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CONSTANTS OF STRONG UNIQUENESS OF MINIMAL NORM-ONE PROJECTIONS

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Abstract. In this paper we calculate the constants of strong uniqueness of minimal norm-one projections on subspaces of codimension k in the space $l_\infty^{(n)}$. This generalizes a main result of W. Odyniec and M. P. Prophet [J. Approx. Theory 145 (2007), 111–121]. We applied in our proof Kolmogorov's type theorem (see A. Wójcik [*Approximation and Function Spaces* (Gdańsk, 1979), PWN, Warszawa / North-Holland, Amsterdam, 1981, 854–866]) for strongly unique best approximation.

1. Introduction. Let X be a normed space and let $Y \subset X$ be a nonempty subset. An element $y \in Y$ is called a strongly unique best approximation (briefly SUBA) to $x \in X$ if and only if there exists $r > 0$ such that for every $v \in Y$

$$\|x - v\| \geqslant \|x - y\| + r\|v - y\|. \tag{1}$$

The largest possible constant r satisfying the above inequality is called the strong unicity constant.

The concept of strong unicity was introduced by Newman and Shapiro in 1963 (see [20]). The main classical example of this notion are spaces $\mathcal{P}_n$ of polynomials of degree not greater than n treated as subspaces of $\mathcal{C}[0,1]$. More precisely, for any $f \in \mathcal{C}[0,1]$ there exists $r_n(f) > 0$ such that

$$\|f - p\| \geqslant \|f - p_n(f)\| + r_n(f)\|p - p_n(f)\|,$$

where $p_n(f)$ denotes the best approximation of f in $\mathcal{P}_n$.

Generally, this property is stronger than uniqueness of the best approximation. Strong unicity implies local Lipschitz continuity of the best approximation operator (see e.g. [6]).

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Another important application of this notion is the estimate of the error of the Remez algorithm ([9, 25, 26]).

Strong unicity has been studied by many authors and there are a large number of papers devoted to it, for example [1, 2, 3, 5, 8, 15, 18, 19, 21, 22, 23, 29, 31]. Also many authors have described the asymptotic behaviour of $r_n(f)$ as n tends to infinity for a given function $f \in \mathcal{C}[0,1]$ (see e.g. [10, 12, 24]). The Kolmogorov criterion for strongly unique best approximation has been introduced by Bartelt and McLaughlin in [3].

Let us denote by $\mathcal{L}(X,Y)$ the set of all linear, continuous operators from X into Y. We use the notation $\mathcal{L}(X)$ as an abbreviation for the space $\mathcal{L}(X,X)$. By $\mathcal{P}(X,Y)$ let us denote the set of all linear, continuous projections going from X onto Y, i.e.,

$$\mathcal{P}(X,Y) = \{P \in \mathcal{L}(X,Y) : Py = y \text{ for any } y \in Y\}.$$

In the case of projections the notion of strong unicity reduces to

Definition 1.1. Let $P_0 \in \mathcal{P}(X,Y)$. Then P_0 is called a *strongly unique minimal projection* (we will write a SUM-projection for brevity) if and only if there exists $r > 0$ such that for any $P \in \mathcal{P}(X,Y)$

$$\|P\| \geqslant \|P_0\| + r\|P - P_0\|. \tag{2}$$

The largest possible constant for which the inequality in (2) holds is called a *strongly unique projection constant* (briefly SUP-constant).

It is clear that any SUM-projection is the unique minimal projection in $\mathcal{P}(X,Y)$.

Now, let us introduce some notation. By S_X we denote the unit sphere in a normed space X and by $\operatorname{ext} S_X$ the set of its extreme points. The symbol e^s, $s = 1,\dots,n$, stands for the linear functional on $l_\infty^{(n)}$ such that $e^s(x) = x_s$ for $x \in l_\infty^{(n)}$. Let X be a Banach space and $Y \subset X$ be its closed subspace. Set

$$E(x) = \{f \in \operatorname{ext} S_{X^*} : f(x) = \|x\|\} \tag{3}$$

and

$$\mathcal{L}_Y = \{L \in \mathcal{L}(X,Y) : L_{|Y} = 0\}. \tag{4}$$

Lemma 1.2 (See e.g. [4]). *Assume X is a normed space and let Y be its subspace of codimension k, $Y = \bigcap_{i=1}^k \ker f^i$, where $f^i \in X^*$ are linearly independent. Then there exist $y^1,\dots,y^k \in X$ satisfying $f^i(y^j) = \delta_{ij}$ for $i,j = 1,\dots,k$ such that*

$$Px = x - \sum_{i=1}^k f^i(x)y^i \quad \textit{for} \quad x \in X.$$

Theorem 1.3 (See e.g. [27], [7, Prop. 2.1, p. 55]). *Let X be a finite-dimensional normed space. Then*

$$\operatorname{ext} S_{\mathcal{L}^*(X)} = \operatorname{ext} S_{X^*} \otimes \operatorname{ext} S_X,$$

where $(x^* \otimes x)(L) = x^*(Lx)$ for $x \in X$, $x^* \in X^*$ and $L \in \mathcal{L}(X)$.

