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# ASYMPTOTIC FORMULAS FOR THE ERROR IN LINEAR INTERPOLATION

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**Abstract.** We give the asymptotic formula for the error in linear interpolation with arbitrary knots.

**1. Main result.** In this paper we calculate the asymptotic formula for the error in linear interpolation for an arbitrary set of knots in $L^p$ norm $1 \le p < \infty$. It is considered on the interval $[0,1]$ only but this is not essential. The operators which were examined before were constructed in shift invariant spaces (see [3]–[5] or [1]–[2], [6]). From the structure of the shift invariant spaces it follows that if such an operator $Q$ has polynomial order $r$, i.e. $Q(p) = p$ for all polynomials $p$, $\deg p < r$, then $Q(x^\beta)(x) - x^\beta$ is a periodic function with period 1 for all $|\beta| = r$. Consequently, we can use the (Fejér) Mazur–Orlicz Theorem for periodic functions, [7]. The linear interpolation for an arbitrary set of knots is not connected with such structure.

Let $I = [0,1]$ and $\pi$ be a partition of $I$, i.e.

$$\pi = \{t_0, t_1, \dots, t_n\}$$

where

$$t_{j-1} < t_j, \quad j = 1, \dots, n, \quad t_0 = 0, \quad t_n = 1.$$

Let us define a function on $I$

$$h(\pi, x) = t_j - t_{j-1}, \quad x \in [t_{j-1}, t_j),$$

where $j = 1, \dots, n$. At $x = 1$ let the function $h(\pi, x)$ be left continuous. The size of the partition is denoted by

$$\delta(\pi) = \max_{1 \le j \le n} |t_j - t_{j-1}| = \max_{x \in I} h(\pi, x).$$

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Let us consider a linear interpolation with knots of the set $\pi$, denoted by $I_\pi$. It is known that if $f \in C^2(I)$, then for $x \in [t_{j-1}, t_j]$

$$\begin{aligned} I_\pi f(x) - f(x) &= \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{x} ds \int_{t_{j-1}}^{t_j} (f'(\tau) - f'(s)) d\tau \\ &= \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} d\tau \int_{t_{j-1}}^{x} ds \int_{s}^{\tau} f''(u) du = \frac{1}{t_j - t_{j-1}} \int_{\Delta_{j,x}} f''(u) du, \end{aligned} \tag{1.1}$$

where we use the following abbreviation

$$\int_{\Delta_{j,x}} g(u) du = \int_{t_{j-1}}^{t_j} d\tau \int_{t_{j-1}}^{x} ds \int_{s}^{\tau} g(u) du. \tag{1.2}$$

Let us define a positive function $\Delta(x)$ for $x \in [t_{j-1}, t_j]$, where $j = 1, \dots, n$ on $I$ by

$$\Delta(x) = \int_{\Delta_{j,x}} du = \int_{t_{j-1}}^{t_j} d\tau \int_{t_{j-1}}^{x} ds \int_{s}^{\tau} du = \frac{1}{2}(t_j - t_{j-1})(x - t_{j-1})(t_j - x). \tag{1.3}$$

Let $1 \le p < \infty$. Using a change of variables for the function $\Delta$ it is easy to prove the following formula:

$$\frac{1}{|t_j - t_{j-1}|^{3p}} \int_{t_{j-1}}^{t_j} |\Delta(x)|^p dx = \frac{1}{2^p}(t_j - t_{j-1}) \int_0^1 (z(1-z))^p dz = \frac{1}{2^p}(t_j - t_{j-1}) B(p+1, p+1), \tag{1.4}$$

where $B$ is the beta function.

THEOREM 1. *Let $1 \le p < \infty$ and $f \in C^2(I)$. Then*

$$\lim_{\delta(\pi) \to 0} \int_I \left| \frac{I_\pi f(x) - f(x)}{h^2(\pi, x)} \right|^p dx = \frac{1}{2^p} B(p+1, p+1) \int_I |f''|^p \tag{1.5}$$

*and if $\delta(\pi) \to 0$, then*

$$\frac{I_\pi f - f}{h^2(\pi, \cdot)} \to \frac{f''}{12} \tag{1.6}$$

*weakly in $L^p(I)$, $1 \le p < \infty$.*

*Proof.* First we prove (1.5). By (1.1), (1.2)

$$\int_I \left| \frac{I_\pi f(x) - f(x)}{h^2(\pi, x)} \right|^p dx = \sum_{j=1}^{n} \frac{1}{|t_j - t_{j-1}|^{3p}} \int_{t_{j-1}}^{t_j} \left| \int_{\Delta_{j,x}} f''(u) du \right|^p dx. \tag{1.7}$$

