ORLICZ CENTENARY VOLUME II
BANACH CENTER PUBLICATIONS, VOLUME 68
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

EXTENSION VIA INTERPOLATION

A. GONCHAROV
Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail: goncha@fen.bilkent.edu.tr

Abstract. We suggest a modification of the Pawłucki and Pleśniak method to construct a continuous linear extension operator by means of interpolation polynomials. As an illustration we present explicitly the extension operator for the space of Whitney functions given on the Cantor ternary set.

1. Introduction. Given a compact subset K of $\mathbb{R}$, let $\mathcal{E}(K)$ denote the space of Whitney jets on K with the topology τ defined by the norms

$$\|f\|_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K, x \neq y, k = 0, 1, \dots, q\right\},$$

$q = 0, 1, \dots,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K, k \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. Each function $f \in \mathcal{E}(K)$ is extendable to a C^∞-function on the line.

Whitney's extension theorem ([8]) gives an extension operator (here and in what follows it means a continuous linear extension operator) from the space $\mathcal{E}^m(K)$ of Whitney jets of m-th order on K to the space of m-times differentiable functions on the whole space, provided $m \in \mathbb{N}$. In the case $m = \infty$ such an operator generally does not exist, and several authors have considered the extension problem in different situations (see e.g. [4], [5] for the bibliography). We restrict our attention to the approach of Pawłucki and Pleśniak. In [5] (see also [6], [7]) they gave an extension operator in the form of a telescoping series containing Lagrange interpolation polynomials with the Fekete-Leja system of knots. The basic assumption for their method was the Markov Property of a compact set K (see [5]–[7] for the definitions). Here we modify slightly the construction, namely we interpolate functions locally. This modification permits one to give explicitly the extension operator for the space of Whitney functions given on the Cantor set (it satisfies the Markov Property by [2], but we do not know the distribution of the Fekete points there). Moreover, the modified version can be applied for generalized Cantor-type

2000 *Mathematics Subject Classification*: Primary 46E10; Secondary 41A05, 41A10.
The paper is in final form and no version of it will be published elsewhere.

sets without Markov's Property ([1]). We also hope that the explicit form of the extension operator will give a hint how to construct a basis in the space of Whitney functions on Cantor-type sets.

2. Divided differences. Let I denote a closed interval; let $|f|_k^{(I)} = \sup\{|f^{(j)}(x)| : x \in I, j \leq k\}$ for $f \in C^\infty(I)$ and $k = 0, 1, \ldots$.

Given a function f, a natural number k with $1 \leq k \leq N$ and distinct points $(x_j)_1^N \subset I$, we are looking for the coefficients $A_j^{(k)}$ in the expansion $[x_1, \ldots, x_N]f = \sum_{j=1}^{N-k+1} A_j^{(k)}[x_j, \ldots, x_{j+k-1}]f$ (see e.g. [3] for the definition and properties of divided differences). Let us fix $j, 1 \leq j \leq N-k+1$. There are $\binom{N-k}{j-1}$ different ways to obtain $[x_j, \ldots, x_{j+k-1}]f$ from $[x_1, \ldots, x_N]f$ using the recurrence relation for the divided differences. Every way corresponds to a certain *possible route* from $[1, \ldots, N]$ to $[j, \ldots, j+k-1]$, that is the chain of truncations of one of the end elements of the interval. We can draw all possible routes from $[1, \ldots, N]$ to $[j, \ldots, j+k-1]$ in the form of a parallelogram containing $(N-j-k+2)j$ subintervals of $[1, \ldots, N]$ as elements. Using the recurrence relation $A_j^{(k)} = -(x_{j-1} - x_{j+k-1})^{-1} A_{j-1}^{(k+1)} + (x_j - x_{j+k})^{-1} A_j^{(k+1)}$ for $j \geq 2, k \leq N-1$ and $A_1^{(N-1)} = -A_2^{(N-1)} = (x_1 - x_N)^{-1}$, we represent $A_j^{(k)}$ as a fraction with denominator equal to the product $\prod_{m=1}^M (x_{a(m)} - x_{b(m)})$ with $[a(m), \ldots, b(m)] \supset [j, \ldots, j+k-1]$ properly. Clearly, the value M is just the number of elements in the parallelogram of possible routes minus one, that is $M = (N-k)j - (j-1)^2$. The numerator of the fraction is the sum of $\binom{N-k}{j-1}$ products, where every product contains $M - (N-k)$ terms. The last is the number of elements in the parallelogram outside of the fixed route. Moreover, we have to include in the result the coefficient $(-1)^{j-1}$, as after any truncation from the left the sign will change. Thus, $A_j^{(k)} = (-1)^{j-1} \frac{\sum P_i}{Q}$, where $Q = \prod_{m=1}^M (x_{a(m)} - x_{b(m)})$ and $P_i = \prod_{m=1}^{M-N+k} (x_{a_i(m)} - x_{b_i(m)})$ corresponds to the complement of the i-th route, $i = 1, \ldots, \binom{N-k}{j-1}$. From this we get the bound