Now we present Kolmogorov's type criterion, concerning strong unicity.

Theorem 1.4 (See e.g. [30, Th. 2.1, p. 855]). *Let X be a normed space and let $Y \subset X$ be one of its subspaces. Assume that $x \in X \setminus Y$. Then $y_0 \in Y$ is a strongly unique best* *approximation to x with a constant $r > 0$ if and only if for any $y \in Y$ there exists an $f \in E(x - y_0)$ such that* $\operatorname{Re} f(y) \leqslant -r\|y\|$.

As a consequence of Theorem 1.4 we obtain the following

LEMMA 1.5 (See [15, Th. 1.3, p. 84]). *Let $V \subset X$, $\dim V = n$, $x_0 \in X \setminus V$, $v_0 \in V$. If v_0 is a SUBA to x_0 in V, then*

$$r = \inf_{v \in S_V} \big(\sup_{g \in E(x_0 - v_0)} g(v) \big)$$

is the strong unicity constant.

Lemma 1.5 permits us to calculate the strong unicity constants of some subspaces of $\mathcal{L}(l_\infty^{(n)})$.

2. The main result. Let $n > k \geqslant 3$. Define

$$\begin{aligned}
f^{(1)} &= (f_1^1, \ldots, f_{n-k}^1, f_{n-k+1}^1, 0, \ldots, 0),\\
f^{(2)} &= (f_1^2, \ldots, f_{n-k}^2, 0, f_{n-k+2}^2, 0, \ldots, 0),\\
&\vdots\\
f^{(k)} &= (f_1^k, \ldots, f_{n-k}^k, 0, \ldots, 0, f_n^k)
\end{aligned}$$

under the constraints

$$0 < f_i^j < \frac{1}{2} \quad \text{for} \quad i = 1, \ldots, n-k,\ j = 1, \ldots, k,$$

$$\sum_{i=1}^{n-k} f_i^j + f_{n-k+j}^j = 1, \qquad f_{n-k+j}^j \geqslant \frac{1}{2} \quad \text{for} \quad j = 1, \ldots, k. \tag{5}$$

Let

$$H = \bigcap_{j=1}^{k} \ker f^{(j)}. \tag{6}$$

The aim of this paper is to calculate the strongly unique minimal projection constant of the space $\mathcal{P}(l_\infty^{(n)}, H)$.

W. Odyniec and M. P. Prophet have given a lower bound for the above strongly unique minimal projection constant in [22]. Using Lemma 1.2 and (8) they have directly calculated all norms occurring in the formula (2). Next, with the help of arduous estimations they have proved that the SUP-constant s_0 of the space $\mathcal{P}(l_\infty^{(n)}, H)$ satisfies the inequality

$$\min\{s_0^1, \ldots, s_0^{n-k}\} \leqslant s_0 < 1, \tag{7}$$

where

$$s_0^i = \min\left\{ \frac{f_{n-k+1}^1 - f_i^1}{f_{n-k+1}^1 + f_i^1}, \frac{f_{n-k+2}^2 - f_i^2}{f_{n-k+2}^2 + f_i^2}, \ldots, \frac{f_n^k - f_i^k}{f_n^k + f_i^k} \right\}, \quad i = 1, \ldots, n-k.$$

In our paper we improve their result and we calculate the SUP-constant of the space $\mathcal{P}(l_\infty^{(n)}, H)$. We use the Kolmogorov criterion for the strong unicity.

It is well known (see [14, Remark 1.12], [21, Th. III.3.1]) that there exists exactly one minimal projection $P_0 \in \mathcal{P}(l_\infty^{(n)}, H)$, $\|P_0\| = 1$ and P_0 has the form

$$P_0(\cdot) = \mathrm{Id} - \sum_{j=1}^{k} f^{(j)}(\cdot) y^{(j)},$$

where

$$y^{(j)} = (0_1, \dots, 0_{n-k+j-1}, 1/f^j_{n-k+j}, 0_{n-k+j+1}, \dots, 0_n), \quad j = 1, \dots, k. \tag{8}$$

Now we find the set $E(P_0)$ (see (3)).

