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Envelopes and refinements in categories, with applications to functional analysis

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Contents

Abstract

An envelope in a category is a construction that generalizes the operations of "exterior completion", like completion of a locally convex space, or the Stone–Čech compactification of a topological space, or the universal enveloping algebra of a Lie algebra. Dually, a refinement generalizes the operations of "interior enrichment", like bornologification (or saturation) of a locally convex space, or simply connected covering of a Lie group. In this paper we define envelopes and refinements in abstract categories and discuss conditions under which these constructions exist and are functors. The aim of the exposition is to lay the foundations for duality theories of non-commutative groups based on the idea of envelope. The advantage of this approach is that in the arising theories the analogs of group algebras are Hopf algebras. At the same time the classical Fourier and Gelfand transforms are interpreted as envelopes with respect to certain classes of algebras.

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1. Introduction

In 1972 J. L. Taylor [45] introduced an operation which associates to an arbitrary topological algebra A a new topological algebra $\mathsf{Env}\, A$ later called by A. Ya. Helemskii [16] "the Arens–Michael envelope of A". In his next paper [46] Taylor gave an amusing formula ([1]) which suggests an unexpectedly simple way to formalize the heuristically evident connection between algebraic geometry and complex analysis:

$$\mathsf{Env}\, \mathcal{P}(\mathbb{C}^n) = \mathcal{O}(\mathbb{C}^n) \tag{1.1}$$

(here $\mathcal{P}(\mathbb{C}^n)$ and $\mathcal{O}(\mathbb{C}^n)$ are the algebras of polynomials and, respectively, of holomorphic functions on the complex space $\mathbb{C}^n$). Despite this promising application, up to the end of the century Taylor's construction did not manifest itself in mathematical literature, and only recently did the interest in the operation $A \mapsto \mathsf{Env}\, A$ appear again in A. Yu. Pirkovskii's papers [31], [32] on "holomorphic non-commutative geometry". In particular, in [32] formula (1.1) was generalized to the case of an arbitrary affine algebraic variety M:

$$\mathsf{Env}\, \mathcal{P}(M) = \mathcal{O}(M). \tag{1.2}$$

This identity was very soon applied by the author [3] to the construction of a generalization of Pontryagin's duality from the category of commutative compactly generated Stein groups to the category of arbitrary (not necessarily commutative) compactly generated Stein groups with the algebraic connected component of identity. The idea of the duality suggested in [3] is illustrated by the diagram

$$\begin{tikzcd} \mathcal{O}^\star(G) \arrow[r, mapsto, "\mathsf{Env}"] & \mathcal{O}^\star_{\exp}(G) \arrow[d, mapsto, "\star"] \\ \mathcal{O}(G) \arrow[u, mapsto, "\star"] & \mathcal{O}_{\exp}(G) \arrow[l, mapsto, "\mathsf{Env}"'] \end{tikzcd} \tag{1.3}$$

where G is a group of the above described class, $\mathcal{O}(G)$ the algebra of holomorphic functions on G, $\mathcal{O}_{\exp}(G)$ its subalgebra consisting of functions of exponential type, $A \mapsto \mathsf{Env}\, A$ the operation of taking the Arens–Michael envelope, and $X \mapsto X^\star$ the operation of passage to the dual stereotype space in the sense of [2], i.e. to the space of continuous linear functionals with the topology of uniform convergence on totally bounded sets (in this case this is equivalent to uniform convergence on compact sets).

One can call the duality presented in diagram (1.3) the *complex geometry duality*, having in mind the class of objects under consideration. The theory obtained for this

([1]) Taylor mentions this fact in passing on pp. 207 and 251 of [46].

class of groups contrasts with other existing theories in the following two points. First, its enveloping category (to which the group algebras belong) consists of Hopf algebras. And second, diagram (1.3) suggests a natural way for constructing analogous dualities for "other geometries", in particular, for differential geometry and for topology: one should just replace the Arens–Michael envelope in diagrams analogous to (1.3) with some other envelopes (and this automatically leads to replacing the constructions in the corners of the diagram with some proper analogs from analysis).

This alleged connection between different dualities in geometry and different envelopes of topological algebras was recently supported by other examples:

1) In the work by Yu. N. Kuznetsova [24] the Arens–Michael envelope was replaced by the envelope generated by the C^*-quotient maps (2), and this immediately led to a variant of *topological duality* where the Stein groups are replaced by the Moore groups, and the algebras $\mathcal{O}(G)$ and $\mathcal{O}_{\exp}(G)$, respectively, by the algebra $\mathcal{C}(G)$ of continuous functions on G and the algebra $\mathcal{K}(G)$ of coefficients of norm-continuous representations of G.
2) In the author's work [5] a notion of smooth envelope $\mathsf{Env}_\infty A$ of a topological algebra A was introduced. This construction replaces the Arens–Michael envelope in the passage from complex analysis to differential geometry, and an analogue of the Pirkovskii theorem (1.2) was proved in the differential-geometric context: if a subalgebra A of the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a smooth variety M has the same spectrum and the same tangent space at each point, then
$$\mathsf{Env}_\infty A = \mathcal{C}^\infty(M).$$
This result gives hope that a similar *duality theory in differential geometry* will be constructed in the near future with a proper class of real Lie groups.

It is interesting (and predictable) that in these theories the classical Fourier and Gelfand transforms are interpreted as envelopes with respect to some class of algebras (see e.g. Theorems 5.44, 5.53 and 5.52 below).

It is clear to the author that the results obtained are just first observations in the field, but they already show the validity of the common philosophical idea which justifies and guides the investigations in this area: in each standard mathematical discipline where certain classes of symmetries play a role (classes of groups, including those understood in a generalized way, like quantum groups), a certain duality theory works (and apparently, is not unique). This idea was suggested in the author's work [3], and among such disciplines the following four were mentioned:

– general topology,
– differential geometry,
– complex analysis,
– algebraic geometry.

This paper is planned as a part of the program described in [3]. We discuss here the question (which has remained open until recently) how one should define envelopes

(2) On p. 179 we define this construction as the Kuznetsova envelope.

in general category theory, and under what conditions they exist and are functors. We suggest a natural definition (from our point of view) and establish some wide necessary and sufficient conditions for the existence of envelopes and their dual constructions, which we call refinements. As applications, we show that in the categories $\mathtt{Ste}$ of stereotype spaces and $\mathtt{Ste}^{\circledast}$ of stereotype algebras the envelopes and the refinements exist in a very wide class of situations.

Notations and conventions. Everywhere in category theory we use the terminology of the textbooks [11], [47] and of the handbook [6], and as a set-theoretic foundation for the notion of category we choose the Morse–Kelley theory [19].

Everywhere $\mathsf{Mono}(\mathtt{K})$, $\mathsf{Epi}(\mathtt{K})$, $\mathsf{SMono}(\mathtt{K})$ and $\mathsf{SEpi}(\mathtt{K})$ mean the classes of monomorphisms, epimorphisms, strong monomorphisms and strong epimorphisms (the last two are defined on p. 18) respectively in the category $\mathtt{K}$. We say that a category $\mathtt{K}$ is

- *injectively* (*projectively*) *complete* if each functor $K : \mathtt{M} \to \mathtt{K}$ from a *small* category $\mathtt{M}$ (i.e. one where the class of morphisms is a set) has an injective (projective) limit,
- *complete* if it is injectively and projectively complete,
- *finitely injectively* (*projectively*) *complete* if each functor $K : \mathtt{M} \to \mathtt{K}$ from a *finite* category $\mathtt{M}$ (i.e. one where the class of morphisms is a finite set) has an injective (projective) limit,
- *finitely complete* if it is finitely injectively complete and finitely projectively complete,
- *linearly complete* if any functor from a linearly ordered set to $\mathtt{K}$ has injective and projective limits.

For any morphism $\varphi : X \to Y$ in an arbitrary category the symbols $\mathsf{Dom}\,\varphi$ and $\mathsf{Ran}\,\varphi$ mean respectively the domain and the range of φ, i.e. $\mathsf{Dom}\,\varphi = X$ and $\mathsf{Ran}\,\varphi = Y$. If $\mathtt{L}$ and $\mathtt{M}$ are two classes of objects in $\mathtt{K}$, then $\mathsf{Mor}(\mathtt{L}, \mathtt{M})$ means the class of morphisms with domains in $\mathtt{L}$ and ranges in $\mathtt{M}$.

Let $\varPhi$ be a class of morphisms and $\mathtt{L}$ a class of objects in a category $\mathtt{K}$. We say that:

- $\varPhi$ *goes from* $\mathtt{L}$ if for any object $X \in \mathtt{L}$ there is a morphism $\varphi \in \varPhi$ with $\mathsf{Dom}\,\varphi = X$; in the special case when $\mathtt{L}$ consists of only one object X, we say that $\varPhi$ *goes from* X.
- $\varPhi$ *goes to* $\mathtt{L}$ if for any object $X \in \mathtt{L}$ there is a morphism $\varphi \in \varPhi$ with $\mathsf{Ran}\,\varphi = X$; in the special case when $\mathtt{L}$ consists of only one object X, we say that $\varPhi$ *goes to* X.

In the theory of topological vector spaces we follow the textbook [38] by H. H. Schaefer, and in the theory of stereotype spaces and algebras the author's papers [2] and [3]. In particular, following [38] we assume that *all locally convex spaces* (LCS for short) *are Hausdorff*. By a *topological algebra* we mean a locally convex topological algebra in the spirit of the textbook [27], i.e. a locally convex space A over the field $\mathbb{C}$, endowed with associative multiplication which is *separately continuous* and has a unit.

We also use the following notation. First, for any locally convex space X the symbol $\mathcal{U}(X)$ denotes the system of all neighborhoods of zero in X. Second, for each neighborhood U of zero in X the set

$$\mathsf{Ker}\,U = \bigcap_{\varepsilon>0} \varepsilon \cdot U$$

will be called the *kernel* of this neighborhood of zero. If U is an absolutely convex neighborhood of zero, then $\mathsf{Ker}\, U$ is a closed subspace in X. And third, if a topological space Y is imbedded into a topological space X (injectively, but the topology of Y is not necessarily inherited from X), and A is a subset in Y, then to distinguish the closure of A in Y from its closure in X, we denote the first one by $\overline{A}^Y$, and the second by $\overline{A}^X$.

Moreover, we say that a subset M in a locally convex space X is *total* (*in* X) if its linear span, $\mathsf{span}\, M$, is dense in X.

2. Nodal decomposition and factorizations

2.1. Skeletally small graphs

2.1.1. Graphs. Recall that an *oriented graph* is a set V with a given subset $\varGamma \subset V \times V$. The elements of V are called *vertices*, and the elements of $\varGamma$ *edges* of this graph. An oriented graph is said to be *reflexive* if (x,x) belongs to $\varGamma$ for each $x \in V$, and *transitive* if for any two edges (x,y) and (y,z) from $\varGamma$ the pair (x,z) also belongs to $\varGamma$. Obviously, every reflexive transitive oriented graph is a (small) category, where the objects and the morphisms are respectively the vertices and the edges (the composition of edges (x,y) and (y,z) is (x,z), and the local identities 1_x are (x,x)). A characteristic property of such categories (apart from their being small) is that the sets of morphisms, $\mathsf{Mor}(A,B)$, always contain at most one element. This justifies the following definition.

- A *graph* is a category $\mathtt{K}$ (not necessarily small) where
$$\forall A,B \in \mathsf{Ob}(\mathtt{K}) \quad \mathsf{card}\,\mathsf{Mor}(A,B) \le 1. \tag{2.1}$$
Clearly, this is equivalent to the structure of (reflexive and transitive) oriented graph on the class $\mathsf{Ob}(\mathtt{K})$ (with the observation that $\mathsf{Ob}(\mathtt{K})$ is not necessarily a set, but just a class).

PROPERTIES OF GRAPHS.

1° *In any graph a morphism $\varphi : A \to B$ is an isomorphism iff there exists a morphism in the reverse direction, $\psi : A \leftarrow B$:*
$$\forall \varphi \in \mathsf{Mor}(A,B) \quad (\varphi \in \mathsf{Iso} \Leftrightarrow \exists \psi \in \mathsf{Mor}(B,A)). \tag{2.2}$$

2° *In any graph a composition of morphisms is an identity iff the same remains true after switching the factors:*
$$\psi \circ \varphi = 1 \;\Leftrightarrow\; \varphi \circ \psi = 1. \tag{2.3}$$

3° *In any graph a composition of morphisms $\psi \circ \varphi$ is an isomorphism iff both ψ and φ are isomorphisms:*
$$\psi \circ \varphi \in \mathsf{Iso} \;\Leftrightarrow\; \psi \in \mathsf{Iso} \,\&\, \varphi \in \mathsf{Iso}. \tag{2.4}$$

Proof. 1° If $\varphi : A \to B$ and $\psi : A \leftarrow B$, then $\psi \circ \varphi$ acts from A into A, so it must coincide with 1_A. Similarly, $\varphi \circ \psi$ acts from B into B, so it must coincide with 1_B.

2° From $\psi \circ \varphi = 1$ it follows that $\mathsf{Ran}\,\varphi = \mathsf{Dom}\,\psi$ and $\mathsf{Ran}\,\psi = \mathsf{Dom}\,\varphi$, and we apply the same reasoning as in part 1°.

3° If $\omega = \psi \circ \varphi \in \mathsf{Iso}$, then $\psi \circ \varphi \circ \omega^{-1} = 1$, so by (2.3), $\varphi \circ \omega^{-1} \circ \psi = 1$, hence $\psi \in \mathsf{Iso}$, and finally $\varphi = \psi^{-1} \circ \omega \in \mathsf{Iso}$. ■

2.1.2. Partially ordered classes. Every partially ordered set I can be considered as a category, where the objects are the elements of this set, and the morphisms are the pairs (i, j) for which $i \leq j$. Such categories $\mathtt{K}$ are, of course, special cases of graphs, since every set $\mathsf{Mor}(A, B)$ contains at most one element (i.e. (2.1) holds). But in addition (and this property distinguishes the partially ordered sets among all graphs), for $A \neq B$ the existence of a morphism $\varphi : A \to B$ automatically excludes the existence of any morphisms $\psi : A \leftarrow B$. This justifies the following definition.

- A *partially ordered class* is a graph where the existence of opposite morphisms $\varphi : A \to B$ and $\psi : A \leftarrow B$ is possible only if $A = B$ (and then $\varphi = \psi = 1_A$). In other words,
$$\forall A \neq B \in \mathsf{Ob}(\mathtt{K}) \quad (\mathsf{Mor}(A, B) \neq \emptyset \;\Rightarrow\; \mathsf{Mor}(B, A) = \emptyset). \tag{2.5}$$
Obviously, this is equivalent to the structure of partial order on $\mathsf{Ob}(\mathtt{K})$ (as in the previous definition, with the difference that $\mathsf{Ob}(\mathtt{K})$ is not necessarily a set, but just a class).

Example 2.1. *Category* $\mathtt{Ord}$. The class $\mathtt{Ord}$ of all ordinal numbers with its natural order (see e.g. [19]) is an example of a partially ordered class which is not a set.

Proposition 2.2. *In a partially ordered class only local identities are isomorphisms.*

Proof. The identity $A = B$ follows from the fact that $\mathsf{Mor}(A, B) \neq \emptyset$ and $\mathsf{Mor}(B, A) \neq \emptyset$, and the identity $\varphi = 1_A$ from the fact that φ and 1_A are collinear arrows in a graph. ∎

2.1.3. Skeleton. A class S of objects of a category $\mathtt{K}$ is called a *skeleton* of $\mathtt{K}$ if every object in $\mathtt{K}$ is isomorphic to exactly one object of S. In other words, S satisfies the following two requirements:

1) elements of S are isomorphic only if they coincide;
2) there exists a map $G : \mathsf{Ob}(\mathtt{K}) \to \mathtt{S}$ such that
$$\forall X \in \mathsf{Ob}(\mathtt{K}) \quad X \cong G(X).$$

The skeleton S is usually endowed with the structure of a full subcategory in $\mathtt{K}$.

Properties of skeletons.

1° *Each category* $\mathtt{K}$ *has a skeleton.*
2° *Any two skeletons in* $\mathtt{K}$ *are isomorphic* (*as categories*).
3° *Each category* $\mathtt{K}$ *is equivalent to its skeleton* S.
4° *Two categories* $\mathtt{K}$ *and* $\mathtt{L}$ *are equivalent if and only if their skeletons are isomorphic* (*as categories*).

Proof. Only the first property is not obvious. It follows from the fact that the class $\mathtt{Set}$ of all sets can be well-ordered (see [25, V, 4.1]): $\mathsf{Ob}(\mathtt{K})$ is a subclass in $\mathtt{Set}$, so it can also be well-ordered, and we can assign to each $X \in \mathsf{Ob}(\mathtt{K})$ the minimal among all objects isomorphic to X in $\mathtt{K}$ with respect to this order. ∎

- A category $\mathtt{K}$ is said to be
  - *skeletal* if any two isomorphic objects coincide in $\mathtt{K}$ (equivalently, $\mathtt{K}$ is a skeleton for itself),
  - *skeletally small* if it has a skeleton which is a set.

EXAMPLE 2.3. Each partially ordered class is a skeletal category (since as already noted, only local identities are isomorphisms), but not vice versa. For instance, the category of all finite sets of the form $\{0,\dots,n\}$, $n \in \mathbb{Z}_+$, with arbitrary maps as morphisms, is skeletal, but it is not a partially ordered class, since $\{0,\dots,n\}$ has many bijections onto itself.