$$|A_j^{(k)}| \leq \binom{N-k}{j-1} \max \prod_{m=1}^{N-k} |x_{a_i(m)} - x_{b_i(m)}|^{-1},$$

where max is taken over all possible routes from $[1, \ldots, N]$ to $[j, \ldots, j+k-1]$. Of course, this formula is valid also when $j = 1, k \leq N-1$ and in the case $k = N, j = 1$, if we adopt the convention that $\prod_{m=1}^0 (\cdots) := 1$.

Finally,

$$|[x_1, \ldots, x_N]f| \leq 2^{N-k} |f|_{k-1}^{(I)} \max \prod_{m=1}^{N-k} |x_{a_i(m)} - x_{b_i(m)}|^{-1}, \tag{1}$$

where max is taken over all possible routes from $[1, \ldots, N]$ to some $[j, \ldots, j+k-1]$ with $1 \leq j \leq N-k+1$.

3. Extension operator for Cantor-type sets. Let $(l_s)_{s=0}^\infty$ be a sequence such that $l_0 = 1, 0 < 2l_{s+1} < l_s, s \in \mathbb{N}$. Let K be the Cantor set associated with the sequence (l_s), that is, $K = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{1,0} = [0,1], E_s$ is the union of 2^s closed *basic* intervals

$I_{j,s}=[a_{j,s},b_{j,s}]$ of length l_s and E_{s+1} is obtained by deleting open concentric subinterval of length l_s-2l_{s+1} from each $I_{j,s}, j=1,2,\dots,2^s$.

Let us fix $s,m\in\mathbb{N}$, let $N=2^m-1$. The interval $I_{1,s}$ covers 2^{m-1} basic intervals of length l_{s+m-1}. The endpoints of these intervals give us $N+1$ points (x_k). Let $\Omega(x)=\prod_{k=1}^{N+1}(x-x_k)$ and $\omega_k(x)=\frac{\Omega(x)}{(x-x_k)\Omega'(x_k)}$. Then $L_N(f,x,I_{1,s})=\sum_{k=1}^{N+1}f(x_k)\omega_k(x)$ is the Lagrange interpolation polynomial, corresponding to the interval $I_{1,s}$. In case $2^m<N+1<2^{m+1}$ we use the following procedure to include new $N+1-2^m$ endpoints of the basic intervals of length l_{s+m} into the interpolation set: the first new point is l_{s+m}, the next newcomer is l_s-l_{s+m}, then $l_{s+1}-l_{s+m}, l_s-l_{s+1}+l_{s+m}, l_{s+2}-l_{s+m}, l_s-l_{s+2}+l_{s+m}$, etc. In the same manner we choose nodes for polynomials $L_N(f,x,I_{j,s})$, corresponding to other basic intervals.