Lemma 2.1. *Let us define for $s = 1, \dots, n-k$ the sets*

$$A_s = \{(x_1, \dots, x_{s-1}, 1_s, x_{s+1}, \dots, x_n) : x_i \in \{-1, 1\} \text{ for } i = 1, \dots, n,\ i \neq s\}$$

and the set

$$B = \{(-1_1, \dots, -1_{n-k}, x_{n-k+1}, \dots, x_n) : \quad x_i \in \{-1, 1\} \text{ for } \quad i = n-k+1, \dots, n\}.$$

Let $P_0 \in \mathcal{P}(l_\infty^{(n)}, H)$ be the minimal projection. Then the set $E(P_0)$ has the form

$$E(P_0) = F \cup G,$$
$$F = \{e^s \otimes x : x \in A_s,\ s = 1, \dots, n-k\},$$
$$G = \Big\{e^s \otimes x : x \in B,\ s \text{ satisfying } f_s^{s-(n-k)} = \frac{1}{2}\Big\},$$

where $e^s \in (l_\infty^{(n)})^*$ is such that $e^s(x) = x_s$ for $x \in l_\infty^{(n)}$, $s = 1, \dots, n$.

Proof. By Theorem 1.3

$$E(P_0) \subseteq \operatorname{ext} l_1^{(n)} \otimes \operatorname{ext} l_\infty^{(n)} \subseteq \{e^s \otimes x : x \in \operatorname{ext} l_\infty^{(n)}\}.$$

Notice that by Lemma 1.2

$$(e^s \otimes x)(P_0) = x_s - \sum_{j=1}^{k} f^{(j)}(x) y_s^{(j)}. \tag{9}$$

The equality $(e^s \otimes x)(P_0) = \|P_0\| = 1$ for $s \in \{1, \dots, n\}$ is true if and only if

$$x_s - \sum_{j=1}^{k} f^{(j)}(x) y_s^{(j)} = 1. \tag{10}$$

Since $y_s^{(j)} = 0$ for $j = 1, \dots, k$, $s = 1, \dots, n-k$ (see (8)), the equality (10) implies $x_s = 1$ for $s \in \{1, \dots, n-k\}$. Hence we get the form of the set F.

Let us consider now $s \in \{n-k+1, \dots, n\}$. By (8) and (9)

$$(e^s \otimes x)(P_0) = x_s - f^{(s-(n-k))}(x) y_s^{(s-(n-k))} = x_s - \frac{f^{(s-(n-k))}(x)}{f_s^{s-(n-k)}}.$$

Hence $e^s \otimes x \in E(P_0)$ if and only if

$$x_s - 1 = \frac{f^{(s-(n-k))}(x)}{f_s^{s-(n-k)}} \quad \text{and} \quad x \in \operatorname{ext} l_\infty^{(n)}. \tag{11}$$

It is easy to see that if $f_s^{s-(n-k)} > 1/2$ then $0 < \frac{|f^{(s-(n-k))}(x)|}{f_s^{s-(n-k)}} < 2$, so (11) cannot hold.

Assume now that $s \in \{n-k+1,\dots,n\}$ satisfies $f_s^{s-(n-k)} = \frac{1}{2}$. Then the equality (11) is equivalent to the following alternative

$$f^{(s-(n-k))}(x) = -1, \quad x_s = -1, \quad \text{and} \quad x_i \in \{-1,1\} \quad \text{for} \quad i = 1,\dots,n$$

or

$$f^{(s-(n-k))}(x) = 0, \quad x_s = 1, \quad \text{and} \quad x_i \in \{-1,1\} \quad \text{for} \quad i = 1,\dots,n.$$

By (5)

$$x_i = -1 \text{ for } i = 1,\dots,n-k, \quad x_s = -1, \quad x_i \in \{-1,1\} \quad \text{for} \quad i = n-k+1,\dots,n$$

or

$$x_i = -1 \text{ for } i = 1,\dots,n-k, \quad x_s = 1, \quad x_i \in \{-1,1\} \quad \text{for} \quad i = n-k+1,\dots,n. \ \blacksquare$$

REMARK 2.2. Let $L \in \mathcal{L}_H$ (see (4)). Then

$$L(\cdot) = \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)},$$

where $z^{(j)} \in H$ for $j = 1,\dots,k$. Since $H = \bigcap_{j=1}^{k} \ker f^{(j)}$, we additionally get that

$$z_{n-k+i}^{(j)} = \Big(-\sum_{s=1}^{n-k} f_s^i z_s^{(j)}\Big)/f_{n-k+i}^i \quad \text{for all} \quad i,j = 1,\dots,k.$$

LEMMA 2.3. *Let $L \in \mathcal{L}_H$, $L(\cdot) = \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)}$, where $z^{(j)} \in H$ for all $j = 1,\dots,k$. Then*

$$\|L\| = \max_{s=1,\dots,n-k} \Big(\sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(i)}|\Big). \tag{12}$$

Proof. Let $x \in l_\infty^{(n)}$. By Remark 2.2 we get

$$L(x) = \sum_{i=1}^{k} f^{(i)}(x) z^{(i)} = \sum_{i=1}^{k} f^{(i)}(x)\Big(z_1^{(i)},\dots,z_{n-k}^{(i)}, -\frac{\sum_{p=1}^{n-k} f_p^1 z_p^{(i)}}{f_{n-k+1}^1},\dots,-\frac{\sum_{p=1}^{n-k} f_p^k z_p^{(i)}}{f_n^k}\Big).$$