2.1.4. Transfinite chain condition

- Let us say that a (covariant or contravariant) functor $F : \mathtt{Ord} \to \mathtt{K}$ *stabilizes* if it satisfies the following two equivalent conditions:
  (i) there exists $k \in \mathtt{Ord}$ such that
$$\forall l \geq k \quad F(k,l) \in \mathsf{Iso};$$
  (ii) there exists $k \in \mathtt{Ord}$ such that
$$\forall l,m \quad \big(k \leq l \leq m \ \Rightarrow\ F(l,m) \in \mathsf{Iso}\big).$$

Proof of equivalence. The implication (i)⇐(ii) is obvious, so we only need to prove (i)⇒(ii). Let F be a covariant functor (the case of a contravariant functor is similar). If (i) holds, then for $k \leq l \leq m$ we have
$$\underbrace{F(k,m)}_{\substack{\Cap \\ \mathsf{Iso}}} = F(l,m) \circ \underbrace{F(k,l)}_{\substack{\Cap \\ \mathsf{Iso}}} \ \Rightarrow\ \underbrace{F(k,m)}_{\substack{\Cap \\ \mathsf{Iso}}} \circ \underbrace{F(k,l)^{-1}}_{\substack{\Cap \\ \mathsf{Iso}}} = F(l,m) \ \Rightarrow\ F(l,m) \in \mathsf{Iso}. \ \blacksquare$$

REMARK 2.4. If a category $\mathtt{K}$ is a partially ordered class, then by Proposition 2.2, for a functor $F : \mathtt{Ord} \to \mathtt{K}$ the isomorphisms in (i) and (ii) become local identities:

(i)′ there exists $k \in \mathtt{Ord}$ such that
$$\forall l \geq k \quad F(k,l) = 1_{F(k)};$$
(ii)′ there exists $k \in \mathtt{Ord}$ such that
$$\forall l,m \quad \big(k \leq l \leq m \ \Rightarrow\ F(l,m) = 1_{F(l)}\big).$$

THEOREM 2.5 (Transfinite chain condition). *Every functor $F : \mathtt{Ord} \to \mathtt{K}$ into an arbitrary skeletally small graph $\mathtt{K}$ stabilizes.*

We will need the following

LEMMA 2.6. *In the class $\mathtt{Ord}$ there is no cofinal subclass which is a set.*

Proof. If K is a cofinal subclass in $\mathtt{Ord}$, then
$$\mathtt{Ord} = \bigcup_{k \in K} \{i \in \mathtt{Ord} : i \leq k\}.$$
Hence if K is a set, then $\mathtt{Ord}$ must also be a set, which is not true. $\blacksquare$

Corollary 2.7. *For any directed set I each monotone map $F : I \to \mathtt{Ord}$ has a least upper bound in $\mathtt{Ord}$.*

Proof. It is sufficient to note that $F(I)$ is bounded in $\mathtt{Ord}$: this follows from the fact that $F(I)$ is a set, and thus cannot be a cofinal subclass in $\mathtt{Ord}$. ∎

Proof of Theorem 2.5. Let $F : \mathtt{Ord} \to \mathtt{K}$ be a (covariant or contravariant) functor into a skeletally small graph $\mathtt{K}$. Suppose that it is not stabilized, i.e. for any $i \in \mathtt{Ord}$ there is $j \in \mathtt{Ord}$ such that $F(i,j) \notin \mathsf{Iso}$. Let us construct a transfinite sequence $\{k_i;\ i \in \mathtt{Ord}\} \subseteq \mathtt{Ord}$ according to the following rules:

0) We set $k_0 = 0$.
1) If for some $j \in \mathtt{Ord}$ all the $\{k_i;\ i < j\}$ are already chosen, then we consider two cases:

- if j is an isolated ordinal, i.e. $j = i + 1$ for some $i < j$, then we take k_j with
$$k_i < k_{i+1} = k_j, \qquad F(k_i, k_{i+1}) = F(k_i, k_j) \notin \mathsf{Iso}$$
(k_j exists due to our assumption that F is not stabilized),
- if j is a limit ordinal, i.e. $j \neq i + 1$ for any $i < j$, then we take
$$k_j = \lim_{i \to j} k_i = \sup_{i<j} k_i$$
(it exists due to Corollary 2.7).

We obtain a transfinite sequence $i \in \mathtt{Ord} \mapsto k_i \in \mathtt{Ord}$ with the following properties:

(i) It is cofinal in $\mathtt{Ord}$, since $i \leq k_i$ for any $i \in \mathtt{Ord}$.
(ii) For $i < j$ we have $F(k_i, k_j) \notin \mathsf{Iso}$, since
$$i < j \ \Rightarrow\ i + 1 \leq j \ \Rightarrow\ F(k_i, k_j) = F(k_{i+1}, k_j) \circ \underbrace{F(k_i, k_{i+1})}_{\notin \mathsf{Iso}} \underset{(2.4)}{\Longrightarrow} F(k_i, k_j) \notin \mathsf{Iso}$$
(we assume here that F is a covariant functor, but for a contravariant one the reasoning is the same).

Now let $S \subseteq \mathtt{K}$ be a skeleton of $\mathtt{K}$. For any $i \in \mathtt{Ord}$ we consider $G(i) \in S$ such that
$$G(i) \cong F(k_i).$$
Suppose that $G(i) = G(j)$ for some $i \leq j$. Then the morphism $F(k_i, k_j) : G(i) \to G(j)$ must coincide with the local identity $1_{G(i)} = 1_{G(j)}$, since the category $\mathtt{S}$ is a graph, and therefore it cannot have two different collinear morphisms. Thus, $F(k_i, k_j)$ must be an isomorphism, and, by (ii), this is possible only if $i = j$. So $G : \mathtt{Ord} \to S$ is injective. But $\mathtt{Ord}$ is a proper class, while S is a set, and this is impossible. ∎

2.2. Some classes of monomorphisms and epimorphisms. The notions of monomorphism and epimorphism, widely used in category theory, have several variations, and two of them, immediate and strong mono- and epimorphisms, will be important for us. As the reader will see, we will stress the analogy between mono/epimorphisms on the one hand and strong mono/epimorphisms on the other. In the cases where due to this analogy the proofs become identical (up to the insertion of "strong" in appropriate places, as in the results about the categories SMono_X and SEpi^X), as well as in the elementary propositions, we omit the proofs.

2.2.1. Monomorphisms and epimorphisms. Recall that a morphism $\varphi : X \to Y$ is called

– a *monomorphism* if $\varphi \circ \alpha = \varphi \circ \beta$ implies $\alpha = \beta$;
– an *epimorphism* if $\alpha \circ \varphi = \beta \circ \varphi$ implies $\alpha = \beta$;
– a *bimorphism* if it is a monomorphism and an epimorphism.

EXAMPLE 2.8. *In any graph* $\mathtt{K}$ *every morphism is a bimorphism.* Indeed, if $\varphi \circ \alpha = \varphi \circ \beta$, then since α and β are collinear, $\alpha = \beta$. So φ is a monomorphism. Similarly, it is an epimorphism.

PROPOSITION 2.9. *A composition of monomorphisms (respectively, epimorphisms) is a monomorphism (respectively, an epimorphism).*

PROPERTIES OF MONO- AND EPIMORPHISMS.

1° *If $\varphi \circ \mu$ is a monomorphism, then so is μ.*
2° *If $\mu \circ \varphi$ is an isomorphism and μ a monomorphism, then μ and φ are isomorphisms.*
3° *If $\varepsilon \circ \varphi$ is an epimorphism, then so is ε.*
4° *If $\varphi \circ \varepsilon$ is an isomorphism and ε an epimorphism, then φ and ε are isomorphisms.*

By a *covariant system* (respectively, *contravariant system*) in a category $\mathtt{K}$ over a partially ordered set $(I, \leq)$ we mean an arbitrary covariant (respectively, contravariant) functor from I into $\mathtt{K}$.

PROPOSITION 2.10. *If a covariant system $\{X^j; \iota_i^j\}$ over a directed set $(I, \leq)$ has projective limit $\{X; \pi^j\}$ and all the morphisms ι_i^j are monomorphisms, then all the morphisms π^j are monomorphisms as well.*

Proof. Assume that I is decreasingly directed. Take $k \in I$, and let $Y \overset{\alpha}{\to} X$ and $Y \overset{\beta}{\to} X$ be morphisms such that

$$\pi^k \circ \alpha = \pi^k \circ \beta.$$

Then for any $j \leq k$ we have

$$\underbrace{\iota_j^k \circ \pi^j}_{\pi^k} \circ \alpha = \underbrace{\iota_j^k \circ \pi^j}_{\pi^k} \circ \beta.$$

Here ι_j^k is a monomorphism, so we can cancel it:

$$\pi^j \circ \alpha = \pi^j \circ \beta, \qquad j \leq k.$$

Set $\sigma^j = \pi^j \circ \alpha = \pi^j \circ \beta$. Then the morphisms $Y \overset{\alpha}{\to} X$ and $Y \overset{\beta}{\to} X$ generate the same cone of the covariant system $\{X^j; \iota_i^j;\ i \leq j \leq k\}$:

$$
\begin{tikzcd}
 & Y \arrow[d, dashed, "\alpha"] \arrow[ddl, bend right, "\sigma^i"'] \arrow[ddr, bend left, "\sigma^j"] & \\
 & X \arrow[dl, "\pi^i"] \arrow[dr, "\pi^j"'] & \\
X^i \arrow[rr, bend right, "\iota_i^j"] & & X^j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[d, dashed, "\beta"] \arrow[ddl, bend right, "\sigma^i"'] \arrow[ddr, bend left, "\sigma^j"] & \\
 & X \arrow[dl, "\pi^i"] \arrow[dr, "\pi^j"'] & \\
X^i \arrow[rr, bend right, "\iota_i^j"] & & X^j
\end{tikzcd}
$$

(the projective limit of a covariant system over a cofinal interval $\{j \in I;\ j \le k\}$ is the same as over I, so we substitute X into this place). This implies that α and β coincide by the uniqueness of the corresponding arrow in the definition of projective limit. ∎

The dual proposition is the following:

Proposition 2.11. *If a covariant system $\{X^j; \iota_i^j\}$ over a directed set $(I, \le)$ has injective limit $\{X; \rho_i\}$ and all the morphisms ι_i^j are epimorphisms, then all the morphisms ρ_i are epimorphisms as well.*

Remark 2.12. If the set I of indices is not directed, then the projective (injective) limit of a covariant system of monomorphisms (epimorphisms) over I is not necessarily a cone of monomorphisms (epimorphisms). For example if the order in I is discrete, i.e. $i \le j \iff i = j$, then the projective limit of any covariant system $\{X^i;\ \iota_i^j\}$ over I is the direct product $\prod_{i\in I} X^i$, where the projections

$$\prod_{i\in I} X^i \xrightarrow{\pi^k} X^k$$

are not monomorphisms in general (although the initial morphisms $\iota_i^i = 1_{X^i}$ are monomorphisms). Similarly, the injective limit of $\{X^i;\ \iota_i^j\}$ is the coproduct $\coprod_{i\in I} X_i$, and the corresponding injections

$$X_k \xrightarrow{\rho_k} \coprod_{i\in I} X_i$$

are not epimorphisms in general (although $\iota_i^i = 1_{X_i}$ are epimorphisms).

2.2.2. Immediate monomorphisms and immediate epimorphisms

- A *factorization* of a morphism $X \xrightarrow{\varphi} Y$ is its representation as a composition of an epimorphism and a monomorphism, i.e. any commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"'] & & Y \\
 & M \arrow[ur, "\mu"'] &
\end{tikzcd}
\tag{2.6}
$$

  where ε is an epimorphism and μ a monomorphism.
- A monomorphism $\mu : X \to Y$ is said to be *immediate* if in any of its factorizations $\mu = \mu' \circ \varepsilon$ the epimorphism ε is automatically an isomorphism. Note that for a monomorphism μ in a factorization $\mu = \mu' \circ \varepsilon$ the epimorphism ε is automatically a bimorphism. As a corollary, the condition of μ being an immediate monomorphism is equivalent to the requirement that, in any decomposition $\mu = \mu' \circ \varepsilon$ where ε is a bimorphism and μ' a monomorphism, ε must be an isomorphism. It is natural to call a

monomorphism μ' in the factorization $\mu = \mu' \circ \varepsilon$ a *mediator* of the monomorphism μ; then the qualifier "immediate" for μ will mean that there are no non-trivial mediators for μ (i.e. mediators which are not isomorphic to μ in Mono_Y—see definition (2.9) below; here $\Gamma = \mathsf{Mono}$).

- An epimorphism $\varepsilon : X \to Y$ is said to be *immediate* if ε is an immediate monomorphism in the dual category. In other words, in any factorization $\varepsilon = \mu \circ \varepsilon'$ the monomorphism μ must be automatically an isomorphism. Note that for an epimorphism ε in any of its factorizations $\varepsilon = \mu \circ \varepsilon'$ the monomorphism μ is automatically a bimorphism. As a corollary, the condition of ε being an immediate epimorphism is equivalent to the requirement that, in any decomposition $\varepsilon = \mu \circ \varepsilon'$ where μ is a bimorphism and ε' an epimorphism, μ must be an isomorphism. It is natural to call an epimorphism ε' in the factorization $\varepsilon = \mu \circ \varepsilon'$ a *mediator* of the epimorphism ε; then the qualifier "immediate" for ε will mean that there are no non-trivial mediators for ε (i.e. mediators which are not isomorphic to ε in Epi^X—see definition (2.16) below; here $\Omega = \mathsf{Epi}$).

Remark 2.13. If in the definition of immediate monomorphism we omit the requirement that the morphism μ' in the representation $\mu = \mu' \circ \varepsilon$ is a monomorphism (i.e. if we only require that each epimorphism ε in such a representation must be an isomorphism), then we obtain exactly the definition of *extremal monomorphism*. Similarly, if in the definition of immediate epimorphism we omit the requirement that the morphism ε' in $\varepsilon = \mu \circ \varepsilon'$ is an epimorphism (i.e. if we only require that each monomorphism μ in such a representation must be an isomorphism), then we obtain the definition of *extremal epimorphism* [7, Definition 4.3.2]. Clearly, each extremal monomorphism (respectively, extremal epimorphism) is an immediate monomorphism (respectively, immediate epimorphism). But the converse is not true, as the following example shows ([1]). Consider a monoid $\langle a, b, c \mid ac = bc \rangle$ (generated by three elements a, b, c with the equality $ac = bc$) as a category with one object. In this category:

1) a, b, c are monomorphisms (since they can be canceled in equalities like $a \cdot P = a \cdot Q$);
2) a, b are epimorphisms (since they can be canceled in equalities like $P \cdot a = Q \cdot a$);
3) c is not an epimorphism (since it cannot be canceled in $a \cdot c = b \cdot c$);
4) $ac = bc$ is
   - a monomorphism (since it can be canceled in equalities like $ac \cdot P = ac \cdot Q$),
   - an epimorphism (since it can be canceled in equalities like $P \cdot ac = Q \cdot ac$),
   - an immediate epimorphism (since there is only one possibility to write it in the form (mono) $\circ$ (epi), namely, $ac = 1 \cdot (ac)$, and 1 is an isomorphism), but
   - not an extremal epimorphism (since it can be written in the form (mono) $\circ$ $(\dots)$, namely $ac = a \cdot c$, where the first morphism, i.e. a, is not an isomorphism).

In addition, $acac$ is not an immediate epimorphism, since it can be represented as

$$acac = \underbrace{(ac)}_{\substack{\Cap \\ \mathsf{Mono}}} \cdot \underbrace{(ac)}_{\substack{\Cap \\ \mathsf{Epi}}}$$

([1]) This example was suggested to the author by B. V. Novikov.

where the first morphism is not an isomorphism. This shows that *a composition of immediate monomorphisms* (*respectively, of immediate epimorphisms*) *is not necessarily an immediate monomorphism* (*respectively, an immediate epimorphism*).

PROPERTIES OF IMMEDIATE MONO- AND EPIMORPHISMS.

1° *If $\varphi \circ \mu$ is an immediate monomorphism, then so is μ.*
2° *If μ is an immediate monomorphism, and at the same time an epimorphism, then μ is an isomorphism.*
3° *If $\varepsilon \circ \varphi$ is an immediate epimorphism, then so is ε.*
4° *If ε is an immediate epimorphism, and at the same time a monomorphism, then ε is an isomorphism.*

2.2.3. Strong monomorphisms and strong epimorphisms. The following two definitions are due to M. Sh. Tsalenko and E. G. Shul'geĭfer [47, Chapter 1, §7] and F. Borceux [7, 4.3].

- A monomorphism $C \xrightarrow{\mu} D$ is said to be *strong* if for any epimorphism $A \xrightarrow{\varepsilon} B$ and for any morphisms $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} D$ such that $\beta \circ \varepsilon = \mu \circ \alpha$ there exists a (unique) morphism $B \xrightarrow{\delta} C$ such that the following diagram is commutative ([2]):

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] \\ C \arrow[r, "\mu"'] & D \end{tikzcd} \tag{2.7}$$

- Dually, an epimorphism $A \xrightarrow{\varepsilon} B$ is said to be *strong* if for any monomorphism $C \xrightarrow{\mu} D$ and for any morphisms $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} D$ such that $\beta \circ \varepsilon = \mu \circ \alpha$ there exists a (unique) morphism $B \xrightarrow{\delta} C$ such that diagram (2.7) is commutative.

REMARK 2.14. The uniqueness of δ follows from the monomorphy of μ (or from the epimorphy of ε): if δ' is another morphism with the same property, then

$$\mu \circ \delta = \beta = \mu \circ \delta' \ \Rightarrow \ \delta = \delta'.$$

Moreover, the commutativity of the upper triangle in (2.7) implies the commutativity of the lower one, and vice versa. For example,

$$\alpha = \delta \circ \varepsilon \ \Rightarrow \ \beta \circ \underset{\substack{\Cap \\ \mathsf{Epi}}}{\varepsilon} = \mu \circ \alpha = \mu \circ \delta \circ \underset{\substack{\Cap \\ \mathsf{Epi}}}{\varepsilon} \ \Rightarrow \ \beta = \mu \circ \delta. \tag{2.8}$$

The following facts are proved in [7, Proposition 4.3.6]:

PROPOSITION 2.15. *A composition of strong monomorphisms* (*respectively, of strong epimorphisms*) *is a strong monomorphism* (*respectively, a strong epimorphism*).

PROPERTIES OF STRONG MONO- AND EPIMORPHISMS.

1° *If $\varphi \circ \mu$ is a strong monomorphism, then so is μ.*
2° *Every strong monomorphism μ is an immediate monomorphism.*

([2]) In the following, we will omit in most cases the phrase "the following diagram is commutative" before diagrams.

3° *If* $\varepsilon \circ \varphi$ *is a strong epimorphism, then so is* ε.
4° *Every strong epimorphism* ε *is an immediate epimorphism.*

PROPOSITION 2.16. *If in a covariant system* $\{X^j; \iota_i^j\}$ *over a decreasingly directed set* $(I, \leq)$ *the morphisms* ι_i^j *are strong monomorphisms, then in its projective limit* $\{X; \pi^j\}$ *the morphisms* π^j *are strong monomorphisms as well.*

Proof. Take $k \in I$. By Proposition 2.10, π^k is a monomorphism, so we need only show that it is strong. Consider a diagram

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
X \arrow[r, "\pi^k"'] & X^k
\end{tikzcd}$$

where ε is an epimorphism. For any $j \leq k$ we can construct a diagram

$$\begin{tikzcd}
A \arrow[rr, "\varepsilon"] \arrow[dr, "\pi^j \circ \alpha"] \arrow[dd, "\alpha"'] & & B \arrow[dd, "\beta"] \\
& X^j \arrow[dr, "\iota_j^k"'] & \\
X \arrow[ur, "\pi^j"] \arrow[rr, "\pi^k"'] & & X^k
\end{tikzcd}$$

and consider the fragment

$$\begin{tikzcd}
A \arrow[rr, "\varepsilon"] \arrow[dr, "\pi^j \circ \alpha"'] & & B \arrow[dd, "\beta"] \\
& X^j \arrow[dr, "\iota_j^k"'] & \\
& & X^k
\end{tikzcd}$$

Since ε is an epimorphism and ι_j^k is a strong monomorphism, there exists a (unique) morphism δ^j such that

$$\begin{tikzcd}
A \arrow[rr, "\varepsilon"] \arrow[dr, "\pi^j \circ \alpha"'] & & B \arrow[dd, "\beta"] \arrow[dl, dashed, "\delta^j"] \\
& X^j \arrow[dr, "\iota_j^k"'] & \\
& & X^k
\end{tikzcd}$$

In particular,

$$\iota_j^k \circ \delta^j = \beta, \qquad j \leq k.$$

As a corollary, if we take a new index $i \leq j$, then for the morphisms δ^j and δ^i we get

$$\iota_j^k \circ \delta^j = \beta = \iota_k^i \circ \delta^i = \iota_j^k \circ \iota_i^j \circ \delta^i.$$

Here ι_j^k is a monomorphism, so we can cancel it:

$$\delta^j = \iota_i^j \circ \delta^i.$$

Thus for any $i \le j \le k$ the following diagram is commutative:

$$\begin{tikzcd} & & B \arrow[dll, dashed, "\delta^i"'] \arrow[dl, dashed, "\delta^j"] \\ X^i \arrow[r, "\iota_i^j"'] & X^j & \end{tikzcd}$$

(for $j = k$ we have $\delta^k = \beta$).

