$\cdot\cos\big[(k_\nu+(1+\alpha)/2)t-\pi\alpha/2\big]\,dt\Big|^q\Big\}^{1/q}$$

and by the Hausdorff-Young inequality

$$C_{r,2}\le\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}}\Big\{(r+1)\Big|\frac{\Phi_xf(\delta_{0,1},2\gamma)}{\gamma^{1+\alpha}}\frac{1}{k_0+1}\Big|^q\Big\}^{1/q}$$
$$+\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}(k_0+1)}\Big\{\int_{2\gamma}^{\pi}\Big|\frac{d}{dt}\Big(\frac{\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)\Big|^p\Big\}^{1/p}$$
$$\le\frac{K|\Phi_xf(\delta_0,2\gamma)|}{A^{\alpha}_{k_r}(k_0+1)\gamma^{1+\alpha}}$$
$$+\frac{K\gamma^{1/q}}{A^{\alpha}_{k_r}(k_0+1)}\Big\{\int_{2\gamma}^{\pi}\Big|\frac{\frac{d}{dt}\Phi_xf(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}+\frac{\Phi_xf(\delta_{0,1},t)(-1-\alpha)\cos t/2}{(2\sin t/2)^{2+\alpha}}\Big|^p\,dt\Big\}^{1/p}.$$

Thus, for $f \in L^p(w_x)^\star$ and by the partial integration,

$$\begin{aligned}
C_{r,2} \le{}& \frac{K(w_x(\delta_{0,1})+w_x(2\gamma))}{A^\alpha_{k_r}(k_0+1)\gamma^{1+\alpha}} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)}\Big\{\int_{2\gamma}^{\pi}\Big[\frac{\frac{1}{\delta_{0,1}}|\varphi_x(\delta_{0,1}+t)-\varphi_x(t)|}{t^{1+\alpha}}+\frac{w_x(\delta_{0,1})+w_x(t)}{t^{2+\alpha}}\Big]^p dt\Big\}^{1/p} \\
\le{}& \frac{K(w_x(\delta_{0,1})+w_x(2\gamma))}{A^\alpha_{k_r}(k_0+1)\gamma^{1+\alpha}} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)}\Big\{\frac{1}{(\delta_{0,1})^p}\int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p}}\frac{d}{dt}\Big(\int_0^t|\varphi_x(\delta_{0,1}+u)-\varphi_x(t)|^p\,du\Big)\,dt\Big\}^{1/p} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)}\Big\{\int_{2\gamma}^{\pi}\frac{(w_x(t))^p}{t^{(2+\alpha)p}}\,dt\Big\}^{1/p} \\
\le{}& \frac{K\big(w_x(\delta_{0,1})+2(\frac{\gamma}{\delta_{0,1}}+1)w_x(\delta_{0,1})\big)}{A^\alpha_{k_r}(k_0+1)\gamma^{1+\alpha}}+\frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)}\Big\{\int_{2\gamma}^{\pi}\frac{(w_x(t)/t)^p}{t^{(1+\alpha)p}}\,dt\Big\}^{1/p} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)\delta_{0,1}}\Big\{\Big[\frac{1}{t^{(1+\alpha)p}}\int_0^t|\varphi_x(\delta_{0,1}+u)-\varphi_x(t)|^p\,du\Big]_{t=2\gamma}^{t=\pi} \\
&+ (1+\alpha)p\int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p+1}}\Big(\int_0^t|\varphi_x(\delta_{0,1}+u)-\varphi_x(t)|^p\,du\Big)\,dt\Big\}^{1/p}.
\end{aligned}$$

