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ON SOME LOCAL GEOMETRY OF MUSIELAK-ORLICZ SEQUENCE SPACES EQUIPPED WITH THE LUXEMBURG NORM

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Abstract. Criteria for strong U-points, compactly locally uniformly rotund points, weakly compactly locally uniformly rotund points and locally uniformly rotund points in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm are given.

1. Introduction. Throughout this paper, X denotes a Banach space and X^* denotes its dual space. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively.

DEFINITION 1. A point $x \in S(X)$ is said to be an *extreme point* if for every y, $z \in S(X)$ with $x = \frac{y+z}{2}$, we have $y = z = x$.

A Banach space X is said to be *rotund* ($X \in (R)$ for short) if every point on $S(X)$ is an extreme point.

DEFINITION 2. A point $x \in S(X)$ is said to be a *strong* U-*point* (SU-point for short) if for any $y \in S(X)$ with $\|y + x\| = 2$ we have $x = y$.

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It is obvious that a Banach space X is rotund if and only if every $x \in S(X)$ is a SU-point.

DEFINITION 3. A point $x \in S(X)$ is said to be a *locally uniformly rotund point* (LUR-point for short) if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $S(X)$ with $\lim_{n\to\infty} \|x_n + x\| = 2$, we have $\lim_{n\to\infty} \|x_n - x\| = 0$.

DEFINITION 4. A point $x \in S(X)$ is said to be a *weakly compactly locally uniformly rotund point* (WCLUR-point for short) if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $S(X)$ with $\lim_{n\to\infty} \|x_n+x\|$ $=2$, there exist an $x' \in S(X)$ and a subsequence $\{x'_n\}$ of $\{x_n\}$ such that x'_n convergent to x' weakly ($x'_n \to^w x'$ for short).

DEFINITION 5. A point $x \in S(X)$ is said to be a *compactly locally uniform rotund point* (CLUR-point for short) if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $S(X)$ with $\lim_{n\to\infty} \|x_n + x\| = 2$, the sequence $\{x_n\}$ is compact in $B(X)$.

DEFINITION 6. A Banach space X is said to have H-*property* if the weak convergence and the convergence in norm coincide in $S(X)$.

For these geometric notions and their role in mathematics we refer to the monographs [1] and [2].

The function sequence $M = (M_i)_{i=1}^{\infty}$ is called a *Musielak-Orlicz function* provided that for any $i \in \mathbb{N}$, $M_i : (-\infty, +\infty) \to [0, +\infty)$ is even, convex, left continuous on $[0, +\infty)$, $M_i(0) = 0$ and there exists $u_i > 0$ such that $M_i(u_i) < \infty$. By $N = (N_i)_{i=1}^{\infty}$ we denote the Musielak-Orlicz function conjugate to $M = (M_i)$ in the sense of Young, i.e.

$$N_i(u) = \sup_{v>0}\{|u|\, v - M_i(v)\}$$

for each $u \in \mathbb{R}$ and $i \in \mathbb{N}$. Furthermore, $P = (p_i)$ is the right derivative of $M = (M_i)$, i.e. p_i is the right derivative of M_i for every $i \in \mathbb{N}$.

By l^0 we denote the space of all sequences $x = (x\,(i))_{i=1}^{\infty}$ of reals. For a given Musielak-Orlicz function $M = (M_i)$ we define the *Musielak-Orlicz sequence space* l_M by

$$l_M = \{x \in l^0 : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_M(x) = \sum_{i=1}^{\infty} M_i(x(i)) \text{ for any } x = (x\,(i)) \in l^0.$$

This space equipped with the *Luxemburg norm*

$$\|x\| = \inf\{\lambda > 0 : \rho_M(x/\lambda) \le 1\}$$

or with the *Orlicz norm*

$$\|x\|^0 = \sup\Big\{\sum_i x(i)y(i) : \rho_N(y) \le 1\Big\} = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx))$$

is a Banach space (see [3]).

By h_M we denote the subspace of l_M defined by

$$h_M = \Big\{x \in l_M : \forall\ \lambda > 0,\ \exists\ i_0 \text{ such that} \sum_{i>i_0} M_i(\lambda x(i)) < \infty\Big\}.$$

To simplify notations, we put $l_M = (l_M, \|\cdot\|)$ and $l_M^0 = (l_M, \|\cdot\|^0)$.

We say that the Musielak-Orlicz function $M = (M_i)$ satisfies the δ_2-*condition* ($M \in \delta_2$ for short) if there exist $a > 0$, $k > 0$, $i_0 \in \mathbb{N}$ and a sequence $(c_i)_{i=i_0+1}^{\infty}$ in $[0, +\infty)$ with $\sum_{i>i_0}^{\infty} c_i < \infty$ such that

$$M_i(2u) \le kM_i(u) + c_i$$

for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $M_i(u) \le a$ (see [3]).

We say that the Musielak-Orlicz function $M = (M_i)$ satisfies the $\overline{\delta}_2$-*condition* ($M \in \overline{\delta}_2$ for short) if its complementary function $N = (N_i)$ satisfies the δ_2-condition.