This means that the system of morphisms $\{\delta^j : B \to X^j;\ j \le k\}$ is a projective cone of a covariant system $\{\iota_i^j : X^i \to X^j;\ i \le j \le k\}$. Hence, there exists a unique morphism $\delta : B \to X$ such that all the following diagrams are commutative:

$$\begin{tikzcd} & B \arrow[dl, dashed, "\delta"'] \arrow[d, "\delta^j"] \\ X \arrow[r, "\pi^j"] & X^j \end{tikzcd}$$

(the limit along a cofinal interval $\{j \in I :\ j \le k\}$ coincides with the limit along I).

In particular, for $j = k$ we get a commutative diagram

$$\begin{tikzcd} & B \arrow[dl, dashed, "\delta"'] \arrow[d, "\beta"] \\ X \arrow[r, "\pi^k"] & X^k \end{tikzcd}$$

This implies the following chain:

$$\beta = \pi^k \circ \delta \ \Rightarrow\ \underbrace{\pi^k}_{\substack{\Cap \\ \mathsf{Mono}}} \circ \alpha = \beta \circ \varepsilon = \underbrace{\pi^k}_{\substack{\Cap \\ \mathsf{Mono}}} \circ \delta \circ \varepsilon \ \Rightarrow\ \alpha = \delta \circ \varepsilon.$$

Thus, the following square is commutative:

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\delta"'] \arrow[d, "\beta"] \\ X \arrow[r, "\pi^k"] & X^k \end{tikzcd}$$

■

The dual proposition is the following:

PROPOSITION 2.17. *If in a covariant system $\{X^j;\ \iota_i^j\}$ over an increasingly directed set $(I, \le)$ the morphisms ι_i^j are strong epimorphisms, then in its injective limit $\{X;\ \rho_i\}$ the morphisms ρ_i are strong epimorphisms as well.*

2.3. Categories of monomorphisms and epimorphisms

2.3.1. Categories of monomorphisms $\varGamma_X$ and systems of subobjects. Let $\varGamma$ be a class of monomorphisms in a category $\mathtt{K}$, and suppose all local identities belong to it (the key examples are the classes $\varGamma = \mathsf{Mono}$ and $\varGamma = \mathsf{SMono}$). For each object X in $\mathtt{K}$ let

$$\varGamma_X = \{\sigma \in \varGamma : \ \mathsf{Ran}\, \sigma = X\}. \tag{2.9}$$

It is a category where a morphism $\rho \xrightarrow{\varkappa} \sigma$ from an object $\rho \in \varGamma_X$ into an object $\sigma \in \varGamma_X$, i.e. from a monomorphism $\rho : A \to X$ into a monomorphism $\sigma : B \to X$, is an arbitrary morphism $\varkappa : A \to B$ in $\mathtt{K}$ such that

$$\begin{tikzcd} A \arrow[dd, "\varkappa"'] \arrow[dr, "\rho"] & \\ & X \\ B \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{2.10}$$

Actually, this diagram in the initial category $\mathtt{K}$ can be considered as a morphism $\rho \xrightarrow{\varkappa} \sigma$ in the category $\varGamma_X$. A composition of such morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \tau$, i.e. of diagrams

$$\begin{tikzcd} A \arrow[dd, "\varkappa"'] \arrow[dr, "\rho"] & \\ & X \\ B \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[dd, "\lambda"'] \arrow[dr, "\sigma"] & \\ & X \\ C \arrow[ur, "\tau"'] & \end{tikzcd}$$

is a morphism $\rho \xrightarrow{\lambda \circ \varkappa} \tau$, i.e. a diagram

$$\begin{tikzcd} A \arrow[dd, "\lambda\circ\varkappa"'] \arrow[dr, "\rho"] & \\ & X \\ C \arrow[ur, "\tau"'] & \end{tikzcd}$$

One can view it as a result of splicing the initial diagrams along the common edge σ, adding the arrow $\varkappa \circ \lambda$, and then throwing away the vertex B together with all the incident edges:

$$\begin{tikzcd} A \arrow[dd, "\lambda\circ\varkappa"'] \arrow[dr, dashed, "\varkappa"'] \arrow[drr, bend left, "\rho"] & & \\ & B \arrow[r, dashed, "\sigma"'] \arrow[dl, dashed, "\lambda"'] & X \\ C \arrow[urr, bend right, "\tau"'] & & \end{tikzcd}$$

Of course, local identities in $\varGamma_X$ are diagrams of the form

$$\begin{tikzcd} A \arrow[dd, "1_X"'] \arrow[dr, "\rho"] & \\ & X \\ A \arrow[ur, "\rho"'] & \end{tikzcd}$$

Remark 2.18. The composition of morphisms in $\varGamma_X$ can be defined in two ways. In our definition this operation is connected with the composition in $\mathtt{K}$ through the identity

$$\lambda \underset{\varGamma_X}{\circ} \varkappa = \lambda \underset{\mathtt{K}}{\circ} \varkappa.$$

Theorem 2.19. *For any object X the category $\varGamma_X$ is a graph.*

Proof. We should verify that for any two objects $\rho : A \to X$ and $\sigma : B \to X$ there exists at most one morphism $\rho \xrightarrow{\varkappa} \sigma$. Indeed, a morphism $\varkappa$ in diagram (2.10) is unique, since the monomorphy of σ gives the implication $\sigma \circ \varkappa = \rho = \sigma \circ \varkappa' \Rightarrow \varkappa = \varkappa'$. ■

Remark 2.20. By Example 2.8 this means that *in the category Γ_X all morphisms are bimorphisms.* The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in Γ_X and the properties of the same morphism $\varkappa : A \to B$ in the initial category $\mathtt{K}$ is expressed in the following observations:

1) *Every morphism $\rho \xrightarrow{\varkappa} \sigma$ in Γ_X is a monomorphism in* $\mathtt{K}$.
2) *A morphism $\rho \xrightarrow{\varkappa} \sigma$ in Γ_X is an isomorphism in Γ_X iff $\varkappa$ is an isomorphism in* $\mathtt{K}$.

Proof. 1) A morphism $\varkappa$ in (2.10) must be a monomorphism due to property 1° on p. 15, since $\sigma \circ \varkappa$ is a monomorphism.

2) If a morphism $\varkappa : A \to B$ in (2.10) is an isomorphism in $\mathtt{K}$, then we can set $\lambda = \varkappa^{-1} : A \leftarrow B$, and the diagrams

$$\begin{tikzcd}
A \arrow[dd, "1_A"'] \arrow[dr, dashed, "\varkappa"'] \arrow[drr, bend left, "\rho"] & & & B \arrow[dd, "1_B"'] \arrow[dr, dashed, "\lambda"'] \arrow[drr, bend left, "\sigma"] & & \\
& B \arrow[r, dashed, "\sigma"'] \arrow[dl, dashed, "\lambda"] & X & & A \arrow[r, dashed, "\rho"'] \arrow[dl, dashed, "\varkappa"] & X \\
A \arrow[urr, bend right, "\rho"'] & & & B \arrow[urr, bend right, "\sigma"'] & &
\end{tikzcd} \tag{2.11}$$

will be commutative, since ρ and σ are monomorphisms. This means that the morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \rho$ in Γ_X are inverse to each other. Conversely, if $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \rho$ are inverse to each other in Γ_X, then diagrams (2.11) are commutative. Hence, $\varkappa$ and λ are inverse to each other in $\mathtt{K}$, and thus $\varkappa$ is an isomorphism in $\mathtt{K}$. ■

It is convenient to introduce a special notation, $\to$, for the pre-order in Γ_X:

$$\rho \to \sigma \iff \exists \varkappa \in \mathsf{Mor}(\mathtt{K}) \;\; \rho = \sigma \circ \varkappa. \tag{2.12}$$

Here the morphism $\varkappa$, if it exists, is unique, and it is a monomorphism (because σ is). As a corollary, there is an operation which to any pair of morphisms $\rho, \sigma \in \Gamma_X$ with the property $\rho \to \sigma$ assigns the morphism $\varkappa = \varkappa_\rho^\sigma$ in (2.12):

$$\rho = \sigma \circ \varkappa_\rho^\sigma. \tag{2.13}$$

If $\rho \to \sigma \to \tau$, then the chain

$$\tau \circ \varkappa_\rho^\tau = \rho = \sigma \circ \varkappa_\rho^\sigma = \tau \circ \varkappa_\sigma^\tau \circ \varkappa_\rho^\sigma$$

implies, due to monomorphy of τ, the equality

$$\varkappa_\rho^\tau = \varkappa_\sigma^\tau \circ \varkappa_\rho^\sigma. \tag{2.14}$$

- A *system of subobjects of class Γ* in an object X of a category $\mathtt{K}$ is an arbitrary skeleton S of the category Γ_X such that the morphism 1_X belongs to S. In other words, a subclass S in Γ_X is a system of subobjects in X if

  (a) the local identity of X belongs to S;
  (b) every monomorphism $\mu \in \Gamma_X$ has an isomorphic monomorphism in S;
  (c) in S, isomorphism (in the sense of Γ_X) is equivalent to equality.

Due to property 1° on p. 12, such a class S always exists. The elements of S are called *subobjects* of X (of class $\varGamma$). The class S is endowed with the structure of a full subcategory in $\varGamma_X$.

THEOREM 2.21. *Any system of subobjects S of an object X is a partially ordered class.*

Proof. Let $\rho \in S$ and $\sigma \in S$ have mutually inverse morphisms $\varkappa : A \leftarrow B$ and $\lambda : A \to B$, i.e.

$$\rho = \sigma \circ \varkappa, \qquad \sigma = \rho \circ \lambda.$$

Then

$$\rho \circ \lambda \circ \varkappa = \rho = \rho \circ 1_A, \qquad \sigma \circ \varkappa \circ \lambda = \sigma = \sigma \circ 1_B,$$

and since ρ and σ are monomorphisms in $\mathtt{K}$, one can cancel them:

$$\lambda \circ \varkappa = 1_A, \qquad \varkappa \circ \lambda = 1_B,$$

Thus, $\varkappa$ and λ are isomorphisms. We obtain $\rho \cong \sigma$, and by property (c), $\rho = \sigma$. ■

THEOREM 2.22. *If S is a system of subobjects in X, then for any $\sigma \in S$, $\sigma : Y \to X$, the class of monomorphisms*

$$A = \{\alpha \in \varGamma_Y : \sigma \circ \alpha \in S\}$$

is a system of subobjects in Y. If in addition S is a set, then A is a set as well.

Proof. STEP 1: Property (a). This is obvious: since $\sigma \circ 1_Y = \sigma \in S$, we have $1_Y \in A$.

STEP 2: Property (b). Let $\beta : B \to Y$ be a monomorphism. The composition $\sigma \circ \beta : B \to X$ is a monomorphism from $\varGamma_X$, and since S is a system of subobjects in X, there exists $\tau \in S$ such that

$$\tau \cong \sigma \circ \beta.$$

This means that

$$\tau = \sigma \circ \beta \circ \iota$$

for some isomorphism ι. Now we see that the monomorphism $\alpha = \beta \circ \iota$ is isomorphic to β and lies in A, since $\sigma \circ \alpha = \tau \in S$.

STEP 3: Property (c). Let $\alpha, \beta \in A$ be isomorphic monomorphisms, i.e.

$$\alpha = \beta \circ \iota$$

for some isomorphism ι. Then, first, the morphisms $\sigma \circ \alpha$ and $\sigma \circ \beta$ are isomorphic as well, since

$$\sigma \circ \alpha = \sigma \circ \beta \circ \iota.$$

And second, they lie in S, since α and β lie in A. But S satisfies (c), hence

$$\sigma \circ \alpha = \sigma \circ \beta.$$

In addition σ is a monomorphism, so $\alpha = \beta$.

STEP 4: It remains to check that if S is a set, then so is A. This follows from the fact that the map $\alpha \in A \mapsto \sigma \circ \alpha \in S$ is injective. Indeed, if for some $\alpha, \alpha' \in A$ we have

$$\sigma \circ \alpha = \sigma \circ \alpha',$$

then, since σ is a monomorphism, $\alpha = \alpha'$. ■

- We say that a category $\mathtt{K}$ is *well-powered in the class $\varGamma$* if each object X has a system of subobjects S of class $\varGamma$ which is a set (i.e. not a proper class); in other words, each category $\varGamma_X$ must be a skeletally small graph.

EXAMPLE 2.23. The standard categories frequently used as examples, as the category of sets, groups, vector spaces, algebras (over a given field), topological spaces, topological vector spaces, topological algebras etc., are obviously well-powered in the class Mono.

THEOREM 2.24. *If a category $\mathtt{K}$ is well-powered in a class $\varGamma$, then there is a map $X \mapsto S_X$ which assigns to each object X in $\mathtt{K}$ its system of subobjects S_X of class $\varGamma$ (and S_X is a set).*

Proof. The class of all sets can be well-ordered [25, V, 4.1]; this allows us to assign to each X the system of subobjects S which is minimal with respect to this well-ordering. ■

2.3.2. Categories of epimorphisms $\varOmega^X$ and systems of quotient objects. Let $\varOmega$ be a class of epimorphisms in a category $\mathtt{K}$, and suppose all local identities belong to it (the key examples are $\varOmega = \mathsf{Epi}$ and $\varOmega = \mathsf{SEpi}$). For each object X in $\mathtt{K}$ we denote

$$\varOmega^X = \{\sigma \in \varOmega : \mathsf{Dom}\,\sigma = X\}. \tag{2.15}$$

This class forms a category where a morphism $\rho \xrightarrow{\varkappa} \sigma$ from $\rho \in \varOmega^X$ into $\sigma \in \varOmega^X$, i.e. from an epimorphism $\rho : X \to A$ into an epimorphism $\sigma : X \to B$, is an arbitrary morphism $\varkappa : A \to B$ in $\mathtt{K}$ such that

$$\begin{tikzcd} & A \arrow[dd, "\varkappa"] \\ X \arrow[ur, "\rho"] \arrow[dr, "\sigma"'] & \\ & B \end{tikzcd} \tag{2.16}$$

Actually, this diagram in $\mathtt{K}$ can be considered as a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\varOmega^X$. A composition of such morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \tau$, i.e. diagrams

$$\begin{tikzcd} & A \arrow[dd, "\varkappa"] \\ X \arrow[ur, "\rho"] \arrow[dr, "\sigma"'] & \\ & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dd, "\lambda"] \\ X \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & \\ & C \end{tikzcd}$$

is a morphism $\rho \xrightarrow{\lambda\circ\varkappa} \tau$, i.e. a diagram

$$\begin{tikzcd} & A \arrow[dd, "\lambda\circ\varkappa"] \\ X \arrow[ur, "\rho"] \arrow[dr, "\tau"'] & \\ & C \end{tikzcd}$$

One can view it as a result of splicing the initial diagrams along the common edge σ, adding the arrow $\lambda \circ \varkappa$, and then throwing away the vertex B together with all the incident edges:

$$\begin{tikzcd} & & A \arrow[dl, dashed, "\varkappa"'] \arrow[dd, "\lambda\circ\varkappa"] \\ X \arrow[urr, bend left, "\rho"] \arrow[r, dashed, "\sigma"'] \arrow[drr, bend right, "\tau"'] & B \arrow[dr, dashed, "\lambda"] & \\ & & C \end{tikzcd}$$

Of course, local identities in Ω^X are diagrams of the form

$$\begin{tikzcd} & A \arrow[dd, "1_A"] \\ X \arrow[ur, "\rho"] \arrow[dr, "\sigma"'] & \\ & A \end{tikzcd}$$

Remark 2.25. The composition of morphisms in Ω^X can be defined in two ways. In our definition this operation is connected with the composition in $\mathtt{K}$ through

$$\lambda \underset{\Omega^X}{\circ} \varkappa = \lambda \underset{\mathtt{K}}{\circ} \varkappa.$$

By analogy with Γ_X the following properties of Ω^X are proved.

Theorem 2.26. *For any object X the category Ω^X is a graph.*

Remark 2.27. By Example 2.8 this means that *in the category Ω^X all morphisms are bimorphisms*. The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in Ω^X and the properties of the same morphism $\varkappa : A \to B$ in the initial category $\mathtt{K}$ is expressed in the following observations:

– *every morphism $\rho \xrightarrow{\varkappa} \sigma$ in Ω^X is an epimorphism in* $\mathtt{K}$*;*
– *a morphism $\rho \xrightarrow{\varkappa} \sigma$ in Ω^X is an isomorphism in Ω^X $\Leftrightarrow$ $\varkappa$ is an isomorphism in* $\mathtt{K}$*.*

It is convenient to introduce a special notation, $\to$, for the pre-order in Ω^X:

$$\rho \to \sigma \;\Leftrightarrow\; \exists \iota \in \mathsf{Mor}(\mathtt{K}) \;\; \sigma = \iota \circ \rho. \tag{2.17}$$

Here the morphism ι, if it exists, must be unique, and it is an epimorphism (since ρ and σ are). As a corollary, there is an operation which to each pair of morphisms $\rho, \sigma \in \Omega^X$ with the property $\rho \to \sigma$ assigns the morphism $\iota = \iota_\rho^\sigma$ in (2.17):

$$\sigma = \iota_\rho^\sigma \circ \rho. \tag{2.18}$$

If $\pi \to \rho \to \sigma$, then the chain

$$\iota_\pi^\sigma \circ \pi = \sigma = \iota_\rho^\sigma \circ \rho = \iota_\rho^\sigma \circ \iota_\pi^\rho \circ \pi$$

implies by epimorphy of π the equality

$$\iota_\pi^\sigma = \iota_\rho^\sigma \circ \iota_\pi^\rho. \tag{2.19}$$

- A *system of quotient objects of class Ω* on an object X in a category $\mathtt{K}$ is an arbitrary skeleton Q of the category Ω^X such that 1_X belongs to Q. In other words, a subclass Q in Ω^X is called a system of quotient objects on X if

  (a) the local identity of X belongs to Q;

(b) every epimorphism $\varepsilon \in \Omega^X$ has an isomorphic epimorphism in Q;
(c) in Q, isomorphism (in the sense of Ω^X) is equivalent to equality.