If $r \le k_0$ then $\gamma \ge \delta_{0,1}$ and therefore

$$\begin{aligned}
C_{r,2} \le{}& \frac{Kw_x(\delta_{0,1})\gamma}{A^\alpha_{k_r}(k_0+1)\gamma^{1+\alpha}\delta_{0,1}}+\frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)}\frac{w_x(2\gamma)}{\gamma}\Big\{\int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p}}\,dt\Big\}^{1/p} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}(k_0+1)\delta_{0,1}}\Big\{\frac{1}{\pi^{(1+\alpha)p}}\int_0^{\pi}\Big|\varphi_x(\delta_{0,1}+u)-\varphi_x(t)\Big|^p du \\
&+ \int_{2\gamma}^{\pi}\frac{1}{t^{(1+\alpha)p}}(w_x(\delta_{0,1}))^p\,dt\Big\}^{1/p} \\
\le{}& \frac{Kw_x(\delta_{0,1})}{A^\alpha_{k_r}\gamma^\alpha}+\frac{K\gamma^{1/q}w_x(\delta_{0,1})}{A^\alpha_{k_r}(k_0+1)\delta_{0,1}}\gamma^{-1/q-\alpha}+\frac{K\gamma^{1/q}w_x(\delta_{0,1})}{A^\alpha_{k_r}(k_0+1)\delta_{0,1}}\big[1+\gamma^{1/p-(1+\alpha)}\big] \\
\le{}& \frac{Kw_x(\delta_{0,1})}{A^\alpha_{k_r}\gamma^\alpha}+\frac{K\gamma^{1/q}}{A^\alpha_{k_r}}w_x(\delta_{0,1})\gamma^{-1/q-\alpha} \le \frac{Kw_x(\delta_{0,1})}{A^\alpha_{k_r}\gamma^\alpha} \le K\Big(\frac{r+1}{k_r+1}\Big)^\alpha w_x(\delta_{0,1}).
\end{aligned}$$

Now, let us consider the case $r \ge k_0$. Then, by the Hausdorff-Young inequality and the partial integration,

$$\begin{aligned}
C_{r,2} ={}& \frac{K\gamma^{1/q}}{A^\alpha_{k_r}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma}^{\pi}\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\sin\big((k_\nu+(1+\alpha)/2)t-\pi\alpha/2\big)\,dt\Big|^q\Big\}^{1/q} \\
\le{}& \frac{K\gamma^{1/q}}{A^\alpha_{k_r}}\Big\{\int_{2\gamma}^{2\delta_{0,1}}\Big|\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big|^p dt\Big\}^{1/p} \\
&+ \frac{K\gamma^{1/q}}{A^\alpha_{k_r}}\Big\{\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\delta_{0,1}}^{\pi}\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\frac{d}{dt}\Big(\frac{\cos((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)}{k_\nu+(1+\alpha)/2}\Big)\,dt\Big|^q\Big\}
\end{aligned}$$