For convenience, we introduce the following notions. For every $x \in l_M$ and $i \in \mathbb{N}$, we put

$$\begin{aligned} \xi(x) &= \inf\Big\{\lambda > 0 : \text{ there exists } i_0 \text{ such that } \sum_{i>i_0} M_i(x(i)/\lambda) < \infty\Big\}, \\ e(i) &= \sup\{u \ge 0 : M_i(u) = 0\}, \\ B(i) &= \sup\{u > 0 : \ M_i(u) < \infty\}. \end{aligned}$$

For every $i \in \mathbb{N}$, we say that a point $x \in \mathbb{R}$ is a *strictly convex point* of M_i if $M_i(\frac{u+v}{2}) < \frac{1}{2}(M_i(u) + M_i(v))$ whenever $x = \frac{u+v}{2}$ and $u \ne v$. We write then $x \in SC_{M_i}$. An interval $[a, b]^{(i)}$ is called a *structurally affine interval* for M_i (or simply SAI of M_i) provided that M_i is affine on $[a, b]^{(i)}$ and it is not affine on $[a - \varepsilon, b]^{(i)}$or $[a, b + \varepsilon]^{(i)}$ for any $\varepsilon > 0$. Let $SAI(M_i) = \{[a_n, b_n]^{(i)}\}_{n=1}^{\infty}$. It is obvious that $SC_{M_i} = \mathbb{R} \setminus \cup_n [a_n, b_n]^{(i)}$, where $[a_n, b_n]^{(i)} \in SAI(M_i)$ for $n = 1, 2, \ldots$.

For every $i \in \mathbb{N}$, denote

$$\begin{aligned} SC_{M_i}^- &= \{u \in SC_{M_i} : \ \exists\, \varepsilon > 0 \text{ such that } M_i \text{ is affine on } [u, u + \varepsilon]\}, \\ SC_{M_i}^+ &= \{u \in SC_{M_i} : \ \exists\, \varepsilon > 0 \text{ such that } M_i \text{ is affine on } [u - \varepsilon, u]\}, \\ SC_{M_i}^0 &= SC_{M_i} \setminus (SC_{M_i}^+ \cup SC_{M_i}^-). \end{aligned}$$

We first formulate several lemmas.

LEMMA 1 ([5]). $(h_M)^* = l_N^0$, $(h_M^0)^* = l_N$.

LEMMA 2 ([5]). $h_M = l_M$ (or $h_M^0 = l_M^0$) *if and only if* $M \in \delta_2$.

LEMMA 3 ([4]). *If* $M \notin \overline{\delta}_2$, *then there exist a sequence* $0 = m_0 < m_1 < m_2 < \cdots$ *and* $u_i^n > 0$ $(i = m_{n-1} + 1, \ldots, m_n)$ *such that* $M_i(u_i^n) \le 1/n$ *and*

$$M_i\Big(\frac{u_i^n}{2}\Big) > \Big(1 - \frac{1}{n}\Big)\frac{M_i(u_i^n)}{2}, \qquad \sum_{i=m_{n-1}+1}^{m_n} M_i(u_i^n) > 1, \qquad n = 1, 2, \ldots.$$

LEMMA 4 ([4]). $M \in \delta_2$ *if and only if* $\|x\| = 1 \Leftrightarrow \rho_M(x) = 1$.

LEMMA 5. *If* $M \in \delta_2$, $\|x\| = 1, \|x_n\| \le 1$ *and* $\|x_n + x\| \to 2$ $(n \to \infty)$, *then*

$$\lim_{n\to\infty} \rho_M(x_n) = \lim_{n\to\infty} \rho_M\Big(\frac{x + x_n}{2}\Big) = 1.$$

Proof. We suppose that there exists $\varepsilon_0 > 0$ such that $\rho_M(x_n) \le 1 - \varepsilon_0$ $(n = 1, 2, \ldots)$. Since $\frac{\|x_n + x\|}{2} \to 1$, for any $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \frac{(1+\eta)(x + x_n)}{2} \right\| > 1 \tag{1}$$

when $n \ge n_0$.

For any $\varepsilon > 0$, by $M \in \delta_2$, there exist $\lambda_0 > 1$, $a > 0$ and $c_i > 0$ $(i = 1, 2, \ldots)$ such that $\sum_{i=1}^{\infty} c_i < \infty$ and $M_i(\lambda_0 u) \le (1+\varepsilon)M_i(u) + c_i$ $(\forall i \in \mathbb{N},\ M_i(u) \le a)$.

Take $i_0 \in \mathbb{N}$ such that $\sum_{i > i_0} c_i < \varepsilon$ and $M_i(x(i)) \le a \quad (i > i_0)$.

Take $\lambda_0' > 0$ with $1 < \lambda_0' < \lambda_0$ such that $\sum_{i=1}^{i_0}(M_i(\lambda x(i)) - M_i(x(i))) < \varepsilon$ $(1 \le \lambda \le \lambda_0')$. Therefore when $1 \le \lambda \le \lambda_0'$, it follows that

$$\begin{aligned} \rho_M(\lambda x) &= \sum_{i=1}^{i_0} M_i(\lambda x(i)) + \sum_{i > i_0} M_i(\lambda x(i)) \\ &\le \sum_{i=1}^{i_0} M_i(\lambda x(i)) + \varepsilon + \sum_{i > i_0}((1+\varepsilon)M_i(x(i)) + c_i) \\ &\le (1+\varepsilon)\rho_M(x) + 2\varepsilon \end{aligned}$$

i.e.

