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INTERPOLATION OF THE ESSENTIAL SPECTRUM AND THE ESSENTIAL NORM

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Abstract. The behavior of the essential spectrum and the essential norm under (complex/real) interpolation is investigated. We extend an example of Albrecht and Müller for the spectrum by showing that in complex interpolation the essential spectrum $\sigma_e(S_{[\theta]})$ of an interpolated operator is also in general a discontinuous map of the parameter θ. We discuss the logarithmic convexity (up to a multiplicative constant) of the essential norm under real interpolation, and show that this holds provided certain compact approximation conditions are satisfied. Some evidence supporting a counterexample is presented.

Introduction. This note is concerned with the behaviour of the essential spectrum of an operator under complex interpolation and that of the essential norm under real interpolation. Let (E_0, E_1), (F_0, F_1) be Banach interpolation couples and $T \in L(E_0+E_1, F_0+F_1)$ a compatible bounded linear operator, that is, the restrictions $T_0 : E_0 \to F_0$ and $T_1 : E_1 \to F_1$ are bounded operators. Let $(E_0, E_1)_{\theta,p}$ be the corresponding real interpolation space, where $0 < \theta < 1$ and $1 \le p \le \infty$. Then the restriction of T defines a bounded operator $T_{\theta,p} : (E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}$ that satisfies the logarithmically convex bound

$$\|T_{\theta,p}\| \le \|T_0\|^{1-\theta}\|T_1\|^{\theta}. \tag{1}$$

Estimate (1) also holds for $T_{[\theta]} : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ between the corresponding complex interpolation spaces for $0 < \theta < 1$. We refer to e.g. [BL], [BS] or [KM] for the definitions and the basic properties of real and complex interpolation.

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Let $K(E,F)$ denote the compact operators $E \to F$, where E, F are Banach spaces. The *essential norm*

$$\|S\|_e = \operatorname{dist}(S, K(E,F)), \quad S \in L(E,F),$$

is the quotient norm in $L(E,F)/K(E,F)$. If E is a complex Banach space, then the *essential spectrum* of $S \in L(E)$ is

$$\sigma_e(S) = \sigma(S + K(E)),$$

the spectrum of $S + K(E)$ in the Calkin algebra $L(E)/K(E)$.

Recently Albrecht and Müller [AM] gave an example of a couple (X_0, X_1) and a compatible operator $T \in L(X_0 + X_1)$ for which the map $\theta \mapsto \sigma(T_{[\theta]})$ is discontinuous at an interior point $\theta \in (0,1)$ in complex interpolation. It is important to have an analogous example for the essential spectrum, which is a useful subset of the spectrum of a bounded operator that is related to Fredholm theory. It turns out (Proposition 1 below) that the example of Albrecht and Müller already exhibits a similar discontinuity for the essential spectrum.

We also study the logarithmic convexity (up to a multiplicative constant) of $\|T_{\theta,p}\|_e$ under real interpolation, and establish positive results under certain compact approximation conditions. One purpose of this note is to draw attention to the intriguing open problem whether $\|T_{\theta,p}\|_e$ is logarithmically convex in general. We discuss some facts supporting a counterexample, and include some examples of the surprising behaviour of $\|\cdot\|_e$ under isometric embeddings and metric surjections.

Discontinuity of the essential spectrum in complex interpolation. We start by recalling the required details from [AM, pp. 808–809]. The spaces $(X_j, \|\cdot\|_j)$ for $j = 0, 1$ are the (weighted) Hilbert spaces consisting of the scalar sequences $x = (a_j)_{j \in \mathbf{Z}}$ for which

$$\|(a_j)\|_0 = \Big(\sum_{j \in \mathbf{Z}} 2^{-2j} |a_j|^2\Big)^{1/2} < \infty \quad \text{and} \quad \|(a_j)\|_1 = \Big(\sum_{j \in \mathbf{Z}} 2^{2j} |a_j|^2\Big)^{1/2} < \infty.$$