and by Lemma 1

$$
\begin{aligned}
C_{r,2} \leq{} & \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{2\delta_{0,1}}\Big|\frac{w_x(\delta_{0,1})+w_x(t)}{t^{1+\alpha}}\Big|^p\,dt\Big\}^{1/p} \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\sum_{\nu=0}^{r}\Big|\Big[\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\,\frac{\cos((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)}{k_\nu+(1+\alpha)/2}\Big]_{2\delta_0}^{\pi} \\
& - \int_{2\delta_{0,1}}^{\pi}\frac{d}{dt}\Big(\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)\frac{\cos((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)}{k_\nu+(1+\alpha)/2}\,dt\Big|^q\Big\}^{1/q} \\
\leq{} & \frac{K\gamma^{1/q}w_x(\delta_{0,1})}{A_{k_r}^{\alpha}}\Big\{\int_{2\gamma}^{2\delta_{0,1}}\frac{1}{t^{(1+\alpha)p}}\,dt\Big\}^{1/p} + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}\frac{1}{(k_0+1)^{1/p}}\Big|\frac{\Phi_x f(\delta_{0,1},2\delta_{0,1})}{(2\sin\delta_0)^{1+\alpha}}\Big| \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}\Big\{\sum_{\nu=0}^{r}\Big|\int_{2\delta_0}^{\pi}\frac{d}{dt}\Big(\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)\frac{\cos((k_\nu+(1+\alpha)/2)t-\pi\alpha/2)}{k_0+(1+\alpha)/2}\,dt\Big|^q\Big\}^{1/q} \\
\leq{} & \frac{K\gamma^{1/q}w_x(\delta_{0,1})}{A_{k_r}^{\alpha}}\gamma^{(1-(1+\alpha)p)/p} + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}\frac{1}{(k_0+1)^{1/p}}\Big[\frac{w_x(\delta_{0,1})+w_x(2\delta_{0,1})}{(\delta_0)^{1+\alpha}}\Big] \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)}\Big\{\int_{2\delta_{0,1}}^{\pi}\Big|\frac{d}{dt}\Big(\frac{\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}}\Big)\Big|^p\,dt\Big\}^{1/p} \\
\leq{} & K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha}w_x(\delta_{0,1}) \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)}\Big\{\int_{2\delta_{0,1}}^{\pi}\Big|\Big(\frac{\frac{d}{dt}\Phi_x f(\delta_{0,1},t)}{(2\sin t/2)^{1+\alpha}} - (1+\alpha)\frac{\Phi_x f(\delta_{0,1},t)\cos t/2}{(2\sin t/2)^{2+\alpha}}\Big)\Big|^p\,dt\Big\}^{1/p} \\
\leq{} & K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha}w_x(\delta_{0,1}) \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)}\Big[\frac{1}{\delta_{0,1}^{p}}\int_{2\delta_{0,1}}^{\pi}t^{-(1+\alpha)p}\frac{d}{dt}\Big(\int_0^t|\varphi_x(\delta_{0,1}+u)-\varphi_x(u)|^p\,du\Big)\,dt\Big]^{1/p} \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)}\Big\{\Big[\int_{2\delta_{0,1}}^{\pi}\Big|\frac{w_x(\delta_{0,1})}{t^{1+\alpha}}\Big|^p\,dt\Big]^{1/p} + \Big[\int_{2\delta_{0,1}}^{\pi}\Big|\frac{w_x(t)}{t^{1+\alpha}}\Big|^p\,dt\Big]^{1/p}\Big\} \\
\leq{} & K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha}w_x(\delta_{0,1}) \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)\delta_{0,1}}\Big[\frac{1}{\pi^{(1+\alpha)p}}\int_0^{\pi}|\varphi_x(\delta_{0,1}+u)-\varphi_x(u)|^p\,du\Big]^{1/p} \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)\delta_{0,1}}\Big[\int_{2\delta_{0,1}}^{\pi}t^{-(1+\alpha)p-1}\Big(\int_0^t|\varphi_x(\delta_{0,1}+u)-\varphi_x(u)|^p\,du\Big)\,dt\Big]^{1/p} \\
& + \frac{K\gamma^{1/q}w_x(\pi)}{A_{k_r}^{\alpha}(k_0+1)}\Big[\int_{2\delta_{0,1}}^{\pi}\Big|\frac{1}{t^{1+\alpha}}\Big|^p\,dt\Big]^{1/p} \\
\leq{} & K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha}w_x(\delta_{0,1}) + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}}w_x(\delta_{0,1}) \\
& + \frac{K\gamma^{1/q}}{A_{k_r}^{\alpha}(k_0+1)\delta_{0,1}}\Big[\int_{2\delta_0}^{\pi}\Big|\frac{w_x(\delta_{0,1})}{t^{1+\alpha}}\Big|^p\,dt\Big]^{1/p} + \frac{K\gamma^{1/q}w_x(\pi)}{A_{k_r}^{\alpha}(k_0+1)}(\delta_{0,1})^{(1-(1+\alpha)p)/p}
\end{aligned}
$$

$$
\begin{aligned}
&\le K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha} w_x(\delta_{0,1}) + \frac{K\gamma^{-\alpha}}{A^{\alpha}_{k_r}} w_x(\delta_{0,1}) \\
&\quad + \frac{K\gamma^{1/q} w_x(\delta_{0,1})}{A^{\alpha}_{k_r}} (\delta_{0,1})^{(1-(1+\alpha)p)/p} + \frac{K\gamma^{1/q} w_x(\pi)}{A^{\alpha}_{k_r}(k_0+1)} (\delta_{0,1})^{(1-(1+\alpha)p)/p} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha} w_x(\delta_{0,1}) + \frac{K\gamma^{1/q} w_x(\delta_{0,1})}{A^{\alpha}_{k_r}} \gamma^{(1-(1+\alpha)p)/p} \\
&\quad + \frac{K\gamma^{1/q} w_x(\delta_{0,1})(\frac{\pi}{\delta_{0,1}}+1)}{A^{\alpha}_{k_r}(k_0+1)} \gamma^{(1-(1+\alpha)p)/p} \\
&\le K\Big(\frac{r+1}{k_r+1}\Big)^{\alpha} w_x(\delta_{0,1}).
\end{aligned}
$$