$$\lim_{\lambda \to 1} \rho_M(\lambda x) = \rho_M(x). \tag{2}$$

Combining (1) with (2) we have

$$\begin{aligned} 1 &< \rho_M\left(\frac{(1+\eta)(x + x_n)}{2}\right) = \rho_M\left(\frac{1+\eta}{2}x_n + \frac{1-\eta}{2}\frac{1+\eta}{1-\eta}x\right) \\ &\le \frac{1+\eta}{2}\rho_M(x_n) + \frac{1-\eta}{2}\rho_M\left(\frac{1+\eta}{1-\eta}x\right) \\ &\le \frac{1+\eta}{2}(1 - \varepsilon_0) + \frac{1-\eta}{2}(1 + o(\eta)). \end{aligned}$$

Let $\eta \to 0$ to get $1 \le \frac{1-\varepsilon_0}{2} + \frac{1}{2}$. This is a contradiction. So $\rho_M(x_n) \to 1$ $(n \to \infty)$.

Using $\left\|\frac{x + \frac{x + x_n}{2}}{2}\right\| = \left\|\frac{3}{4}x + \frac{1}{4}x_n\right\| \to 1$, by the same argument as above we have $\rho_M(\frac{x + x_n}{2}) \to 1$ $(n \to \infty)$.

Lemma 6. *If $M \in \delta_2$ and $x_n(i) \to 0$ $(i = 1, 2 \ldots)$, then $\|x_n\| \to 0 \Leftrightarrow \rho_M(x_n) \to 0$.*

Proof. Since it is obvious that $\|x_n\| \to 0$ implies $\rho_M(x_n) \to 0$, we only need to prove that $\rho_M(x_n) \to 0$ implies $\|x_n\| \to 0$ $(n \to \infty)$. For any $\varepsilon > 0$, by $M \in \delta_2$, there exist $k > 0$, $a > 0$, $i_0 \in \mathbb{N}$ and $\{c_i\}_{i=i_0+1}^{\infty}$ with $\sum_{i=i_0+1}^{\infty} c_i < \infty$ which satisfy

$$M_i(u/\varepsilon) \le kM_i(u) + c_i \quad (i > i_0,\ M_i(u) \le a).$$

Since $\sum_{i=i_0+1}^{\infty} c_i < \infty$, there exists $i_1 \in \mathbb{N}$ such that $\sum_{i=i_1+1}^{\infty} c_i < 1/3$. By $x_n(i) \to 0$ $(i = 1, 2, \ldots, i_1)$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{i_1} M_i(x_n(i)/\varepsilon) < 1/3$ when $n \ge n_0$. Moreover, since $\rho_M(x_n) \to 0$, there exists $n_1 \in \mathbb{N}$ such that $\rho_M(x_n) < \min\{1/3k, a\}$ when

$n \geq n_1$. Therefore, when $n \geq \max\{n_0, n_1\}$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} M_i\Big(\frac{x_n(i)}{\varepsilon}\Big) &= \sum_{i=1}^{i_1} M_i\Big(\frac{x_n(i)}{\varepsilon}\Big) + \sum_{i=i_1+1}^{\infty} M_i\Big(\frac{x_n(i)}{\varepsilon}\Big)\\ &\leq \frac{1}{3} + \sum_{i=i_1+1}^{\infty} (kM_i(x_n(i)) + c_i)\\ &\leq \frac{1}{3} + k \cdot \frac{1}{3k} + \frac{1}{3} = 1.\end{aligned}$$

It follows that $\|x_n\| < \varepsilon$, i.e. $\|x_n\| \to 0$ $(n \to \infty)$.

Lemma 7 ([1]). *If $M \in \delta_2$, then $B(i) = \infty$.*

2. Results

Theorem 1. *A point $x \in S(l_M)$ is a strongly U-point if and only if*

(1) $|x(i)| = B(i)$ $(i \in \mathbb{N})$ *or* $\rho_M(x) = 1$,

(2) $\xi(x) < 1$,

(3) (i) *If for any $i \in \mathbb{N}$, $|x(i)| \in SC_{M_i}$, then there do not exist i, $j \in \mathbb{N}$ with $i \neq j$ such that $|x(i)| \in SC^{+}_{M_i}$ and $|x(j)| \in SC^{-}_{M_j}$,*

(ii) *If there exists $i_0 \in \mathbb{N}$ such that $|x(i_0)| \notin SC_{M_{i_0}}$, then $|x(j)| \in SC^{0}_{M_j}$ for any $j \in \mathbb{N}$ with $j \neq i_0$,*

(4) *If $e(i) > 0$, then $e(i) < |x(i)|$* $(i = 1, 2, \ldots)$.

Proof. Without loss of generality, we may assume that $x(i) \geq 0$ $(i \in \mathbb{N})$.

We suppose (1) does not hold, then there exists $i_0 \in \mathbb{N}$ such that $x(i_0) < B(i_0)$ and $\rho_M(x) < 1$. Furthermore, we can find a real number $\lambda > 0$ such that

$$M_{i_0}(x(i_0) + \lambda) \leq 1 - \sum_{i \neq i_0} M_i(x(i)).$$

Put

$$y(i) = \begin{cases} x(i), & i \neq i_0,\\ x(i_0) + \lambda, & i = i_0,\end{cases} \qquad z(i) = \begin{cases} x(i), & i \neq i_0,\\ x(i_0) - \lambda, & i = i_0.\end{cases}$$

It is obvious that $y+z = 2x$ and $y \neq z$. But $\rho_M(y) = \sum_{i \neq i_0} M_i(x(i)) + M_{i_0}(x(i_0)+\lambda) \leq 1$, hence $\|y\| \leq 1$. Similarly, we also have $\|z\| \leq 1$. Using $\|y + z\| = 2$, we get $\|y\| = \|z\| = 1$. This means that x is not an extreme point. Since a strong U-point must be an extreme point, this is a contradiction.