Let us define the constant $A := \max_{i=1,\dots,n-k} \|e^i \circ L\|$. Now we show that $\|e^s \circ L\| \leqslant A$ for any $s \in \{n-k+1,\dots,n\}$. Let $s \in \{n-k+1,\dots,n\}$ and let $x \in l_\infty^{(n)}$, $\|x\| = 1$. We find

$$\begin{aligned}
|(e^s \circ L)(x)| &= \Big|-\sum_{i=1}^{k} f^{(i)}(x) \frac{\sum_{p=1}^{n-k} f_p^{s-(n-k)} z_p^{(i)}}{f_s^{s-(n-k)}}\Big| \\
&\leqslant \sum_{p=1}^{n-k} \frac{f_p^{s-(n-k)}}{f_s^{s-(n-k)}}\Big|\sum_{i=1}^{k} f^{(i)}(x) z_p^{(i)}\Big| = \sum_{p=1}^{n-k} \frac{f_p^{s-(n-k)}}{f_s^{s-(n-k)}} |(e^p \circ L)(x)| \\
&\leqslant \sum_{p=1}^{n-k} \frac{f_p^{s-(n-k)}}{f_s^{s-(n-k)}} \|e^p \circ L\| \leqslant \sum_{p=1}^{n-k} \frac{f_p^{s-(n-k)}}{f_s^{s-(n-k)}} \cdot A = \frac{1 - f_s^{s-(n-k)}}{f_s^{s-(n-k)}} \cdot A.
\end{aligned}$$

Taking into account that $f_s^{s-(n-k)} \geqslant 1/2$ for $s = n-k+1,\dots,n$ we find

$$\|(e^s \circ L)\| \leqslant \max_{i=1,\dots,n-k} \|e^i \circ L\| \quad \text{for} \quad s = n-k+1,\dots,n.$$

Hence

$$\|L\| = \max_{i=1,\dots,n-k} \|e^i \circ L\|. \tag{13}$$

To calculate the norm of the operator L we find $\|e^s \circ L\|$ for $s = 1, \dots, n-k$. Let $x \in l_\infty^{(n)}$ and $s \in \{1, \dots, n-k\}$. Then

$$\begin{aligned}|(e^s \circ L)(x)| &= \Big|\sum_{i=1}^{k} f^{(i)}(x) z_s^{(i)}\Big| = \Big|\sum_{i=1}^{k}\Big(\sum_{p=1}^{n-k} f_p^i x_p + f_{n-k+i}^i x_{n-k+i}\Big) \cdot z_s^{(i)}\Big| \\ &= \Big|\sum_{p=1}^{n-k}\Big(\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big) x_p + \sum_{i=1}^{k} (f_{n-k+i}^i z_s^{(i)}) x_{n-k+i}\Big|.\end{aligned}$$

By the above formula for $s \in \{1, \dots, n-k\}$

$$\|e^s \circ L\| = \max_{x \in \operatorname{ext} l_\infty^{(n)}} \Big|\sum_{p=1}^{n-k}\Big(\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big) x_p + \sum_{i=1}^{k} (f_{n-k+i}^i z_s^{(i)}) x_{n-k+i}\Big|. \tag{14}$$

Let $s_0 \in \{1, \dots, n-k\}$ be such that the maximum (13) is attained on the coordinate s_0. Then by (14)

$$\|L\| = \max_{x \in \operatorname{ext} l_\infty^{(n)}} \Big|\sum_{p=1}^{n-k}\Big(\sum_{i=1}^{k} f_p^i z_{s_0}^{(i)}\Big) x_p + \sum_{i=1}^{k} (f_{n-k+i}^i z_{s_0}^{(i)}) x_{n-k+i}\Big|.$$

Taking

$$x_p = \operatorname{sgn}\Big(\sum_{i=1}^{k} f_p^i z_{s_0}^{(i)}\Big) \text{ for } p = 1, \dots, n-k \text{ and } x_{n-k+i} = \operatorname{sgn}(z_{s_0}^{(i)}) \text{ for } i = 1, \dots, k$$

we get

$$\|L\| = \sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_{s_0}^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{s_0}^{(i)}|.$$

This completes the proof. ∎

LEMMA 2.4. *Let*

$$\begin{aligned}U = \Big\{&(z^{(1)}, \dots, z^{(k)}) \in (l_\infty^{(n)})^k : z^{(j)} \in H,\ j = 1, \dots, k, \\ &\max_{s=1,\dots,n-k}\Big(\sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(i)}|\Big) = 1\Big\}.\end{aligned} \tag{15}$$

