NONCOMMUTATIVE GEOMETRY
AND QUANTUM GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 61
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2003

AN IDEMPOTENT FOR A JORDANIAN QUANTUM COMPLEX SPHERE

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Abstract. A new Jordanian quantum complex 4-sphere together with an instanton-type idempotent is obtained as a suspension of the Jordanian quantum group $SL_h(2)$.

Recently a number of examples of 4-dimensional noncommutative spheres have been constructed in [3], [4], [5], [6], [7], [8], [9]. Each such example consists of a noncommutative algebra (of functions on a quantum sphere) with an associated projective module (vector bundle) given in terms of a projector. In this short note we present a noncommutative algebra and an idempotent based on the Jordanian deformation of the group $SL(2)$ (as described in [2]).

The construction presented in this work is based on the idea of [1, Section 3] to use the R-matrix form of the defining relations of a quantum group G to define a projector for a noncommutative bundle E. Here we take G to be the Jordanian deformation of the group $SL(2)$, and the idempotent comes out as a 4×4 matrix with elements from a Jordanian algebra A, given in the block form as

$$e = \frac{1}{2}\begin{pmatrix} 1+z & \mathbf{t} \\ \tilde{\mathbf{t}} & 1-z \end{pmatrix}, \tag{1}$$

where $\mathbf{t}$, $\tilde{\mathbf{t}}$ are 2×2 matrices (of generators of G) such that

$$\mathbf{t}\tilde{\mathbf{t}} = \tilde{\mathbf{t}}\mathbf{t} = 1 - z^2 \tag{2}$$

and z is central in A. The equation (2) guarantees that $e^2 = e$.

2000 *Mathematics Subject Classification*: Primary 81R50; Secondary 81R60, 58B34.
Supported by Łódź University Grant no. 795 and EPSRC grant GR/S01078/01.
The paper is in final form and no version of it will be published elsewhere.

Here we lift to a higher dimension an algebra defined by relations (2.3) and (2.4) in [2]. Explicitly we set:

$$\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{\mathbf{t}} = \begin{pmatrix} d - hc & -b + h(a-d) + h^2 c \\ -c & a + hc \end{pmatrix}, \tag{3}$$

where a, b, c, d are generators of the algebra and h is complex parameter. The algebra of the Jordanian quantum complex three-sphere $SL_h(2)$ is then extended by a central element z. The matrix $\tilde{\mathbf{t}}$ is the antipode of $\mathbf{t}$ in [2]. The original algebra needs to be modified so that $\mathbf{t}$ and $\tilde{\mathbf{t}}$ satisfy (2). Finally the Jordanian quantum complex 4-sphere obtained as a suspension of $SL_h(2)$ is given by an algebra $A = A(S_h^4)$ with generators a, b, c, d, z and relations

$$\begin{array}{ll} [a,b] = h(a^2 - D), & [d,b] = h(d^2 - D), \\ [a,c] = -hc^2, & [d,c] = -hc^2, \\ [a,d] = h(ac - dc), & [b,c] = -h(ac + cd) \end{array} \tag{4}$$

and

$$D = 1 - z^2, \tag{5}$$

where $D = ad - bc - hac$. The element z is central in A. The algebra A has an associated finitely generated projective module with the idempotent e given by equation (1).

The first component of the Chern character of e vanishes, $\mathrm{ch}_0 = 0$, the next two are different from zero and in particular:

$$\begin{aligned} \mathrm{ch}_1 \propto 2h[&z \otimes (c \otimes a - a \otimes c + d \otimes c - c \otimes d) \\ &+ (c \otimes a - a \otimes c + d \otimes c - c \otimes d) \otimes z \\ &- (c \otimes z \otimes a - a \otimes z \otimes c + d \otimes z \otimes c - c \otimes z \otimes d)]. \end{aligned} \tag{6}$$

As a next step one can consider $*$-structures on the Jordanian algebra A. We have found several ways of defining $*$ on A to make it a $*$-algebra, but it is possible that there are other ways of defining $*$ consistent with (4), (5).

If h is purely imaginary, we can set:

$$a^* = a, \quad b^* = b, \quad c^* = c, \quad d^* = d, \quad z^* = \pm z, \tag{7}$$

or

$$a^* = d, \quad b^* = b, \quad c^* = c, \quad d^* = a, \quad z^* = \pm z. \tag{8}$$

If h is real we can set:

$$a^* = -d, \quad b^* = b, \quad c^* = c, \quad d^* = -a, \quad z^* = \pm z. \tag{9}$$

For a general $h = re^{i\phi}$, where r, ϕ are real, we have:

$$a^* = -e^{2i\phi} d, \quad b^* = b, \quad c^* = e^{4i\phi} c, \quad d^* = -e^{2i\phi} a, \quad D^* = e^{4i\phi} D. \tag{10}$$

If $\phi = n\pi/2$ this reduces to the previous two cases. If $e^{4i\phi} \neq 1$, D is not selfadjoint and we cannot see any simple way of defining z^*. Apart from this, one would rather wish z to be selfadjoint because then we can interpret it as an additional (real) coordinate raising the dimension of the quantum complex sphere.

If $h = 0$ (classical case), the above definitions of $*$ together with relations (4) do not define spheres but other hypersurfaces (such as hyperboloids).

Unfortunately, the idempotent e defined by (1) is not hermitian regardless of which of the above definitions of $*$ we would choose. Moreover, for the cases (7)-(9) one can prove that it is impossible to make e hermitian by a similarity transformation using only numerical matrices.

Indeed, when we define $*$ by either (7), (8) or (9), we can find a set x_i, $i = 1, \ldots, 5$ of selfadjoint ($x_i^* = x_i$) operators generating A. In the case (7) a, b, c, d, z is an appropriate set, in the case (8) we can take $a + d, i(a - d), b, c, z$ and finally in the case (9) we can take $a - d, i(a + d), b, c, z$. Then we can write an idempotent $e = X_0 + \sum_{i=1}^{5} x_i X_i$, where $X_i, i = 0, \ldots, 5$ are numerical matrices. Suppose that we can find a numerical matrix U such that UeU^{-1} is hermitian. This means that all UX_iU^{-1} are hermitian. (The generators $1, x_1, \ldots, x_5$ are linearly independent.) Hence UX_iU^{-1}'s, and consequently also X_i's have to have 4 linearly independent eigenvectors. However, if we consider the matrix corresponding to b (b is one of the x_i's in the cases considered), we obtain:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, this matrix has only two linearly independent eigenvectors. Hence we are led to a contradiction.

Our construction leaves several interesting open questions which we would like to address in the future. For example, is it possible to find a $*$-structure on A such that the idempotent e (or its similarity transform) is hermitian? What are hermitian projectors on A with prescribed $*$-structures? Finally, the representations of A should also be studied.

Acknowledgements. I would like to thank Dr. T. Brzeziński and Prof. P. Maślanka for guidance and helpful discussions.

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