Let $(e_n)_{n \in \mathbf{Z}}$ be the unit coordinate basis, so that $\|e_n\|_0 = 2^{-n}$ and $\|e_n\|_1 = 2^n$ for $n \in \mathbf{Z}$. The complex interpolation spaces $(X_0, X_1)_{[\theta]}$ are the (weighted) Hilbert spaces consisting of the scalar sequences $x = (a_j) = \sum_{j \in \mathbf{Z}} a_j e_j$, for which

$$\Big\| \sum_{j \in \mathbf{Z}} a_j e_j \Big\|_{[\theta]} = \Big(\sum_{j \in \mathbf{Z}} r_\theta^{2j} |a_j|^2\Big)^{1/2} < \infty,$$

where $r_\theta = 2^{-(1-\theta)} 2^\theta = 2^{2\theta - 1}$. Note that $r_{1/2} = 1$, so that $(X_0, X_1)_{[1/2]} = \ell^2(\mathbf{Z})$. Let $H_0 = \ell^2(\mathbf{Z}, X_0)$ and $H_1 = \ell^2(\mathbf{Z}, X_1)$ be the vector-valued direct ℓ^2-sums indexed by $\mathbf{Z}$. It follows that $(H_0, H_1)_{[\theta]} = \ell^2(\mathbf{Z}, (X_0, X_1)_{[\theta]})$ for $0 < \theta < 1$.

Let $S : X_j \to X_j$ be the (weighted) right shift operator $Se_k = e_{k+1}$ for $k \in \mathbf{Z}$ and $j = 0, 1$. Note that $\|Sx\|_0 = \frac{1}{2}\|x\|_0$ for $x \in X_0$ and $\|Sx\|_1 = 2\|x\|_1$ for $x \in X_1$. Thus $S_{[\theta]} : (X_0, X_1)_{[\theta]} \to (X_0, X_1)_{[\theta]}$ is defined by the same condition, and $\|S_{[\theta]} x\|_{[\theta]} = r_\theta \|x\|_{[\theta]}$ for $x \in (X_0, X_1)_{[\theta]}$ and $0 < \theta < 1$. Define $T \in L(H_0 + H_1)$ by

$$T(x_j) = (\ldots, Sx_{-2}, Sx_{-1}, (S - I)x_0, Sx_1, \ldots),$$

where Sx_{-1} is the 0-th coordinate. It was verified in [AM, Example 1] that $\sigma(T_{[\theta]}) = \{\lambda \in \mathbf{C} : |\lambda| = r_\theta\}$ for $0 < \theta < 1$ and $\theta \neq \frac{1}{2}$, and that $\sigma(T_{[1/2]}) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. The aim of this section is to show that there is a similar discontinuity for the essential spectrum.

Let E, F be Banach spaces. Recall that $S \in L(E, F)$ is a Fredholm operator, denoted by $S \in \Phi(E, F)$, if its kernel $Ker(S)$ is finite dimensional and its image $Im(S)$ has finite codimension in F. We will use the basic fact that $\sigma_e(S) = \{\lambda \in \mathbf{C} : \lambda - S \notin \Phi(E)\}$ for $S \in L(E)$. We first state the following simple fact:

Lemma 1. *Let $0 < \theta < 1$. Then*

$$e_j \notin Im(r_\theta - S_{[\theta]}), \quad j \in \mathbf{Z},$$

so that $r_\theta - S_{[\theta]} \notin \Phi((X_0, X_1)_{[\theta]})$.

Proof. Let $n \in \mathbf{Z}$ and suppose that $(a_j) \in (X_0, X_1)_{[\theta]}$ satisfies $(r_\theta - S_{[\theta]})(a_j) = (r_\theta a_j - a_{j-1})_{j \in \mathbf{Z}} = e_n$. Thus

$$r_\theta a_m - a_{m-1} = 0 \quad \text{for } m \neq n, \qquad r_\theta a_n - a_{n-1} = 1. \tag{2}$$

We get by iteration from (2) that $a_{n+k} = r_\theta^{-k} a_n$ for $k = 1, 2, \ldots$. Since $\sum_{k=1}^\infty r_\theta^{2(n+k)} |a_{n+k}|^2 = |a_n|^2 \sum_{k=1}^\infty r_\theta^{2n}$ is finite we must have $0 = a_n = a_{n+k}$ for all $k = 1, 2, \ldots$. By substituting $a_n = 0$ into (2) we get that $a_{n-k} = -r_\theta^{k-1}$ for $k = 1, 2, \ldots$. This yields a contradiction since

$$\sum_{k=n+1}^{\infty} r_\theta^{2(n-k)} |a_{n-k}|^2 = \sum_{k=n+1}^{\infty} r_\theta^{2(n-k)} r_\theta^{2(k-1)} = \infty. \ \blacksquare$$