Let us prove the necessity of condition (2). Otherwise, $\xi(x) = 1$ i.e. $\rho_M(\lambda x) = \infty$ for any $\lambda > 1$. Since $\|x\| = 1$, there exists $i_0 \in \mathbb{N}$ such that $x(i_0) \neq 0$. Put

$$y(i) = \begin{cases} x(i), & i \neq i_0,\\ 0, & i = i_0.\end{cases}$$

It is obvious that $\rho_M(\lambda y) = \infty$ for any $\lambda > 1$, whence $\|y\| \geq 1$. On the other hand, clearly $\|y\| \leq \|x\| = 1$. So we have $\|y\| = 1$. Consequently, $1 \geq \big\|\frac{1}{2}(x + y)\big\| \geq \big\|\frac{1}{2}(y + y)\big\| = 2$, hence $\|x + y\| = 2$. But $x \neq y$, which contradicts that x is a strong U-point.

If the condition (i) of (3) does not hold, then there exist $i,\, j \in \mathcal{N}$ such that $x(i) \in SC^+_{M_i}$ and $x(j) \in SC^-_{M_j}$. For convenience we may assume $i = 1$, $j = 2$ and $x(1) = b_1$, $x(2) = a_2$ where $b_1 \in SC^+_{M_1}$, $a_2 \in SC^-_{M_2}$, then there exist $a_1 > 0$ and $b_2 > 0$ such that

$$M_1(u) = A_1 u + B_1 \qquad \text{for } u \in [a_1, b_1]$$

and

$$M_2(u) = A_2 u + B_2 \qquad \text{for } u \in [a_2, b_2].$$

Take $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $b_1 - \varepsilon_1 \in (a_1, b_1)$, $a_2 + \varepsilon_2 \in (a_2, b_2)$ and $A_1\varepsilon_1 = A_2\varepsilon_2$. Let

$$y = (x(1) - \varepsilon_1, x\,(2) + \varepsilon_2, x(3), x(4), \ldots)$$

Then

$$\begin{aligned}\rho_M(y) &= M_1(x(1) - \varepsilon_1) + M_2(x\,(2) + \varepsilon_2) + \sum_{i\ge 3} M_i(x(i))\\ &= A_1(x(1) - \varepsilon_1) + B_1 + A_2(x\,(2) + \varepsilon_2) + B_2 + \sum_{i\ge 3} M_i(x(i))\\ &= M_1(x(1)) + M_2(x\,(2)) + \sum_{i\ge 3} M_i(x(i)) = \rho_M(x) = 1.\end{aligned}$$

So by the definition of the Luxemburg norm, we have $\|y\| = 1$. Similarly,

$$\begin{aligned}\rho_M\left(\frac{x+y}{2}\right) &= M_1\left(x(1) - \frac{\varepsilon_1}{2}\right) + M_2\left(x\,(2) + \frac{\varepsilon_2}{2}\right) + \sum_{i\ge 3} M_i(x(i))\\ &= M_1(x(1)) + M_2(x\,(2)) + \sum_{i\ge 3} M_i(x(i)) = \rho_M(x) = 1,\end{aligned}$$

i.e. $\left\|\frac{x+y}{2}\right\| = 1$. Since $x \ne y$, x is not a strong U-point. A contradiction.

We suppose the condition (ii) of (3) is not true. Then there exists $i_0 \in \mathbb{N}$ such that $|x(i_0)| \notin SC_{M_{i_0}}$ and $j \in \mathbb{N}$, $j \ne i_0$ such that $x(j) \notin SC^0_{M_j}$. i.e. $x(j) \notin SC_{M_j}$ or $x(j) \in SC^+_{M_j}$ or $x(j) \in SC^-_{M_j}$. So, we can repeat the procedure from the proof of the necessity of the condition (i) of (3).

Let us finally prove the necessity of (4). Otherwise, there exists $i_0 \in \mathbb{N}$ such that $e(i_0) > 0$ and $x(i_0) \le e(i_0)$. Let us consider two cases:

Case I: $x(i_0) = e(i_0)$. Put

$$y(i) = \begin{cases} x(i), & i \ne i_0,\\ \frac{x(i_0)}{2}, & i = i_0.\end{cases}$$

Since $x(i_0) = e(i_0) < B(i_0)$, in virtue of (1) we have $\rho_M(x) = 1$. Therefore, we have the following equality

$$\rho_M(y) = \sum_{i\ne i_0} M_i(x(i)) + M_{i_0}\left(\frac{x(i_0)}{2}\right) = \sum_{i\ne i_0} M_i(x(i)) + M_{i_0}(x(i_0)) = \rho_M(x) = 1.$$

So $\|y\| = 1$. Similarly,

$$\rho_M\left(\frac{x+y}{2}\right) = \sum_{i\ne i_0} M_i(x(i)) + M_{i_0}\left(\frac{3}{4}x(i_0)\right) = \sum_{i\ne i_0} M_i(x(i)) + M_{i_0}(x(i_0)) = \rho_M(x) = 1,$$

i.e. $\|x+y\| = 2$. But obviously $x \ne y$, which contradicts the fact that x is a strong U-point.