Let us next fix $\delta>0$, a compact set E and take the C^∞-function $u(\cdot,\delta,E)$ with the properties: $u(\cdot,\delta,E)\equiv 1$ on E, $u(x,\delta,E)=0$ for $dist(x,E)>\delta$ and $|u|_p\le c_p\delta^{-p}$, where the constant c_p depends only on p (see e.g. [7], L.2.1).

Suppose we have a sequence of natural numbers $(n_s)_0^\infty$ and a sequence of positive values $(\delta_{N,s})_{N=1,s=0}^{\infty,\infty}$. Let $N_s=2^{n_s}-1$ and $M_s=2^{n_{s-1}-1}-1$ for $s\ge 1, M_0=1$. Consider the operator

$$
\begin{aligned}
L(f,x)&=L_{M_0}(f,x,I_{1,0})u(x,\delta_{M_0+1,0},I_{1,0}\cap K)\\
&\quad+\sum_{s=0}^{\infty}\Big\langle\sum_{j=1}^{2^s}\sum_{N=M_s+1}^{N_s}[L_N(f,x,I_{j,s})-L_{N-1}(f,x,I_{j,s})]u(x,\delta_{N,s},I_{j,s}\cap K)\\
&\quad+\sum_{j=1}^{2^{s+1}}[L_{M_{s+1}}(f,x,I_{j,s+1})-L_{N_s}(f,x,I_{[\frac{j+1}{2}],s})]u(x,\delta_{N_s,s},I_{j,s+1}\cap K)\Big\rangle.
\end{aligned}
$$

Here $[\frac{j+1}{2}]$ is the greatest integer in $\frac{j+1}{2}$. We see that for fixed j the sum $\sum_{N=M_s+1}^{N_s}\cdots$ gives accumulation of degree of interpolation polynomials on the corresponding basic interval of length l_s, whereas the term in the last sum is the passage from 2^{n_s} points on the basic interval of length l_s to 2^{n_s-1} points on its subinterval of length l_{s+1}.

Let us rearrange the terms in angular brackets. Suppose that supports of the smoothing functions u, corresponding to different basic intervals of equal length, are disjoint. The sums $\sum_{j=1}^{2^s}L_{N_s}(f,x,I_{j,s})\cdot[u(x,\delta_{N_s,s},I_{j,s}\cap K)-u(x,\delta_{N_s,s},I_{2j-1,s+1}\cap K)-u(x,\delta_{N_s,s},I_{2j,s+1}\cap K)]$ vanish, since the expression in square brackets is 0. Therefore,

$$L(f,x)=\sum_{s=0}^{\infty}\sigma_s \tag{2}$$

with

$$
\begin{aligned}
\sigma_0=&\sum_{N=M_0+1}^{N_0-1}L_N(f,x,I_{1,0})[u(x,\delta_{N,0},I_{1,0}\cap K)-u(x,\delta_{N+1,0},I_{1,0}\cap K)]\\
&+\sum_{j=1}^{2}L_{M_1}(f,x,I_{j,1})u(x,\delta_{N_0,1},I_{j,1}\cap K)
\end{aligned}
$$

and for $s \in \mathbb{N}$

$$\sigma_s = -\sum_{j=1}^{2^s} L_{M_s}(f, x, I_{j,s}) u(x, \delta_{M_{s+1},s}, I_{j,s} \cap K)$$
$$+ \sum_{j=1}^{2^s} \sum_{N=M_s+1}^{N_s-1} L_N(f, x, I_{j,s}) [u(x, \delta_{N,s}, I_{j,s} \cap K) - u(x, \delta_{N+1,s}, I_{j,s} \cap K)]$$
$$+ \sum_{j=1}^{2^{s+1}} L_{M_{s+1}}(f, x, I_{j,s+1}) u(x, \delta_{N_s,s+1}, I_{j,s+1} \cap K).$$

In the general case we can only state that L is a linear operator.