Proposition 1. $\sigma_e(T_{[\theta]}) = \{\lambda \in \mathbf{C} : |\lambda| = r_\theta\}$ *for* $0 < \theta < 1$ *and* $\theta \neq \frac{1}{2}$, *and* $\sigma_e(T_{[1/2]}) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. *Hence the map $\theta \mapsto \sigma_e(T_{[\theta]})$ is discontinuous at $\theta = \frac{1}{2}$.*

Proof. Case $\theta = \frac{1}{2}$. Suppose first that $0 < |\lambda| \leq 1$. Let $e_n \in (X_0, X_1)_{[1/2]} = \ell^2(\mathbf{Z})$ be the n-th (non-normalized) unit coordinate vector for $n \in \mathbf{Z}$. Assume that $n \in \mathbf{Z}$ and $(x_j) \in (H_0, H_1)_{[1/2]} = \ell^2(\mathbf{Z}, (X_0, X_1)_{[1/2]})$ satisfies

$$\begin{aligned}(\lambda - T_{[1/2]})(x_j) &= (\ldots, \lambda x_0 - Sx_{-1}, \lambda x_1 - (S - I)x_0, \lambda x_2 - Sx_1, \ldots) \\ &= (\ldots, 0, e_n, 0, 0, \ldots),\end{aligned}$$

where e_n occupies the 1-st coordinate in $\ell^2(\mathbf{Z}, (X_0, X_1)_{[1/2]})$. Here we put $S = S_{[1/2]}$ for notational simplicity. Hence we get

$$\lambda x_j - Sx_{j-1} = 0 \quad \text{for } j \neq 1, \quad \lambda x_1 - (S - I)x_0 = e_n. \tag{3}$$

By iteration in (3) starting from $x_2 = \lambda^{-1} Sx_1$ we get $x_{k+1} = \lambda^{-k} S^k x_1$ for $k = 1, 2, \ldots$. Thus $\|x_{k+1}\| = |\lambda|^{-k} \|x_1\| \geq \|x_1\|$ since the right shift operator S is an isometry on $\ell^2(\mathbf{Z})$. This yields that $x_1 = 0$, so that $x_j = 0$ for $j \geq 1$. Hence we are left with the condition $(S - I)x_0 = -e_n$ in (3). However, this is not possible according to Lemma 1. This means that the image $Im(\lambda - T_{[1/2]})(\ell^2(\mathbf{Z}, (X_0, X_1)_{[1/2]})$ has infinite codimension in $\ell^2(\mathbf{Z}, (X_0, X_1)_{[1/2]})$, and $\lambda \in \sigma_e(T_{[1/2]})$ whenever $0 < |\lambda| \leq 1$. The case $\lambda = 0$ is obvious, since (3) yields directly that $(S - I)x_0 = -e_n$.

Case $\theta \neq \frac{1}{2}$. Suppose that $|\lambda| = r_\theta$ and let $(e_n)_{n\in\mathbf{Z}}$ be the (non-normalized) coordinate basis in $(X_0, X_1)_{[\theta]}$. Let $n \in \mathbf{N}$ and assume that $(x_j) \in (H_0, H_1)_{[\theta]} = \ell^2(\mathbf{Z}, (X_0, X_1)_{[\theta]})$ satisfies

$$\begin{aligned}(\lambda - T_{[\theta]})(x_j) &= (\ldots, \lambda x_0 - S_{[\theta]}x_{-1}, \lambda x_1 - (S_{[\theta]} - I)x_0, \lambda x_2 - S_{[\theta]}x_1, \ldots)\\ &= (\ldots, 0, e_n, 0, 0, \ldots),\end{aligned}$$

with e_n in the 1-st coordinate in $\ell^2(\mathbf{Z}, (X_0, X_1)_{[\theta]})$. Hence,

$$\lambda x_j - S_{[\theta]}x_{j-1} = 0 \quad \text{for } j \neq 1, \quad \lambda x_1 - (S_{[\theta]} - I)x_0 = e_n. \tag{4}$$