Case II: $x(i_0) < e(i_0)$. We put

$$y(i) = \begin{cases} x(i), & i \neq i_0, \\ x(i_0) + \frac{e(i_0) - x(i_0)}{2}, & i = i_0, \end{cases} \qquad z(i) = \begin{cases} x(i), & i \neq i_0, \\ x(i_0) - \frac{e(i_0) - x(i_0)}{2}, & i = i_0. \end{cases}$$

It is obvious that $y + z = 2x$ and $y \neq z$. In the same way as in case I, it is easy to prove that $\|y\| = \|z\| = 1$. Therefore, x is not an extreme point, which leads to a contradiction.

Sufficiency. Let $x,\ y \in S(l_M)$ with $\|x + y\| = 2$, we consider the following two cases:

Case I: $|x(i)| = B(i)$ for all $i \in \mathbb{N}$. Without loss of generality, we may assume $x(i) \geq 0$ and $y(i) \geq 0$ $(i = 1, 2, \ldots)$. In this case we have $\|(B(1),\ B(2),\ \ldots)\| = \|(x(1),\ x(2),\ \ldots)\| = \|x\| = 1$. Using

$$x(i) + y(i) \leq 2B(i) \qquad (i = 1, 2, \ldots)$$

and

$$2 = \|x + y\| \leq 2\,\|(B(1),\ B(2),\ \ldots)\| = 2$$

we have the equality $x(i) = B(i)$ $(i = 1, 2, \ldots)$. Therefore $y(i) = x(i) = B(i)$ for all $i \in \mathbb{N}$ i.e. $x = y$.

Case II: $\rho_M(x) = 1$. First, we will prove that $\rho_M(\frac{x+y}{2}) = 1$.

For any $\varepsilon \in (0, \frac{1-\xi(x)}{1+\xi(x)})$ we have $\left\|(1+\varepsilon)\frac{x+y}{2}\right\| = 1 + \varepsilon$ and $\rho_M\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) < \infty$. Hence there exists $\alpha > 0$ such that

$$\rho_M\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) = \rho_M(x) + \alpha\varepsilon.$$

Therefore

$$\begin{aligned} 1 < \rho_M\left((1+\varepsilon)\frac{x+y}{2}\right) &= \rho_M\left(\frac{1+\varepsilon}{2}y + \frac{1-\varepsilon}{2}\frac{1+\varepsilon}{1-\varepsilon}x\right) \\ &\leq \frac{1+\varepsilon}{2}\rho_M(y) + \frac{1-\varepsilon}{2}\rho_M\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) \\ &= \frac{1+\varepsilon}{2}\rho_M(y) + \frac{1-\varepsilon}{2}(\rho_M(x) + \alpha\varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$, we get $\rho_M(y) = 1$. Since $\left\|\frac{x+y}{2}\right\| = 1$ and the norm $\|\cdot\|_M$ is a convex function, it follows that $\|\cdot\|_M$ is an affine function on the segment between x and y. Therefore

$$\left\|\frac{(\frac{1}{2}(x+y) + x)}{2}\right\| = \left\|\frac{1}{4}y + \frac{3}{4}x\right\| = 1.$$

Hence we can get in the same way as above (with $\frac{1}{2}(x+y)$ in place of y) that $\rho_M(\frac{x+y}{2}) = 1$. Hence

$$\begin{aligned} 0 &= \frac{\rho_M(x) + \rho_M(y)}{2} - \rho_M\left(\frac{x+y}{2}\right) \\ &= \sum_{i=1}^{\infty}\left[\left(\frac{M_i(x(i) + M_I(Y(i)}{2}\right) - M_i\left(\frac{x(i) + y(i)}{2}\right)\right] \geq 0. \end{aligned}$$

Thus we have

$$\frac{M_i(x(i)) + M_i(y(i))}{2} = M_i\left(\frac{x(i)+y(i)}{2}\right), \qquad i = 1, 2, 3, \ldots.$$

This means that $x(i) = y(i)$ or $x(i)$ and $y(i)$ belong to the same intervals of $SAI(M_i)$ for all $i \in \mathbb{N}$.

If the condition (i) of (3) holds true, we may assume without loss of generality that x, $y \geq 0$ and either $x(i) \in SC^+_{M_i}$ or $x(i) \in SC^0_{M_i}$ for all $i \in \mathbb{N}$. Define

$$N_1 = \{i \in \mathbb{N} : x(i) \in SC^+_{M_i}\}.$$

In view of condition (4), we get, for any $i \in \mathbb{N}$, that there exist $A_i > 0$, $B_i \in \mathbb{R}$ and $\varepsilon_i > 0$ such that $M_i(u) = A_i u + B_i$ for all $u \in [x(i) - \varepsilon_i, x(i)]$. Therefore by the above properties of x and y, we have

$$\begin{aligned} y(i) &= x(i) \qquad (\forall i \in \mathbb{N} \backslash N_1), \\ y(i) &\leq x(i) \qquad (\forall i \in N_1). \end{aligned}$$

The equality $\rho_M(\frac{x+y}{2}) = \rho_M(x)$ implies that

$$\sum_{i \in N_1} M_i\left(\frac{x(i)+y(i)}{2}\right) = \sum_{i \in N_1} M_i(x(i)),$$

i.e.

$$\sum_{i \in N_1} \left(A_i \frac{x(i)+y(i)}{2} + B_i\right) = \sum_{i \in N_1} (A_i x(i) + B_i),$$

Hence

$$\sum_{i \in N_1} A_i\left(\frac{y(i)-x(i)}{2}\right) = 0.$$

Consequently, $y(i) = x(i)$ for all $i \in \mathbb{N}$, i.e. $x = y$.