By property 1° on p. 12 such a class Q always exists. The elements of Q are called *quotient objects* on X. The class Q is endowed with the structure of a full subcategory in Ω^X.

By analogy with Theorems 2.21 and 2.22 we have:

THEOREM 2.28. *Any system Q of quotient objects on an object X is a partially ordered class.*

THEOREM 2.29. *If Q is a system of quotient objects on an object X, then for any quotient object $\pi \in Q$, $\pi : X \to Y$, the class of epimorphisms*

$$A = \{\alpha \in \Omega^Y : \alpha \circ \pi \in Q\}$$

is a system of quotient objects on Y. If in addition Q is a set, then A is a set as well.

- We say that a category $\mathtt{K}$ is *co-well-powered in the class Ω* if each object X has a system of quotient objects Q of class Ω which is a set (i.e. not a proper class); in other words, each category Ω^X must be a skeletally small graph.

EXAMPLE 2.30. Among the standard categories—of sets, groups, vector spaces, algebras over a given field, topological spaces, topological vector spaces, topological algebras—some are co-well-powered in the class Epi, but sometimes this is not easy to prove (see [1]). In contrast, the co-well-poweredness in the class SEpi is much easier to verify.

By analogy with Theorem 2.24 the following fact is proved:

THEOREM 2.31. *If a category $\mathtt{K}$ is co-well-powered in a class Ω, then there exists a map $X \mapsto Q_X$ which assigns to any object X in $\mathtt{K}$ a system of its quotient objects Q_X of class Ω (and Q_X is a set).*

2.4. Nodal decomposition

2.4.1. Strong decompositions

- A representation of a morphism φ as a composition

$$\varphi = \iota \circ \rho \circ \gamma,$$

where ι is a strong monomorphism and γ a strong epimorphism, will be called a *strong decomposition* of φ.

THEOREM 2.32. *If $\varphi = \iota \circ \rho \circ \gamma$ is a strong decomposition of φ, then for any other decomposition*

$$\varphi = \mu \circ \varepsilon$$

we have:

– *the epimorphy of* ε *implies the existence of a unique morphism* μ' *such that*

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dd, "\gamma"'] \arrow[dr, "\varepsilon"] & & Y \\
& M \arrow[ur, "\mu"'] \arrow[dr, dashed, "\mu'"] & \\
X' \arrow[rr, "\rho"] & & Y' \arrow[uu, "\iota"']
\end{tikzcd} \tag{2.20}$$

(*in this case if* μ *is a monomorphism, then so is* μ');

– *the monomorphy of* μ *implies the existence of a unique morphism* ε' *such that*

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dd, "\gamma"'] \arrow[dr, "\varepsilon"] & & Y \\
& M \arrow[ur, "\mu"'] & \\
X' \arrow[ur, dashed, "\varepsilon'"] \arrow[rr, "\rho"] & & Y' \arrow[uu, "\iota"']
\end{tikzcd} \tag{2.21}$$

(*in this case if* ε *is an epimorphism, then so is* ε').

Proof. Let ε be an epimorphism. Consider the diagram

$$\begin{tikzcd}
X \arrow[dd, "\gamma"'] \arrow[dr, "\varepsilon"] & & Y \\
& M \arrow[ur, "\mu"'] & \\
X' \arrow[rr, "\rho"] & & Y' \arrow[uu, "\iota"']
\end{tikzcd}$$

and transform it into

$$\begin{tikzcd}
X \arrow[r, "\varepsilon"] \arrow[d, "\rho\circ\gamma"'] & M \arrow[d, "\mu"] \\
Y' \arrow[r, "\iota"] & Y
\end{tikzcd}$$

Here ε is an epimorphism, and ι a strong monomorphism, hence there exists a (unique) morphism μ' such that

$$\begin{tikzcd}
X \arrow[r, "\varepsilon"] \arrow[d, "\rho\circ\gamma"'] & M \arrow[d, "\mu"] \arrow[dl, dashed, "\mu'"'] \\
Y' \arrow[r, "\iota"] & Y
\end{tikzcd}$$

This is the morphism for (2.20). By property 1° on p. 15, if in addition $\mu = \iota \circ \mu'$ is a monomorphism, then so is μ'. The second case is dual. ■

Suppose we have two strong decompositions $\varphi = \iota \circ \rho \circ \gamma$ and $\varphi = \iota' \circ \rho' \circ \gamma'$ of a morphism φ:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & Y \\ P \arrow[r, "\rho"'] & Q \arrow[u, "\iota"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma'"'] & Y \\ P' \arrow[r, "\rho'"'] & Q' \arrow[u, "\iota'"'] \end{tikzcd}$$

If there exist (necessarily unique by Theorem 2.32) morphisms $\sigma : P \to P'$ and $\tau : Q' \to Q$ such that

$$\begin{tikzcd} X \arrow[rrr, "\varphi"] \arrow[dr, "\gamma'"] \arrow[dd, "\gamma"'] & & & Y \\ & P' \arrow[r, "\rho'"] & Q' \arrow[ur, "\iota'"] \arrow[dr, dashed, "\tau"] & \\ P \arrow[ur, dashed, "\sigma"] \arrow[rrr, "\rho"'] & & & Q \arrow[uu, "\iota"'] \end{tikzcd} \tag{2.22}$$

then we say that the strong decomposition $\varphi = \iota \circ \rho \circ \gamma$ *is subordinated* to the strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$, and we write

$$(\iota, \rho, \gamma) \le (\iota', \rho', \gamma').$$

If in addition σ and τ are isomorphisms, then we say that the decompositions $\varphi = \iota \circ \rho \circ \gamma$ and $\varphi = \iota' \circ \rho' \circ \gamma'$ are *isomorphic*, and we write

$$(\iota, \rho, \gamma) \cong (\iota', \rho', \gamma').$$

PROPOSITION 2.33. *The double inequality*

$$(\iota, \rho, \gamma) \le (\iota', \rho', \gamma') \le (\iota, \rho, \gamma)$$

is equivalent to the isomorphism of strong decompositions:

$$(\iota, \rho, \gamma) \cong (\iota', \rho', \gamma').$$

Proof. The first inequality implies the existence of the (unique) dotted arrows in (2.22), and the second one means that the reverse arrows exist as well (and again are unique). In addition the epimorphy of γ and γ' implies that σ and its reverse arrow are mutually inverse isomorphisms, while the monomorphy of ι and ι' implies that the same is true for τ and its reverse arrow. ∎

2.4.2. Nodal decomposition. If in a strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ the middle morphism ρ' is a bimorphism, then we call this a *nodal decomposition.* We also say that $\mathtt{K}$ is a *category with nodal decomposition* if every morphism φ in $\mathtt{K}$ has a nodal decomposition.

PROPOSITION 2.34. *Each nodal decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ subordinates each strong decomposition $\varphi = \iota \circ \rho \circ \gamma$:*

$$(\iota, \rho, \gamma) \le (\iota', \rho', \gamma').$$

As a corollary, a nodal decomposition is unique up to isomorphism.

Proof. Let $\varphi = \iota \circ \rho \circ \gamma$ be a strong decomposition. If we transform the diagram

$$\begin{tikzcd}
X \arrow[dr, "\gamma'"] \arrow[dd, "\gamma"'] & & & Y \\
& P' \arrow[r, "\rho'"] & Q' \arrow[ur, "\iota'"] & \\
P \arrow[rrr, "\rho"'] & & & Q \arrow[uu, "\iota"']
\end{tikzcd} \tag{2.23}$$

into

$$\begin{tikzcd}
X \arrow[dr, "\rho' \circ \gamma'"] \arrow[ddrr, bend right, "\rho \circ \gamma"'] & & Y \\
& Q' \arrow[ur, "\iota'"] & \\
& & Q \arrow[uu, "\iota"']
\end{tikzcd}$$

then one can recognize here a quadrangle of the form (2.7), since ι is a strong monomorphism, and $\rho' \circ \gamma'$ an epimorphism (as a composition of an epimorphism γ' and a bimorphism ρ'). Hence, there is a unique morphism τ such that

$$\begin{tikzcd}
X \arrow[dr, "\rho' \circ \gamma'"] \arrow[ddrr, bend right, "\rho \circ \gamma"'] & & Y \\
& Q' \arrow[ur, "\iota'"] \arrow[dr, dashed, "\tau"'] & \\
& & Q \arrow[uu, "\iota"']
\end{tikzcd}$$

Similarly, one can transform diagram (2.23) into

$$\begin{tikzcd}
X \arrow[dr, "\gamma'"] \arrow[dd, "\gamma"'] & & Y \\
& P' \arrow[ur, "\iota' \circ \rho'"] & \\
P \arrow[uurr, bend right, "\iota \circ \rho"'] & &
\end{tikzcd}$$

and this is again a quadrangle of the form (2.7), since γ is a strong epimorphism, and $\iota' \circ \rho'$ a monomorphism (as a composition of a bimorphism ρ' and a monomorphism ι'). Hence, there exists a unique morphism σ such that

$$\begin{tikzcd}
X \arrow[dr, "\gamma'"] \arrow[dd, "\gamma"'] & & Y \\
& P' \arrow[ur, "\iota' \circ \rho'"] & \\
P \arrow[ur, dashed, "\sigma"'] \arrow[uurr, bend right, "\iota \circ \rho"'] & &
\end{tikzcd}$$

These two morphisms together give diagram (2.22). ∎

- From the uniqueness (up to isomorphism) of the nodal decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ it follows that one can assign symbols to its components. We will further depict a nodal decomposition of a morphism $\varphi : X \to Y$ as a diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathsf{coim}_\infty \varphi"'] & Y \\ \mathsf{Coim}_\infty \varphi \arrow[r, "\mathsf{red}_\infty \varphi"] & \mathsf{Im}_\infty \varphi \arrow[u, "\mathsf{im}_\infty \varphi"'] \end{tikzcd} \tag{2.24}$$

(where elements are defined up to isomorphism). The proof of Theorem 2.36 below and Remark 2.43 justify these symbols, since they show that coim_∞, red_∞ and im_∞ can be conceived as a sort of "transfinite induction" of the usual operations coim, red and im in pre-abelian categories:

$$\mathsf{coim}_\infty = \lim_{n\to\infty} \underbrace{\mathsf{coim} \circ \cdots \circ \mathsf{coim}}_{n \text{ factors}}, \qquad \mathsf{red}_\infty = \lim_{n\to\infty} \underbrace{\mathsf{red} \circ \cdots \circ \mathsf{red}}_{n \text{ factors}},$$
$$\mathsf{im}_\infty = \lim_{n\to\infty} \underbrace{\mathsf{im} \circ \cdots \circ \mathsf{im}}_{n \text{ factors}}.$$

We will call

- $\mathsf{im}_\infty \varphi$ the *nodal image*,
- $\mathsf{red}_\infty \varphi$ the *nodal reduced part*,
- $\mathsf{coim}_\infty \varphi$ the *nodal coimage*

of the morphism φ.

Remark 2.35. By Theorem 2.32:

- for any decomposition $\varphi = \mu \circ \varepsilon$ where ε is an epimorphism, there is a unique morphism μ' such that

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"] \arrow[dd, "\mathsf{coim}_\infty \varphi"'] & & Y \\ & M \arrow[ur, "\mu"] \arrow[dr, dashed, "\mu'"] & \\ \mathsf{Coim}_\infty \varphi \arrow[rr, "\mathsf{red}_\infty \varphi"] & & \mathsf{Im}_\infty \varphi \arrow[uu, "\mathsf{im}_\infty \varphi"'] \end{tikzcd} \tag{2.25}$$

(and if μ is a monomorphism, then so is μ'),
- for any decomposition $\varphi = \mu \circ \varepsilon$ where μ is a monomorphism, there is a unique morphism ε' such that

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"] \arrow[dd, "\mathsf{coim}_\infty \varphi"'] & & Y \\ & M \arrow[ur, "\mu"] & \\ \mathsf{Coim}_\infty \varphi \arrow[ur, dashed, "\varepsilon'"] \arrow[rr, "\mathsf{red}_\infty \varphi"] & & \mathsf{Im}_\infty \varphi \arrow[uu, "\mathsf{im}_\infty \varphi"'] \end{tikzcd} \tag{2.26}$$

(and if ε is an epimorphism, then so is ε').

2.4.3. On existence of a nodal decomposition. Let us note that if μ is a monomorphism in a category $\mathtt{K}$, then for any decomposition $\mu = \mu' \circ \varepsilon$, if ε is a strong epimorphism, then ε must be an isomorphism. Indeed, by 1° on p. 15, the equality $\mu = \mu' \circ \varepsilon$ means that ε is an monomorphism, and since in addition ε is a strong epimorphism, so (by 4° on p. 19) an immediate epimorphism, we deduce by 4° on p. 18 that ε is an isomorphism.

- Let us say that in a category $\mathtt{K}$ *strong epimorphisms discern monomorphisms* if the converse is true: from the fact that a morphism μ is not a monomorphism it follows that μ can be represented as a composition $\mu = \mu' \circ \varepsilon$ where ε is a strong epimorphism which is not an isomorphism.

Dually, if ε is an epimorphism in a category $\mathtt{K}$, then for any decomposition $\varepsilon = \mu \circ \varepsilon'$, if μ is a strong monomorphism, then μ must be an isomorphism.

- Let us say that in a category $\mathtt{K}$ *strong monomorphisms discern epimorphisms* if the converse is true: from the fact that a morphism ε is not an epimorphism it follows that ε can be represented as a composition $\varepsilon = \mu \circ \varepsilon'$ where μ is a strong monomorphism which is not an isomorphism.

Recall that the notion of linearly complete category was introduced on p. 9.

THEOREM 2.36. *Let* $\mathtt{K}$ *be a linearly complete category, well-powered in strong monomorphisms and co-well-powered in strong epimorphisms, where strong epimorphisms discern monomorphisms, and dually, strong monomorphisms discern epimorphisms. Then* $\mathtt{K}$ *is a category with nodal decomposition.*

Before proving this theorem let us introduce the following auxiliary construction. Take a morphism $\varphi : X \to Y$ in a category $\mathtt{K}$. Since $\mathtt{K}$ is co-well-powered in strong epimorphisms, in the category SEpi^X of strong epimorphisms going from X there exists a *set* of strong quotient objects $Q \subseteq \mathsf{SEpi}^X$, and in the category SMono_Y of strong monomorphisms coming to Y there exists a *set* of strong subobjects $S \subseteq \mathsf{SMono}_Y$. We fix these sets Q and S.

- A decomposition $\varphi = \iota \circ \rho \circ \gamma$ of a morphism φ is said to be *admissible* if $\gamma \in Q$ and $\iota \in S$. Clearly, any strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ of a morphism φ is isomorphic to some admissible decomposition $\varphi = \iota \circ \rho \circ \gamma$.
- A *local basic decomposition* of a morphism φ in a category $\mathtt{K}$ is an arbitrary map $\rho \mapsto (\mathsf{coim}\,\rho, \mathsf{red}\,\rho, \mathsf{im}\,\rho)$ that to each admissible decomposition (ι, ρ, γ) of φ assigns some strong decomposition $(\mathsf{im}\,\rho, \mathsf{red}\,\rho, \mathsf{coim}\,\rho)$ of ρ:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & Y \\ \mathsf{Dom}\,\rho \arrow[r, "\rho"] \arrow[d, "\mathsf{coim}\,\rho"'] & \mathsf{Ran}\,\rho \arrow[u, "\iota"'] \\ \mathsf{Coim}\,\rho \arrow[r, "\mathsf{red}\,\rho"] & \mathsf{Im}\,\rho \arrow[u, "\mathsf{im}\,\rho"'] \end{tikzcd} \tag{2.27}$$

in such a way that the following conditions are fulfilled:

(a) the decomposition $(\iota \circ \mathsf{im}\,\rho, \mathsf{red}\,\rho, \mathsf{coim}\,\rho \circ \gamma)$ of φ is admissible (i.e. $\mathsf{coim}\,\rho \circ \gamma \in Q$ and $\iota \circ \mathsf{im}\,\rho \in S$),

(b) ρ is a monomorphism $\Leftrightarrow$ $\mathsf{coim}\,\rho$ is an isomorphism $\Leftrightarrow$ $\mathsf{coim}\,\rho = 1$,

(c) ρ is an epimorphism $\Leftrightarrow$ $\mathsf{im}\,\rho$ is an epimorphism $\Leftrightarrow$ $\mathsf{im}\,\rho = 1$.

LEMMA 2.37. *Let* $\mathtt{K}$ *be a category well-powered in strong monomorphisms and co-well-powered in strong epimorphisms, where strong epimorphisms discern monomorphisms, and strong monomorphisms discern epimorphisms. Then each morphism* φ *in* $\mathtt{K}$ *has a local basic decomposition.*

Proof. First of all, it is clear that admissible decompositions always exist, for example one can take $\varphi = 1\circ\varphi\circ 1$. Let us now show that for any admissible decomposition (ι,ρ,γ) of φ a diagram (2.27) satisfying (a)–(c) exists. Let us fix such a decomposition (ι,ρ,γ) and consider several cases.

1. If ρ is not a monomorphism, then there exists a decomposition $\rho = \rho'\circ\varepsilon$ where ε is a strong epimorphism, but not an isomorphism. Set $\mathsf{coim}\,\rho = \varepsilon$ and consider the morphism ρ'.
   - 1.1. If ρ' is not an epimorphism, then there exists a decomposition $\rho' = \mu\circ\rho''$ where μ is a strong monomorphism, but not an isomorphism. Then we set $\mathsf{im}\,\rho = \mu$ and $\mathsf{red}\,\rho = \rho''$.
   - 1.2. If ρ' is an epimorphism, then we set $\mathsf{im}\,\rho = 1_{\mathsf{Ran}\,\rho}$ and $\mathsf{red}\,\rho = \rho'$.
2. If ρ is a monomorphism, then we set $\mathsf{coim}\,\rho = 1_{\mathsf{Dom}\,\rho}$ and again consider ρ.
   - 2.1. If ρ is not an epimorphism, then there exists a decomposition $\rho = \mu\circ\rho'$ where μ is a strong monomorphism, but not an isomorphism. We set $\mathsf{im}\,\rho = \mu$ and $\mathsf{red}\,\rho = \rho'$.
   - 2.2. If ρ is an epimorphism, then we set $\mathsf{im}\,\rho = 1_Y$ and $\mathsf{red}\,\rho = \rho$.

In any case we obtain a decomposition $\rho = \mathsf{im}\,\rho\circ\mathsf{red}\,\rho\circ\mathsf{coim}\,\rho$ where $\mathsf{im}\,\rho$ is a strong monomorphism, $\mathsf{coim}\,\rho$ is a strong epimorphism, and (b) and (c) are fulfilled. Now to prove (a) we have to replace (if necessary) the epimorphism $\mathsf{coim}\,\rho$ with an isomorphic epimorphism $\pi\circ\mathsf{coim}\,\rho$ in such a way that $\pi\circ\mathsf{coim}\,\rho\circ\gamma\in Q$, and this can be done due to Theorem 2.29. Similarly, the monomorphism $\mathsf{im}\,\rho$ should be replaced with an isomorphic monomorphism $\mathsf{im}\,\rho\circ\sigma$ in such a way that $\iota\circ\mathsf{im}\,\rho\circ\sigma\in S$, and this can be done due to Theorem 2.22.

