

ON THE EXISTENCE OF PULLBACK ATTRACTOR FOR A TWO-DIMENSIONAL SHEAR FLOW WITH TRESKA'S BOUNDARY CONDITION

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Abstract. We consider a two-dimensional Navier-Stokes shear flow with time dependent boundary driving and subject to Tresca law. We establish the existence of a unique global in time solution and then, using a recent method based on the concept of the Kuratowski measure of noncompactness of a bounded set, we prove the existence of the pullback attractor for the associated cocycle. This research is motivated by a problem from lubrication theory.

1. Introduction. In this article we study the problem of existence of the pullback attractor for a class of two-dimensional turbulent *boundary driven* flows subject to the Tresca law which naturally appears in lubrication theory.

The problem is as follows. The flow of an incompressible fluid in a two-dimensional domain Ω is described by the equation of motion

$$u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \Omega \quad (1.1)$$

and the incompressibility condition

$$\operatorname{div} u = 0 \quad \text{in } \Omega. \quad (1.2)$$

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To define the domain of the flow Ω consider first the channel

$$\Omega_\infty = \{x = (x_1, x_2) : -\infty < x_1 < \infty, 0 < x_2 < h(x_1)\},$$

where h is a positive function, smooth, and L -periodic in x_1 . Then

$$\Omega = \{x = (x_1, x_2) : 0 < x_1 < L, 0 < x_2 < h(x_1)\}$$

and $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_L \cup \bar{\Gamma}_1$, where Γ_0 and Γ_1 are the bottom and the top, and Γ_L is the lateral part of the boundary of Ω .

We are interested in solutions of (1.1)-(1.2) in Ω which are L -periodic with respect to x_1 . We assume that

$$u = 0 \quad \text{at} \quad \Gamma_1. \quad (1.3)$$

Moreover, we assume that there is no flux condition across Γ_0 so that the normal component of the velocity on Γ_0 satisfies

$$u \cdot n = 0 \quad \text{at} \quad \Gamma_0, \quad (1.4)$$

and that the tangential component of the velocity u_η on Γ_0 is unknown and satisfies the Tresca law with a constant and positive friction coefficient k . This means that, c.f., e.g., [7],

$$\left. \begin{aligned} |\sigma_\eta(u, p)| < k &\Rightarrow u_\eta = U_0(t)e_1 \\ |\sigma_\eta(u, p)| = k &\Rightarrow \exists \lambda \geq 0 \text{ such that } u_\eta = U_0(t)e_1 - \lambda \sigma_\eta(u, p) \end{aligned} \right\} \quad \text{at} \quad \Gamma_0 \quad (1.5)$$

where σ_η is the tangential component of the stress tensor on Γ_0 and

$$t \mapsto U_0(t)e_1 = (U_0(t), 0)$$

is the time dependent velocity of the lower surface, producing the driving force of the flow. We suppose that U_0 is a locally Lipschitz continuous function of time t .

If $n = (n_1, n_2)$ is the unit outward normal to Γ_0 , and $\eta = (\eta_1, \eta_2)$ is the unit tangent vector to Γ_0 then, using the summation convention, we have

$$\sigma_\eta(u, p) = \sigma(u, p) \cdot n - ((\sigma(u, p) \cdot n) \cdot n)n, \quad (1.6)$$

where $\sigma_{ij}(u, p) = -p\delta_{ij} + \nu(u_{i,j} + u_{j,i})$ is the stress tensor.

In the end, the initial condition for the velocity at time $\tau \in \mathbb{R}$ is

$$u(x, \tau) = u_0(x) \quad \text{for} \quad x \in \Omega. \quad (1.7)$$

The problem is motivated by a flow in an infinite (rectified) journal bearing $\Omega \times (-\infty, +\infty)$, where $\Gamma_1 \times (-\infty, +\infty)$ represents the outer cylinder, and $\Gamma_0 \times (-\infty, +\infty)$ represents the inner, rotating cylinder. In the lubrication problems the gap h between cylinders is never constant. We can assume that the rectification does not change the equations as the gap between cylinders is very small with respect to their radii.

The system (1.1)-(1.2) with boundary conditions: (1.3) at Γ_1 , for $h = \text{const}$, and $u = \text{const}$ on Γ_0 instead of (1.4)-(1.5), was intensively studied in several contexts, some of them mentioned in the introduction of our paper [4]. The autonomous case with $h \neq \text{const}$ and with $u = \text{const}$ on Γ_0 was considered in [2], [3]. See also [4] where the case $h \neq \text{const}$, $u = U(t)e_1$ on Γ_0 , was considered. On the other hand, the important for applications

dynamical problem we consider in this paper has not been studied earlier, neither in the autonomous nor in the nonautonomous case.

Our plan is as follows. First, we homogenize the boundary condition (1.5) by defining a smooth background flow, a simple version of the Hopf construction. In section 2 we study the variational formulation of the homogenized problem. In section 3 we prove existence and uniqueness of a global in time solution of our problem. In section 4 we study the existence of the pullback attractor.