We define

$$M_1^s(z^{(1)}, \dots, z^{(k)}) = \sum_{\substack{p=1\\ p\neq s}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(i)}| + \sum_{i=1}^{k} f_s^i z_s^{(i)} \tag{16}$$

$$\textit{for} \quad (z^{(1)}, \dots, z^{(k)}) \in U,\ s = 1, \dots, n-k,$$

$$M_1(z^{(1)}, \dots, z^{(k)}) = \max_{s=1,\dots,n-k} M_1^s(z^{(1)}, \dots, z^{(k)}) \quad \textit{for} \quad (z^{(1)}, \dots, z^{(k)}) \in U$$

and

$$M_2^s(z^{(1)},\dots,z^{(k)}) = \sum_{j=1}^{k}(-1+f^j_{n-k+j})z_s^{(j)} + \sum_{j=1}^{k} f^j_{n-k+j}|z_s^{(j)}| \tag{17}$$

$$\textit{for}\quad (z^{(1)},\dots,z^{(k)})\in U,\ s\in\{1,\dots,n-k\}\ \textit{satisfying}\ f_s^{s-(n-k)}=1/2,$$

$$M_2(z^{(1)},\dots,z^{(k)}) = \max_{\substack{s=n-k+1,\dots,n\\ \textit{satisfying}\ f_s^{s-(n-k)}=1/2}} M_2^s(z^{(1)},\dots,z^{(k)})\quad \textit{for}\quad (z^{(1)},\dots,z^{(k)})\in U.$$

Then the SUP-constant of $\mathcal{P}(l_\infty^{(n)},H)$ *is equal to*

$$r = \min_{(z^{(1)},\dots,z^{(k)})\in U} \max\big\{M_1(z^{(1)},\dots,z^{(k)}), M_2(z^{(1)},\dots,z^{(k)})\big\}. \tag{18}$$

Proof. Let $P_0\in\mathcal{P}(l_\infty^{(n)},H)$ be a minimal projection. Then $0\in\mathcal{L}_H$ is a best approximation to P_0 in $\mathcal{L}_H$. Assume that $L\in\mathcal{L}_H$ and $\|L\|=1$, i.e.,

$$L(\cdot)=\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\quad\text{and}\quad \max_{s=1,\dots,n-k}\Big(\sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f^i_{n-k+i}|z_s^{(i)}|\Big)=1$$

(see Lemma 2.3). Applying Lemma 1.5, Remark 2.2 and Lemma 2.1 we get

$$r = \min_{\substack{L\in\mathcal{L}_H\\ \|L\|=1}}\big(\max_{g\in E(P_0)} g(L)\big) = \min_{(z^{(1)},\dots,z^{(k)})\in U}\Big[\max\Big\{\max_{\substack{x\in A_s\\ s=1,\dots,n-k}}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big),$$

$$\max_{\substack{x\in B, s=n-k+1,\dots,n\\ \text{satisfying}\ f_s^{s-(n-k)}=1/2}}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big)\Big\}\Big]. \tag{19}$$

Our proof will be completed if we calculate the quantities

$$\max_{x\in A_s}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big)\quad\text{for } s=1,\dots,n-k,$$

$$\max_{x\in B}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big)\quad\text{for } s=n-k+1,\dots,n\quad\text{satisfying } f_s^{s-(n-k)}=1/2.$$

For $s\in\{1,\dots,n-k\}$ and $x\in A_s$ we obtain

$$(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big) = \sum_{j=1}^{k} f^j(x)z_s^{(j)} = \sum_{j=1}^{k}\Big(\sum_{p=1}^{n-k} f_p^j x_p z_s^{(j)} + f^j_{n-k+j}x_{n-k+j}z_s^{(j)}\Big)$$

$$= \sum_{p=1}^{n-k}\Big(\sum_{j=1}^{k} f_p^j z_s^{(j)}\Big)x_p + \sum_{j=1}^{k} f^j_{n-k+j}z_s^{(j)}x_{n-k+j}.$$

Taking $x_p=\operatorname{sgn}\big(\sum_{i=1}^k f_p^i z^{(j)}\big)$ for $p\in\{1,\dots,n-k\}\setminus\{s\}$ and $x_{n-k+j}=\operatorname{sgn}(z_s^{(j)})$ for $j=1,\dots,k$ we get

$$\max_{x\in A_s}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big) = \sum_{\substack{p=1\\ p\neq s}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(j)}\Big| + \sum_{i=1}^{k} f^i_{n-k+i}|z_s^{(j)}| + \sum_{i=1}^{k} f_s^i z_s^{(j)},$$

hence

$$M_1^s(z^{(1)},\dots,z^{(k)}) = \sum_{\substack{p=1\\ p\neq s}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(j)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(j)}| + \sum_{i=1}^{k} f_s^i z_s^{(j)}.$$