Thus, for an arbitrary admissible decomposition (ι,ρ,γ) of φ a diagram (2.27) satisfying (a)–(c) exists. Note now that from 2.29 and 2.22 it follows that for a given admissible decomposition (ι,ρ,γ) of a morphism φ the class of decompositions $(\mathsf{im}\,\rho,\mathsf{red}\,\rho,\mathsf{coim}\,\rho)$ of ρ which satisfy (a)–(c) is a set. Indeed, every such $(\mathsf{im}\,\rho,\mathsf{red}\,\rho,\mathsf{coim}\,\rho)$ is uniquely defined by the morphisms $\mathsf{im}\,\rho$ and $\mathsf{coim}\,\rho$ (since from monomorphy of $\mathsf{im}\,\rho$ and epimorphy of $\mathsf{coim}\,\rho$ it follows that $\mathsf{red}\,\rho$, if it exists, is unique). So the class of decompositions $(\mathsf{im}\,\rho,\mathsf{red}\,\rho,\mathsf{coim}\,\rho)$ can be viewed as a subclass in the cartesian product $A\times B$ of sets, where $A = \{\alpha\in\mathsf{SMono}_{\mathsf{Ran}\,\rho} : \iota\circ\alpha\in S\}$ is a class of monomorphisms through which $\mathsf{im}\,\rho$ runs, and which is a set by Theorem 2.22, and $B = \{\beta\in\mathsf{SEpi}^{\mathsf{Dom}\,\rho} : \beta\circ\varepsilon\in Q\}$ is a class of epimorphisms through which $\mathsf{coim}\,\rho$ runs, and which is a set by Theorem 2.29).

We deduce that for any admissible decomposition (ι,ρ,γ) of φ the class of decompositions $(\mathsf{coim}\,\rho,\mathsf{red}\,\rho,\mathsf{im}\,\rho)$ satisfying (2.27) and (a)–(c) is a (non-empty) set. Hence we can apply the axiom of choice and construct a map which to each admissible decomposition

(ι, ρ, γ) of φ assigns a decomposition $(\mathsf{coim}\,\rho, \mathsf{red}\,\rho, \mathsf{im}\,\rho)$ satisfying (2.27) and (a)–(c). This is the required map $\rho \mapsto (\mathsf{coim}\,\rho, \mathsf{red}\,\rho, \mathsf{im}\,\rho)$. ∎

Proof of Theorem 2.36. Take a morphism $\varphi : X \to Y$, find a *set* of strong quotient objects $Q \subseteq \mathsf{SEpi}^X$ and a *set* of strong subobjects $S \subseteq \mathsf{SMono}_Y$, and construct a local basic decomposition as in Lemma 2.37. The proof consists in constructing a transfinite system of objects and morphisms, indexed by $i \in \mathtt{Ord}$,

$$X^i \xrightarrow{\varphi^i} Y^i, \quad X^i \xrightarrow{\varepsilon^i_j} X^j, \quad Y^i \xleftarrow{\mu^i_j} Y^j \qquad (i \leq j),$$

the idea of which is illustrated by the following diagram (extended infinitely below):

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[d, equal, "1_X"'] & & Y \arrow[d, equal, "1_Y"] \\
X^0 \arrow[rr, "\varphi^0=\varphi"] \arrow[d, "\varepsilon^0_1=\mathsf{coim}\,\varphi^0"'] & & Y^0 \\
X^1 \arrow[rr, "\varphi^1=\mathsf{red}\,\varphi^0"] \arrow[d, "\varepsilon^1_2=\mathsf{coim}\,\varphi^1"'] & & Y^1 \arrow[u, "\mu^0_1=\mathsf{im}\,\varphi^0"'] \\
X^2 \arrow[rr, "\varphi^2=\mathsf{red}\,\varphi^1"] \arrow[d, "\varepsilon^2_3=\mathsf{coim}\,\varphi^2"'] & & Y^1 \arrow[u, "\mu^1_2=\mathsf{im}\,\varphi^1"'] \\
\dots & & \dots \arrow[u, "\mu^2_3=\mathsf{im}\,\varphi^2"']
\end{tikzcd} \tag{2.28}$$

Here is how we do this.

0) Initially, we set

$$X^0 = X, \quad Y^0 = Y, \quad \varphi^0 = \varphi, \quad \varepsilon^0_1 = \mathsf{coim}\,\varphi^0, \quad \mu^0_1 = \mathsf{im}\,\varphi^0, \quad \varphi^1 = \mathsf{red}\,\varphi^0.$$

1) Then for an arbitrary ordinal k we set

$$\varepsilon^k_k = 1_{X^k}, \quad \mu^k_k = 1_{Y^k}$$

and:

– If $k = j+1$ for some j, then we set

$$X^k = X^{j+1} = \mathsf{Coim}\,\varphi^j, \quad Y^k = Y^{j+1} = \mathsf{Im}\,\varphi^j,$$
$$\varepsilon^j_k = \varepsilon^j_{j+1} = \mathsf{coim}\,\varphi^j, \quad \mu^j_k = \mu^j_{j+1} = \mathsf{im}\,\varphi^j, \quad \varphi^k = \varphi^{j+1} = \mathsf{red}\,\varphi^j,$$

and then, for any other ordinal $i < j$,

$$\varepsilon^i_k = \varepsilon^i_{j+1} = \varepsilon^j_{j+1} \circ \varepsilon^i_j, \quad \mu^i_k = \mu^i_{j+1} = \mu^i_j \circ \mu^j_{j+1}.$$

– If k is a limit ordinal, then X^k is defined as the injective limit of the covariant system $\{X^j, \varepsilon^i_j;\ i \leq j < k\}$, and Y^k as the projective limit of the contravariant system $\{Y^j, \mu^i_j; i \leq j < k\}$:

$$X^k = \lim_{j \to k} X^j, \quad Y^k = \lim_{k \leftarrow j} Y^j,$$

the system of morphisms $\{\varepsilon^i_k;\ i < k\}$ is the corresponding injective cone of morphisms going to X^k, and $\{\mu^i_k; i < k\}$ is the corresponding projective cone of morphisms going from Y^k,

$$\varepsilon_k^i = \lim_{j\to k} \varepsilon_j^i, \quad \mu_k^i = \lim_{k\leftarrow j} \mu_j^i, \qquad i \le k.$$

This automatically implies

$$\varepsilon_k^i = \varepsilon_k^j \circ \varepsilon_j^i, \quad \mu_k^i = \mu_j^i \circ \mu_k^j, \qquad i \le j \le k,$$

and by Proposition 2.17 all the morphisms ε_k^i are strong epimorphisms, while by Proposition 2.16 all the morphisms μ_j^i are strong monomorphisms. As a corollary, X^k can be chosen in such a way that the epimorphism ε_k^0 lies in Q (for this we just need to multiply the system $\{\varepsilon_k^i;\ i < k\}$ from the left by a morphism so that the property of being an injective cone is preserved); similarly, Y^k can be chosen in such a way that the monomorphism μ_k^0 lies in S (for this we just need to multiply the system $\{\mu_k^i;\ i < k\}$ from the right so that the property of being a projective cone is preserved). Then φ^k can be defined by two equivalent formulas:

$$\varphi^k = \lim_{k\leftarrow i} \lim_{j\to k} \mu_j^i \circ \varphi^j = \lim_{i\to k} \lim_{k\leftarrow j} \varphi^j \circ \varepsilon_j^i.$$

Here the first double limit should be understood as follows: for a given $i < k$ the family $\{\mu_j^i \circ \varphi^j;\ i \le j < k\}$ is an injective cone of the covariant system $\{\varepsilon_j^l;\ i \le l, j < k\}$, so the limit

$$\lim_{j\to k} \mu_j^i \circ \varphi^j$$

exists; then $\{\lim_{j\to k} \mu_j^i \circ \varphi^j;\ i < k\}$ turns out to be a projective cone of the contravariant system $\{\mu_j^l;\ i \le l, j < k\}$, so the limit

$$\lim_{k\leftarrow i} \lim_{j\to k} \mu_j^i \circ \varphi^j$$

exists. Similarly, in the second double limit for a given $i < k$ the family $\{\varphi^j \circ \varepsilon_j^i;\ i \le j < k\}$ is a projective cone of the contravariant system $\{\mu_j^l;\ i \le l, j < k\}$, so the limit

$$\lim_{k\leftarrow j} \varphi^j \circ \varepsilon_j^i$$

exists; then $\{\lim_{k\leftarrow j} \varphi^j \circ \varepsilon_j^i;\ i < k\}$ turns out to be an injective cone of the covariant system $\{\varepsilon_j^l;\ i \le l, j < k\}$, so the limit

$$\lim_{i\to k} \lim_{k\leftarrow j} \varphi^j \circ \varepsilon_j^i$$

exists. Each of these double limits gives an arrow from X^k into Y^k which makes all the necessary diagrams commutative, and since this arrow is unique (this follows from the fact that the μ_k^i are monomorphisms and the ε_k^i are epimorphisms), those double limits (arrows) coincide.

Eventually we obtain a system of morphisms such that for any $i \le j$ the following diagram is commutative:

$$\begin{tikzcd} X^i \arrow[r, "\varphi^i"] \arrow[d, "\varepsilon_j^i"'] & Y^i \\ X^j \arrow[r, "\varphi^j"] & Y^j \arrow[u, "\mu_j^i"'] \end{tikzcd}$$

and for any $i \le j \le k$ the following diagrams are commutative:

$$
\begin{tikzcd}
 & X^i \arrow[dl, "\varepsilon^i_j"'] \arrow[dd, "\varepsilon^i_k"] \\
X^j \arrow[dr, "\varepsilon^j_k"'] & \\
 & X^k
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y^i \\
Y^j \arrow[ur, "\mu^i_j"] & \\
 & Y^k \arrow[ul, "\mu^j_k"] \arrow[uu, "\mu^i_k"']
\end{tikzcd}
$$

and moreover the ε^i_j are strong epimorphisms, and the μ^i_j are strong monomorphisms. From the last two diagrams it follows that the formulas

$$
\begin{cases} F(i) = \varepsilon^0_i, & i \in \mathtt{Ord}, \\ F(i,j) = \varepsilon^i_j, & i \le j \in \mathtt{Ord}, \end{cases} \qquad \begin{cases} G(i) = \mu^0_i, & i \in \mathtt{Ord}, \\ G(i,j) = \mu^i_j, & i \le j \in \mathtt{Ord}, \end{cases}
$$

define a covariant functor $F : \mathtt{Ord} \to Q$ and a contravariant functor $G : \mathtt{Ord} \to S$. Since Q and S are sets, by Theorem 2.5 these functors must stabilize, i.e. starting from some ordinal k (which can be chosen common for F and G) the morphisms $F(i,j)$ and $G(i,j)$ become isomorphisms. Since in addition the categories Q and S are partially ordered classes (and as a corollary, only local identities are isomorphisms, by Proposition 2.2), we conclude (following Remark 2.4) that diagram (2.28) stabilizes in the sense that, starting from some k,

– the objects X^l become the same, and the morphisms ε^l_m become local identities of X^k;
– the objects Y^l become the same and the morphisms μ^l_m become local identities of Y^k.

Now let us consider the diagram

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\varepsilon^0_k"'] & Y \\
X^k \arrow[r, "\varphi^k"] & Y^k \arrow[u, "\mu^0_k"']
\end{tikzcd}
\tag{2.29}
$$

Here ε^0_k is a strong epimorphism, and μ^0_k a strong monomorphism. From the equality $\varepsilon^k_{k+1} = \mathsf{coim}\,\varphi^k = 1_{X^k}$ (which holds since the sequence ε^0_j stabilizes for $j \ge k$) it follows by condition (b) on p. 31 that φ^k is a monomorphism. On the other hand, from $\mu^k_{k+1} = \mathsf{im}\,\varphi^k = 1_{Y^k}$ (which holds since the sequence μ^0_j stabilizes for $j \ge k$) it follows by condition (c) on p. 32 that φ^k is an epimorphism. Thus, φ^k is a bimorphism, hence (2.29) is a nodal decomposition for φ. ■

2.4.4. Connection with the basic decomposition in pre-abelian categories. Let us discuss the obvious analogy between nodal decomposition and the decomposition of a morphism φ in a pre-abelian category K into a coimage $\mathsf{coim}\,\varphi$, an image $\mathsf{im}\,\varphi$ and a morphism between them which we denote by $\mathsf{red}\,\varphi$.

Recall (see definition in [11] or in [6]) that a *pre-abelian category* is an enriched category K over the category Ab of abelian groups, which is finitely complete and has a zero object. In such a category every morphism $\varphi : X \to Y$ has a kernel and a cokernel. Hence

φ can be represented as a composition

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathsf{coim}\,\varphi"'] & Y \\ \mathsf{Coim}\,\varphi \arrow[r, dashed, "\mathsf{red}\,\varphi"] & \mathsf{Im}\,\varphi \arrow[u, "\mathsf{im}\,\varphi"'] \end{tikzcd} \tag{2.30}$$

where $\mathsf{coim}\,\varphi = \mathsf{coker}(\mathsf{ker}\,\varphi)$ is called the *coimage* of φ, $\mathsf{im}\,\varphi = \mathsf{ker}(\mathsf{coker}\,\varphi)$ the *image* of φ, and $\mathsf{red}\,\varphi$ the *reduced part* of φ; its existence and uniqueness are proved separately.

• The representation (2.30) of φ will be called the *basic decomposition* of φ.

It is known (see [7, Proposition 4.3.6(4)]) that in a pre-abelian category (in fact, in a category with zero) every kernel $\mathsf{ker}\,\varphi$ (and thus every image $\mathsf{im}\,\varphi$) is always a strong monomorphism, and every cokernel $\mathsf{coker}\,\varphi$ (and thus every coimage $\mathsf{coim}\,\varphi$) is a strong epimorphism. As a corollary, we have

Theorem 2.38. *In a pre-abelian category every basic decomposition is strong.*

This implies that *if a category* $\mathtt{K}$ *is abelian, then every basic decomposition in* $\mathtt{K}$ *is nodal.* But if $\mathtt{K}$ is not abelian, then these decompositions do not necessarily coincide: see Example 4.98 below.

The following two propositions are obvious:

Proposition 2.39. *In a pre-abelian category for a morphism $\varphi : X \to Y$ the following conditions are equivalent:*

(i) *φ is a monomorphism,*
(ii) *the zero morphism $0_{0,X}$ is the kernel for φ,*
(iii) *the identity morphism 1_X is the coimage for φ,*
(iv) *$\mathsf{coim}\,\varphi$ is an isomorphism.*

Proposition 2.40. *In a pre-abelian category for a morphism $\varphi : X \to Y$ the following conditions are equivalent:*

(i) *φ is an epimorphism,*
(ii) *$0_{Y,0} = \mathsf{coker}\,\varphi$,*
(iii) *$1_Y = \mathsf{im}\,\varphi$,*
(iv) *$\mathsf{im}\,\varphi$ is an isomorphism.*

They imply

Proposition 2.41. *In a pre-abelian category $\mathtt{K}$ strong epimorphisms discern monomorphisms and strong monomorphisms discern epimorphisms.*

Proof. Consider the basic decomposition of $\varphi : X \to Y$:

$$\varphi = \mathsf{im}\,\varphi \circ \mathsf{red}\,\varphi \circ \mathsf{coim}\,\varphi.$$

If $\varphi : X \to Y$ is not a monomorphism, then by Proposition 2.39, $\mathsf{coim}\,\varphi$ is not an isomorphism. On the other hand, by Theorem 2.38, $\mathsf{coim}\,\varphi$ is a strong epimorphism. So, if we set $\varphi' = \mathsf{im}\,\varphi \circ \mathsf{red}\,\varphi$, then in the decomposition $\varphi = \varphi' \circ \mathsf{coim}\,\varphi$ the morphism $\mathsf{coim}\,\varphi$ is a strong epimorphism, but not an isomorphism. This means that strong epimorphisms discern monomorphisms in $\mathtt{K}$. The statement about strong monomorphisms is proved similarly. ∎

Proposition 2.41 implies that if a pre-abelian category $\mathtt{K}$ is well-powered in strong monomorphisms and co-well-powered in strong epimorphisms, then $\mathtt{K}$ has a local basic decomposition (defined on p. 31): the map $(\iota,\rho,\gamma)\mapsto(\mathsf{coim}\,\rho,\mathsf{red}\,\rho,\mathsf{im}\,\rho)$ that to each admissible decomposition (ι,ρ,γ) (again see p. 31) of a given morphism φ assigns the basic decomposition of ρ, is a local basic decomposition of φ. Hence, the sufficient condition for the existence of nodal decomposition (Theorem 2.36) becomes simpler:

Theorem 2.42. *If a pre-abelian category* $\mathtt{K}$ *is well-powered in strong monomorphisms and co-well-powered in strong epimorphisms, then every morphism* $\varphi : X\to Y$ *in* $\mathtt{K}$ *has a nodal decomposition* (2.24).

Remark 2.43. From Proposition 2.41 and diagram (2.28) it follows that:

- the nodal reduced part $\mathsf{red}_\infty\,\varphi$ in diagram (2.24) can be viewed as a "limit" of the transfinite sequence of "usual" reduced morphisms $\varphi^{i+1}=\mathsf{red}\,\varphi^i$;
- the nodal coimage $\mathsf{coim}_\infty\,\varphi$ is an injective limit of the transfinite sequence of "usual" coimages $\mathsf{coim}\,\varphi^i$ of this system of morphisms;
- the nodal image $\mathsf{im}_\infty\,\varphi$ is a projective limit of the transfinite sequence of "usual" images $\mathsf{im}\,\varphi^i$ of this system of morphisms.