The facts that $x_{k+1} = \lambda^{-k}S^k_{[\theta]}x_1$ for $k = 1, 2, \ldots$ by (4), and $\|S_{[\theta]}x\|_{[\theta]} = r_\theta\|x\|_{[\theta]}$ for $x \in (X_0, X_1)_{[\theta]}$, yield that $\|x_{k+1}\|_{[\theta]} = |\lambda|^{-k}r_\theta^k\|x_1\|_{[\theta]} = \|x_1\|_{[\theta]}$. Hence $x_1 = 0$ and $(S_{[\theta]} - I)x_0 = -e_n$ for $n \in \mathbf{Z}$ in (4). Thus $\lambda - T_{[\theta]} \notin \Phi(\ell^2(\mathbf{Z}, (X_0, X_1)_{[\theta]}))$ by Lemma 1 for $|\lambda| = r_\theta$. This implies the claim, since $\sigma(T_{[\theta]}) \subset \{\lambda \in \mathbf{C} : |\lambda| = r_\theta\}$ for $\theta \neq \frac{1}{2}$ by [AM, Example 1]. ■

By following the outline of [AM, Thm. 2] one may prove the following stronger discontinuity property for $\sigma_e(T_{[\theta]})$ (the details are left to the interested reader):

FACT. *Let $M \subset (0, 1)$ be a dense G_δ-set. Then there is a couple (H_0, H_1) consisting of non-separable Hilbert spaces and $T \in L(H_0 + H_1)$ so that M is the set of continuity of the map $\theta \mapsto \sigma_e(T_{[\theta]})$.*

Log-convexity of the essential norm in real interpolation. Cwikel [C] showed, solving a longstanding problem, that $T_{\theta,p}$ is a compact operator $(E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}$ whenever $T_0 : E_0 \to F_0$ or $T_1 : E_1 \to F_1$ is compact. Recently, Cobos, Fernandez-Martinez and Martinez [CMM] established a quantitative strengthening of his result. Let

$$\gamma(S) = \inf\{\varepsilon > 0 : SB_E \subset D + \varepsilon B_F, D \subset F \text{ compact}\}$$

be the measure of non-compactness of $S \in L(E, F)$, where B_E is the closed unit ball of E. They showed [CMM, Thm. 1.2] that $\gamma(T_{\theta,p})$ is logarithmically convex up to a multiplicative constant, that is,

$$\gamma(T_{\theta,p}) \leq 16\delta \cdot \gamma(T_0)^{1-\theta}\gamma(T_1)^\theta \tag{5}$$

for $T \in L(E_0 + E_1, F_0 + F_1)$, where $\delta = \delta(\theta) = \frac{2^\theta}{3-2^\theta-2^{1-\theta}}$. The constant 16δ in (5) cannot be replaced by 1, see [CMM, Example 1.1]. Note also that $\delta(\theta) \to \infty$ as $\theta \to 0+$.

Equation (5) raises the problem whether the essential norm is also logarithmically convex in real interpolation. Here $\gamma(S) \leq \|S\|_e$ for any S, but $\|\cdot\|_e$ and $\gamma(\cdot)$ are not equivalent in general, see [AT, Thms. 2.3 and 2.5] and [T2, Thm. 1.2]. We note also that $r_e(T_{\theta,p}) \leq r_e(T_0)^{1-\theta}r_e(T_1)^\theta$ for $0 < \theta < 1$ by [CMM, Cor. 1.3], where $r_e(S) = \max\{|\lambda| : \lambda \in \sigma_e(S)\}$ is the essential spectral radius of $S \in L(E)$. We refer to e.g. [LS], [AT], [T2] and [A] for further results related to measures of non-compactness and the essential spectral radius.

We show that there is an analogue of (5) for $\|T_{\theta,p}\|_e$ provided that certain compact approximation conditions are satisfied. A Banach space X is said to have the *inner compact approximation property* (abbreviated inner CAP) if there is a constant $C < \infty$ so that

$$\inf\{\|U - UV\| : V \in K(X), \|I - V\| \le C\} = 0$$

for any compact operator $U \in K(X, Z)$ (where Z is an arbitrary Banach space). Moreover, X has the *bounded compact approximation property* (BCAP) if there is a constant $C < \infty$ so that for any compact subset $D \subset X$ and $\varepsilon > 0$ there is a compact operator $V \in K(X)$ satisfying

$$\sup_{x \in D} \|x - Vx\| < \varepsilon \text{ and } \|I - V\| \le C.$$

The preceding compact approximation properties differ from the standard approximation properties defined in terms of finite rank operators. For instance, Willis [W] constructed a space X so that X has the BCAP but X fails to have the approximation property AP. There is also a space Y having a Schauder basis which fails to have the inner CAP, see [GW, 4.3], [T2, Example 2.5] or [CJ, Thm. 2.5]. We refer e.g. to [GW], [S], [CJ] or [T2] for further examples of this kind.