Let $s \in \{n-k+1,\dots,n\}$ be such that $f_s^{s-(n-k)} = 1/2$ and let $x \in B$. Then we calculate

$$\begin{aligned}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big) &= \sum_{j=1}^{k} f^j(x)z_s^{(j)} = \sum_{j=1}^{k}\Big(\Big(-\sum_{p=1}^{n-k} f_p^j\Big)z_s^{(j)} + f_{n-k+j}^j x_{n-k+j} z_s^{(j)}\Big)\\ &= \sum_{j=1}^{k}(-1+f_{n-k+j}^j)z_s^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j x_{n-k+j} z_s^{(j)}.\end{aligned}$$

Taking $x_{n-k+j} = \operatorname{sgn}(z_s^{(j)})$ for $j=1,\dots,k$ we get

$$\max_{x\in B}(e^s\otimes x)\Big(\sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}\Big) = \sum_{j=1}^{k}(-1+f_{n-k+j}^j)z_s^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j |z_s^{(j)}|,$$

hence

$$M_2^s(z^{(1)},\dots,z^{(k)}) = \sum_{j=1}^{k}(-1+f_{n-k+j}^j)z_s^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j |z_s^{(j)}|.$$

Consequently, by (19) we obtain our result. ■

THEOREM 2.5. *The SUP-constant of the space $\mathcal{P}(l_\infty^{(n)},H)$ satisfies the inequality*

$$r \leqslant 1 - 2\max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j. \tag{20}$$

Proof. Let $i_0 \in \{1,\dots,n-k\}$ and $j_0 \in \{1,\dots,k\}$ be such that

$$f_{i_0}^{j_0} = \max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j. \tag{21}$$

Let us define

$$z^{(j_0)} = \Big(0_1,\dots,,0_{i_0-1},-1_{i_0},0_{i_0+1},\dots,0_{n-k},\frac{f_{i_0}^1}{f_{n-k+1}^1},\dots,\frac{f_{i_0}^k}{f_n^k}\Big)$$

and

$$z^{(j)} = (0_1,\dots,0_n) \quad \text{for} \quad j\neq j_0.$$

Now we show that $(z^{(1)},\dots,z^{(k)}) \in U$ (see (15)). Since

$$f^{(p)}(z^{(j_0)}) = -f_{i_0}^p + f_{n-k+p}^p\cdot\frac{f_{i_0}^p}{f_{n-k+p}^p} = -f_{i_0}^p + f_{i_0}^p = 0$$

and by the definition of $z^{(j)}$ for $j\neq j_0$ we obtain that $z^{(j)}\in H$ for $j=1,\dots,k$. Taking into account the equalities

$$\sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_{i_0}^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^{(i)}| = \sum_{p=1}^{n-k} |f_p^{j_0}\cdot(-1)| + (f_{n-k+j_0}^{j_0})\cdot|-1|$$

$$= \Big(\sum_{p=1}^{n-k} f_p^{j_0}\Big) + \Big(1-\sum_{j=1}^{n-k} f_j^{j_0}\Big) = 1,$$

$$\sum_{p=1}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(i)}| = 0 \quad \text{for} \quad s\in\{1,\dots,n-k\}\setminus\{i_0\},$$

we demonstrate that $(z^{(1)},\dots,z^{(k)})\in U$. Now we calculate constants $M_1(z^{(1)},\dots,z^{(k)})$ and $M_2(z^{(1)},\dots,z^{(k)})$. First note that

$$M_1^{i_0}(z^{(1)},\dots,z^{(k)}) = \sum_{\substack{p=1\\p\neq i_0}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_{i_0}^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^{(i)}| + \sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)}$$

$$= \sum_{\substack{p=1\\p\neq i_0}}^{n-k} f_p^{j_0} + f_{n-k+j_0}^{j_0} - f_{i_0}^{j_0} = 1 - 2f_{i_0}^{j_0}.$$

For $s\in\{1,\dots,n-k\}\setminus\{i_0\}$ we get

$$M_1^s(z^{(1)},\dots,z^{(k)}) = \sum_{\substack{p=1\\p\neq s}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_s^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_s^{(i)}| + \sum_{i=1}^{k} f_s^i z_s^{(i)} = 0.$$

Hence

$$M_1(z^{(1)},\dots,z^{(k)}) = 1 - 2f_{i_0}^{j_0}. \tag{22}$$

Let us consider $s\in\{n-k+1,\dots,n\}$ such that $f_s^{s-(n-k)} = 1/2$. Then

$$z_s^{(j_0)} = \frac{f_{i_0}^{s-(n-k)}}{f_s^{s-(n-k)}} = 2f_{i_0}^{s-(n-k)}$$

and

$$M_2^s(z^{(1)},\dots,z^{(k)}) = \sum_{j=1}^{k}(-1+f_{n-k+j}^j)z_s^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j |z_s^{(j)}|$$

$$= (-1+f_{n-k+j_0}^{j_0})z_s^{(j_0)} + f_{n-k+j_0}^{j_0}|z_s^{(j_0)}| = (2f_{n-k+j_0}^{j_0}-1)z_s^{(j_0)}$$

$$= 2(2f_{n-k+j_0}^{j_0}-1)f_{i_0}^{s-(n-k)}.$$

We show that

$$M_2^s(z^{(1)},\dots,z^{(k)}) \leqslant 1-2f_{i_0}^{j_0} \quad \text{for } s\in\{n-k+1,\dots,n\} \text{ such that } f_s^{s-(n-k)} = 1/2. \tag{23}$$