Remark 2.44. Since the basic decomposition $\varphi=\mathsf{im}\,\varphi\circ\mathsf{red}\,\varphi\circ\mathsf{coim}\,\varphi$ is strong, and thus, by Proposition 2.34, is subordinated to the nodal decomposition, there must exist unique morphisms σ and τ such that

$$\begin{tikzcd}
X \arrow[rrr, "\varphi"] \arrow[dr, "\mathsf{coim}_\infty\varphi"] \arrow[dd, "\mathsf{coim}\,\varphi"'] & & & Y \\
& \mathsf{Coim}_\infty\varphi \arrow[r, "\mathsf{red}_\infty\varphi"'] & \mathsf{Im}_\infty\varphi \arrow[ur, "\mathsf{im}_\infty\varphi"] \arrow[dr, dashed, "\tau"] & \\
\mathsf{Coim}\,\varphi \arrow[ur, dashed, "\sigma"] \arrow[rrr, "\mathsf{red}\,\varphi"'] & & & \mathsf{Im}\,\varphi \arrow[uu, "\mathsf{im}\,\varphi"']
\end{tikzcd} \tag{2.31}$$

At the same time, by Theorem 2.32:

- for any decomposition $\varphi=\mu\circ\varepsilon$ where ε is an epimorphism, there exists a unique morphism μ' such that

$$\begin{tikzcd}
X \arrow[rrrr, "\varphi"] \arrow[rr, "\varepsilon"] \arrow[dr, "\mathsf{coim}_\infty\varphi"] \arrow[ddd, "\mathsf{coim}\,\varphi"'] & & & & Y \\
& & M \arrow[urr, "\mu"] \arrow[dr, dashed, "\mu'"] & & \\
& \mathsf{Coim}_\infty\varphi \arrow[rr, "\mathsf{red}_\infty\varphi"'] & & \mathsf{Im}_\infty\varphi \arrow[uur, "\mathsf{im}_\infty\varphi"] \arrow[dr, "\tau"] & \\
\mathsf{Coim}\,\varphi \arrow[ur, "\sigma"] \arrow[rrrr, "\mathsf{red}\,\varphi"'] & & & & \mathsf{Im}\,\varphi \arrow[uuu, "\mathsf{im}\,\varphi"']
\end{tikzcd} \tag{2.32}$$

(in addition, if μ is a monomorphism, then so is μ');

– for any decomposition $\varphi = \mu \circ \varepsilon$ where μ is a monomorphism, there exists a unique morphism ε' such that

$$\begin{tikzcd}
X \arrow[rrrr, "\varphi"] \arrow[drr, "\varepsilon"] \arrow[ddr, "\mathsf{coim}_\infty \varphi"'] \arrow[ddd, "\mathsf{coim}\,\varphi"'] & & & & Y \\
& & M \arrow[urr, "\mu"] & & \\
& \mathsf{Coim}_\infty \varphi \arrow[ur, dashed, "\varepsilon'"] \arrow[rr, "\mathsf{red}_\infty \varphi"'] & & \mathsf{Im}_\infty \varphi \arrow[uur, "\mathsf{im}_\infty \varphi"'] \arrow[dr, "\tau"] & \\
\mathsf{Coim}\,\varphi \arrow[ur, "\sigma"'] \arrow[rrrr, "\mathsf{red}\,\varphi"'] & & & & \mathsf{Im}\,\varphi \arrow[uuu, "\mathsf{im}\,\varphi"']
\end{tikzcd} \tag{2.33}$$

(in addition, if ε is an epimorphism, then so is ε');

– in particular, for any factorization $\varphi = \mu \circ \varepsilon$ of φ there exist unique morphisms $\mathsf{Coim}\,\varphi \xrightarrow{\varepsilon'} M$ and $M \xrightarrow{\mu'} \mathsf{Im}\,\varphi$ such that

$$\begin{tikzcd}
X \arrow[rrrr, "\varphi"] \arrow[drr, "\varepsilon"] \arrow[ddr, "\mathsf{coim}_\infty \varphi"'] \arrow[ddd, "\mathsf{coim}\,\varphi"'] & & & & Y \\
& & M \arrow[urr, "\mu"] \arrow[dr, dashed, "\mu'"] & & \\
& \mathsf{Coim}_\infty \varphi \arrow[ur, dashed, "\varepsilon'"] \arrow[rr, "\mathsf{red}_\infty \varphi"'] & & \mathsf{Im}_\infty \varphi \arrow[uur, "\mathsf{im}_\infty \varphi"'] \arrow[dr, "\tau"] & \\
\mathsf{Coim}\,\varphi \arrow[ur, "\sigma"'] \arrow[rrrr, "\mathsf{red}\,\varphi"'] & & & & \mathsf{Im}\,\varphi \arrow[uuu, "\mathsf{im}\,\varphi"']
\end{tikzcd} \tag{2.34}$$

and in addition, ε' is an epimorphism and μ' a monomorphism.

2.5. Factorizations of a category

2.5.1. Factorizations in a category with nodal decomposition. Recall that the notion of a factorization of a morphism was defined in (2.6). From (2.25) and (2.26) we immediately have

Proposition 2.45. *If $X \xrightarrow{\varepsilon} M \xrightarrow{\mu} Y$ is a factorization of a morphism $X \xrightarrow{\varphi} Y$ in a category $\mathtt{K}$ with a nodal decomposition, then there are unique morphisms $\mathsf{Coim}_\infty \varphi \xrightarrow{\varepsilon'} M$ and $M \xrightarrow{\mu'} \mathsf{Im}_\infty \varphi$ such that*

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon"] \arrow[dd, "\mathsf{coim}_\infty \varphi"'] & & Y \\
& M \arrow[ur, "\mu"] \arrow[dr, dashed, "\mu'"] & \\
\mathsf{Coim}_\infty \varphi \arrow[ur, dashed, "\varepsilon'"] \arrow[rr, "\mathsf{red}_\infty \varphi"] & & \mathsf{Im}_\infty \varphi \arrow[uu, "\mathsf{im}_\infty \varphi"']
\end{tikzcd} \tag{2.35}$$

Moreover, ε' is an epimorphism and μ' a monomorphism.

Let (ε, μ) and (ε', μ') be factorizations of φ. We say that (ε, μ) *is subordinated* to (ε', μ') (or (ε', μ') *subordinates* (ε, μ)), and write

$$(\varepsilon, \mu) \leq (\varepsilon', \mu'),$$

if there exists a morphism β such that

$$\varepsilon' = \beta \circ \varepsilon, \qquad \mu = \mu' \circ \beta,$$

that is,

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] \arrow[dr, "\varepsilon'" near end] & Y \\ M \arrow[r, dashed, "\beta"'] \arrow[ur, "\mu" near end] & M' \arrow[u, "\mu'"'] \end{tikzcd}$$

From properties 1° and 3° on p. 15 it follows that β, if it exists, must be a bimorphism, and from the fact that μ' is a monomorphism (or from the fact that ε is an epimorphism) that β is unique.

THEOREM 2.46. *In a category* $\mathtt{K}$ *with nodal decomposition:*

(i) *every morphism* φ *has a factorization;*
(ii) *among all factorizations of* φ *there is a minimal one* $(\varepsilon_{\min}, \mu_{\min})$ *and a maximal one* $(\varepsilon_{\max}, \mu_{\max})$, *i.e. for any other factorization* (ε, μ),

$$(\varepsilon_{\min}, \mu_{\min}) \leq (\varepsilon, \mu) \leq (\varepsilon_{\max}, \mu_{\max}).$$

Proof. Part (i) follows from (ii). To prove (ii), let

$$\varepsilon_{\min} = \mathsf{coim}_\infty \varphi, \qquad \mu_{\min} = \mathsf{im}_\infty \varphi \circ \mathsf{red}_\infty \varphi$$

and

$$\varepsilon_{\max} = \mathsf{red}_\infty \varphi \circ \mathsf{coim}_\infty \varphi, \qquad \mu_{\max} = \mathsf{im}_\infty \varphi.$$

Then these will be factorizations of φ, and from (2.35) it follows that the first is minimal, and the second is maximal. ■

2.5.2. Strong morphisms in a category with nodal decomposition

THEOREM 2.47. *In a category with nodal decomposition:*

(a) μ *is an immediate monomorphism* $\Leftrightarrow$ μ *is a strong monomorphism* $\Leftrightarrow$ $\mu \cong \mathsf{im}_\infty \mu \Leftrightarrow$ $\mathsf{coim}_\infty \mu$ *and* $\mathsf{red}_\infty \mu$ *are isomorphisms,*
(b) ε *is an immediate epimorphism* $\Leftrightarrow$ ε *is a strong epimorphism* $\Leftrightarrow$ $\varepsilon \cong \mathsf{coim}_\infty \varepsilon \Leftrightarrow$ $\mathsf{im}_\infty \mu$ *and* $\mathsf{red}_\infty \mu$ *are isomorphisms.*

Proof. By the duality principle it is sufficient to prove (a).

If $\mu : X \to Y$ is an immediate monomorphism, then in its maximal factorization $\mu = \mu_{\max} \circ \varepsilon_{\max}$ the morphism $\varepsilon_{\max} = \mathsf{red}_\infty \mu \circ \mathsf{coim}_\infty \mu$ must be an isomorphism. This implies

$$1_X = (\varepsilon_{\max})^{-1} \circ \mathsf{red}_\infty \mu \circ \mathsf{coim}_\infty \mu,$$

from which one can conclude that $\mathsf{coim}_\infty \mu$ is a coretraction. On the other hand, $\mathsf{coim}_\infty \mu$ is an epimorphism, hence an isomorphism. This implies that $\mathsf{red}_\infty \mu = \varepsilon_{\max} \circ (\mathsf{coim}_\infty \mu)^{-1}$ is an isomorphism.

If $\mathsf{coim}_\infty \mu$ and $\mathsf{red}_\infty \mu$ are isomorphisms, then $\chi = \mathsf{red}_\infty \mu \circ \mathsf{coim}_\infty \mu$ is an isomorphism as well, and at the same time $\mu = \mathsf{im}_\infty \mu \circ \chi$. This means that $\mu \cong \mathsf{im}_\infty \mu$.

If $\mu \cong \mathsf{im}_\infty \mu$, then since $\mathsf{im}_\infty \mu$ is a strong monomorphism, so is μ.

If μ is a strong monomorphism, then by property 2° on p. 18, μ is an immediate monomorphism. ∎

2.5.3. Factorization of a category

- A pair of morphisms (μ, ε) is said to be *diagonizable* [6, 47] if for all morphisms $\alpha : \mathsf{Dom}\,\varepsilon \to \mathsf{Dom}\,\mu$ and $\beta : \mathsf{Ran}\,\varepsilon \to \mathsf{Ran}\,\mu$ such that $\mu \circ \alpha = \beta \circ \varepsilon$ there exists a morphism $\delta : B \to C$ such that diagram (2.7) is commutative:

$$\begin{tikzcd} \mathsf{Dom}\,\varepsilon \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & \mathsf{Ran}\,\varepsilon \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] \\ \mathsf{Dom}\,\mu \arrow[r, "\mu"] & \mathsf{Ran}\,\mu \end{tikzcd}$$

This is denoted by writing $\mu \downarrow \varepsilon$.

Example 2.48. The following example shows that in contrast to the situation considered above (in particular on p. 18) the relation $\mu \downarrow \varepsilon$ does not necessarily mean that $\mu \in \mathsf{Mono}$ and $\varepsilon \in \mathsf{Epi}$: in the category of vector spaces over $\mathbb{C}$ the pair of morphisms $\mu = 0 : \mathbb{C} \to 0$ and $\varepsilon = 0 : 0 \to \mathbb{C}$ is diagonizable:

$$\begin{tikzcd} 0 \arrow[r, "\varepsilon=0"] \arrow[d, "\alpha"'] & \mathbb{C} \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] \\ \mathbb{C} \arrow[r, "\mu=0"'] & 0 \end{tikzcd}$$

- For any class Λ of morphisms in $\mathtt{K}$:
  - its *epimorphic conjugate class* is the class
$$\Lambda^{\downarrow} = \{\varepsilon \in \mathsf{Epi}(\mathtt{K}) : \forall \lambda \in \Lambda \ \ \lambda \downarrow \varepsilon\}.$$
  - its *monomorphic conjugate class* is the class
$$^{\downarrow}\Lambda = \{\mu \in \mathsf{Mono}(\mathtt{K}) : \forall \lambda \in \Lambda \ \ \mu \downarrow \lambda\}.$$

Clearly, for each class Λ of morphisms,

$$\mathsf{Iso} \subseteq \Lambda^{\downarrow} \subseteq \mathsf{Epi}, \qquad \mathsf{Iso} \circ \Lambda^{\downarrow} \subseteq \Lambda^{\downarrow}, \tag{2.36}$$
$$\mathsf{Iso} \subseteq {}^{\downarrow}\Lambda \subseteq \mathsf{Mono}, \qquad {}^{\downarrow}\Lambda \circ \mathsf{Iso} \subseteq {}^{\downarrow}\Lambda. \tag{2.37}$$

- Let us say that classes Γ and Ω of morphisms *define a factorization of the category* (3) $\mathtt{K}$ if:

  F.1. Ω is the epimorphic conjugate class for Γ: $\Gamma^{\downarrow} = \Omega$;
  F.2. Γ is the monomorphic conjugate class for Ω: $\Gamma = {}^{\downarrow}\Omega$;
  F.3. the composition of the classes Γ and Ω covers the class of all morphisms: $\Gamma \circ \Omega = \mathsf{Mor}(\mathtt{K})$ (this means that each morphism $\varphi \in \mathsf{Mor}(\mathtt{K})$ can be represented as a composition $\mu \circ \varepsilon$ where $\mu \in \Gamma$, $\varepsilon \in \Omega$).

(3) This construction is also called a *bicategory* [6, 47].

If these conditions are fulfilled, we write

$$\mathtt{K} = \Gamma \circledcirc \Omega. \tag{2.38}$$

EXAMPLE 2.49. In a category $\mathtt{K}$ with nodal decomposition the following classes of morphisms define factorizations:

$$\mathtt{K} = \mathsf{Mono} \circledcirc \mathsf{SEpi} = \mathsf{SMono} \circledcirc \mathsf{Epi}.$$

The following is proved in [47, Theorem 8.2]:

THEOREM 2.50. *Classes Γ and Ω define a factorization of $\mathtt{K}$ if and only if the following conditions hold:*

(i) $\Gamma \subseteq \mathsf{Mono}(\mathtt{K})$ *and* $\Omega \subseteq \mathsf{Epi}(\mathtt{K})$;
(ii) $\mathsf{Iso}(\mathtt{K}) \subseteq \Omega \cap \Gamma$;
(iii) *for each* $\varphi \in \mathsf{Mor}(\mathtt{K})$ *there is a decomposition*

$$\varphi = \mu_\varphi \circ \varepsilon_\varphi, \qquad \mu_\varphi \in \Gamma,\ \varepsilon_\varphi \in \Omega; \tag{2.39}$$

(iv) *for any other decomposition with the same properties*

$$\varphi = \mu' \circ \varepsilon', \qquad \mu' \in \Gamma,\ \varepsilon' \in \Omega,$$

there is $\theta \in \mathsf{Iso}(\mathtt{K})$ *such that*

$$\mu' = \mu_\varphi \circ \theta, \qquad \varepsilon' = \theta^{-1} \circ \varepsilon_\varphi.$$

- Let us say that a class Ω of morphisms in $\mathtt{K}$ is *monomorphically complementable* if

$$\mathtt{K} = {}^{\downarrow}\Omega \circledcirc \Omega. \tag{2.40}$$

In other words, Ω must be the epimorphic conjugate to its monomorphic conjugate class: $\Omega = ({}^{\downarrow}\Omega)^{\downarrow}$, and the composition of ${}^{\downarrow}\Omega$ and Ω must cover the class of all morphisms: ${}^{\downarrow}\Omega \circ \Omega = \mathsf{Mor}(\mathtt{K})$. In this case ${}^{\downarrow}\Omega$ will be called the *monomorphic complement* to Ω.

REMARK 2.51. From (2.36) it follows that if a class Ω of morphisms is monomorphically complementable, then

$$\mathsf{Iso} \subseteq \Omega \subseteq \mathsf{Epi}, \qquad \mathsf{Iso} \circ \Omega \subseteq \Omega. \tag{2.41}$$

- Similarly, we say that a class Γ of morphisms in $\mathtt{K}$ is *epimorphically complementable* if

$$\mathtt{K} = \Gamma \circledcirc \Gamma^{\downarrow}. \tag{2.42}$$

In other words, Γ must be the monomorphic conjugate to its epimorphic conjugate class: $\Gamma = {}^{\downarrow}(\Gamma^{\downarrow})$, and the composition of Γ and $\Gamma^{\downarrow}$ must cover the class of all morphisms: $\Gamma \circ \Gamma^{\downarrow} = \mathsf{Mor}(\mathtt{K})$. In this case $\Gamma^{\downarrow}$ will be called the *epimorphic complement* to Γ.

REMARK 2.52. From (2.37) it follows that if Γ is epimorphically complementable, then

$$\mathsf{Iso} \subseteq \Gamma \subseteq \mathsf{Mono}, \qquad \Gamma \circ \mathsf{Iso} \subseteq \Gamma. \tag{2.43}$$

3. Envelope and refinement

3.1. Envelope

3.1.1. Envelope in a class of morphisms with respect to a class of morphisms. Suppose we have:

– a category $\mathtt{K}$ called an *enveloping category*,
– a category $\mathtt{T}$ called an *attracting category*,
– a covariant functor $F : \mathtt{T} \to \mathtt{K}$,
– two classes Ω and Φ of morphisms in $\mathtt{K}$, taking values in objects of the class $F(\mathtt{T})$, with Ω called the *class of realizing morphisms*, and Φ the *class of test morphisms*.

Then:

- For $X \in \mathsf{Ob}(\mathtt{K})$ and $X' \in \mathsf{Ob}(\mathtt{T})$ a morphism $\sigma : X \to F(X')$ is called an *extension of the object* $X \in \mathtt{K}$ *over the category* $\mathtt{T}$ *in the class* Ω *of morphisms with respect to the class* Φ *of morphisms* if $\sigma \in \Omega$, and for any object B in $\mathtt{T}$ and any morphism $\varphi : X \to F(B)$ in Φ there exists a unique morphism $\varphi' : X' \to B$ in $\mathtt{T}$ such that

$$\begin{tikzcd} & X \arrow[dl, "\Omega\ni\sigma"'] \arrow[dr, "\varphi\in\Phi"] & \\ F(X') \arrow[rr, dashed, "F(\varphi')"] & & F(B) \end{tikzcd} \tag{3.1}$$

- An extension $\rho : X \to F(E)$ of an object $X \in \mathtt{K}$ over a category $\mathtt{T}$ in the class Ω with respect to the class Φ is called an *envelope of* X *over the category* $\mathtt{T}$ *in the class* Ω *with respect to the class* Φ if for each extension $\sigma : X \to F(X')$ (of X over the category $\mathtt{T}$ in the class Ω with respect to the class Φ) there exists a unique morphism $\upsilon : X' \to E$ in $\mathtt{T}$ such that

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & \\ F(X') \arrow[rr, dashed, "F(\upsilon)"] & & F(E) \end{tikzcd} \tag{3.2}$$

In what follows, we are almost exclusively interested in the case when $\mathtt{T} = \mathtt{K}$ and $F : \mathtt{K} \to \mathtt{K}$ is the identity functor. It is useful to give the definitions for this case separately:

- A morphism $\sigma : X \to X'$ in a category $\mathtt{K}$ is called an *extension of* $X \in \mathsf{Ob}(\mathtt{K})$ *in the class* Ω *with respect to the class* Φ if $\sigma \in \Omega$, and for any morphism $\varphi : X \to B$ in Φ

there exists a unique morphism $\varphi' : X' \to B$ in $\mathtt{K}$ such that

$$\begin{tikzcd} & X \arrow[dl, "\Omega\ni\sigma"'] \arrow[dr, "\forall\varphi\in\Phi"] & \\ X' \arrow[rr, dashed, "\exists!\varphi'"] & & B \end{tikzcd} \tag{3.3}$$

- An extension $\rho : X \to E$ of an object $X \in \mathsf{Ob}(\mathtt{K})$ in the class Ω with respect to the class Φ is called an *envelope of X in Ω with respect to Φ* if for any other extension $\sigma : X \to X'$ (of X in Ω with respect to Φ) there is a unique morphism $\upsilon : X' \to E$ in $\mathtt{K}$ such that

$$\begin{tikzcd} & X \arrow[dl, "\forall\sigma"'] \arrow[dr, "\rho"] & \\ X' \arrow[rr, dashed, "\exists!\upsilon"] & & E \end{tikzcd} \tag{3.4}$$

For an envelope $\rho : X \to E$ we use the notation

$$\rho = \mathsf{env}^{\Omega}_{\Phi} X. \tag{3.5}$$

The very object E is also called an *envelope* of X (in Ω with respect to Φ), and we write

$$E = \mathsf{Env}^{\Omega}_{\Phi} X. \tag{3.6}$$

REMARK 3.1. Clearly, the object $\mathsf{Env}^{\Omega}_{\Phi} X$ (if it exists) is unique up to isomorphism. The question when the correspondence $X \mapsto \mathsf{Env}^{\Omega}_{\Phi} X$ can be defined as a functor is discussed below starting from p. 69.