To homogenize the boundary condition (1.5) let

$$u(x_1, x_2, t) = U(x_2, t)e_1 + v(x_1, x_2, t) \quad (1.8)$$

with

$$U(0, t) = U_0(t), \quad U(h(x_1), t) = 0, \quad x \in (0, L), \quad t \in (-\infty, \infty). \quad (1.9)$$

The new vector field v is L -periodic in x_1 and satisfies the equation of motion

$$v_t - \nu \Delta v + (v \cdot \nabla)v + \nabla p = G(v) \quad (1.10)$$

with

$$G(v) = -Uv_{,x_1} - (v)_2 U_{,x_2} e_1 + \nu U_{,x_2 x_2} e_1 - U_{,t} e_1$$

where by $(v)_2$ we denoted the second component of v . As $\operatorname{div}(Ue_1) = 0$ we get

$$\operatorname{div} v = 0 \quad \text{in } \Omega. \quad (1.11)$$

From (1.8)–(1.9) we obtain

$$v = 0 \quad \text{on } \Gamma_1, \quad (1.12)$$

and

$$v \cdot n = 0 \quad \text{on } \Gamma_0. \quad (1.13)$$

Moreover, we have,

$$\sigma_\eta(v, p) = \sigma_\eta(u, p) + \left(\nu \frac{\partial U(x_2, t)}{\partial x_2} \Big|_{x_2=0}, 0 \right).$$

Since we can define the extension U in such a way that

$$\frac{\partial U(x_2, t)}{\partial x_2} \Big|_{x_2=0} = 0$$

the Tresca condition (1.5) transforms to

$$\left. \begin{array}{l} |\sigma_\eta(v, p)| < k \Rightarrow v_\eta = 0 \\ |\sigma_\eta(v, p)| = k \Rightarrow \exists \lambda \geq 0 \text{ such that } v_\eta = -\lambda \sigma_\eta(v, p) \end{array} \right\} \quad \text{at } \Gamma_0 \quad (1.14)$$

In the end the initial condition becomes

$$v(x, \tau) = v_0(x) = u_0(x) - U(x_2, \tau)e_1. \quad (1.15)$$

2. Variational formulation of the problem. In this section we present the variational formulation of the homogenized problem (1.10)–(1.15). Then, for the convenience of the readers, we describe the relations between the classical and the weak formulations.

We begin with some basic definitions of the paper.

Let

$$\tilde{V} = \{v \in \mathcal{C}^\infty(\Omega)^2 : \operatorname{div} v = 0 \text{ in } \Omega, v \text{ is L-periodic in } x_1, \\ v = 0 \text{ at } \Gamma_1, \quad v \cdot n = 0 \text{ at } \Gamma_0\}$$

and

$$V = \text{closure of } \tilde{V} \text{ in } H^1(\Omega)^2, \quad H = \text{closure of } \tilde{V} \text{ in } L^2(\Omega)^2.$$

We define scalar products in H and V , respectively, by

$$(u, v) = \int_{\Omega} u(x)v(x) dx \quad \text{and} \quad (\nabla u, \nabla v)$$

and their associated norms by

$$|v| = (v, v)^{\frac{1}{2}} \quad \text{and} \quad \|v\| = (\nabla v, \nabla v)^{\frac{1}{2}}.$$

Let, for u, v and w in V

$$a(u, v) = (\nabla u, \nabla v) \quad \text{and} \quad b(u, v, w) = ((u \cdot \nabla)v, w).$$

In the end, let us define the functional j on V by

$$j(u) = \int_{\Gamma_0} k|u(x_1, 0)|dx_1.$$

The variational formulation of the homogenized problem (1.10)–(1.15) is as follows.

PROBLEM 2.1. *Given $\tau \in \mathbb{R}$ and $v_0 \in H$, find $v : (\tau, \infty) \rightarrow H$ such that:*

(i) *for all $T > \tau$,*

$$v \in \mathcal{C}([\tau, T]; H) \cap L^2(\tau, T; V), \quad \text{with} \quad v_t \in L^2(\tau, T; V')$$

(ii) *for all Θ in V , all $T > \tau$, and for almost all t in the interval $[\tau, T]$, the following variational inequality holds*

$$\langle v_t(t), \Theta - v(t) \rangle + \nu a(v(t), \Theta - v(t)) + b(v(t), v(t), \Theta - v(t)) \\ + j(\Theta) - j(v(t)) \geq (\mathcal{L}(v(t)), \Theta - v(t)) \quad (2.1)$$

(iii) *the initial condition*

$$v(x, \tau) = v_0(x). \quad (2.2)$$

holds.

In (2.1) the functional $\mathcal{L}(v(t))$ is defined for almost all $t \geq \tau$ by,

$$(\mathcal{L}(v(t)), \Theta) = -\nu a(\xi, \Theta) - (\xi, v_t(t), \Theta) \\ - b(\xi(t), v(t), \Theta) - b(v(t), \xi(t), \Theta), \quad (2.3)$$

where $\xi = Ue_1$ is a suitable smooth background flow.

