FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

CONVERGENCE THEOREMS BY HYBRID PROJECTION METHODS FOR LIPSCHITZ-CONTINUOUS MONOTONE MAPPINGS AND A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce two iterative schemes for finding a common element of the set of a common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping in a Hilbert space by using the hybrid projection methods in the mathematical programming. Then we prove strong convergence theorems by the hybrid projection methods for a monotone, Lipschitz-continuous mapping and a countable family of nonexpansive mappings. Moreover, we apply our result to the problem for finding a common fixed point of two mappings, such that one of these mappings is nonexpansive and the other is taken from the more general class of Lipschitz pseudocontractive mappings. Our results extend and improve the results of Nadezhkina and Takahashi [SIAM J. Optim. 16 (2006), 1230–1241], Zeng and Yao [Taiwanese J. Math. 10 (2006), 1293–1303] and many authors.

1. Introduction. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of

2010 *Mathematics Subject Classification*: Primary 47H09; 47H10; Secondary 46C05, 47D03.
Key words and phrases: fixed point, hybrid projection method, monotone mapping, nonexpansive mapping, variational inequality.
The paper is in final form and no version of it will be published elsewhere.

H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{1}$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{2}$$

for all $u, v \in C$. A is called α-*inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{3}$$

for all $u, v \in C$. A is called k-*Lipschitz-continuous* if there exists a positive constant k such that for all $u, v \in C$

$$\|Au - Av\| \leq k\|u - v\|. \tag{4}$$

Obviously, it is easy to see that every α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au\rangle \geq 0$ for all $v \in C$. We denote by $\mathrm{VI}(A, C)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g. [17, 18] and the references therein.

Construction of fixed points of nonexpansive mapping is an important subject in the theory of nonexpansive mappings. However, the sequence $\{S^n x\}_{n=0}^{\infty}$ of iterates of the mapping S at a point $x \in C$ may not converge even in weak topology. More precisely, Mann's iterated procedure is a sequence $\{x_n\}$ which is generated in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \quad n \geq 0, \tag{5}$$

where the initial guess $x_0 \in C$ is chosen arbitrary. However, we note that Mann's iterations have only weak convergence even in a Hilbert space [9].

For finding an element of $F(S) \cap \mathrm{VI}(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive, and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \end{aligned} \tag{6}$$

for every $n \geq 0$ where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (6) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, A)$.

In 2006, motivated by the idea of Korpelevich's extragradient method [8], Nadezhkina and Takahashi [11] introduced an iterative scheme for finding an element of $F(S) \cap \mathrm{VI}(C, A)$. They proved the following weak convergence result.

THEOREM 1.1 ([11, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H.*

Let S be a nonexpansive mappings from C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n) \quad \forall n \geq 0, \end{aligned} \tag{7}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \mathrm{VI}(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap \mathrm{VI}(C,A)} x_n$.

Recently, Zeng and Yao [20] proved the following strong convergence theorem:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n) \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\{\lambda_n k\} \subset (0, 1 - \delta)$ for some $\delta \in (0, 1)$;
(ii) $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

They proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{F(S)\cap \mathrm{VI}(C,A)} x_0$ provided that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

On the other hand, motivated by the idea of Nakajo and Takahashi [12], Nadezhkina and Takahashi [10] introduced the following iterative scheme for finding an element of $F(S) \cap \mathrm{VI}(C, A)$ and proved the following strong convergence theorem by using the CQ hybrid method. Recently, Takahashi, Takeuchi and Kubota [15] proved the following strong convergence theorem by using the new hybrid method in mathematical programming.