Let E, F be Banach spaces. It is convenient to put

$$\begin{aligned}\beta(S) = \inf\{\varepsilon > 0 : \|Sx\| \le \|Ux\| + \varepsilon\|x\| \text{ for all } x \in E,\\ U \in K(E, Z), Z \text{ an arbitrary Banach space}\},\end{aligned}$$

for $S \in L(E, F)$. It is known [GM] that $\beta(S) = \gamma(S^*)$ for $S \in L(E, F)$. We will need the estimates

$$\frac{1}{2}\gamma(S) \le \beta(S) = \gamma(S^*) \le 2\gamma(S), \quad S \in L(E, F), \tag{6}$$

due to Goldenstein and Markus (see e.g. [T2, Prop. 2.3.(i)]).

THEOREM 1. *Let $p \in [1, \infty]$ be fixed. Assume that (E_0, E_1) and (F_0, F_1) are interpolation couples, so that either*

(i) the interpolation spaces $(F_0, F_1)_{\theta,p}$ have the BCAP with a uniformly bounded constant C for $0 < \theta < 1$, or

(ii) the interpolation spaces $(E_0, E_1)_{\theta,p}$ have the inner CAP with a uniformly bounded constant C for $0 < \theta < 1$.

Then,

$$\|T_{\theta,p}\|_e \le 32C\delta \cdot \|T_0\|_e^{1-\theta}\|T_1\|_e^{\theta} \tag{7}$$

for $T \in L(E_0 + E_1, F_0 + F_1)$ and $0 < \theta < 1$, where $\delta = \delta(\theta) > 0$ is the constant from (5).

Proof. (i) If $(F_0, F_1)_{\theta,p}$ has the BCAP with constant C, then $\|S\|_e \le C \cdot \gamma(S)$ for any $S \in L(Z, (F_0, F_1)_{\theta,p})$ and any Z, see [LS, Thm. 3.6]. By applying this to $T_{\theta,p} : (E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}$ we get from (5) that

$$\|T_{\theta,p}\|_e \le C \cdot \gamma(T_{\theta,p}) \le 16C\delta \cdot \gamma(T_0)^{1-\theta}\gamma(T_1)^{\theta} \le 16C\delta \cdot \|T_0\|_e^{1-\theta}\|T_1\|_e^{\theta}$$

for $T \in L(E_0 + E_1, F_0 + F_1)$.

(ii) If $(E_0, E_1)_{\theta,p}$ has the inner CAP with constant C, then $\|S\|_e \le C \cdot \beta(S)$ for any $S \in L((E_0, E_1)_{\theta,p}, Z)$ and any Z by [T2, Thm. 1.2]. Hence, by applying (5) to $T_{\theta,p} : (E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}$ and using (6) we get

$$\begin{aligned}\|T_{\theta,p}\|_e &\le C \cdot \beta(T_{\theta,p}) \le 2C \cdot \gamma(T_{\theta,p}) \le 32C\delta \cdot \gamma(T_0)^{1-\theta}\gamma(T_1)^{\theta}\\ &\le 32C\delta \cdot \|T_0\|_e^{1-\theta}\|T_1\|_e^{\theta}. \ \blacksquare\end{aligned} \tag{8}$$

Theorem 1 is much simpler to state in special cases where one of the couples is trivial (that is, $E_0 = E_1$ or $F_0 = F_1$). Here (5) was known earlier (see [TE, Thm. 1]) with a uniformly bounded constant $c = c(\theta)$.

Corollary 1. *Let $p \in [1, \infty]$ be fixed, let E be a Banach space and (F_0, F_1) a Banach couple. Suppose that either*

(i) $(F_0, F_1)_{\theta,p}$ have the BCAP with a uniformly bounded constant C for $0 < \theta < 1$, or
(ii) E has the inner CAP with constant C.