By (5) and definitions of i_0 and j_0 (see (21)) we obtain

$$2(2f_{n-k+j_0}^{j_0}-1)f_{i_0}^{s-(n-k)} \leqslant 2(2(1-f_{i_0}^{j_0})-1)f_{i_0}^{j_0} = 2(1-2f_{i_0}^{j_0})f_{i_0}^{j_0}.$$

Hence, to demonstrate inequality (23) it is sufficient to verify that

$$2(1-2f_{i_0}^{j_0})f_{i_0}^{j_0} \leqslant 1-2f_{i_0}^{j_0}. \tag{24}$$

Note that (24) follows immediately from the fact that $f_{i_0}^{j_0} < 1/2$. By Lemma 2.4 and conditions (22)–(23) we get

$$r \leqslant \max\{M_1(z^{(1)},\dots,z^{(k)}), M_2(z^{(1)},\dots,z^{(k)})\} = 1-2\max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j. \ \blacksquare \tag{25}$$

Now we prove the main result of this paper.

THEOREM 2.6. *Let $f^{(1)},\dots,f^{(k)}$ satisfy conditions* (5) *and $n > k \geqslant 3$. Then the SUP-constant r of $\mathcal{P}(l_\infty^{(n)}, H)$ is equal to*

$$1-2\max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j.$$

Proof. Let $(z^{(1)},\dots,z^{(k)}) \in U$ realize minimum in (18). Then by (15) there exists $i_0 \in \{1,\dots,n-k\}$ such that

$$\sum_{p=1}^{n-k}\Big|\sum_{j=1}^{k} f_p^j z_{i_0}^{(j)}\Big| + \sum_{j=1}^{k} f_{n-k+j}^j |z_{i_0}^{(j)}| = 1. \tag{26}$$

By (18), (20) and (26) we get

$$\sum_{j=1}^{k} f_{i_0}^j z_{i_0}^{(j)} < 0. \tag{27}$$

We show that j satisfying $z_{i_0}^{(j)} > 0$ does not exist. Assume to the contrary that for $l_1,\dots,l_k$ such that $\{l_1,\dots,l_k\} = \{1,\dots,k\}$

$$z_{i_0}^{(l_1)} > 0,\ \dots,\ z_{i_0}^{(l_s)} > 0,\ z_{i_0}^{(l_{s+1})} \leqslant 0,\ \dots,\ z_{i_0}^{(l_k)} \leqslant 0. \tag{28}$$

Taking into account (20) it is sufficient to verify that inequalities (28) imply

$$M_1(z^{(1)},\dots,z^{(k)}) > 1-2\max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j. \tag{29}$$

If we show that the inequality

$$\sum_{\substack{p=1\\ p\neq i_0}}^{n-k}\Big|\sum_{j=1}^{k} f_p^j z_{i_0}^{(j)}\Big| + \sum_{j=1}^{k} f_{n-k+j}^j |z_{i_0}^{(j)}| + \sum_{j=1}^{k} f_{i_0}^j z_{i_0}^{(j)} \leqslant 1-2\max_{j=1,\dots,k} f_{i_0}^j \tag{30}$$

cannot be true then this will establish

$$M_1^{i_0}(z^{(1)},\dots,z^{(k)}) > 1-2\max_{j=1,\dots,k} f_{i_0}^j \geqslant 1-2\max_{\substack{j=1,\dots,k\\ i=1,\dots,n-k}} f_i^j,$$

which implies (29). Assume to the contrary that for $(z^{(1)},\dots,z^{(k)})$ satisfying (28) the inequality (30) is true. Then

$$\sum_{\substack{p=1\\p\neq i_0}}^{n-k}\Big|\sum_{i=1}^{k} f_p^i z_{i_0}^{(i)}\Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^{(i)}| + \sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)}$$
$$= 1 - \Big|\sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)}\Big| + \sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)} = 1 + 2\Big(\sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)}\Big).$$

Hence (30) is equivalent to the following inequalities

$$1 + 2\Big(\sum_{j=1}^{k} f_{i_0}^j z_{i_0}^{(j)}\Big) \leqslant 1 - 2\max_{j=1,\ldots,k} f_{i_0}^j$$

and

$$\max_{j=1,\ldots,k} f_{i_0}^j + \sum_{j=1}^{k} f_{i_0}^j z_{i_0}^{(j)} \leqslant 0. \tag{31}$$