THEOREM 1.2 ([15, Theorem 3.3]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence as follows:*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1, \end{aligned} \tag{8}$$

where $0 \leq \alpha_n < \alpha < 1$ for all $n \geq 1$. Let T be a mapping of C into itself such that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Suppose that for each bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_{n+1} - T_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - T_m z_n\| = 0$ for each $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

Inspired and motivated by the previously mentioned results, the purpose of this paper is to improve and generalize the processes (7) and (8) to the new general processes for finding a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for a monotone Lipschitz-continuous mapping. Let C be nonempty closed convex subset of a

Hilbert space H, A be a monotone and k-Lipschitz-continuous mapping of C into H and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap \mathrm{VI}(C,A) \neq \emptyset$. Define $\{x_n\}$ in two ways:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0,1,2,\ldots, \end{aligned} \tag{9}$$

and

$$\begin{aligned} x_0 \in H, \quad C_1 = C, \quad & x_1 = P_{C_1} x_0, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A y_n), \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1, \end{aligned} \tag{10}$$

where $0 \leq \alpha_n < c < 1$ and $0 < a < \lambda_n < b < \frac{1}{k}$ for all $n = 1,2,3\ldots$.

We shall prove that both iterations (9) and (10) converge strongly to a point z in $\bigcap_{n=1}^{\infty} F(S_n) \cap \mathrm{VI}(C,A)$. Our results extend and improve the corresponding ones announced by Nadezhkina and Takahashi [10] and Zeng and Yao [20].

2. Preliminaries. Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{11}$$

and

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{12}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the *Opial condition* [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{13}$$

holds for every $y \in H$ with $y \neq x$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{14}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{15}$$

$$\|x-y\|^2 \geq \|x-P_Cx\|^2 + \|y-P_Cx\|^2 \tag{16}$$

for all $x \in H, y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in \mathrm{VI}(A,C) \Leftrightarrow u = P_C(u-\lambda Au), \text{ for all } \lambda > 0. \tag{17}$$

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-h\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let N_Cv be the normal cone to C at $v \in C$, i.e.,

$$N_Cv = \{w \in H : \langle v-u, w\rangle \geq 0 \ \forall u \in C\},$$

and define

$$Tv = \begin{cases} Av + N_Cv, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{18}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C,A)$; see [14].

The following lemma will be useful for proving the convergence result of this paper.

Lemma 2.1 (Browder [2]). *Let C be a non-empty closed convex subset of a uniformly convex Banach space E, and suppose $S : C \to E$ is nonexpansive. Then the mapping $I - S$ is demiclosed at zero, i.e.,*

$$x_n \rightharpoonup x, \quad x_n - Sx_n \to 0 \quad \textit{implies} \quad x = Sx. \tag{19}$$

Lemma 2.2 ([1], Lemma 3.2). *Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_ny\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_ny \qquad \textit{for all } y \in C.$$

Then $\lim_{n\to\infty} \sup\{\|T_nz - Tz\| : z \in C\} = 0$.

3. Main theorems. In this section, we prove strong convergence theorems by hybrid methods for finding a common fixed points of k-Lipschitz-continuous monotone mappings and a family of nonexpansive mappings in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and k-Lipschitz-continuous mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap \mathrm{VI}(C,A)$ is nonempty. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in C defined as follows:*

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n Ax_n), \\ z_n &= \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n Ay_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \end{aligned}$$

$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,$$

where $0 \leq \alpha_n < c < 1$ *and* $0 < a < \lambda_n < b < \frac{1}{k}$ *for all* $n = 1, 2, 3 \ldots$. *Let* $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C *and* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_{F(S)\cap \mathrm{VI}(C,A)} x_0$.

Proof. We divide the proof into four steps.

Step 1. We claim that C_n and Q_n are closed and convex for all $n \geq 0$, and $F(S) \cap \mathrm{VI}(C, A) \subset C_n \cap Q_n$, for all $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for all $n \geq 0$. Since $C_n = \{z \in C : \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0\}$, we deduce that C_n is convex for all $n \geq 0$.

Next, we show that

$$F(S) \cap \mathrm{VI}(C, A) \subset C_n, \quad \forall n \geq 0. \tag{20}$$

Put $v_n = P_C(x_n - \lambda_n A y_n)$ for all $n \geq 0$. Let $u \in F(S) \cap \mathrm{VI}(C, A)$. Thus, we have $u = P_C(u - \lambda_n A u)$. From (16) and the monotonicity of A, we have

$$\begin{aligned}
\|v_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - v_n\|^2 \\
&= \|x_n - u\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle A y_n, u - v_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - v_n\|^2 \\
&\quad + 2\lambda_n(\langle A y_n - A u, u - y_n\rangle + \langle A u, u - y_n\rangle) + \langle A y_n, y_n - v_n\rangle \\
&\leq \|x_n - u\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n A y_n - y_n, v_n - y_n\rangle.
\end{aligned}$$