Then (7) holds with a uniformly bounded constant for $T \in L(E, F_0 + F_1)$.

Corollary 2. *Let $p \in [1, \infty]$ be fixed, let F be a Banach space and (E_0, E_1) a Banach couple. Suppose that either*

(i) $(E_0, E_1)_{\theta,p}$ have the inner CAP with a uniformly bounded constant C for $0 < \theta < 1$, or
(ii) F has the BCAP with constant C.

Then (7) holds with a uniformly bounded constant for $T \in L(E_0 + E_1, F)$.

The preceding approximation assumptions on the interpolation spaces have some drawbacks.

Remark 1. Neither the BCAP nor the inner CAP passes in general to real interpolation spaces, so that the conditions on $(E_0, E_1)_{\theta,p}$ or $(F_0, F_1)_{\theta,p}$ in Theorem 1 cannot be ensured by assuming that the spaces E_0, E_1 or F_0, F_1 have these properties. Indeed, it follows from [DS, Thm. 1] (see also [GMS, p. 505]) that there exists a Banach couple (E_0, E_1) so that E_0 and E_1 have the BCAP, but $(E_0, E_1)_{\theta,p}$ fail to have the BCAP for any $0 < \theta < 1$ and $1 \le p \le \infty$. Similarly, there is (E_0, E_1) so that E_0 and E_1 have the inner CAP, but $(E_0, E_1)_{\theta,p}$ fails to have the inner CAP for any $0 < \theta < 1$ and $1 \le p \le \infty$.

Remark 2. Another convexity estimate for $\|T_{\theta,p}\|_e$ can be found by modifying an argument of Teixeira and Edmunds [TE, Thm. 2]. Here one assumes that the image couple (F_0, F_1) satisfies a technical approximation condition (H) (we refer to [TE, p. 133] for the definition).

Fact. *Let $0 < \theta < 1$ and $1 \le p \le \infty$. Assume that (E_0, E_1) and (F_0, F_1) are Banach couples, where (F_0, F_1) satisfies condition (H) with the constants c_0 and c_1. Then*

$$\|T_{\theta,p}\|_e \le c_0^{1-\theta} c_1^{\theta} \|T_0\|_e^{1-\theta} \|T_1\|_e^{\theta}$$

for $T \in L(E_0 + E_1, F_0 + F_1)$.

We omit the details, since condition (H) appears to be quite cumbersome to verify and its connection to Theorem 1 remains unclear. Note that at least $(L^{p_0}(0,1), L^{p_1}(0,1))$ and (ℓ^{p_0}, ℓ^{p_1}) satisfy condition (H) for $1 \le p_0, p_1 < \infty$, see [TE, p. 135].

Remark 3. The inequality $\|S_{\theta,p}\|_e \le c \cdot \|S_0\|_e^{1-\theta} \|S_1\|_e^{\theta}$ cannot always hold with $c = 1$. This fact is a simple modification of [CMM, Example 1].

Let $1 < q < \infty$, $\theta = 1 - \frac{1}{q}$ and consider the couples (ℓ^1, ℓ^1), (ℓ^1, ℓ^∞) and the identity operator $S = I$. We need the fact that $\|I : \ell^1 \to \ell^\infty\|_e \le \frac{1}{2}$. Indeed, define the rank-1 operator $U : \ell^1 \to \ell^\infty$ by $Ux = \frac{1}{2}(\sum_{j=1}^\infty x_j)(1, 1, \ldots)$ for $x = (x_j) \in \ell^1$, and note that

$\|Ix - Ux\|_\infty = \sup_{n\in\mathbf{N}} |x_n - \frac{1}{2}\sum_{j\neq n}^{\infty} x_j| \leq \frac{1}{2}$ for $x = (x_k) \in B_{\ell^1}$. Since Corollary 1 applies to these couples, the estimates in [CMM, pp. 27–28] yield then that

$$1 = \|I : \ell^1 \to \ell^q\|_e \leq c \cdot \theta^{-(1-\theta)} \|I : \ell^1 \to \ell^\infty\|_e^\theta \leq c \cdot \theta^{-(1-\theta)} 2^{-\theta}, \tag{9}$$