Finally, we estimate the last term.

$$
\begin{aligned}
D_r &= \Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big|\frac{1}{\pi}\int_{2\gamma_r}^{\pi} \varphi_x(t) K^{\alpha}_{k_\nu,2}(t)\,dt\Big|^q\Big\}^{1/q} \\
&\le K\Big\{\frac{1}{r+1}\Big(\sum_{\nu=0}^{\nu_0-1} + \sum_{\nu=\nu_0}^{r}\Big)\Big[\Big(\int_{2\gamma_r}^{2\delta_\nu} + \int_{2\delta_\nu}^{\pi}\Big)|\varphi_x(t)|\big|K^{\alpha}_{k_\nu,2}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\le K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\Big(\int_{2\gamma_r}^{2\delta_\nu} + \int_{2\delta_\nu}^{\pi}\Big)|\varphi_x(t)|\big|K^{\alpha}_{k_\nu,2}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\quad + K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\int_{2\delta_\nu}^{\pi} |\varphi_x(t)|\big|K^{\alpha}_{k_\nu,2}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\le K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\int_{2\gamma_r}^{2\delta_\nu} |\varphi_x(t)|\big|K^{\alpha}_{k_\nu}(t) + K^{\alpha}_{k_\nu,1}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\quad + K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\int_{2\delta_\nu}^{\pi} |\varphi_x(t)|\big|K^{\alpha}_{k_\nu,2}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\quad + K\Big\{\frac{1}{r+1}\sum_{\nu=\nu_0}^{r}\Big[\int_{2\delta_\nu}^{\pi} |\varphi_x(t)|\big|K^{\alpha}_{k_\nu,2}(t)\big|\,dt\Big]^q\Big\}^{1/q} \\
&\le K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\frac{1}{(k_\nu+1)^{\alpha}}\int_{2\gamma_r}^{2\delta_\nu} \frac{|\varphi_x(t)|}{t^{1+\alpha}}\,dt\Big]^q\Big\}^{1/q} \\
&\quad + K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{1}{k_\nu+1}\int_{2\delta_\nu}^{\pi} \frac{|\varphi_x(t)|}{t^2}\,dt\Big]^q\Big\}^{1/q} \\
&\le K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\frac{1}{(k_\nu+1)^{\alpha}}\int_{2\gamma_r}^{2\delta_\nu} \frac{1}{t^{1+\alpha}}\frac{d}{dt}\Big(\int_0^t |\varphi_x(u)|\,du\Big)\,dt\Big]^q\Big\}^{1/q} \\
&\quad + K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{1}{k_\nu+1}\int_{2\delta_\nu}^{\pi} \frac{1}{t^2}\frac{d}{dt}\Big(\int_0^t |\varphi_x(u)|\,du\Big)\,dt\Big]^q\Big\}^{1/q}
\end{aligned}
$$