Since $y_n = P_C(x_n - \lambda_n A x_n)$ and A is k-Lipschitz-continuous, it follows that

$$\begin{aligned}
\langle x_n - \lambda_n A y_n - y_n, v_n - y_n\rangle &= \langle x_n - \lambda_n A x_n - y_n, v_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, v_n - y_n\rangle \\
&\leq \langle \lambda_n A x_n - \lambda_n A y_n, v_n - y_n\rangle \\
&\leq \lambda_n k\|x_n - y_n\|\|v_n - y_n\|.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|v_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n k\|x_n - y_n\|\|v_n - y_n\| \\
&\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + \lambda_n k(\|x_n - y_n\|^2 - \|v_n - y_n\|^2) \\
&= \|x_n - u\|^2 + (\lambda_n k - 1)\|x_n - y_n\|^2 - (1 + \lambda_n k)\|y_n - v_n\|^2 \qquad (21) \\
&\leq \|x_n - u\|^2.
\end{aligned}$$

Therefore, from (21), $z_n = \alpha_n x_n - (1-\alpha_n)S_n v_n$ and $u = S_n u$, we have

$$\begin{aligned}
\|z_n - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(S_n v_n - u)\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n v_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|v_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)(\|x_n - u\|^2 + (\lambda_n k - 1)\|x_n - y_n\|^2) \\
&\le \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n k - 1)\|x_n - y_n\|^2 \qquad (22) \\
&\le \|x_n - u\|^2
\end{aligned}$$

for all $n \ge 0$ and hence $u \in C_n$. So $F(S) \cap \mathrm{VI}(C, A) \subset C_n$, for all $n \ge 0$.

Next, we show that

$$F(S) \cap \mathrm{VI}(C, A) \subset Q_n \quad \text{for all } n \ge 0. \tag{23}$$

We prove this by induction. For $n = 0$, we have $F(S) \cap \mathrm{VI}(C, A) \subset CQ_0$. Suppose that $F(S) \cap \mathrm{VI}(C, A) \subset Q_n$. Then $\emptyset \ne F(S) \cap \mathrm{VI}(C, A) \subset C_n \cap Q_n$ and there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. Then

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0$$

for each $z \in C_n \cap Q_n$. In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1}\rangle \ge 0$$

for each $p \in F(S) \cap \mathrm{VI}(C, A)$. It follows that $F(S) \cap \mathrm{VI}(C, A) \subset Q_{n+1}$ and hence (23) holds. Therefore

$$F(S) \cap \mathrm{VI}(C, A) \subset C_n \cap Q_n \quad \text{for all } n \ge 0.$$

This implies that $\{x_n\}$ is well-defined.

Step 2. We claim that the following statements hold:

1. $\{x_n\}$ is bounded;
2. $\|x_{n+1} - x_n\| \to 0$.

It follows from the definition of Q_n that $x_n = P_{Q_n} x_0$. Therefore

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } z \in Q_n \text{ and all } n \ge 0.$$

Let $z \in F(S) \cap \mathrm{VI}(C, A)$. Then

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } n \ge 0.$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \quad \text{for all } n \ge 0.$$

Therefore $\{\|x_n - x_0\|\}$ is nondecreasing and bounded. So $\lim_{n\to\infty} \|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. From (21) and (22), we also obtain that $\{z_n\}$ and $\{v_n\}$ are bounded.

Since $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have $\langle x_n - x_{n+1}, x_0 - x_n\rangle \ge 0$. It follows from (11) that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle
\end{aligned}$$

$$\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2$$

for all $n = 0, 1, 2 \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{24}$$

Step 3. We claim that the following statements hold:

1. $\lim_{n\to\infty} \|x_n - z_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
2. $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|S_n x_n - x_n\| = 0$.