4. Cantor set. From now on, we deal with the Cantor set K, that is, $l_s = 3^{-s}$. We take $n_s = 2, s \le 3$ and $n_s = [\log_2 s]$ for $s \ge 4$, so $N_{2^m} = N_{2^m+1} = \cdots = N_{2^{m+1}-1} = 2^m - 1$.

Let us show that L is in fact an extension operator. Since the values of interpolation polynomials on compact sets do not depend on parameters of smoothing functions, we can specify the sequence $(\delta_{N,s})_{N=1,s=0}^{\infty,\infty}$ later.

LEMMA 1. *For any $f \in \mathcal{E}(K), x \in K$, we have $L(f,x) = f(x)$.*

Proof. We want to show that the series (2) converges for $x \in K$ and gives the value $f(x)$. By telescoping effect summation in (2) will give only the expression in the form of the last sum in the definition of σ_s. Moreover, for fixed $x \in K$ only one term out of 2^{s+1} is not zero here. Therefore,

$$L(f,x) = \lim_{s\to\infty} L_{M_{s+1}}(f, x, I_{j,s+1}), \tag{3}$$

where $j = j(s)$ is chosen in a such way that $x \in I_{j,s+1}$.

Now, for $N = 2^n - 1$, let us consider the difference $L_N(f, x, I_{1,s}) - f(x)$. Since $\sum_1^{2^n} \omega_k(x) \equiv 1$ and $\sum_1^{2^n} (x - x_k)^i \omega_k(x) \equiv 0, i = 1, 2, \ldots, N$, we get $L_N(f, x, I_{1,s}) - f(x) = \sum_{k=1}^{2^n} [f(x_k) - f(x) - f'(x)(x_k - x) - \cdots - f^{(q)}(x)(x_k - x)^q/q!] \omega_k(x) = \sum_{k=1}^{2^n} R_x^q f(x_k) \omega_k(x)$. Then for $q \le N$

$$|L_N(f, x, I_{1,s}) - f(x)| \le ||f||_q \sum_{k=1}^{2^n} |x - x_k|^q |\omega_k(x)|. \tag{4}$$

For the denominator of $|\omega_k(x)|$ since $1 - 2l_{i+1}/l_i = 1/3$, we get the bound

$$|x_k - x_1| \cdots |x_k - x_{k-1}| \cdot |x_k - x_{k+1}| \cdots |x_k - x_{N+1}|$$
$$\ge l_{n+s-1} \cdot (l_{n+s-2} - 2l_{n+s-1})^2 \cdots (l_s - 2l_{s+1})^{2^{n-1}}$$
$$= 3^{-n-s+1} \cdot 3^{2(-n-s+2)} \cdots 3^{2^{n-1}(-s)} \cdot (1/3)^{2+4+\cdots+2^{n-1}} = 3^{-\mu},$$

where $\mu = 2^n(s+2) - (n+s) - 3$, as is easy to check.

On the other hand, the numerator of $|\omega_k(x)|$ multiplied by $|x - x_k|^q$ gives the bound

$$|x - x_k|^{q-1} \prod_1^{N+1} |x - x_k| \le l_s^{q-1} \cdot l_{n+s} \cdot l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{2^{n-1}} = 3^{-\lambda}$$

with $\lambda = 2^n(s+1) + (q-1)s - 1$.

Hence, the sum in (4) may be estimated from above by $2^n 3^{2^n-n-qs-2}$, which approaches the limit 0 for $q \geq 1$ because $2^n \leq s$. The same arguments are valid for $L_{M_{s+1}}(f, x, I_{j,s+1})$. In fact, $M_{s+1} + 1 = 2^{n_s-1} \leq \frac{1}{2}s < s+1$. Thus the limit in (3) equals $f(x)$. ■

5. Estimation for the model case. It remains to prove that the operator L is well-defined and continuous. First let us consider the situation when the number of interpolation points is just 2^n.