By (26)

$$1 = \sum_{p=1}^{n-k}\Big|\sum_{j=1}^{k} f_p^j z_{i_0}^{(j)}\Big| + \sum_{j=1}^{k} f_{n-k+j}^j |z_{i_0}^{(j)}|$$
$$\geqslant -\sum_{p=1}^{n-k}\sum_{j=1}^{k} f_p^j z_{i_0}^{(j)} + \sum_{j=1}^{s} f_{n-k+l_j}^{l_j} z_{i_0}^{(l_j)} - \sum_{j=s+1}^{k} f_{n-k+l_j}^{l_j} z_{i_0}^{(l_j)}$$
$$= \sum_{j=1}^{s}\Big(-\sum_{p=1}^{n-k} f_p^{l_j} + f_{n-k+l_j}^{l_j}\Big) z_{i_0}^{(j)} + \sum_{j=s+1}^{k}\Big(-\sum_{p=1}^{n-k} f_p^{l_j} - f_{n-k+l_j}^{l_j}\Big) z_{i_0}^{(l_j)}$$
$$= \sum_{j=1}^{s}(2f_{n-k+l_j}^{l_j} - 1) z_{i_0}^{(l_j)} - \sum_{j=s+1}^{k} z_{i_0}^{(l_j)}.$$

From the above calculations we get

$$\sum_{j=s+1}^{k} z_{i_0}^{(l_j)} \geqslant \sum_{j=1}^{s}(2f_{n-k+l_j}^{l_j} - 1) z_{i_0}^{(l_j)} - 1. \tag{32}$$

Applying (28) and (32) we can write

$$\max_{j=1,\ldots,k} f_{i_0}^j + \sum_{j=1}^{k} f_{i_0}^j z_{i_0}^{(j)} = \max_{j=1,\ldots,k} f_{i_0}^j + \sum_{j=1}^{s} f_{i_0}^{l_j} z_{i_0}^{(l_j)} + \sum_{j=s+1}^{k} f_{i_0}^{l_j} z_{i_0}^{(l_j)}$$
$$\geqslant \max_{j=1,\ldots,k} f_{i_0}^j + \sum_{j=1}^{s} f_{i_0}^{l_j} z_{i_0}^{(l_j)} + \max_{j=1,\ldots,k} f_{i_0}^j \cdot \sum_{j=s+1}^{k} z_{i_0}^{(l_j)}$$
$$\geqslant \max_{j=1,\ldots,k} f_{i_0}^j + \sum_{j=1}^{s} f_{i_0}^{l_j} z_{i_0}^{(l_j)} + \max_{j=1,\ldots,k} f_{i_0}^j \cdot \Big(\sum_{j=1}^{s}(2f_{n-k+l_j}^{l_j} - 1) z_{i_0}^{(l_j)} - 1\Big)$$

$$= \sum_{j=1}^{s} \Big[\max_{j=1,\dots,k} f_{i_0}^{l_j} \cdot (2 f_{n-k+l_j}^{l_j} - 1) + f_{i_0}^{l_j} \Big] z_{i_0}^{(l_j)} > 0;$$

a contradiction with (31). Hence by (27) for $i_0 \in \{1, \dots, k\}$ satisfying (26)

$$z_{i_0}^{(j)} \leqslant 0 \quad \text{for} \quad j = 1, \dots, k. \tag{33}$$

Now (26) is equivalent to

$$\sum_{j=1}^{k} z_{i_0}^{(j)} = -1. \tag{34}$$

By (16), (33) and (34) we get

$$M_1^{i_0}(z^{(1)}, \dots, z^{(k)}) = \sum_{\substack{p=1 \\ p \neq i_0}}^{n-k} \Big| \sum_{i=1}^{k} f_p^i z_{i_0}^{(i)} \Big| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^{(i)}| + \sum_{i=1}^{k} f_s^i z_{i_0}^{(i)}$$

$$= 1 + 2\Big(\sum_{i=1}^{k} f_{i_0}^i z_{i_0}^{(i)} \Big) \geqslant 1 + 2 \max_{\substack{j=1,\dots,k \\ i=1,\dots,n-k}} f_i^j \cdot \Big(\sum_{i=1}^{k} z_{i_0}^{(i)} \Big) = 1 - 2 \max_{\substack{j=1,\dots,k \\ i=1,\dots,n-k}} f_i^j.$$

Hence

$$r = \max\{M_1(z^{(1)}, \dots, z^{(k)}), M_2(z^{(1)}, \dots, z^{(k)})\} \geqslant M_1^{i_0}(z^{(1)}, \dots, z^{(k)}) \geqslant 1 - 2 \max_{\substack{j=1,\dots,k \\ i=1,\dots,n-k}} f_i^j.$$

By (20)

$$r = 1 - 2 \max_{\substack{j=1,\dots,k \\ i=1,\dots,n-k}} f_i^j.$$

The proof is complete. ∎

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