Let

$$\begin{aligned} S_\pi &= \sum_{j=1}^{n} \frac{1}{|t_j - t_{j-1}|^{3p}} \int_{t_{j-1}}^{t_j} \left| \int_{\Delta_{j,x}} f''(t_j) du \right|^p dx \\ &= \sum_{j=1}^{n} \frac{1}{|t_j - t_{j-1}|^{3p}} |f''(t_j)|^p \int_{t_{j-1}}^{t_j} \left| \int_{\Delta_{j,x}} du \right|^p dx. \end{aligned}$$

By (1.4)

$$S_\pi = \frac{1}{2^p} \int_0^1 (z(1-z))^p dz \sum_{j=1}^{n} (t_j - t_{j-1}) |f''(t_j)|^p.$$

Note that

$$\lim_{\delta(\pi)\to 0} S_\pi = \frac{1}{2^p} B(p+1,p+1) \int_I |f''|^p.$$

To prove (1.5) it is sufficient to show that

$$\lim_{\delta(\pi)\to 0} \left( S_\pi - \int_I \left| \frac{I_\pi f(x) - f(x)}{h^2(\pi,x)} \right|^p dx \right) = 0. \tag{1.8}$$

Let us fix $\varepsilon > 0$. There is $\delta > 0$ such that if $|u_1 - u_2| < \delta$ then

$$|f''(u_1) - f''(u_2)| < \varepsilon.$$

Consequently for $x \in [t_{j-1}, t_j]$ if $t_j - t_{j-1} < \delta$ then

$$\begin{aligned} &\left| \int_{\Delta_{j,x}} f''(u)du - \int_{\Delta_{j,x}} f''(t_j)du \right| \\ &\qquad < \int_{t_{j-1}}^{t_j} d\tau \int_{t_{j-1}}^{x} ds \left| \int_s^\tau f''(u) - f''(t_j)du \right| < \varepsilon (t_j - t_{j-1})^3. \end{aligned} \tag{1.9}$$

Let us take the partition $\pi$ such that $\delta(\pi) < \delta$. Then using the known inequality

$$||a|^p - |b|^p| \le p(\max\{|a|,|b|\})^{p-1}|a-b| \tag{1.10}$$

and (1.7), (1.9), we have

$$\begin{aligned} &\left| S_\pi - \int_I \left| \frac{I_\pi f(x) - f(x)}{h^2(\pi,x)} \right|^p dx \right| \\ &\quad < \left| \sum_{j=1}^n \frac{1}{|t_j - t_{j-1}|^{3p}} \int_{t_{j-1}}^{t_j} \left( \left| \int_{\Delta_{j,x}} f''(u)du \right|^p - \left| \int_{\Delta_{j,x}} f''(t_j)du \right|^p \right) dx \right| \\ &\quad < \left| \sum_{j=1}^n \frac{1}{|t_j - t_{j-1}|^{3p}} \int_{t_{j-1}}^{t_j} p(\max_{x\in I} |f''(x)|(t_j - t_{j-1})^3)^{p-1} \varepsilon (t_j - t_{j-1})^3 \right| \\ &\quad = \max_{x\in I} |f''(x)|^{p-1} p\varepsilon \sum_{j=1}^n \int_{t_{j-1}}^{t_j} = \max_{x\in I} |f''(x)|^{p-1} p\varepsilon. \end{aligned}$$

This finishes the proof of (1.5). To prove (1.6) it is sufficient to show that

$$\int_I \frac{I_\pi f(x) - f(x)}{h^2(\pi,x)} \chi_A(x) dx \to \int_A \frac{f''(u)}{12} du,$$

where $\chi_A$ is the characteristic function of the measurable set $A = (a,b) \subset I$. Note that

$$\int_I \frac{I_\pi f(x) - f(x)}{h^2(\pi,x)} \chi_A(x) dx = \sum_{j=1}^n \frac{1}{|t_j - t_{j-1}|^3} \int_{(t_{j-1},t_j)\cap A} \int_{\Delta_{j,x}} f''(u) du dx.$$

Moreover

$$\frac{1}{|t_j - t_{j-1}|^3} \int_{t_{j-1}}^{t_j} \Delta(x) dx = \frac{1}{2} \int_0^1 (z(1-z)) dz (t_j - t_{j-1}) = 1/12(t_j - t_{j-1}).$$

By a similar argument we get (1.6).

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