LEMMA 2. *Let $N = 2^n - 1, \delta = \frac{1}{2}l_{n+s-1}$. Enumerate the endpoints $(x_k)_1^{N+1}$ of first basic intervals of length l_{s+n-1} in increasing order. Suppose two natural numbers $p, (p < N)$, and $q = 2^v - 1$ with $v < N$ are given. Then for $f \in \mathcal{E}(K)$ and $x \in \mathbb{R}$ we have the bound*

$$|\{[L_N(f,x,I_{1,s}) - L_{N-1}(f,x,I_{1,s})]u(x,\delta,I_{1,s}\cap K)\}^{(p)}| \leq C2^N N^p \delta^{-p} \prod_{k=2}^{q+1} x_k |\tilde{f}|_q^{([0,1])},$$

where the constant C depends only on p and $\tilde{f} \in C^\infty[0,1]$ is any extension of f on $[0,1]$.

Proof. By Newton's form of the interpolation operator we deduce the representation $L_N(f,x,I_{1,s}) - L_{N-1}(f,x,I_{1,s}) = [x_1,\ldots,x_{N+1}]f \cdot \Omega(x)$ with $\Omega(x) = \prod_{k=1}^N (x - x_k)$. The i-th derivative of Ω represents the sum of $\frac{N!}{(N-i)!}$ products, where every product contains $N-i$ terms of the type $(x - x_k)$. Hence for any x with $dist(x, I_{1,s}\cap K) \leq \delta$ we get $|\Omega^{(i)}(x)| \leq \frac{N!}{(N-i)!}\prod_{k=i+1}^N(\delta + x_k)$. Then for some constant C_p we get

$$|(\Omega\cdot u)^{(p)}| \leq \sum_{i=0}^p \binom{p}{i}\frac{c_{p-i}}{\delta^{p-i}}N^i \prod_{k=i+1}^N (\delta + x_k) \leq C_p\delta^{-p}\prod_{k=1}^N(\delta + x_k)\max_{i\leq p} B_i,$$

with $B_0 = 1, B_i = \frac{(N\delta)^i}{(\delta+x_1)\cdots(\delta+x_i)}$. In our case $x_k = D_k\delta$, where the sequence $(D_k)_{k=1} = (0,2,4,6,12,14,16,18,36,\ldots)$ is defined by the structure of the Cantor set. Clearly, $D_k + 1 \geq k$. For this reason, $B_i = \frac{N^i}{(D_1+1)\cdots(D_i+1)} \leq N^i/i!, \max B_i \leq N^p$.

On the other hand, $\delta + x_k \leq x_{k+1}$, because 2δ is a mesh of the net $(x_k)_1^{N+1}$. Therefore,

$$\sup_{x\in\mathbb{R}} |(\Omega\cdot u)^{(p)}(x)| \leq C_p N^p \delta^{-p}\prod_{k=2}^{N+1} x_k. \tag{5}$$

To complete the proof we return to (1):

$$|[x_1,\ldots,x_{N+1}]f| \leq 2^{N-q}|\tilde{f}|_q^{([0,1])}\Big(\min\prod_{m=1}^{2^n-2^v}|x_{a_i(m)} - x_{b_i(m)}|\Big)^{-1},$$

where min is taken over all possible routes from $[1,\ldots,N+1]$ to some $[j,\ldots,j+q]$ with $1 \leq j \leq N+1-q$.