We have the following relations between classical and weak formulations.

PROPOSITION 2.1. *Every classical solution of problem (1.10)–(1.15) is also a solution of Problem 2.1. On the other hand, every solution of Problem 2.1 which is smooth enough is also a classical solution of problem (1.10)–(1.15).*

Proof. Let v be a classical solution of problem (1.10)–(1.15). As it is (by assumption) sufficiently regular, we have to check only (2.1). Remark first that (1.10) can be written as

$$v_t - \sigma_{ij,j}(v, p) + (v \cdot \nabla)v = G(v(t)). \quad (2.4)$$

Let $\Theta \in V$. Mutiplying (2.4) by $\Theta - v(t)$ and using Green's formula we obtain

$$\begin{aligned} & \langle v_t, \Theta - v(t) \rangle + \int_{\Omega} \sigma_{ij}(v, p)(\Theta - v(t))_{i,j} dx + b(v(t), v(t), \Theta - v(t)) \\ &= \int_{\partial\Omega} \sigma_{ij}(v, p)n_j(\Theta - v(t))_i + \langle G(v(t)), \Theta - v(t) \rangle \text{ for almost all } t \in [\tau, T]. \end{aligned} \quad (2.5)$$

As $v(t)$ and Θ are in V , we have

$$\int_{\Omega} \sigma_{ij}(v, p)(\Theta - v(t))_{i,j} dx = \nu a(v(t), \Theta - v(t)) \quad (2.6)$$

and using (1.6) we obtain

$$\int_{\partial\Omega} \sigma_{ij}(v, p)n_j(\Theta - v(t))_i = \int_{\Gamma_0} \sigma_{\eta}(v, p) \cdot (\Theta - v_{\eta}(t)) - \int_{\Gamma_0} (\sigma_{ij}n_jn_i)n_i(\Theta - v_{\eta}(t))_i$$

As $n_i(\Theta - v_{\eta}(t))_i = 0$ on Γ_0 , we get

$$\begin{aligned} \int_{\partial\Omega} \sigma_{ij}(v, p)n_j(\Theta - v(t))_i &= \int_{\Gamma_0} \sigma_{\eta}(v, p) \cdot (\Theta - v_{\eta}(t)) = \int_{\Gamma_0} (\sigma_{\eta} \cdot \Theta + k|\Theta|) \\ &\quad - \int_{\Gamma_0} k(|\Theta| - |v_{\eta}(t)|) - \int_{\Gamma_0} (k|v_{\eta}(t)| + \sigma_{\eta} \cdot v_{\eta}(t)). \end{aligned}$$

Remark that $\sigma_{\eta} \cdot \Theta + k|\Theta| \geq 0$, and the Tresca condition (1.14) is equivalent [7] to

$$k|v_{\eta}(t)| + \sigma_{\eta} \cdot v_{\eta}(t) = 0 \quad \text{a.e. on } \Gamma_0.$$

Thus

$$\int_{\partial\Omega} \sigma_{ij}(v, p)n_j(\Theta - v(t))_i \geq - \int_{\Gamma_0} k(|\Theta| - |v_{\eta}(t)|) = -j(\Theta) + j(v(t)). \quad (2.7)$$

As

$$\langle G(v(t)), \Theta - v(t) \rangle = (\mathcal{L}(v(t)), \Theta - v(t))$$

from (2.6) and (2.7) we see that (2.5) becomes (2.1), and (2.2) is the same as (1.15).

Conversely, suppose that v is a solution to Problem 2.1 and let φ be in the space $(H_{div}^1(\Omega))^2 = \{\varphi \in V : \varphi = 0 \text{ on } \Gamma\}$. We take $\Theta = v(t) \pm \varphi$ in (2.1), and using the Green formula, we obtain

$$\langle v_t - \nu\Delta v(t) + v(t) \cdot \nabla v(t) - G(v(t)), \varphi \rangle = 0 \quad \forall \varphi \in (H_{div}^1(\Omega))^2.$$

Thus, there exists $p \in H^{-1}(\Omega)$ such that

$$v_t - \nu\Delta v(t) + v(t) \cdot \nabla v(t) - G(v(t)) = \nabla p \quad \text{a.e. in } \Omega.$$

so that (1.10) holds. We obtain (1.14) as in [1], and we have immediately (1.11)–(1.13) and (1.15).

3. Existence and uniqueness of a global in time solution. In this section we establish the existence and uniqueness of a global in time solution for Problem 2.1. First, we present three lemmas.