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n + x_{n+1}\|$. This implies that

$$\begin{aligned}\|x_n - S_n v_n\| &= \frac{1}{1-\alpha_n}\|z_n - x_n\| \\ &\leq \frac{1}{1-\alpha_n}\left(\|z_n - x_{n+1}\| + \|x_n - x_{n+1}\|\right) \\ &\leq \frac{2}{1-\alpha_n}\|x_n - x_{n+1}\|\end{aligned}$$

for all $n \geq 0$. From (24) and $0 \leq \alpha_n < c < 1$, we get

$$\lim_{n\to\infty} \|x_n - S_n v_n\| = 0. \tag{25}$$

Since $\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|, \quad \forall n \geq 0$, it follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{26}$$

For each $u \in F(S) \cap \mathrm{VI}(C, A)$, from (22), we obtain

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n k - 1)\|x_n - y_n\|^2.$$

Thus, we have

$$\begin{aligned}\|x_n - y_n\|^2 &\leq \frac{1}{(1-\alpha_n)(\lambda_n k)}(\|x_n - u\|^2 - \|z_n - u\|^2) \\ &\leq \frac{1}{(1-\alpha_n)(\lambda_n k)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$, $\{z_n\}$ are bounded, we obtain $\|x_n - y_n\| \to 0$.

As A is k-Lipschitz-continuous, we have

$$\begin{aligned}\|y_n - v_n\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x_n - \lambda_n Ay_n)\| \\ &\leq \lambda_n \|Ax_n - Ay_n\| \\ &\leq \lambda_n k \|x_n - y_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{27}$$

Moreover, we note that

$$\begin{aligned}\|S_n x_n - x_n\| &\leq \|S_n x_n - S_n y_n\| + \|S_n y_n - S_n v_n\| + \|S_n v_n - x_n\| \\ &\leq \|x_n - y_n\| + \|y_n - v_n\| + \|S_n v_n - x_n\|.\end{aligned}$$

From (25), (26) and (27), we obtain

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{28}$$

Since $\sum_{n=1}^{\infty} \sup\{\|Sz - S_n z\| : z \in \{x_n\}\} < \infty$, and

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \\ &\leq \sup\{\|Sz - S_n z\| : z \in \{x_n\}\} + \|S_n x_n - x_n\|,\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{29}$$

Step 4. We claim that $\{x_n\}$ converge strongly to z_0, where $z_0 = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to z. By [11, Theorem 3.1 pp. 197–198], we can show that $z \in \mathrm{VI}(C, A)$. Next, we show that $z \in F(S)$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ which converges weakly to z. Since $I - S$ is demiclosed, it follows by Lemma 2.1 that $z \in F(S)$. Hence $z \in F(S) \cap \mathrm{VI}(C, A)$.

Since $x_n = P_{Q_n} x_0$ and $z_0 \in F(S) \cap \mathrm{VI}(C, A) \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|z_0 - x_0\|.$$

It follows from $z_0 = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$ and the lower semicontinuity of the norm that

$$\|z_0 - x_0\| \leq \|z - x_0\| \leq \liminf_{k\to\infty} \|x_{n_k} - x_0\| \leq \limsup_{k\to\infty} \|x_{n_k} - x_0\| \leq \|z_0 - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|z - x_0\| = \|z_0 - x_0\|$.

Using the Kadec-Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_k} = z = z_0.$$

Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z_0, where $z_0 = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$. So, from $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$, we infer that both $\{y_n\}$ and $\{z_n\}$ converge to $z_0 \in P_{F(T)\cap \mathrm{VI}(C,A)} x_0$. ■

Setting $S_n \equiv S$ in Theorem 3.1, we have the following result.

Theorem 3.2 (Nadezhkina and Takahashi [10, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and k-Lipschitz-continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in C defined as follows*:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{30}$$

where $0 \leq \alpha_n < c < 1$ and $0 < a < \lambda_n < b < \frac{1}{k}$ for all $n = 1, 2, 3 \ldots$. Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S)\cap \mathrm{VI}(C,A)} x_0$.

Setting $P_C(I - \lambda_n A) = I$ in Theorem 3.2, we have the following theorem.

THEOREM 3.3 (Nakajo and Takahashi [12]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping from C into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) S x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{31}$$

where $0 \leq \alpha_n < c < 1$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x$.