Let us consider $q+1$ points in succession from $(x_k)_1^{N+1}$. We see that in order to minimize the product above, one has to include intervals containing large gaps of the Cantor set in the chain $[x_j,\ldots,x_{j+q}] \subset \cdots \subset [x_1,\ldots,x_{N+1}]$ as late as possible, that is, all $q+1$ points must belong to the same basic interval of length $l_{s+n-v-1}$. And what is more, the position of $q+1$ successive points is not important, since all intervals in the chain will contain a gap of length $l_{s+n-v-1} - 2l_{s+n-v}$. Therefore,

$$\min \prod_{m=1}^{2^n-2^v} |x_{a_i(m)} - x_{b_i(m)}| = x_{q+2} \cdots x_{N+1}.$$

Combining this with (5) we get the desired result. ■

6. Boundedness of the operator. A slight change of the proof of Lemma 2 can be applied for any term in the sum representing the operator L. Let us fix $\delta_{N,s}$ to be half of the shortest distance between the points of interpolation for the polynomial $L_N(f,x,I_{j,s})$.

THEOREM 1. *Let* $n_s = 2, s \le 3$ *and* $n_s = [\log_2 s]$ *for* $s \ge 4, \delta_{N,s} = \frac{1}{2} l_{s+[\log_2 N]}$. *Then* $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$, *given in Section 3, is a continuous linear extension operator.*

Proof. Lemma 1 implies that L is a linear extension operator. In order to get its continuity, let us fix any natural number p and take $q = 2^v - 1 > p+1$. Given $x \in \mathbb{R}$, we will estimate $|(L(f,x))^{(p)}|$. Fix $s \ge 4$. First, we examine the accumulation sums. We specify $\delta_{N,s}$ in a such way that for given x only one term of the sum with respect to j does not vanish. Let us fix the corresponding value of $j = j(s,x)$.

Represent the term

$$g_N(x) := [L_N(f,x,I_{j,s}) - L_{N-1}(f,x,I_{j,s})]u(x,\delta_{N,s},I_{j,s} \cap K)$$

in the form $[x_1,\ldots,x_{N+1}]f \cdot \Omega_N(x)u(x)$. Here $2^m \le N < 2^{m+1}$ for some m with $n_{s-1} \le m+1 \le n_s$, so $\delta := \delta_{N,s} = \frac{1}{2} l_{s+m} = \frac{1}{2} 3^{-s-m}$. Arguing as in the proof of Lemma 2, we get the bound $|(\Omega_N \cdot u)^{(p)}| \le CN^p \delta^{-p} \prod_{k=2}^{N+1} x_k$. Here and in what follows we denote by C any constant that depends only on p and q.

Similarly, $\big|[x_1,\ldots,x_{N+1}]f\big| \le C2^N (x_{q+2}\cdots x_{N+1})^{-1} |\tilde{f}|_q^{([0,1])}$. The vector space $\mathcal{E}(K)$ can be identified with the quotient space $C^\infty[0,1]/Z$, where $Z = \{f \in C^\infty[0,1] : f|_K = 0\}$. The quotient topology τ_Q is given by the norms $(inf|\tilde{f}|_q^{([0,1])})_{q=0}^\infty$, where the infimum is taken for all possible extensions of f to $\tilde{f}$. Clearly, this topology is complete. Using the Lagrange form of the Taylor remainder, we see that $\tau_Q \succeq \tau$. Hence, by the open mapping theorem $\tau_Q \sim \tau$ and for given q there exists $r \in \mathbb{N}, C > 0$ such that $\inf |\tilde{f}|_q^{([0,1])} \le C||f||_r$ for any $f \in \mathcal{E}(K)$. Therefore,

$$|g_N^{(p)}(x)| \le C2^N N^p \delta^{-p} \prod_{k=2}^{q+1} x_k ||f||_r.$$

But $N < 2^{n_s} \le s$ and the number of terms in the sum is $N_s - M_s$, which is less than s. Therefore we get

$$\sum_{N=M_s+1}^{N_s} |g_N^{(p)}(x)| \le C2^s s^{p+1} \delta^{-p} \prod_{k=2}^{q+1} x_k ||f||_r.$$

The condition $2^m \le N$ implies that all endpoints of the basic subintervals of length $l_{s+m-1} = 6\delta$ on $I_{j,s}$ are included in the interpolation set. Therefore we can estimate the product above roughly: $\prod_{k=2}^{q+1} x_k \le D_2(6\delta)\cdot D_3(6\delta)\cdots D_{q+1}(6\delta) = C\delta^q$. Since $2^s s^{p+1} \delta^{q-p} < s^{p+1}(\frac{2}{3})^s$ we see that the part of the general series, corresponding to the accumulation sums, is convergent.