LEMMA 3.1 ([4]). *For any $t \in \mathbb{R}$ consider the smooth extension*

$$\xi(x_2, t) = U(x_2, t)e_1 = U_0(t)\rho(x_2/(h_0\varepsilon(t)))e_1.$$

of $U_0(t)e_1$ from Γ_0 to Ω , where

$$\varepsilon = \varepsilon(t) = \begin{cases} 2 & \text{if } |U_0(t)| \leq \nu/(8h_0), \\ \nu/(4h_0|U_0(t)|) & \text{if } |U_0(t)| \geq \nu/(8h_0), \end{cases}$$

$h_0 = \min_{0 \leq x_1 \leq L} h(x_1)$, and $\rho : [0, \infty) \rightarrow [0, 1]$ is a smooth function such that

$$\rho(0) = 1, \quad \rho'(0) = 0, \quad \text{supp } \rho \subset [0, 1/2], \quad \max |\rho'(s)| \leq \sqrt{8}.$$

Then we have

$$|b(v, \xi(t), v)| \leq \frac{\nu}{4} \|v\|^2 \quad \text{for all } v \in V.$$

and

$$\frac{\partial U(x_2, t)}{\partial x_2} \Big|_{x_2=0} = 0.$$

LEMMA 3.2 ([4]). *Let U be as in Lemma 3.1. Then for almost all t ,*

$$\begin{aligned} |\xi(t)|^2 &= \int_{\Omega} |U(x_2, t)|^2 dx_1 dx_2 \leq \frac{1}{2} L h_0 U_0^2(t) \varepsilon(t), \\ |\nabla \xi(t)|^2 &= \int_{\Omega} |U_{,x_2}(x_2, t)|^2 dx_1 dx_2 \leq \frac{4L U_0^2(t)}{h_0} \frac{1}{\varepsilon(t)}, \end{aligned}$$

and

$$|\xi_{,t}|^2 = \int_{\Omega} |U_{,t}(x_2, t)|^2 dx_1 dx_2 \leq \left(|U_0'(t)| + \sqrt{2} |U_0(t)| \frac{|\varepsilon'(t)|}{\varepsilon(t)} \right)^2 \frac{L h_0 \varepsilon(t)}{2}.$$

LEMMA 3.3. *For all v in V we have the anisotropic Ladyzhenskaya inequality*

$$\|v\|_{L^4(\Omega)} \leq C(\Omega) \|v\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}}. \quad (3.1)$$

Proof. Let $v \in V$ and $\xi \in C^1(\cdot - L, L)$ such that $\xi = 1$ on $[0, L]$ and $\xi = 0$ at $x_1 = -L$. Define $\varphi = \xi v$, and extend φ by 0 to $\Omega_1 =]-L, L[\times]0, h[$, where $h = \max_{0 \leq x_1 \leq L} h(x_1)$. We obtain

$$\varphi^2(x_1, x_2) = 2 \int_{-L}^{x_1} \varphi(t_1, x_2) \frac{\partial \varphi}{\partial t_1}(t_1, x_2) dt_1 \leq 2 \int_{-L}^L |\varphi(x_1, x_2)| \left| \frac{\partial \varphi}{\partial x_1}(x_1, x_2) \right| dx_1$$

and

$$\varphi^2(x_1, x_2) = -2 \int_{x_2}^h \varphi(x_1, t_2) \frac{\partial \varphi}{\partial t_2}(x_1, t_2) dt_2 \leq 2 \int_0^h |\varphi(x_1, x_2)| \left| \frac{\partial \varphi}{\partial x_2}(x_1, x_2) \right| dx_2,$$

whence

$$\begin{aligned} \|\varphi\|_{L^4(\Omega_1)}^4 &= \int_{\Omega_1} \varphi^2(x_1, x_2) \varphi^2(x_1, x_2) dx_1 dx_2 \\ &\leq \left(\int_0^h \sup_{-L \leq x_1 \leq L} \varphi^2(x_1, x_2) dx_2 \right) \left(\int_{-L}^L \sup_{0 \leq x_2 \leq h} \varphi^2(x_1, x_2) dx_1 \right) \\ &\leq 4 \left(\int_0^h \int_{-L}^L |\varphi| \left| \frac{\partial \varphi}{\partial x_1} \right| dx_1 dx_2 \right) \times \left(\int_0^h \int_{-L}^L |\varphi| \left| \frac{\partial \varphi}{\partial x_2} \right| dx_2 dx_1 \right) \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\varphi\|_{L^4(\Omega_1)}^4 &\leq 4 \|\varphi\|_{L^2(\Omega_1)}^2 \left| \frac{\partial \varphi}{\partial x_1} \right|_{L^2(\Omega_1)} \left| \frac{\partial \varphi}{\partial x_2} \right|_{L^2(\Omega_1)} \\ &\leq 2 \|\varphi\|_{L^2(\Omega_1)}^2 \left(\left| \frac{\partial \varphi}{\partial x_1} \right|_{L^2(\Omega_1)}^2 + \left| \frac{\partial \varphi}{\partial x_2} \right|_{L^2(\Omega_1)}^2 \right) \\ &\leq 2 \|\varphi\|_{L^2(\Omega_1)}^2 \|\nabla \varphi\|_{L^2(\Omega_1)}^2 \end{aligned}$$

We use $|\xi| \leq 1$ and the Poincaré inequality to get

$$\|v\|_{L^4(\Omega)} \leq \|\varphi\|_{L^4(\Omega_1)}, \quad |\varphi|_{L^2(\Omega_1)} \leq 2\|v\|_{L^2(\Omega)} \quad \text{and} \quad \|\nabla \varphi\|_{L^2(\Omega_1)} \leq C\|v\|_V$$

for some constant C , whence (3.1) holds.