THEOREM 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and k-Lipschitz-continuous mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap \mathrm{VI}(C, A) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A y_n), \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n = 1, 2, 3 \ldots, \end{aligned}$$

where $0 \leq \alpha_n < c < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n = 1, 2, 3 \ldots$. Let $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C and S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges strongly to $w = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$.

Proof. We first show by induction that $F(S) \cap \mathrm{VI}(C, A) \subset C_n$ for all $n = 1, 2, 3 \ldots$. It is obvious that $F(S) \cap \mathrm{VI}(C, A) \subset C = C_1$. Suppose that $F(S) \cap \mathrm{VI}(C, A) \subset C_k$ for each $k = 1, 2, 3 \ldots$. Hence, for $u \in F(S) \cap \mathrm{VI}(C, A) \subset C_k$ we have $u = P_C(u - \lambda_{k+1} A u)$. Putting $v_n = P_C(x_n - \lambda_n A y_n)$ for all $n \geq 0$, as in the proof of Theorem 3.1, we can show that

$$\|v_n - u\|^2 \leq \|x_n - u\|^2 + (\lambda_n k - 1)\|x_n - y_n\|^2 \leq \|x_n - u\|^2$$

and

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n k - 1)\|x_n - y_n\|^2 \leq \|x_n - u\|^2$$

for all $n \geq 0$. Thus $u \in C_n$, $n \geq 0$ and hence $F(S) \cap \mathrm{VI}(C, A) \subset C_n$, for all $n \geq 0$.

Next, we prove that C_n is closed and convex for all $n = 1, 2, 3 \ldots$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k = 1, 2, 3 \ldots$. For $z \in C_k$, we know that $\|z_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z \rangle \geq 0.$$

So, C_{k+1} is closed and convex. Then, for any $n = 1, 2, 3 \ldots$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y \rangle \geq 0$$

for each $y \in C_n$. Using $F(S) \cap \mathrm{VI}(C, A) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u \rangle \geq 0 \text{ for each } u \in F(S) \cap \mathrm{VI}(C, A) \quad \text{and} \quad n = 1, 2, 3 \ldots.$$

So, for $u \in F(S) \cap \mathrm{VI}(C, A)$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } \; u \in F(S) \cap \mathrm{VI}(C, A) \quad \text{and} \quad n = 1, 2, 3 \ldots.$$

By the same as in the proof of [15, Theorem 3.3], we can show that $\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0$ and hence $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

On the other hand, from $x_{n+1} \in C_{n+1} \subset C$, we have

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{32}$$

Further, we have

$$\begin{aligned} \|z_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n) S_n v_n - x_n\| \\ &= (1 - \alpha_n) \|S_n v_n - x_n\|. \end{aligned}$$

As in the proof of Theorem 3.1 (Step 3), we obtain

$$\lim_{n\to\infty} \|x_n - S_n v_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - S x_n\| = 0, \tag{33}$$

since $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_0$. By the same argument as in the proof of [6, Theorem 3.1, pp. 346–347], we can show that $z_0 \in \mathrm{VI}(C, A)$. Since $I - S$ is demiclosed, it follows by Lemma 2.1 that $z_0 \in F(S)$. Hence $z_0 \in F(S) \cap \mathrm{VI}(C, A)$.

Finally, we show that $x_n \to w$, where $w = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. Since $\{x_{n_k}\} \subset C$ and C is closed and convex, we obtain $w' \in C$.

Since $x_n = P_{C_n} x_0$ and $w \in F(S) \cap \mathrm{VI}(C, A) \subset C_n$, we have

$$\|x_n - x_0\| \leq \|w - x_0\|.$$

It follows from $w = P_{F(S)\cap \mathrm{VI}(C,A)} x_0$ and the lower semicontinuity of the norm that

$$\|w - x_0\| \leq \|w' - x_0\| \leq \liminf_{k\to\infty} \|x_{n_k} - x_0\| \leq \limsup_{k\to\infty} \|x_{n_k} - x_0\| \leq \|w - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_k} = w' = w.$$

Therefore $\{x_n\}$ converges strongly to w, where $w = P_{F(T)\cap \mathrm{VI}(C,A)} x_0$. ■

Setting $S_n \equiv S$ in Theorem 3.4, we have the following result.