If (ii) of (3) holds, then $x(i) = y(i)$ for $i \neq i_0$. Moreover, by condition (4), there exist $A_0 > 0, B_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$M_{i_0}(u) = A_0 u + B_0, \qquad u \in [x(i_0) - \varepsilon_0, x(i_0) + \varepsilon_0].$$

The equality $\rho_M(\frac{x+y}{2}) = \rho_M(x)$ implies $M_{i_0}(\frac{x(i_0)+y(i_0)}{2}) = M_{i_0}(x(i_0))$, i.e.

$$A_0\left(\frac{x(i_0)+y(i_0)}{2}\right) + B_0 = A_0(x(i_0)) + B_0.$$

Hence $x(i_0) = y(i_0)$ and so $x = y$. This finishes the proof of the theorem.

THEOREM 2. *If $x \in S(l_M)$, then the following statements are equivalent*:

1. *x is a CLUR-point*,
2. *x is a WCLUR-point*,
3. (i) *$M \in \delta_2$*
   (ii) *$M \in \overline{\delta}_2$ or $\{i \in \mathbb{N} : |x(i)| \in (a, b]\} = \emptyset$ where $[a, b] \in SAI(M_i)$.*

Proof. The implication 1⇒2 is obvious.

2⇒3. We suppose (i) does not hold, i.e. $M \notin \delta_2$. By Lemma 2, there exist $z \in l_M$ and a singular function Φ with $\rho_M(z) < \infty$ and $\Phi(x-z) \neq 0$. Set

$$x_n = (x(1), \ldots, x(n), z(n+1), z(n+2), \ldots) \quad (n = 1, 2, \ldots).$$

Then

$$\rho_M(x_n) \leq \rho_M(x) + \sum_{i=n+1}^{\infty} M_i(z(i)) \to \rho_M(x) \leq 1,$$

so $\limsup_{n\to\infty} \|x_n\| \leq 1$. Notice $\|x_n + x\| \geq 2\,\|(x(1), \ldots, x(n), 0, \ldots)\| \to 2$, we have $\liminf_{n\to\infty} \|x_n + x\| \geq 2$. Hence $\|x_n\| \to 1$ and $\|x_n + x\| \to 2$ $(n \to \infty)$. Since $x_n \to x$ coordinatewise, we may assume without loss of generality that $x_n \xrightarrow{w} x$ (passing to a subsequence if necessary). But $\Phi(x - x_n) = \Phi(x - z) \neq 0$, which contradicts $x_n \xrightarrow{w} x$. This contradiction shows that $M \in \delta_2$.

Without loss of generality, we assume $x(i) \geq 0$ for all $i \in \mathbb{N}$.

If the condition (ii) of (3) does not hold, then there exists $j \in \mathbb{N}$ such that $x(j) \in (a, b]$, without loss of generality we may assume $j = 1$ and $M \notin \overline{\delta}_2$ where $[a, b] \in SAI(M_1)$ satisfies $M_1(u) = Au + B \quad$ for $u \in [a, b]$. Take $\varepsilon > 0$, such that $x(1) - \varepsilon \in (a, b]$. Since $M \notin \overline{\delta}_2$, by Lemma 3, there exist $u_i^n > 0$ satisfying

$$M_i(u_i^n) \leq \frac{1}{n}, \qquad M_i\left(\frac{u_i^n}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{M_i(u_i^n)}{2} \qquad (i = m_{n-1}+1, \ldots, m_n)$$

and

$$\sum_{i=m_{n-1}+1}^{m_n} M_i(u_i^n) > 1.$$

Without loss of generality, we may assume $A\varepsilon < 1$. For every sufficiently large n, take $m_{n-1} < m'_n \leq m_n$ such that

$$A\varepsilon - \frac{1}{2^n} \leq \sum_{i=m_{n-1}+1}^{m'_n} M_i(u_i^n) < A\varepsilon, \qquad n = 1, 2, \ldots.$$

Let $\{e_n\}_n$ be the natural basis of l^1 and $\{p_n\}_n$ the projections $p_n(x) = \sum_{i=1}^{n} x(i)e_i$ for $x = (x(i))_i \in l_M$. Put

$$x_n = P_n x - P_1 x + (x(1) - \varepsilon)e_1 + \sum_{i=m_{n-1}+1}^{m'_n} u_i^n e_i$$