THEOREM 3.1. *Let $v_0 \in H$ and the function $s \mapsto |U_0(s)|^3 + |U'_0(s)|^2$ be locally integrable on the real line. Then there exists a solution of Problem 2.1.*

Proof. We provide only the main steps of the proof as it is quite standard and, on the other hand, long. The estimates we obtain will be used further in the paper.

Observe that the functional j is convex but nondifferentiable. To overcome this difficulty we use the following approach (see, i.e., [8]). For $\delta > 0$ let $j_\delta : V \rightarrow \mathbb{R}$ be a functional defined by

$$\varphi \mapsto j_\delta(\varphi) = \frac{1}{1+\delta} \int_{\Gamma_0} k |\varphi|^{1+\delta} dx$$

which is convex, lower continuous and finite on V , and has the following properties

- $\exists \chi \in V'$ and $\mu \in \mathbb{R}$ such that $j_\delta(\varphi) \geq \langle \chi, \varphi \rangle + \mu \quad \forall \varphi \in V$,
- $\lim_{\delta \rightarrow 0^+} j_\delta(\varphi) = j(\varphi) \quad \forall \varphi \in V$,
- $v_\delta \rightharpoonup v$ (weakly) in $V \Rightarrow \lim_{\delta \rightarrow 0^+} j_\delta(v_\delta) \geq j(v)$.

The functional j_δ is Gâteaux differentiable in V , with

$$(j'_\delta(v), \Theta) = \int_{\Gamma_0} k |v|^{\delta-1} v \Theta dx \quad \forall \Theta \in V.$$

Let us consider the following equation

$$\begin{aligned} \left(\frac{dv_\delta}{dt}, \Theta \right) + \nu a(v_\delta(t), \Theta) + b(v_\delta(t), v_\delta(t), \Theta) + (j'_\delta(v_\delta), \Theta) \\ = -\nu a(\xi(t), \Theta) - (\xi, t, \Theta) - b(\xi(t), v_\delta(t), \Theta) - b(v_\delta(t), \xi(t), \Theta) \end{aligned} \quad (3.2)$$

with initial condition

$$v_\delta(\tau) = v_0. \quad (3.3)$$

For $\delta > 0$, we establish an a priori estimate of v_δ . Since $(j'_\delta(v_\delta), v_\delta) \geq 0$, $v_\delta(t) \in V$, and $b(v_\delta(t), v_\delta(t), v_\delta(t)) = b(\xi, v_\delta(t), v_\delta(t)) = 0$ then taking $\Theta = v_\delta(t)$ in (3.2) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |v_\delta(t)|^2 + \nu \|v_\delta(t)\|^2 &\leq (-\nu a(\xi(t), v_\delta(t)) \\ &\quad - (\xi, v_\delta(t)) - b(v_\delta(t), \xi(t), v_\delta(t))) \end{aligned} \quad (3.4)$$

In view of estimate (3.1) and the Poincaré inequality we obtain from (3.4)

$$\frac{1}{2} \frac{d}{dt} |v_\delta(t)|^2 + \frac{\nu}{2} \|v_\delta(t)\|^2 \leq 2\nu \|\xi\|^2 + \frac{1}{\nu \lambda_1} |\xi, v_\delta(t)|^2.$$

where λ_1 is the first eigenvalue of the Stokes operator in our problem. We estimate the right hand side in terms of the data using Lemma 3.2 with ε as in Lemma 3.1 to get

$$\frac{1}{2} \frac{d}{dt} |v_\delta(t)|^2 + \frac{\nu}{2} \|v_\delta(t)\|^2 \leq F(t), \quad (3.5)$$

with

$$\begin{aligned} F(t) &= 2\nu \left(\frac{1}{2} L h_0 U_0^2(t) \varepsilon(t) + \frac{4LU_0^2(t)}{h_0 \varepsilon(t)} \right) \\ &\quad + \frac{1}{\nu \lambda_1} \left(|U_0'(t)| + \sqrt{2} |U_0(t)| \frac{|\varepsilon'(t)|}{\varepsilon(t)} \right)^2 \frac{L h_0 \varepsilon(t)}{2} \\ &= c_1(\nu, \Omega) (|U_0(t)|^3 + |U_0'(t)|^2). \end{aligned} \quad (3.6)$$