where $c < \infty$. By letting $q \to \infty$ in (9) we see that $c > 1$. ■

Theorem 1 raises the problem whether the compact approximation conditions are essential for the logarithmic convexity of $\|T_{\theta,p}\|_e$. The proof of (5) in [CMM, Thm. 1.2] uses (among other things) the facts that

$$\gamma(SQ) = \gamma(S) \quad \text{and} \quad \frac{1}{2}\gamma(S) \leq \gamma(JS) \leq \gamma(S), \quad S \in L(E,F), \tag{10}$$

where $Q : X \to E$ is a linear metric surjection (that is, $\overline{QB_X} = B_E$) and $J : F \to Y$ is an isometric embedding. In addition, a couple of crucial estimates from the argument in [CMM, Thm. 1.2] would also work in the case of $\|\cdot\|_e$ provided that the seminorms $\gamma(\cdot)$ and $\|\cdot\|_e$ are uniformly comparable between suitable sequence spaces.

The equivalence of $\gamma(\cdot)$ and $\|\cdot\|_e$ is closely tied to the BCAP or the inner CAP, see [AT, Thm. 2.5] and [T2, Thm. 1.2] . The same remark also applies to the analogues of (10) for $\|\cdot\|_e$ (see the next section). The preceding facts suggest the following intriguing possibility, which in part motivated this note.

Question. *Find Banach couples* (E_0, E_1), (F_0, F_1) *and a sequence of operators* $(T_n) \subset L(E_0 + E_1, F_0 + F_1)$ *so that*

$$\lim_{n\to\infty} \frac{\|(T_n)_0\|_e^{1-\theta} \cdot \|(T_n)_1\|_e^\theta}{\|(T_n)_{\theta,p}\|_e} = 0$$

holds for any $0 < \theta < 1$ *(or, for some* $0 < \theta < 1$*). Here* $p \in [1,\infty]$ *is fixed.*

Behaviour of $\|\cdot\|_e$ under isometric embeddings and metric surjections. Results in [AT] and [T2] imply that the essential norm may behave quite strangely under isometric embeddings and metric surjections. The preceding compact approximation properties are crucial tools for this, but these facts are poorly documented in the literature. Some non-explicit examples are contained in [AT, Thm. 3.5 and Cor. 3.6] (see also [T3, Example 1.5]). The unpublished thesis [T1, pp. 4–5] contains simpler versions of Examples 2 and 3 below, but other variants are also possible. Hence we take the opportunity to include these results here. Put $S_E = \{x \in E : \|x\| = 1\}$.

Example 2. Let F be a separable Banach space that fails to have the BCAP. Fix a countable subset $\Lambda = \{x_n^* : n \in \mathbf{N}\} \subset S_{F^*}$ so that Λ norms F, and let $J : F \to \ell^\infty$ be the isometry $Jx = (x_n^*(x)), x \in F$. We claim that there is a space E and a sequence $(S_n) \subset L(E,F)$ for which

$$\|S_n\|_e = 1 \quad \text{for } n \in \mathbf{N} \quad \text{and} \quad \|JS_n\|_e \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Indeed, [AT, Thms. 2.3 and 2.5] provide a Banach space E and a sequence $(S_n) \subset L(E,F)$ for which $\|S_n\|_e = 1$ and $\gamma(S_n) < \frac{1}{n}$ for $n \in \mathbf{N}$. Let $J_\infty : F \to \ell^\infty(B_{F^*})$ be the isometry $J_\infty x = (x^*(x))_{x^*\in B_{F^*}}$ for $x \in F$. It is known that $\|J_\infty S\|_e = \gamma(S^*)$ for $S \in L(E,F)$ by

[As, Cor. 5.6]. Hence it follows from (6) that

$$\|J_\infty S_n\|_e = \gamma(S_n^*) \leq 2 \cdot \gamma(S_n) < \frac{2}{n}, \quad n \in \mathbf{N}.$$

We factorize $J_\infty = J_0 \circ J$, where J_0 is a suitable linear isometry $\ell^\infty \to \ell^\infty(B_{F^*})$, by viewing $\ell^\infty = \ell^\infty(\Lambda) \subset \ell^\infty(B_{F^*})$ as a closed one-complemented subspace. Recall that there is a norm one projection $P : \ell^\infty(B_{F^*}) \to J_0(\ell^\infty)$ by the extension property of ℓ^∞. Thus $\|JS_n\|_e = \|PJ_\infty S_n\|_e < \frac{2}{n} \to 0$ as $n \to \infty$, which yields (11). ∎