REMARK 3.2. If $\Omega = \emptyset$, then of course neither extensions nor envelopes in Ω exist. So this construction can be interesting only when Ω is a non-empty class. The following two situations will be of special interest:

– $\Omega = \mathsf{Epi}(\mathtt{K})$ (the class of all epimorphisms in $\mathtt{K}$); then we will use the notation

$$\mathsf{env}^{\mathsf{Epi}}_{\Phi} X := \mathsf{env}^{\mathsf{Epi}(\mathtt{K})}_{\Phi} X, \qquad \mathsf{Env}^{\mathsf{Epi}}_{\Phi} X := \mathsf{Env}^{\mathsf{Epi}(\mathtt{K})}_{\Phi} X. \tag{3.7}$$

– $\Omega = \mathsf{Mor}(\mathtt{K})$ (the class of all morphisms in $\mathtt{K}$); in this case it is convenient to omit Ω from the formulations and notation, so we will be speaking about the *envelope of $X \in \mathtt{K}$ in $\mathtt{K}$ with respect to the class Φ*, and the notation will be simplified:

$$\mathsf{env}_{\Phi} X := \mathsf{env}^{\mathsf{Mor}(\mathtt{K})}_{\Phi} X, \qquad \mathsf{Env}_{\Phi} X := \mathsf{Env}^{\mathsf{Mor}(\mathtt{K})}_{\Phi} X. \tag{3.8}$$

REMARK 3.3. Another degenerate, but this time informative case is when $\Phi = \emptyset$. It is essential that for a given object X, Φ does not contain morphisms going from X. Then, obviously, any morphism $\sigma : X \to X'$ belonging to Ω is an extension of X (in the class Ω with respect to the class $\emptyset$). If in addition $\Omega = \mathsf{Epi}$, then the envelope of X is the terminal object in the category Epi^X (if it exists):

$$\mathsf{Env}^{\Omega}_{\emptyset} X = \max \mathsf{Epi}^X.$$

In particular, if $\mathtt{K}$ is a category with zero 0, and Ω contains all morphisms going to 0, then the envelope of any object with respect to the empty class of morphisms is 0.

Remark 3.4. Another extreme situation is when $\Phi = \mathsf{Mor}(\mathtt{K})$. It is essential that for a given object X the class Φ contains the local identity of X. Then for any extension σ the diagram

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "1_X"'] & & X' \arrow[dl, dashed] \\
& X &
\end{tikzcd}$$

implies that σ must be a coretraction (moreover, the dashed arrow must be unique). When $\Omega \subseteq \mathsf{Epi}$ this is possible only if σ is an isomorphism. As a corollary, in this case the envelope of X coincides with X (up to isomorphism).

Properties of envelopes.

1° *Suppose that $\Sigma \subseteq \Omega$. Then for any object X and any class Φ of morphisms*:

(a) *each extension $\sigma : X \to X'$ in Σ with respect to Φ is an extension in Ω with respect to Φ*;

(b) *if there are envelopes $\mathsf{env}^{\Sigma}_{\Phi} X$ and $\mathsf{env}^{\Omega}_{\Phi} X$, then there is a unique morphism $\rho : \mathsf{Env}^{\Sigma}_{\Phi} X \to \mathsf{Env}^{\Omega}_{\Phi} X$ such that*

$$\begin{tikzcd}
& X \arrow[dl, "\mathsf{env}^{\Sigma}_{\Phi} X"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi} X"] & \\
\mathsf{Env}^{\Sigma}_{\Phi} X \arrow[rr, dashed, "\rho"] & & \mathsf{Env}^{\Omega}_{\Phi} X
\end{tikzcd} \tag{3.9}$$

(c) *if there is $\mathsf{env}^{\Omega}_{\Phi} X \in \Sigma$, then $\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Sigma}_{\Phi} X$.*

2° *Let Σ, Ω, Φ be classes of morphisms, and suppose that, for an object X,*

(a) *every extension $\sigma : X \to X'$ in Ω with respect to Φ belongs to Σ.*

Then:

(b) *an envelope of X with respect to Φ in the class Ω exists if and only if there exists an envelope of X with respect to Φ in the class $\Omega \cap \Sigma$, and $\mathsf{env}^{\Omega}_{\Phi} = \mathsf{env}^{\Omega\cap\Sigma}_{\Phi}$;*

(c) *if $\Sigma \subseteq \Omega$, then an envelope of X with respect to Φ in the (narrower) class Σ exists if and only if there exists an envelope of X with respect to Φ in the (wider) class Ω, and $\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Sigma}_{\Phi} X$.*

3° *Suppose $\Psi \subseteq \Phi$. Then for any object X and for any class Ω of morphisms*:

(a) *each extension $\sigma : X \to X'$ in Ω with respect to Φ is an extension in Ω with respect to Ψ;*

(b) *if there are envelopes $\mathsf{env}^{\Omega}_{\Psi} X$ and $\mathsf{env}^{\Omega}_{\Phi} X$, then there is a unique morphism $\alpha : \mathsf{Env}^{\Omega}_{\Psi} X \leftarrow \mathsf{Env}^{\Omega}_{\Phi} X$ such that*

$$\begin{tikzcd}
& X \arrow[dl, "\mathsf{env}^{\Omega}_{\Psi} X"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi} X"] & \\
\mathsf{Env}^{\Omega}_{\Psi} X & & \mathsf{Env}^{\Omega}_{\Phi} X \arrow[ll, dashed, "\alpha"']
\end{tikzcd} \tag{3.10}$$

4° *Suppose that $\Phi \subseteq \mathsf{Mor}(\mathtt{K}) \circ \Psi$ (i.e. each $\varphi \in \Phi$ can be represented as $\varphi = \chi \circ \psi$, where $\psi \in \Psi$). Then for any object X and any class Ω of morphisms:*

(a) *if an extension $\sigma : X \to X'$ in Ω with respect to Ψ is at the same time an epimorphism in* $\mathtt{K}$*, then it is an extension in Ω with respect to Φ;*

(b) *if there are envelopes $\mathsf{env}^{\Omega}_{\Psi} X$ and $\mathsf{env}^{\Omega}_{\Phi} X$, and $\mathsf{env}^{\Omega}_{\Psi} X$ is at the same time an epimorphism in* $\mathtt{K}$*, then there exists a unique morphism $\beta : \mathsf{Env}^{\Omega}_{\Psi} X \to \mathsf{Env}^{\Omega}_{\Phi} X$ such that*

$$\begin{tikzcd} & X \arrow[dl, "\mathsf{env}^{\Omega}_{\Psi} X"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi} X"] & \\ \mathsf{Env}^{\Omega}_{\Psi} X \arrow[rr, dashed, "\beta"] & & \mathsf{Env}^{\Omega}_{\Phi} X \end{tikzcd} \qquad (3.11)$$

5° *Suppose that Ω and Φ are some classes of morphisms, and $\varepsilon : X \to Y$ is an epimorphism in* $\mathtt{K}$ *such that:*

(a) *there exists an envelope $\mathsf{env}^{\Omega}_{\Phi\circ\varepsilon} X$ (with respect to $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon;\ \varphi \in \Phi\}$);*

(b) *there exists an envelope $\mathsf{env}^{\Omega}_{\Phi} Y$;*

(c) $\mathsf{env}^{\Omega}_{\Phi} Y \circ \varepsilon \in \Omega$.

Then there exists a unique morphism $\upsilon : \mathsf{Env}^{\Omega}_{\Phi\circ\varepsilon} X \leftarrow \mathsf{Env}^{\Omega}_{\Phi} Y$ such that

$$\begin{tikzcd} X \arrow[rr, "\varepsilon"] \arrow[d, "\mathsf{env}^{\Omega}_{\Phi\circ\varepsilon} X"'] \arrow[drr, bend right=10, "\mathsf{env}^{\Omega}_{\Phi} Y\circ\varepsilon"] & & Y \arrow[d, "\mathsf{env}^{\Omega}_{\Phi} Y"] \\ \mathsf{Env}^{\Omega}_{\Phi\circ\varepsilon} X & & \mathsf{Env}^{\Omega}_{\Phi} Y \arrow[ll, dashed, "\upsilon"] \end{tikzcd} \qquad (3.12)$$

Proof. 1° If a morphism σ satisfies (3.3) with Σ instead of Ω, then σ satisfies the initial condition (3.3), since $\Sigma \subseteq \Omega$. This proves (a). From this we moreover see that $\mathsf{env}^{\Sigma}_{\Psi} X$ is an extension in Ω with respect to Φ, so there must exist a unique dashed arrow in (3.9). This means that (b) is also true. Finally, if there exists an envelope $\mathsf{env}^{\Omega}_{\Phi} X$ (in the wider class), and it lies in Σ (in the narrower class), then $\mathsf{env}^{\Omega}_{\Phi} X$ is an extension in Σ. On the other hand, any other extension $\sigma : X \to X'$ in Σ is an extension in Ω due to (a), hence there is a unique morphism υ into the envelope in Ω:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi} X"] & \\ X' \arrow[rr, dashed, "\upsilon"] & & \mathsf{Env}^{\Omega}_{\Phi} X \end{tikzcd}$$

This proves that $\mathsf{env}^{\Omega}_{\Phi} X$ is an envelope in Σ, and we have proved (c).

2° If an object X has an envelope $\mathsf{env}^{\Omega}_{\Phi} X$ in Ω with respect to Φ, then by (a) this will be an extension in the narrower class $\Omega \cap \Sigma$ with respect to Φ. Applying 1°(c), we deduce that $\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Omega\cap\Sigma}_{\Phi} X$.

Conversely, suppose there exists an envelope $\mathsf{env}^{\Omega\cap\Sigma}_{\Phi} X$. Then by 1°(a), it will be an envelope with respect to Φ in Ω. Take another extension $\sigma : X \to X'$ with respect to Φ in Ω. By (a), σ is an extension with respect to Φ in $\Omega \cap \Sigma$. Hence, there exists a unique morphism $\upsilon : X' \to \mathsf{Env}^{\Omega\cap\Sigma}_{\Phi} X$ such that

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\mathsf{env}^{\Omega\cap\Sigma}_{\Phi} X"] & \\ X' \arrow[rr, dashed, "\upsilon"] & & \mathsf{Env}^{\Omega\cap\Sigma}_{\Phi} X \end{tikzcd}$$

This proves that $\mathsf{env}_{\Phi}^{\Omega\cap\Sigma} X$ is (not just an extension, but also) an envelope with respect to Φ in Ω. We see that 2°(b) is true, and 2°(c) is a corollary.

3° Suppose that $\Psi \subseteq \Phi$. Then (a) is obvious: each extension $\sigma : X \to X'$ with respect to Φ is an extension with respect to Ψ. For (b) we have: since $\mathsf{env}_{\Phi}^{\Omega} X$ is an extension with respect to Φ, it must be an extension with respect to Ψ, so there exists a unique morphism from $\mathsf{Env}_{\Phi}^{\Omega} X$ into $\mathsf{Env}_{\Psi}^{\Omega} X$ such that (3.10) is commutative.

4° Suppose $\Phi \subseteq \mathsf{Mor}(\mathtt{K}) \circ \Psi$. For (a) our reasoning is illustrated by the diagram

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\psi"] \arrow[ddr, bend right, "\varphi"'] & & X' \arrow[dl, dashed, "\psi'"'] \arrow[ddl, dashed, bend left, "\varphi'"] \\
& Y \arrow[d, "\chi"] & \\
& B &
\end{tikzcd}$$

If $\sigma : X \to X'$ is an extension of X in Ω with respect to Ψ, then for any morphism $\varphi : X \to B$ in Φ, we take a decomposition $\varphi = \chi \circ \psi$ where $\psi \in \Psi$. There is a morphism ψ' such that $\psi = \psi' \circ \sigma$. Set $\varphi' = \chi \circ \psi'$, and note that

$$\varphi = \chi \circ \psi = \chi \circ \psi' \circ \sigma = \varphi' \circ \sigma.$$

The uniqueness of φ' follows from the epimorphy of $\sigma \in \Omega$, and thus σ is an extension of X in Ω with respect to Φ. Once (a) is proved, (b) becomes a corollary: the morphism $\mathsf{env}_{\Psi}^{\Omega} X : X \to \mathsf{Env}_{\Psi}^{\Omega} X$ is an extension of X in Ω with respect to Ψ, hence, by (a), with respect to Φ as well. So there exists a morphism β from $\mathsf{Env}_{\Psi}^{\Omega} X$ into $\mathsf{Env}_{\Phi}^{\Omega} X$ such that (3.11) is commutative.

5° For any morphism $\varphi : Y \to B$ in Φ we have the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "\mathsf{env}_{\Phi}^{\Omega} Y \circ \varepsilon"] \arrow[dr, "\varepsilon"'] \arrow[ddr, bend right, "\varphi\circ\varepsilon"'] & & \mathsf{Env}_{\Phi}^{\Omega} Y \arrow[ddl, dashed, bend left, "\varphi'"] \\
& Y \arrow[ur, "\mathsf{env}_{\Phi}^{\Omega} Y"'] \arrow[d, "\varphi"] & \\
& B &
\end{tikzcd}$$

It should be understood as follows. On the one hand, since $\mathsf{env}_{\Phi}^{\Omega} Y$ is an extension with respect to Φ, there exists a morphism φ' such that the lower right triangle is commutative, and as a corollary, the perimeter is commutative as well. On the other hand, if φ' is a morphism such that the perimeter is commutative, i.e.

$$\varphi' \circ \mathsf{env}_{\Phi}^{\Omega} Y \circ \varepsilon = \varphi \circ \varepsilon,$$

then, since ε is an epimorphism, we can cancel it:

$$\varphi' \circ \mathsf{env}_{\Phi}^{\Omega} Y = \varphi.$$

So the lower right triangle is commutative as well. This means that φ' is unique (since by the definition of envelope, the dashed arrow in the lower right triangle is unique).

We see that the perimeter has a unique dashed arrow φ'. This is true for any $\varphi \in \varPhi$, and in addition $\mathsf{env}_{\varPhi}^{\varOmega} Y \circ \varepsilon \in \varOmega$. So we come to the conclusion that $\mathsf{env}_{\varPhi}^{\varOmega} Y \circ \varepsilon$ is an extension of X in $\varOmega$ with respect to $\varPhi \circ \varepsilon$. As a corollary, there exists a unique morphism υ from $\mathsf{Env}_{\varPhi}^{\varOmega} Y$ into $\mathsf{Env}_{\varPhi\circ\varepsilon}^{\varOmega} X\ \varPhi \circ \varepsilon$ such that (3.12) is commutative. ■

- Let us say that in a category $\mathtt{K}$ *a class $\varPhi$ of morphisms is generated on the inside by a class $\varPsi$ of morphisms* if

$$\varPsi \subseteq \varPhi \subseteq \mathsf{Mor}(\mathtt{K}) \circ \varPsi. \tag{3.13}$$

THEOREM 3.5. *Suppose that in a category $\mathtt{K}$ a class $\varPhi$ of morphisms is generated on the inside by a class $\varPsi$ of morphisms. Then for any class $\varOmega$ of epimorphisms* (*not necessarily all*) *and any object X the existence of $\mathsf{env}_{\varPsi}^{\varOmega} X$ is equivalent to the existence of $\mathsf{env}_{\varPhi}^{\varOmega} X$, and*

$$\mathsf{env}_{\varPsi}^{\varOmega} X = \mathsf{env}_{\varPhi}^{\varOmega} X. \tag{3.14}$$

Proof. Suppose first that $\mathsf{env}_{\varPsi}^{\varOmega} X$ exists. Since it is an extension with respect to $\varPsi$, and at the same time an epimorphism, by 2°(a) we see that it is an extension with respect to $\varPhi$ as well. If $\sigma : X \to X'$ is another extension with respect to $\varPhi$, then by 3°(a) it is an extension with respect to $\varPsi$ as well, so there exists a unique morphism $\upsilon : \mathsf{Env}_{\varPsi}^{\varOmega} X \leftarrow X'$ such that

$$\begin{tikzcd} & X \arrow[dl, "\mathsf{env}_{\varPsi}^{\varOmega} X"'] \arrow[dr, "\sigma"] & \\ \mathsf{Env}_{\varPsi}^{\varOmega} X & & X' \arrow[ll, dashed, "\exists! \upsilon"'] \end{tikzcd}$$

This means that $\mathsf{env}_{\varPsi}^{\varOmega} X$ is an envelope with respect to $\varPhi$, and (3.14) holds.

Conversely, suppose that $\mathsf{env}_{\varPhi}^{\varOmega} X$ exists. It is an extension with respect to $\varPhi$, so by 2°(a) it is an extension with respect to $\varPsi$ as well. If $\sigma : X \to X'$ is another extension in $\varOmega$ with respect to $\varPsi$, then since $\sigma \in \mathsf{Epi}$, by 3°(a) it must be an extension with respect to $\varPhi$, so there exists a unique morphism $\upsilon : X' \to \mathsf{Env}_{\varPhi}^{\varOmega} X$ such that

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\mathsf{env}_{\varPhi}^{\varOmega} X"] & \\ X' \arrow[rr, dashed, "\exists! \upsilon"] & & \mathsf{Env}_{\varPhi}^{\varOmega} X \end{tikzcd}$$

This means that $\mathsf{env}_{\varPhi}^{\varOmega} X$ is an envelope with respect to $\varPsi$, and again we have (3.14). ■

- Let us say that a class $\varPhi$ of morphisms in a category $\mathtt{K}$ *separates morphisms on the outside* if for any two morphisms $\alpha \neq \beta : X \to Y$ there is a morphism $\varphi : Y \to M$ in $\varPhi$ such that $\varphi \circ \alpha \neq \varphi \circ \beta$.