From (3.5) we conclude that

$$|v_\delta(t)|^2 + \nu \int_\tau^t \|v_\delta(s)\|^2 ds \leq |v(\tau)|^2 + 2 \int_\tau^t F(s) ds. \quad (3.7)$$

As the function F is, by assumption, locally integrable on the real line, we deduce that

$$v_\delta \text{ is bounded in } L^2(\tau, T; V) \cap L^\infty(\tau, T; H), \text{ independently of } \delta. \quad (3.8)$$

The existence of v_δ satisfying (3.2)–(3.3) is based on inequality (3.5), the Galerkin approximations, and the compactness method. Moreover from (3.7) we can deduce that

$$\frac{dv_\delta}{dt} \text{ is bounded in } L^2(\tau, T; V'). \quad (3.9)$$

From (3.8) and from (3.9) we conclude that there exists v such that (possibly for a subsequence)

$$v_\delta \rightharpoonup v \text{ in } L^2(\tau, T; V), \quad \text{and} \quad \frac{dv_\delta}{dt} \rightharpoonup \frac{dv}{dt} \text{ in } L^2(\tau, T; V') \text{ (weakly)} \quad (3.10)$$

In view of (3.10), $v \in \mathcal{C}([\tau, T]; H)$, and

$$v_\delta \rightarrow v \text{ in } L^2(\tau, T; H) \text{ strongly.}$$

We can now pass to the limit $\delta \rightarrow 0$ in (3.2)–(3.3) exactly as in [7] to obtain the variational inequality (2.1) for almost every point $t \in]\tau, T[$. Thus the existence of a solution of Problem 2.1 is established.

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, the solution v of Problem 2.1 is unique and the map $v(\tau) \rightarrow v(t)$, for $t > \tau$, is continuous in H .*

Proof. Let v and w be two solutions of Problem 2.1. Then for $u(t) = w(t) - v(t)$ we have

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 \leq b(u(t), w(t), u(t)) + b(u(t), \xi(t), u(t))$$

By Lemma 3.1 and the anisotropic Ladyzhenskaya inequality (3.1) we obtain

$$\frac{d}{dt} |u(t)|^2 + \frac{\nu}{4} \|u(t)\|^2 \leq \frac{4}{\nu} C(\Omega) \|w(t)\|^2 |u(t)|^2,$$

and in view of the Poincaré inequality we conclude

$$\frac{d}{dt} |u(t)|^2 + \frac{\sigma}{4} |u(t)|^2 \leq \frac{4}{\nu} C(\Omega) \|w(t)\|^2 |u(t)|^2.$$

Using again the Gronwall lemma, we obtain

$$|u(t)|^2 \leq [u(\tau)]^2 \exp\left\{-\int_{\tau}^t \left(\frac{\sigma}{4} - \frac{4}{\nu} C(\Omega) \|w(s)\|^2\right) ds\right\}. \quad (3.11)$$

From (3.10) it follows that the solution w of Problem 2.1 belongs to $L^2(\tau, t; V)$. We have thus proved that the map $v(\tau) \rightarrow v(t)$, $t > \tau$, in H is continuous. In particular, as $u(\tau) = w(\tau) - v(\tau) = 0$, the solution v of Problem 2.1 is unique. This ends the proof of Theorem 3.2.

4. Existence of the pullback attractor. In this section we prove existence of the pullback attractor for the cocycle associated to the considered problem by using a method based on the concept of the Kuratowski measure of noncompactness of a bounded set. This method is very useful when one deals with variational inequalities as it overcomes obstacles coming from the usual methods. One needs neither compactness of the dynamics which results from the second energy inequality nor asymptotic compactness, cf., i.e., [4, 5, 6, 11], which results from the energy equation. In the case of variational inequalities it is much more difficult to obtain the second energy inequality due to presence of boundary functionals, on the other hand, we do not have any energy equation.

First we recall the definition of a cocycle and a pullback attractor, then, following [10], recall the abstract theory, and in the end apply the latter to our problem.

DEFINITION 4.1. Let \mathcal{P} be a metric space and $\theta : \mathbb{R} \times \mathcal{P} \rightarrow \mathcal{P}$ be a mapping such that $\theta_t : \mathcal{P} \rightarrow \mathcal{P}$ form a group, namely,

$$\theta_0 = Id, \quad \text{and} \quad \theta_{t+r} = \theta_t \circ \theta_r \quad \forall t, r \in \mathbb{R}.$$

Let H be a complete metric space. A mapping $\Phi : \mathbb{R}_+ \times \mathcal{P} \times H \rightarrow H$ is said to be a *cocycle* on H with respect to a group θ if

$$\Phi(0, p, v_0) = v_0 \quad \forall v_0 \in H, \quad p \in \mathcal{P},$$

$$\Phi(t+r, p, v_0) = \Phi(t, \theta_r(p), \Phi(r, p, v_0)), \quad \forall t, r \in \mathbb{R}_+, \quad p \in \mathcal{P}.$$