Example 3. Let E be a separable Banach space that fails to have the inner CAP. Fix a countable dense subset $\{x_n : n \in \mathbf{N}\}$ of S_E, and let $Q : \ell^1 \to E$ be the quotient map defined by $Q(\sum_{n=1}^\infty a_n e_n) = \sum_{n=1}^\infty a_n x_n$ for $\sum_{n=1}^\infty a_n e_n \in \ell^1$. By [T2, Thm. 1.2] there is a space F and a sequence $(S_n) \subset L(E,F)$ so that $\|S_n\|_e = 1$ and $\beta(S_n) < \frac{1}{n}$ for $n \in \mathbf{N}$. The fact that Q is a quotient map implies that $\|SQ\|_e = \gamma(S)$ for $S \in L(E,F)$, see [As, Thm. 3.8 and Cor. 3.9]. Hence it follows from (6) that $\|S_n Q\|_e = \gamma(S_n) \leq 2 \cdot \beta(S_n) \to 0$ as $n \to \infty$. ∎

Conversely, compact approximation properties guarantee that $\|SQ\|_e$ (or $\|JS\|_e$) is uniformly comparable to $\|S\|_e$.

Proposition 2. *(i) Let E, F and Z be Banach spaces, and $J : F \to Z$ a fixed linear isometric embedding. Then, there is $c > 0$ so that*

$$c\|S\|_e \leq \|JS\|_e, \quad S \in L(E,F),$$

provided one of the following conditions are satisfied:

(a) E has the inner CAP, (b) F has the BCAP, or (c) there is a bounded linear projection $P : Z \to JE$.

(ii) Let E, F and Z be Banach spaces, and $Q : Z \to E$ a fixed linear metric surjection. Then, there is $c > 0$ so that

$$c\|S\|_e \leq \|SQ\|_e, \quad S \in L(E,F),$$

provided one of the following conditions are satisfied:

(a) E has the inner CAP, (b) F has the BCAP, or (c) there is a bounded linear projection $P : Z \to Ker(Q)$.

Proof. (i) Suppose that either F has the BCAP or that E has the inner CAP. It follows from [LS, Thm. 3.6], respectively [T2, Thm. 1.2] and (6), that there is $c > 0$ so that $\|S\|_e \leq c \cdot \gamma(S) \leq 2c \cdot \gamma(JS) \leq 2c \cdot \|JS\|_e$ for $S \in L(E,F)$. Let P be a bounded projection $Z \to JE$ and assume that $\lambda > \|JS\|_e$. Pick $R \in K(E,Z)$ so that $\|JS - R\| < \lambda$. Then $J^{-1}PR$ defines a compact operator $E \to F$ satisfying

$$\|S - J^{-1}PR\| = \|JS - PR\| = \|P(JS - R)\| < \|P\|\lambda.$$

Thus $\|S\|_e \leq \|P\| \cdot \|JS\|_e$ for $S \in L(E,F)$.

(ii) The argument for the cases (a) and (b) is similar to that of the first part of (i), since $\gamma(SQ) = \gamma(S)$ for $S \in L(E,F)$.

Put $M = Im(I-P)$, where $P : Z \to Ker(Q)$ is a projection. Suppose that $\lambda > \|SQ\|_e$ and pick $R \in K(Z,F)$ satisfying $\|SQ - R\| < \lambda$. Note that $\widetilde{Q} = Q_{|M} : M \to QM = E$ has a bounded inverse $\widetilde{Q}^{-1} : E \to M$. If $J : M \to Z$ denotes the inclusion map, then

$$\begin{aligned} \|S - RJ\widetilde{Q}^{-1}\| &= \|SQJ\widetilde{Q}^{-1} - RJ\widetilde{Q}^{-1}\| \\ &\leq \|SQ - R\| \cdot \|\widetilde{Q}^{-1}\| < \lambda \cdot \|\widetilde{Q}^{-1}\|, \end{aligned}$$

where $RJ\widetilde{Q}^{-1} \in K(E,F)$. ■

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