THEOREM 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continous mapping of C into H. Let S be a nonexpansive* *mappings from C into itself such that $F(S)\cap \mathrm{VI}(C,A)\neq\emptyset$ and let $x_0\in H$. For $C_1=C$ and $x_1=P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows*:

$$\begin{aligned} y_n &= P_C(x_n-\lambda_nAx_n),\\ z_n &= \alpha_nx_n+(1-\alpha_n)SP_C(x_n-\lambda_nAy_n),\\ C_{n+1} &= \{z\in C_n:\|z_n-z\|\le\|x_n-z\|\},\\ x_{n+1} &= P_{C_{n+1}}x_0,\quad n=1,2,3\ldots, \end{aligned}\tag{34}$$

where $0\le\alpha_n<c<1$ and $0<a<\lambda_n<b<2\alpha$ for all $n=1,2,3\ldots$. Then $\{x_n\}$ converges strongly to $z_0=P_{F(S)\cap\mathrm{VI}(C,A)}x_0$.

Setting $P_C(I-\lambda_nA)=I$ in Theorem 3.4, we obtain Theorem 1.2.

4. Applications

4.1. Monotone operator. In this section, we consider the problem of finding a zero of a monotone operator. A multivalued operator $T:H\to 2^H$ with domain $D(T)=\{z\in H:Tz\neq\emptyset\}$ and range $R(T)=\{Tz:z\in D(T)\}$ is said to be monotone if for each $x_i\in D(T)$ and $y_i\in Tx_i$, $i=1,2$, we have $\langle x_1-x_2,y_1-y_2\rangle\ge 0$. A monotone operator T is said to be maximal if its graph $G(T)=\{(x,y):y\in Tx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and let $T:H\to 2^H$ be a maximal monotone operator. Then we can define, for each $r>0$, a nonexpansive single valued mapping $J_r:H\to H$ by $J_r=(I+rT)^{-1}$. It is called the resolvent (or the proximal mapping) of T. We also define the Yosida approximation A_r by $A_r=(I-J_r)/r$. We know that $A_rx\in TJ_rx$ and $\|A_rx\|\le\inf\{\|y\|:y\in Tx\}$ for all $x\in H$. We also know that $T^{-1}0=F(J_r)$ for all $r>0$; see, for instance, Rockafellar [14].

Lemma 4.1 (the resolvent identity). *For $\lambda,\mu>0$, there holds the identity*

$$J_\lambda x=J_\mu\Big(\frac{\mu}{\lambda}+\Big(1-\frac{\mu}{\lambda}\Big)J_\lambda x\Big),\quad x\in H.$$

By using Theorem 3.1 and Lemma 4.1 we may obtain the following improvement.

Lemma 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T\subset H\times H$ be a maximal monotone operator such that $\overline{D(T)}\subset C\subset\bigcap_{r>0}R(I+rT)$ and let J_r be the resolvent of T and $\{r_n\}$ be a sequence in $(0,\infty)$. If $\inf\{r_n:n=1,2,3\ldots\}>0$, and $\sum_{n=1}^\infty|r_{n+1}-r_n|<\infty$, then*

(i) $\sum_{n=1}^\infty\sup\{\|J_{r_{n+1}}z-J_{r_n}z\|:z\in B\}<\infty$ *for any bounded subset B of H,*
(ii) $J_rx=\lim_{n\to\infty}J_{r_n}x$ *for all $x\in C$ and $F(J_r)=\bigcap_{n=1}^\infty F(J_{r_n})$, where $\lim_{n\to\infty}r_n=r$.*