DEFINITION 4.2. A family $\mathcal{A} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$ of nonempty compact sets of H is called a *pullback attractor* of the θ -cocycle Φ if it is Φ -invariant, that is,

$$\Phi(t, p, \mathcal{A}_p) = \mathcal{A}_{\theta_t(p)} \quad \forall (t, p) \in \mathbb{R}_+ \times \mathcal{P},$$

and pullback attracting, that is,

$$\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, \theta_{-t}(p), B), \mathcal{A}_p) = 0 \quad \forall B \in B(H), \quad \forall p \in \mathcal{P}.$$

where $B(H)$ is the set of all bounded subsets of H , and $dist$ is the Hausdorff semidistance between X and Y defined by

$$dist(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y).$$

DEFINITION 4.3. The *pullback ω -limit set* $\omega_p(B)$ of B is defined, for any $t \geq 0$, by the following

$$\omega_p(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \Phi(t, \theta_{-t}(p), B)}.$$

The other definitions and theorems we need are as follows. Let $B \in B(H)$. The Kuratowski measure of noncompactness [9] is defined by

$$\alpha(B) = \inf\{\delta : B \text{ admits a finite cover by sets of diameter } \leq \delta\}.$$

DEFINITION 4.4 ([10]). A cocycle Φ on H is said to be *pullback ω -limit compact* if for any $B \in B(H)$, for any $p \in \mathcal{P}$,

$$\lim_{\tau \rightarrow \infty} \alpha\left(\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}(p), B)\right) = 0.$$

DEFINITION 4.5 ([10]). Let H be a Banach space. A cocycle Φ is said to be *norm-to-weak continuous* on H if for all $(t, p, x) \in \mathbb{R}_+ \times \mathcal{P} \times H$ and for every sequence $(x_n) \in H$,

$$x_n \rightarrow x \text{ strongly in } H \Rightarrow \Phi(t, p, x_n) \rightharpoonup \Phi(t, p, x) \text{ weakly in } H.$$

THEOREM 4.1 ([10]). *Let H be a Banach space. Let Φ be a cocycle on H with respect to a group θ . If Φ is norm-to-weak continuous and possesses a uniformly absorbing set B_0 , then Φ possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$, satisfying*

$$\mathcal{A}_p = \omega_p(B_0), \quad \forall p \in \mathcal{P},$$

if and only if it is pullback ω -limit compact.

THEOREM 4.2 ([10]). *Let H be a Banach space. If the cocycle Φ satisfies the pullback condition **(PC)**: for any $p \in \mathcal{P}$, $B \in B(H)$ and $\varepsilon > 0$, there exists $t_0(p, B, \varepsilon)$ and a finite dimensional subspace E of H such that for a bounded projector $P : H \rightarrow E$,*

$$P\left(\bigcup_{t \geq t_0} \Phi(t, \theta_{-t}(p), B)\right) \text{ is bounded, } \quad \forall t \in \mathbb{R}_+,$$

$$\left|(I - P)\left(\bigcup_{t \geq t_0} \Phi(t, \theta_{-t}(p), x)\right)\right| \leq \varepsilon, \quad \forall x \in B,$$

then Φ is pullback ω -limit compact.

Now, we define a cocycle $\Phi : \mathbb{R}_+ \times \mathbb{R} \times H \rightarrow H$ as the evolutionary process \mathbf{U} associated with Problem 2.1, namely,

$$\Phi(s, t, v_0) = \mathbf{U}(s + t, t)v_0$$

with $\theta_s(t) = t + s$. In particular,

$$\Phi(s, \theta_{-s}(t), v_0) = \mathbf{U}(\theta_{-s}(t) + s, \theta_{-s}(t))v_0 = \mathbf{U}(t, t - s)v_0.$$

LEMMA 4.1. *Let*

$$\sup_{h \in \mathbb{R}} \int_h^{h+1} F(s) ds < R(F) < \infty, \quad (4.1)$$

$\sigma > 0$ and $t \geq \tau$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_\tau^t e^{-\sigma(t-s)} F(s) ds \leq \frac{\varepsilon}{2} + \frac{e^{-\sigma\delta}}{1 - e^{-\sigma}} R(F). \quad (4.2)$$

Proof. Let δ be such that $\int_{t-\delta}^t F(s) ds \leq \frac{\varepsilon}{2}$, $\tau < \delta < t$. Then, by (4.1),

$$\int_\tau^t e^{-\sigma(t-s)} F(s) ds \leq \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} \int_{t-(\delta+k+1)}^{t-(\delta+k)} e^{-\sigma(t-s)} F(s) ds \leq \frac{\varepsilon}{2} + \frac{e^{-\sigma\delta}}{1 - e^{-\sigma}} R(F).$$