Then

$$\begin{aligned}\rho_M(x_n) &= M_1(x(1) - \varepsilon) + \sum_{i=2}^{n} M_i(x(i)) + \sum_{i=m_{n-1}+1}^{m'_n} M_i(u_i^n) \\ &= \alpha x(1) - \alpha\varepsilon + \beta + \sum_{i=2}^{n} M_i(x(i)) + \sum_{i=m_{n-1}+1}^{m'_n} M_i(u_i^n)\end{aligned}$$

$$= M_1(x(1)) - \alpha\varepsilon + \sum_{i=2}^{n} M_i(x(i)) + \sum_{i=m_{n-1}+1}^{m'_n} M_i(u_i^n)$$

$$< \sum_{i=1}^{n} M_i(x(i)) - A\varepsilon + A\varepsilon = \sum_{i=1}^{n} M_i(x(i)) \leq 1.$$

So $\limsup_{n\to\infty} \|x_n\| \leq 1$. Moreover,

$$\rho_M\left(\frac{x+x_n}{2}\right) \geq M_1\left(x(1) - \frac{\varepsilon}{2}\right) + \sum_{i=2}^{n} M_i(x(i)) + \sum_{i=m_{n-1}+1}^{m'_n} M_i\left(\frac{x(i)+u_i^n}{2}\right)$$

$$\geq \sum_{i=1}^{n} M_i(x(i)) - \frac{A\varepsilon}{2} + \sum_{i=m_{n-1}+1}^{m'_n} \left(\left(1-\frac{1}{n}\right)\frac{M_i(u_i^n)}{2}\right)$$

$$\geq \sum_{i=1}^{n} M_i(x(i)) - \frac{A\varepsilon}{2} + \frac{1}{2}\left(1-\frac{1}{n}\right)\left(A\varepsilon - \frac{1}{2^n}\right) \to 1 \ (n\to\infty).$$

Hence $\liminf_{n\to\infty} \left\|\frac{x+x_n}{2}\right\| \geq 1$. Thus we have $\|x_n\| \to 1$ and $\|x_n + x\| \to 2$ $(n\to\infty)$.

Since $\lim_{n\to\infty}(A\varepsilon - 1/2^n) = A\varepsilon > A\varepsilon/2$, there exists n_0 such that $A\varepsilon - 1/2^n > A\varepsilon/2$ when $n \geq n_0$. Therefore

$$\|x_m - x_n\| \geq \left\|\sum_{i=m_{m-1}+1}^{m'_m} u_i^m e_i\right\| \geq \sum_{i=m_{m-1}+1}^{m'_m} M_i(u_i^m) > A\varepsilon - \frac{1}{2^m} > \frac{A\varepsilon}{2}$$

when $m > n \geq n_0$.

This means that $\{x_n\}$ is not compact in $S(l_M)$, hence x is not a CLUR-point. But, by $M \in \delta_2$ and Theorem 2 in [7], we can get that l_M has H-property. Therefore x is not a WCLUR-point. This is a contradiction.

$3 \Rightarrow 1$. Suppose $x \in S(l_M), \{x_n\}_{n=1}^{\infty} \subset S(l_M)$ and $\|x_n + x\| \to 2$ $(n\to\infty)$. In order to complete this proof we distinguish two cases.

(I) $M \in \delta_2 \cap \overline{\delta}_2$. In this case, by Lemma 1 and Lemma 2, we take $\{f_n\} \subset S(l_N^0)$ such that $f_n(x_n + x) = \|x_n + x\| \to 2$ $(n\to\infty)$. Then

$$f_n(x) \to 1 \quad \text{and} \quad f_n(x_n) \to 1 \quad (n\to\infty).$$

In virtue of [6], l_N^0 is reflexive. Then there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and $f \in l_N^0$ such that $f_n \to^w f$. It is obvious that in virtue of $\lim_{n\to\infty} f_n(x) = 1$ this yields $f(x) = 1$. Hence $\|f\|^0 = 1$. By Theorem 1 in [7], we get that l_N^0 has H-property. Hence $\|f_n - f\|^0 \to 0$ $(n\to\infty)$. So

$$f(x_{n_i}) = (f - f_{n_i})(x_{n_i}) + f_{n_i}(x_{n_i}) \to 1 \quad (n\to\infty).$$

Using now the reflexivity of l_M, we can find a subsequence $\{x'_{n_i}\} \subset \{x_{n_i}\}$ and $x' \in l_M$ such that $x'_{n_i} \to^w x' \quad (n\to\infty)$. Obviously $f(x') = 1$, whence $\|x'\| = 1$. By the property H for l_M, we have $\lim_{n\to\infty} \|x'_{n_i} - x'\| = 0$, i.e. $\{x_n\}$ is compact in $S(l_M)$, which implies that x is a CLUR-point.

(II) $M \in \delta_2$ and $\{i \in \mathbb{N} : |x(i)| \in (a,b]\} = \emptyset$ where $[a,b] \in SAI(M_i)$. First, we will prove that $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$. We first show

$$\liminf_n x_n(j) \geq x(j), \quad j = 1, 2, \ldots . \tag{1}$$

If not, there exist $j_0 \in \mathbb{N}$, $\varepsilon_0 > 0$ and a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$x_n(j_0) \leq x(j_0) - \varepsilon.$$

Since $x(j_0) \notin (a,b]$, there exists $\delta > 0$ such that

$$M_{j_0}\left(\frac{x(j_0) + x_n(j_0)}{2}\right) \leq (1-\delta)\frac{M_{j_0}(x(j_0)) + M_{j_0}(x_n(j_0))}{2}.$$

Then by Lemma 4 and Lemma 5, we get

$$\begin{aligned} 0 \leftarrow & \frac{\rho_M(x) + \rho_M(x_n)}{2} - \rho_M\left(\frac{x_n + x}{2}\right) \\ = & \sum_{i=1}^{\infty}\left[\frac{M_i(x_n(i)) + M_i(x(i))}{2} - M_i\left(\frac{x_n(i) + x(i)}{2}\right)\right] \\ \geq & \frac{M_{j_0}(x_n(j_0)) + M_{j_0}(x(j_0))}{2} - M_{j_0}\left(\frac{x_n(j_0) + x(j_0)}{2}\right) \\ \geq & \delta\frac{M_{j_0}(x_n(j_0)) + M_{j_0}(x(j_0))}{2} \geq \frac{\delta}{2}M\left(\frac{\varepsilon}{2}\right) > 0. \end{aligned}$$

This contradiction shows that condition (1) holds.