Proof. We first prove (i). Let B be a bounded subset of H. Since $\{J_{r_n}z:z\in B,\ n=1,2,3\ldots\}$ is bounded, from Lemma 4.1, using the resolvent identity

$$J_{r_{n+1}}z=J_{r_n}\Big(\frac{r_n}{r_{n+1}}z+\Big(1-\frac{r_n}{r_{n+1}}\Big)J_{r_{n+1}}z\Big),\quad z\in H,$$

we obtain

$$\begin{aligned}\|J_{r_{n+1}}z-J_{r_n}z\| &\le\frac{|r_{n+1}-r_n|}{r_{n+1}}\|J_{r_{n+1}}z-z\|\\ &\le M|r_{n+1}-r_n|\end{aligned}$$

for each $z \in B$ and $n = 1, 2, 3 \ldots$ where $M = \frac{\sup\{\|J_{r_{n+1}}z - z\| : z \in B, n=1,2,3\ldots\}}{\inf\{r_n : n=1,2,3\ldots\}}$. Hence we get

$$\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}z - J_{r_n}z\| : z \in B\} \leq M \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Next, we prove (ii). By the assumption for $\{r_n\}$, we know that $r_n \to r$ for some $r > 0$. Since $\|J_r z - J_{r_n} z\| \leq \frac{|r-r_n|}{r}\|z - J_r z\|$, we obtain that $\lim_{n\to\infty} J_{r_n} z = J_r z$ for all $z \in H$. Then $J_r x = \lim_{n\to\infty} J_{r_n} x$ for all $x \in C$ and hence $F(J_r) = \bigcap_{n=1}^{\infty} F(J_{r_n}) = T^{-1}0$. ■

By Lemmas 4.2 and Theorem 3.1, we have the following theorem.

THEOREM 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset H \times H$ be a maximal monotone operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$ and let J_r be the resolvent of T. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{aligned}
x_0 &\in C \quad \text{is arbitrary,} \\
y_n &= \alpha_n x_n + (1 - \alpha_n) J_{r_n}(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}$$

where $0 \leq \alpha_n < a < 1$ for all $n = 1, 2, 3 \ldots, \{\lambda_n\} \subset (a, b) \subset (0, 2\alpha)$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If $\inf\{r_n : n = 1, 2, 3 \ldots\} > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to z, where $z = P_{T^{-1}0 \cap \mathrm{VI}(C,A)} x_0$.

Proof. Since H is a Hilbert space, $C = \overline{D(T)}$ is closed and convex and $F(J_r) = T^{-1}0$ for all $r > 0$. By Lemma 4.2, we have the following

$$F(J_r x) = \bigcap_{n=1}^{\infty} F(J_{r_n}) = T^{-1}0 \neq \emptyset.$$

We note that $F(T) = \mathrm{VI}(A, C)$. Therefore, by Theorem 3.1, we obtain $\{x_n\}$ converges strongly to $z = P_{T^{-1}0 \cap \mathrm{VI}(C,A)} x_0$. ■

4.2. Strictly pseudocontractive mappings. A mapping $T : C \to C$ is called *strictly pseudocontractive* on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. We know that A is $\frac{1-k}{2}$-inverse strongly monotone and $A^{-1}0 = F(T)$ (see [6] and [19]).

Using Theorem 3.1, we have the following theorem.

THEOREM 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself. Let T be a strictly pseudocontractive mapping with constant k of C into itself and let $\{x_n\}$ be a sequence* *generated by*

$$
\begin{aligned}
&x_0 \in C \text{ is arbitrary},\\
&y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C((1-\lambda_n)x_n + \lambda_n T(x_n - \lambda_n A x_n)),\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \dots,
\end{aligned}
$$

where $0 \le \alpha_n < a < 1$ *for all* $n = 1, 2, 3 \dots$ *and* $\{\lambda_n\} \subset (a, b) \subset (0, 2\alpha)$. *Let* $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C *and* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)} x_0$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $F(T)$ is the solution set of $\mathrm{VI}(A, C)$ i.e., $F(T) = \mathrm{VI}(A, C)$ and

$$P_C(x_n - \lambda_n A(x_n - \lambda_n A x_n)) = (1-\lambda_n)x_n + \lambda_n T(x_n - \lambda_n A x_n).$$

Therefore, by Theorem 3.2, $\{x_n\}$ converges strongly to $z = P_{F(S)\cap F(T)} x_0$. ∎

Acknowledgments. The authors would like to thank the Thailand Research Fund and the Commission on Higher Education under Grant No. MRG5380044 for their financial support.

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