Let $F(s) = |U_0(s)|^3 + |U'_0(s)|^2$ be as in (3.6). Then we have

LEMMA 4.2. *Let the initial condition v_0 in Problem 2.1 belong to a ball $B(0, \rho)$ in H . Suppose that (4.1) holds. Then the solution v of Problem 2.1 satisfies*

$$\sup_{h \geq \tau} \int_h^{h+1} \|v(s)\|^2 ds \leq \frac{2}{\nu} \left\{ \rho^2 + \left(1 + \frac{e^{-\sigma\delta}}{1 - e^{-\sigma}} \right) R(F) \right\}. \quad (4.3)$$

Proof. Taking $\Theta = 0$ in (2.1) we obtain, similarly as in the proof of Theorem 3.1,

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 + \frac{\nu}{2} \|v(t)\|^2 \leq F(t) \quad (4.4)$$

and, in consequence,

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 + \frac{\sigma}{2} |v(t)|^2 \leq F(t) \quad (4.5)$$

with $\sigma = \nu\lambda_1$. By Gronwall's inequality and Lemma 4.1 with $\varepsilon = \rho^2$ we conclude from the last inequality that for $t \geq \tau$,

$$|v(t)|^2 \leq 2\rho^2 + \frac{2e^{-\sigma\delta}}{1 - e^{-\sigma}} R(F). \quad (4.6)$$

By integration of (4.4) we obtain the first energy estimate: for $\tau \leq \eta \leq t$,

$$|v(t)|^2 + \nu \int_\eta^t \|v(s)\|^2 ds \leq 2 \int_\eta^t F(s) ds + |v(\eta)|^2. \quad (4.7)$$

Using this estimate and (4.6) we obtain (4.3).

THEOREM 4.3. *Let $v_0 \in H$ and U_0 be such that (4.1) holds, with $F(s) = |U_0(s)|^3 + |U'_0(s)|^2$. Then there exists a pullback attractor \mathcal{A} in the sense of Theorem 4.1 for the cocycle Φ given by the evolutionary process $\mathbf{U}(t, \tau)$ above.*

Proof. From (4.5), (4.6), the Gronwall inequality, and Lemma 4.1 we obtain

$$|\mathbf{U}(s+t, t)v_0|^2 \leq e^{-\sigma s} |v_0|^2 + \varepsilon + \frac{2e^{-\sigma\delta}}{1 - e^{-\sigma}} R(F).$$

For v_0 in $B(0, \rho)$ and s large enough, $\mathbf{U}(s+t, t)v_0 \in B(0, \rho_0)$, where ρ_0 depends only on ε , ρ , and $R(F)$, which means that there exists a uniformly absorbing ball in H .

From (3.11) it follows that the evolutionary process \mathbf{U} is strongly continuous in H , whence, in particular, it is norm-to-weak continuous on H .

Thus, according to Theorem 4.1 and Theorem 4.2, to finish the proof we have to prove that the pullback condition **(PC)** holds.

Let A be the Stokes operator in H . As A^{-1} is continuous and compact in H , there exists a sequence $\{\lambda_j\}_{j=1}^\infty$ such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \quad \text{with} \quad \lim_{j \rightarrow +\infty} \lambda_j = \infty,$$

and a family of elements $\{\varphi_j\}_{j=1}^\infty$ of $D(A)$, which are orthonormal in H such that $A\varphi_j = \lambda_j\varphi_j$.

We define the m -dimensional subspace V_m , of V , and the orthogonal projection operator $P_m : V \rightarrow V_m$ by

$$V_m = \text{span}\{\varphi_1, \dots, \varphi_m\} \quad \text{and} \quad P_m v = \sum_{j=1}^m (v, \varphi_j) \varphi_j$$

For $v \in D(A) \subset V$, we can write it

$$v = P_m v + (I - P_m)v = P_m v + v_2.$$

Set $\Theta = v_1(t)$ in (2.1) to get

$$\frac{1}{2} \frac{d}{dt} |v_2(t)|^2 + \nu \|v_2(t)\|^2 \leq j(v_1(t)) - j(v(t)) - b(v(t), v(t), v_2(t)) + (\mathcal{L}(v(t)), v_2(t)).$$

From the continuity of the trace operator we have

$$j(v_1(t)) - j(v(t)) \leq j(v_2(t)) \leq C + \frac{\nu}{4} \|v_2(t)\|^2$$

Further, using the Ladyzhenskaya inequality (3.1) we easily arrive at

$$\frac{d}{dt} |v_2(t)|^2 + \nu \|v_2(t)\|^2 \leq C_2(1 + F(t) + \|v(t)\|^2).$$

and

$$\frac{d}{dt} |v_2(t)|^2 + \nu \lambda_{m+1} |v_2(t)|^2 \leq C_2(1 + F(t) + \|v(t)\|^2).$$

Now, let $\varepsilon > 0$ be given. Using Lemmas 4.1 and 4.2, and taking m large enough, we obtain

$$|(I - P_m)\mathbf{U}(s + t, t)v_0|^2 \leq \varepsilon$$

uniformly in t , for $v_0 \in B(0, \rho)$ and all $s \geq s_0(\rho, \varepsilon)$ large enough. This ends the proof of the theorem.

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