Now, we will show that

$$\limsup_n x_n(j) \leq x(j), \quad j = 1, 2, \ldots . \tag{2}$$

Otherwise, there exist $j_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $\limsup_n x_n(j_0) \geq x(j_0) + \varepsilon$. Then $\limsup_n M_{j_0}(x_n(j_0)) \geq M_{j_0}(x(j_0)) + \varepsilon'$ for some $\varepsilon' > 0$. Hence

$$\begin{aligned} 1 = \limsup_n \rho_M(x_n) &= \limsup_n \sum_{i \neq j_0} M_i(x_n(i)) + M_{j_0}(x_n(j_0)) \\ &\geq \sum_{i \neq j_0} M_i(x(i)) + M_{j_0}(x(j_0)) + \varepsilon' = \rho_M(x) + \varepsilon' = 1 + \varepsilon'. \end{aligned}$$

This is a contradiction. So $\lim_{n\to\infty} x_n(i) = x(i)$ $(i \in \mathbb{N})$ thanks to (1) and (2).

Next, we will show that $\rho_M(\frac{x_n - x}{2}) \to 0$ $(n \to \infty)$. In fact, for any $\varepsilon > 0$, there exist i_0 and n_0 such that

$$\sum_{i > i_0} M_i(x(i)) < \frac{\varepsilon}{4}, \quad \sum_{i=1}^{i_0} M_i\left(\frac{x_n(i) - x(i)}{2}\right) < \frac{\varepsilon}{4}$$

and

$$\sum_{i=1}^{i_0} |M_i(x_n(i) - M_i(x(i))| < \varepsilon \quad \text{when } n \geq n_0.$$

Hence when $n \geq n_0$,

$$\sum_{i>i_0} M_i(x_n(i)) = \rho_M(x_n) - \sum_{i=1}^{i_0} M_i(x_n(i) \leq 1 - \sum_{i=1}^{i_0} M_i(x(i)) + \varepsilon$$
$$\leq 1 - \left(1 - \sum_{i>i_0} M_i(x(i))\right) + \varepsilon < \frac{5}{4}\varepsilon.$$

Therefore

$$\begin{aligned}\rho_M\left(\frac{x_n - x}{2}\right) &= \sum_{i=1}^{i_0} M_i\left(\frac{x_n(i) - x(i)}{2}\right) + \sum_{i>i_0} M_i\left(\frac{x_n(i) - x(i)}{2}\right) \\ &\leq \sum_{i=1}^{i_0} M_i\left(\frac{x_n(i) - x(i)}{2}\right) + \frac{1}{2}\Big[\sum_{i>i_0} M_i(x_n(i)) + \sum_{i>i_0} M_i(x(i))\Big] \\ &< \frac{\varepsilon}{4} + \frac{1}{2}\left(\frac{5}{4}\varepsilon + \frac{\varepsilon}{4}\right) = \varepsilon,\end{aligned}$$

i.e. $\rho_M(\frac{x_n - x}{2}) \to 0$ $(n \to \infty)$. So by $\lim_{n\to\infty} x_n(i) = x(i)$ $(i \in \mathbb{N})$ and Lemma 6, we get $\|x_n - x\| \to 0$ $(n \to \infty)$. This means x is a CLUR-point. Thus, the proof is finished.

It is obvious that a point $x \in S(X)$ is a LUR-point if and only if it is a CLUR-point and a SU-point. So, in view of Lemma 7, Theorem 1 and Theorem 2, we easily obtain the following criteria for LUR-point of $S(l_M)$.

Theorem 3. *A point $x \in S(l_M)$ is a LUR-point if and only if:*

1. $M \in \delta_2$,
2. *If for any $i \in \mathbb{N}$, $|x(i)| \in SC_{M_i}$, then*
   (i) $\{i \in \mathbb{N} : |x(i)| \in SC^+_{M_i}\} = \emptyset$;
   (ii) *if $\{i \in \mathbb{N} : |x(i)| \in SC^+_{M_i}\} \neq \emptyset$, then $\{\forall i \in \mathbb{N} : |x(i)| \in SC^-_{M_i}\} = \emptyset$ and $M \in \overline{\delta}_2$.*
3. *If there exists $i_0 \in \mathbb{N}$ such that $|x(i_0)| \notin SC_{M_{i_0}}$, then $|x(j)| \in SC^0_{M_j}$ $(j \in \mathbb{N}$, $j \neq i_0)$ and $M \in \overline{\delta}_2$,*
4. *If $e(i) > 0$, then $e(i) < |x(i)|$ for all $i \in \mathbb{N}$.*

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