$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\Big[\frac{(k_\nu+1)^{-\alpha}}{t^{1+\alpha}}\int_0^t|\varphi_x(u)|\,du\Big]_{2\gamma_r}^{2\delta_\nu}$$
$$+\frac{1+\alpha}{(k_\nu+1)^\alpha}\int_{2\gamma_r}^{2\delta_\nu}\frac{1}{t^{2+\alpha}}\Big(\int_0^t|\varphi_x(u)|\,du\Big)\,dt\Big]^q\Big\}^{1/q}$$
$$+K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\Big[\frac{1}{(k_\nu+1)t^2}\int_0^t|\varphi_x(u)|\,du\Big]_{2\delta_\nu}^{\pi}$$
$$+\frac{2}{k_\nu+1}\int_{2\delta_{\nu r}}^{\pi}\frac{1}{t^3}\Big(\int_0^t|\varphi_x(u)|\,du\Big)\,dt\Big]^q\Big\}^{1/q}$$
$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\frac{(k_\nu+1)^{-\alpha}}{(2\delta_\nu)^{1+\alpha}}\int_0^{2\delta_\nu}|\varphi_x(u)|\,du+\frac{1+\alpha}{(k_\nu+1)^\alpha}\int_{2\gamma_r}^{2\delta_\nu}\frac{w_x(t)}{t^{1+\alpha}}\,dt\Big]^q\Big\}^{1/q}$$
$$+K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{1}{(k_\nu+1)\pi^2}\int_0^{\pi}|\varphi_x(u)|\,du+\frac{2}{k_\nu+1}\int_{2\delta_\nu}^{\pi}\frac{w_x(t)}{t^2}\,dt\Big]^q\Big\}^{1/q}$$
$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[\frac{(k_\nu+1)^{-\alpha}}{(\delta_\nu)^{\alpha}}w_x(\delta_\nu)+\frac{w_x(\delta_\nu)}{(k_\nu+1)^\alpha}\int_{2\gamma_r}^{2\delta_\nu}\frac{1}{t^{1+\alpha}}\,dt\Big]^q\Big\}^{1/q}$$
$$+K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{w_x(\pi)}{(k_\nu+1)}+2\delta_\nu\int_{2\delta_\nu}^{\pi}\frac{w_x(t)}{t^2}\,dt\Big]^q\Big\}^{1/q}$$
$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}\Big[w_x(\delta_\nu)+\frac{w_x(\delta_\nu)}{(k_\nu+1)^\alpha}\frac{t^{-\alpha}}{-\alpha}\Big|_{2\gamma_r}^{2\delta_\nu}\Big]^q\Big\}^{1/q}$$
$$+K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{w_x(\pi)}{(k_\nu+1)}+\frac{1}{k_\nu+1}\sum_{k=0}^{k_\nu}w_x\Big(\frac{\pi}{k+1}\Big)\Big]^q\Big\}^{1/q}$$
$$\leq K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{\nu_0-1}[w_x(\delta_\nu)]^q\Big\}^{1/q}+K\Big\{\frac{1}{r+1}\sum_{\nu=0}^{r}\Big[\frac{1}{k_\nu+1}\sum_{k=0}^{k_\nu}w_x\Big(\frac{\pi}{k+1}\Big)\Big]^q\Big\}^{1/q}$$
$$\leq Kw_x(\delta_0)+K\frac{1}{k_0+1}\sum_{k=0}^{k_0}w_x\Big(\frac{\pi}{k+1}\Big).$$

Finally we have

$$D_r\leq K\frac{1}{k_0+1}\sum_{k=0}^{k_0}w_x\Big(\frac{\pi}{k+1}\Big).$$

This completes the proof. ∎

4.2. Proof of Theorem 3. We start with the obvious inequality

$$H_{\lambda,\cdot}^{\phi,\alpha}f(x)\leq K\sum_{m=2}^{\infty}\sum_{\nu=N_{m-2}+1}^{N_m}\lambda_\nu\phi(|\sigma_\nu^\alpha f(x)-f(x)|).$$

Using the Hölder inequality

$$H_{\lambda,\cdot}^{\phi,\alpha} f(x) \le K \sum_{m=1}^{\infty} \Big\{ \sum_{\nu=N_{m-2}+1}^{N_m} (\lambda_\nu)^s \Big\}^{1/s} \Big\{ \sum_{\nu=N_{m-2}+1}^{N_m} \phi^q(|\sigma_\nu^\alpha f(x) - f(x)|) \Big\}^{1/q}$$

with $\frac{1}{s} + \frac{1}{q} = 1$ $(s > 1)$, and by the assumption $(\lambda_\nu) \in \Lambda_s(N_m)$, we obtain

$$H_{\lambda,\cdot}^{\phi,\alpha} f(x) \le K \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \Big\{ \frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} \phi^q(|\sigma_\nu^\alpha f(x) - f(x)|) \Big\}^{1/q}.$$

The second assumption $\phi \in \Phi^\alpha$ implies that also $\phi^q \in \Phi^\alpha$. Thus by Lemma 3,

$$\begin{aligned} H_{\lambda,\cdot}^{\phi,\alpha} f(x) &\le K \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \Big\{ K \phi^q \Big(\frac{1}{N_{m-2}+1} \sum_{k=0}^{N_{m-2}} w_x\Big(\frac{\pi}{k+1}\Big)\Big)\Big\}^{1/q} \\ &\le K \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \phi\Big(\frac{1}{N_{m-2}+1} \sum_{k=0}^{N_{m-2}} w_x\Big(\frac{\pi}{k+1}\Big)\Big). \end{aligned}$$

Thus our result is proved. ∎

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