We now turn to the difference

$$[L_{M_{s+1}}(f,x,I_{j,s+1})-L_{N_s}(f,x,I_{[\frac{j+1}{2}],s})]u(x,\delta_{N_s,s},I_{j,s+1}\cap K).$$

Without loss of generality we can assume $j=1$. We will apply Lemma 2 again. To this end, let us write the expression above in the telescoping form

$$-\sum_{N=2^{n_s-1}}^{2^{n_s}-1}[L_N(f,x,I_{1,s})-L_{N-1}(f,x,I_{1,s})]u(x,l_{s+n_s-1}/2,I_{1,s+1}\cap K). \tag{6}$$

Here the interpolation set for the polynomial $L_N(f,x,I_{1,s})$ consists of all endpoints of the basic subintervals of length l_{s+n_s-1} on $I_{1,s+1}$ and some (from 0 for $N=2^{n_s-1}-1$ to all for $N=2^{n_s}-1$) endpoints of the basic subintervals of the same length on $I_{2,s+1}$. Given N we again represent the term of the sum (6) in the form

$$[x_1,\ldots,x_{N+1}]f\cdot\Omega_N(x)u(x,l_{s+n_s-1}/2,I_{1,s+1}\cap K).$$

We note that x is rather close to $I_{1,s+1}\cap K$. For this reason, in order to maximize the value $|\Omega_N^{(i)}(x)|$, we have to, by differentiation, remove from $\prod_{k=1}^N(x-x_k)$ the terms $(x-x_k)$ for $x_k\in I_{1,s+1}$; and save the terms corresponding to the second part of $I_{1,s}$. Therefore we have the same bound of $||\Omega_N\cdot u||_p$ as in (5). Finally, let us consider $[x_1,\ldots,x_{N+1}]f$. As in the proof of Lemma 2, we want to minimize the product of lengths of intervals, corresponding to the chain $[x_j,\ldots,x_{j+q}]\subset\cdots\subset[x_1,\ldots,x_{N+1}]$. Clearly, we have to take $x_j,\ldots,x_{j+q}$ in the interval $I_{1,s+1}$. Thus we can repeat all previous arguments and obtain the bound

$$\big|[x_1,\ldots,x_{N+1}]f\big|\cdot|(\Omega_N\cdot u)^{(p)}|\le C2^N N^p\delta^{q-p}||f||_r.$$

Taking into account the value of δ and the number of terms in the sum (6), we see that the second part of the general series representing L converges as well. Thus the operator L is well-defined and bounded. ■

References

[1] M. Altun and A. Goncharov, *A local version of the Pawłucki-Pleśniak extension operator*, J. Approx. Theory 132 (2005), 34–41.

[2] L. Białas and A. Volberg, *Markov's property of the Cantor ternary set*, Studia Math. 104 (1993), 259–268.

[3] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag (1993).

[4] A. Goncharov, *On the explicit form of an extension operator for C^∞-functions*, East Journal on Approximation 7 (2001), 179–193.

[5] W. Pawłucki and W. Pleśniak, *Extension of C^∞ functions from sets with polynomial cusps*, Studia Math. 88 (1988), 279–287.

[6] W. Pleśniak, *Markov's inequality and the existence of an extension operator for C^∞ functions*, J. Approx. Th. 61 (1990), 106–117.

[7] W. Pleśniak, *Recent progress in multivariate Markov inequality*, in: Approximation Theory: in memory of A. K. Varma, edited by N. K. Govil, Marcel Dekker, New York, 1998, 449–464.

[8] H. Whitney, *Analytic extension of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.