THEOREM 3.6. *If a class $\varPhi$ of morphisms separates morphisms on the outside, then for any class $\varOmega$ of morphisms:*

(i) *each extension in $\varOmega$ with respect to $\varPhi$ is a monomorphism;*
(ii) *an envelope with respect to $\varPhi$ in $\varOmega$ exists if and only if there exists an envelope with respect to $\varPhi$ in $\varOmega \cap \mathsf{Mono}$; in this case $\mathsf{env}_{\varPhi}^{\varOmega} = \mathsf{env}_{\varPhi}^{\varOmega\cap\mathsf{Mono}}$;*

(iii) *if Ω contains all monomorphisms, then the existence of an envelope with respect to Φ in* Mono *automatically implies the existence of an envelope with respect to Φ in Ω, and* $\mathsf{env}_{\Phi}^{\Omega} = \mathsf{env}_{\Phi}^{\mathsf{Mono}}$.

Proof. (i) Suppose that some extension $\sigma : X \to X'$ is not a monomorphism, i.e. there are parallel morphisms $\alpha \neq \beta : T \to X$ such that

$$\sigma \circ \alpha = \sigma \circ \beta. \tag{3.15}$$

Since Φ separates morphisms on the outside, there exists a morphism $\varphi : X \to M$ in Φ such that

$$\varphi \circ \alpha \neq \varphi \circ \beta. \tag{3.16}$$

As $\sigma : X \to X'$ is an extension with respect to Φ, there is a continuation $\varphi' : X' \to M$ of the morphism $\varphi : X \to M$: $\varphi = \varphi' \circ \sigma$. Now we obtain

$$\varphi \circ \alpha = \varphi' \circ \sigma \circ \alpha \overset{(3.15)}{=} \varphi' \circ \sigma \circ \beta = \varphi \circ \beta,$$

and this contradicts (3.16).

(ii) Suppose for an object X there exists an envelope $\mathsf{env}_{\Phi}^{\Omega} X$. Then, as already proved, it is an extension in $\Omega \cap \mathsf{Mono}$ with respect to Φ. Applying property 1°(c) on p. 44, we deduce that $\mathsf{env}_{\Phi}^{\Omega} X = \mathsf{env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X$.

Conversely, suppose there is an envelope $\mathsf{env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X$. By 1°(a) on p. 44, it is an extension with respect to Φ in Ω. Consider another extension $\sigma : X \to X'$ with respect to Φ in Ω. By (i), σ is an extension with respect to Φ in $\Omega \cap \mathsf{Mono}$. Hence, there is a unique morphism $\upsilon : X' \to \mathsf{Env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X$ such that

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\mathsf{env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X"] & \\ X' \arrow[rr, dashed, "\upsilon"] & & \mathsf{Env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X \end{tikzcd}$$

This proves that $\mathsf{env}_{\Phi}^{\Omega\cap\mathsf{Mono}} X$ is (not only an extension, but also) an envelope with respect to Φ in the class Ω.

(iii) immediately follows from (ii). ∎

• Let us recall that a class Φ of morphisms in a category $\mathtt{K}$ is called a *right ideal* if

$$\Phi \circ \mathsf{Mor}(\mathtt{K}) \subseteq \Phi.$$

THEOREM 3.7. *If a class Φ of morphisms separates morphisms on the outside and is a right ideal in the category* $\mathtt{K}$, *then for any class Ω of morphisms:*

(i) *each extension in Ω with respect to Φ is a bimorphism;*
(ii) *an envelope with respect to Φ in Ω exists if and only if there exists an envelope with respect to Φ in the class $\Omega \cap$* Bim *of bimorphisms belonging to Ω; in this case* $\mathsf{env}_{\Phi}^{\Omega} = \mathsf{env}_{\Phi}^{\Omega\cap\mathsf{Bim}}$;
(iii) *if Ω contains all bimorphisms, then an envelope with respect to Φ in Ω exists if and only if there exists an envelope with respect to Φ in* Bim, *and* $\mathsf{env}_{\Phi}^{\Omega} = \mathsf{env}_{\Phi}^{\mathsf{Bim}}$.

Proof. By property 2° on p. 44, (ii) and (iii) follow from (i). To prove (i), let $\sigma : X \to X'$ be an extension in Ω with respect to Φ. By Theorem 3.6(i), σ is a monomorphism.

Suppose that it is not an epimorphism. This means that there are parallel morphisms $\alpha \neq \beta : X' \to T$ such that
$$\alpha \circ \sigma = \beta \circ \sigma. \tag{3.17}$$
Since $\varPhi$ separates morphisms on the outside, there is $\varphi : T \to M$ in $\varPhi$ such that
$$\varphi \circ \alpha \neq \varphi \circ \beta.$$
In addition, by (3.17),
$$\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma.$$
If we now suppose that $\varPhi$ is a right ideal in $\mathtt{K}$, then $\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma$ is in $\varPhi$. So we can interpret this picture as follows: the test (i.e. belonging to $\varPhi$) morphism $\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma : X \to M$ has two different continuations $\varphi \circ \alpha \neq \varphi \circ \beta : X' \to M$ along $\sigma : X \to X'$. This means that σ cannot be an extension with respect to $\varPhi$. ∎

3.1.2. Envelope in a class of objects with respect to a class of objects. A special case of the construction is when $\varOmega$ and/or $\varPhi$ are classes of all morphisms into the objects from some given subclasses of $\mathsf{Ob}(\mathtt{K})$. A precise formulation for the case when both $\varOmega$ and $\varPhi$ are defined in that way is the following. Suppose we have a category $\mathtt{K}$ and two subclasses $\mathtt{L}$ and $\mathtt{M}$ in $\mathsf{Ob}(\mathtt{K})$.

- A morphism $\sigma : X \to X'$ is called an *extension of the object $X \in \mathtt{K}$ in the class $\mathtt{L}$ with respect to the class $\mathtt{M}$* if $X' \in \mathtt{L}$ and for any object $B \in \mathtt{M}$ and any morphism $\varphi : X \to B$ there exists a unique morphism $\varphi' : X' \to B$ such that
$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\forall\varphi"] & \\ X' \arrow[rr, dashed, "\exists!\varphi'"] \arrow[d, phantom, "\Cap"] & & B \arrow[d, phantom, "\Cap"] \\ \mathtt{L} & & \mathtt{M} \end{tikzcd}$$

- An extension $\rho : X \to E$ of $X \in \mathtt{K}$ in $\mathtt{L}$ with respect to $\mathtt{M}$ is called an *envelope of the object $X \in \mathtt{K}$ in the class $\mathtt{L}$ with respect to the class $\mathtt{M}$*, in symbols
$$\rho = \mathsf{env}^{\mathtt{L}}_{\mathtt{M}} X, \tag{3.18}$$
if for any other extension $\sigma : X \to X'$ (in $\mathtt{L}$ with respect to $\mathtt{M}$) there exists a unique morphism $\upsilon : X' \to E$ such that
$$\begin{tikzcd} & X \arrow[dl, "\forall\sigma"'] \arrow[dr, "\rho"] & \\ X' \arrow[rr, dashed, "\exists!\upsilon"] \arrow[d, phantom, "\Cap"] & & E \arrow[d, phantom, "\Cap"] \\ \mathtt{L} & & \mathtt{L} \end{tikzcd} \tag{3.19}$$
The object E is also called an *envelope* of X (in the class $\mathtt{L}$ with respect to the class $\mathtt{M}$), and we will write
$$E = \mathsf{Env}^{\mathtt{L}}_{\mathtt{M}} X. \tag{3.20}$$

The following two extreme situations in the choice of $\mathtt{L}$ can occur:

– If $\mathtt{L} = \mathsf{Ob}(\mathtt{K})$, then we will speak about an *envelope of an object* $X \in \mathtt{K}$ *in the category* $\mathtt{K}$ *with respect to the class* $\mathtt{M}$ *of objects*, and the notation will be

$$\mathsf{env}_{\mathtt{M}} X := \mathsf{env}_{\mathtt{M}}^{\mathtt{K}} X, \qquad \mathsf{Env}_{\mathtt{M}} X := \mathsf{Env}_{\mathtt{M}}^{\mathtt{K}} X. \tag{3.21}$$

– If $\mathtt{L} = \mathtt{M}$, then the notions of extension and envelope coincide: *each extension of* X *in* $\mathtt{L}$ *with respect to* $\mathtt{L}$ *is an envelope of* X *in* $\mathtt{L}$ *with respect to* $\mathtt{L}$ (indeed, if $\rho : X \to E$ and $\sigma : X \to X'$ are two extensions in $\mathtt{L}$ with respect to $\mathtt{L}$, then in diagram (3.19) the morphism υ exists and is unique just because σ is an extension); for simplicity, in the case of $\mathtt{L} = \mathtt{M}$ we speak about the *envelope of* X *in* $\mathtt{L}$, and our notation simplifies

$$\mathsf{env}^{\mathtt{L}} X := \mathsf{env}_{\mathtt{L}}^{\mathtt{L}} X, \qquad \mathsf{Env}^{\mathtt{L}} X := \mathsf{Env}_{\mathtt{L}}^{\mathtt{L}} X. \tag{3.22}$$

• Let us say that a class $\mathtt{M}$ of objects in a category $\mathtt{K}$ *separates morphisms on the outside*, if the class of morphisms with ranges in $\mathtt{M}$ has this property, i.e. for any morphisms $\alpha \neq \beta : X \to Y$ there is a morphism $\varphi : Y \to M$ in $\mathtt{M}$ such that $\varphi \circ \alpha \neq \varphi \circ \beta$.

From Theorem 3.7 we have

THEOREM 3.8. *If a class* $\mathtt{M}$ *of objects separates morphisms on the outside, then for any class* $\mathtt{L}$ *of objects:*

(i) *each envelope in* $\mathtt{L}$ *with respect to* $\mathtt{M}$ *is a bimorphism;*
(ii) *an envelope in* $\mathtt{L}$ *with respect to* $\mathtt{M}$ *exists if and only if there exists an anvelope in the class of bimorphisms with values in* $\mathtt{L}$ *with respect to* $\mathtt{M}$*; in this case* $\mathsf{env}_{\mathtt{M}}^{\mathtt{L}} = \mathsf{env}_{\mathtt{M}}^{\mathsf{Bim}(\mathtt{K},\mathtt{L})}$.

3.1.3. Examples of envelopes

EXAMPLE 3.9 (Universal enveloping algebra). Let $\mathtt{K} = \mathtt{LieAlg}$ be the category of Lie algebras (say, over $\mathbb{C}$), $\mathtt{T} = \mathtt{Alg}$ the category of associative algebras (again over $\mathbb{C}$) with identity, and $F : \mathtt{Alg} \to \mathtt{LieAlg}$ the functor that represents every associative algebra A as the Lie algebra with Lie bracket

$$[x, y] = x \cdot y - y \cdot x.$$

Then the envelope of a Lie algebra $\mathfrak{g}$ over $\mathtt{Alg}$ in $\mathsf{Mor}(\mathtt{LieAlg}, F(\mathtt{Alg}))$ with respect to $\mathsf{Mor}(\mathtt{LieAlg}, F(\mathtt{Alg}))$ is exactly the universal enveloping algebra $U(\mathfrak{g})$ (cf. [9]): $U(\mathfrak{g}) = \mathsf{Env}^{\mathsf{Mor}(\mathtt{LieAlg}, F(\mathtt{Alg}))} \mathfrak{g}$.

EXAMPLE 3.10 (Stone–Čech compactification). In the category $\mathtt{Tikh}$ of Tikhonov spaces the Stone–Čech compactification $\beta : X \to \beta X$ is an envelope of the space X in the class $\mathtt{Com}$ of compact spaces with respect to the same class $\mathtt{Com}$: $\beta X = \mathsf{Env}^{\mathtt{Com}} X$.

Proof. Here one uses [13, Theorem 3.6.1], which states that any continuous map $f : X \to K$ into an arbitrary compact space K can be extended to a continuous map $F : \beta X \to K$. Since $\beta(X)$ is dense in βX, this extension F is unique, and therefore $\beta : X \to \beta X$ is an extension in $\mathtt{Com}$ with respect to $\mathtt{Com}$. By the remark containing (3.22), in the case $\mathtt{L} = \mathtt{M}$ each extension is an envelope, so β is an envelope. ∎

EXAMPLE 3.11. The *completion* $X^{\blacktriangledown}$ of a locally convex space X is an envelope of X in the category $\mathtt{LCS}$ of all locally convex spaces with respect to the class $\mathtt{Ban}$ of Banach spaces: $X^{\blacktriangledown} = \mathsf{Env}_{\mathtt{Ban}}^{\mathtt{LCS}} X$.

Proof. Let us denote the natural embedding of X into its completion by $\blacktriangledown_X : X \to X^{\blacktriangledown}$ (we use the notation of [2]).

First, each continuous linear map $f : X \to B$ into an arbitrary Banach space B uniquely extends to a continuous linear map $F : X^{\blacktriangledown} \to B$ (here one can refer, for instance, to the general theorem for all uniform spaces [13, Theorem 8.3.10]). Hence, the completion $\blacktriangledown_X : X \to X^{\blacktriangledown}$ is an extension of X in $\mathtt{LCS}$ with respect to the subclass $\mathtt{Ban}$.

Note that $\mathtt{Ban}$ separates morphisms on the outside in $\mathtt{LCS}$. By Theorem 3.8 this means that any extension $\sigma : X \to X'$ with respect to $\mathtt{Ban}$ is a bimorphism in $\mathtt{LCS}$, i.e. σ is injective and $\sigma(X)$ is dense in X'. Let us show that in addition σ is an open map: for any zero neighborhood $U \subseteq X$ there is a zero neighborhood $V \subseteq X'$ such that

$$\sigma(U) \supseteq V \cap \sigma(X). \tag{3.23}$$

We can assume that U is closed and convex. Then $\mathsf{Ker}\, U = \bigcap_{\varepsilon>0} \varepsilon \cdot U$ is a closed subspace in X. Consider the quotient space $X/\mathsf{Ker}\, U$ and endow it with the topology of normed space with unit ball $U + \mathsf{Ker}\, U$. Then $(X/\mathsf{Ker}\, U)^{\blacktriangledown}$ will be a Banach space, and we will denote it by A/U. The natural map (the composition of the quotient map $X \to X/\mathsf{Ker}\, U$ and the completion $X/\mathsf{Ker}\, U \to (X/\mathsf{Ker}\, U)^{\blacktriangledown}$) will be denoted by $\pi_U : X \to X/U$. Since $\sigma : X \to X'$ is an extension with respect to $\mathtt{Ban}$, the map $\pi_U : X \to X/U$ extends to some continuous linear map $(\pi_U)' : X' \to X/U$:

$$\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "\pi_U"'] & & X' \arrow[dl, dashed, "(\pi_U)'"] \\ & X/U & \end{tikzcd}$$

If we denote by W the unit ball in X/U, i.e. the closure of $U + \mathsf{Ker}\, U$ in $(X/\mathsf{Ker}\, U)^{\blacktriangledown} = X/U$, then for the zero neighborhood $V = ((\pi_U)')^{-1}(W)$ we obtain the following chain, which proves (3.23):

$$\begin{aligned} y \in V \cap \sigma(X) \ &\Rightarrow\ \exists x \in X \ \ y = \sigma(x) \ \&\ y \in V \\ &\Rightarrow\ \exists x \in X \ \ y = \sigma(x) \ \&\ (\pi_U)'(y) = (\pi_U)'(\sigma(x)) = \underbrace{\pi_U(x) \in W}_{\substack{\Updownarrow \\ x \in U}} \\ &\Rightarrow\ \exists x \in U \ \ y = \sigma(x) \ \Rightarrow\ y \in \sigma(U). \end{aligned}$$

Thus, $\sigma : X \to X'$ is an open and injective continuous linear map, and $\sigma(X)$ is dense in X'. This means that X' can be perceived as a subspace in the completion $X^{\blacktriangledown}$ with the induced topology. That is, there is a unique continuous linear map $\upsilon : X' \to X^{\blacktriangledown}$ such that

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\blacktriangledown_X"] & \\ X' \arrow[rr, dashed, "\upsilon"] & & X^{\blacktriangledown} \end{tikzcd}$$

We conclude that $\blacktriangledown_X : X \to X^{\blacktriangledown}$ is an envelope of X in $\mathtt{LCS}$ with respect to $\mathtt{Ban}$. ■

3.2. Refinement

3.2.1. Refinement in a class of morphisms by means of a class of morphisms. Suppose we have:

- a category $\mathtt{K}$, called the *enveloping category*,
- a category $\mathtt{T}$, called the *repelling category*,
- a covariant functor $F : \mathtt{T} \to \mathtt{K}$,
- two classes $\varGamma$ and $\varPhi$ of morphisms in $\mathtt{K}$ whose domains are objects of $F(\mathtt{T})$; $\varGamma$ is called a *class of realizing morphisms*, and $\varPhi$ a *class of test morphisms*.

Then:

- For $X \in \mathsf{Ob}(\mathtt{K})$ and $X' \in \mathsf{Ob}(\mathtt{T})$ a morphism $\sigma : F(X') \to X$ is called an *enrichment of the object $X \in \mathtt{K}$ in the class $\varGamma$ over the category $\mathtt{T}$ by means of the class $\varPhi$* if $\sigma \in \varGamma$ and for any object B in $\mathtt{T}$ and any morphism $\varphi : F(B) \to X$ in $\varPhi$ there is a unique morphism $\varphi' : B \to X'$ in $\mathtt{T}$ such that

$$\begin{tikzcd} & X & \\ F(B) \arrow[ur, "\varPhi\ni\varphi"] \arrow[rr, dashed, "F(\varphi')"] & & F(X') \arrow[ul, "\sigma\in\varGamma"'] \end{tikzcd} \tag{3.24}$$

- An enrichment $\rho : F(E) \to X$ of $X \in \mathtt{K}$ in $\varGamma$ over $\mathtt{T}$ by means of $\varPhi$ is called a *refinement of the object $X \in \mathtt{K}$ in the class $\varGamma$ over the category $\mathtt{T}$ by means of the class $\varPhi$* if for any other enrichment $\sigma : F(X') \to X$ (of $X \in \mathtt{K}$ in $\varGamma$ over $\mathtt{T}$ by means of $\varPhi$) there is a unique morphism $\upsilon : E \to X'$ in $\mathtt{T}$ such that

$$\begin{tikzcd} & X & \\ F(E) \arrow[ur, "\rho"] \arrow[rr, dashed, "F(\upsilon)"] & & F(X') \arrow[ul, "\sigma"'] \end{tikzcd} \tag{3.25}$$

In what follows, we are almost exclusively interested in the case when $\mathtt{T} = \mathtt{K}$ and $F : \mathtt{K} \to \mathtt{K}$ is the identity functor. As in the case of envelopes, we formulate the definitions for this situation separately.

- A morphism $\sigma : X' \to X$ in $\mathtt{K}$ is called an *enrichment of the object $X \in \mathsf{Ob}(\mathtt{K})$ in the class $\varGamma$ by means of the class $\varPhi$* if $\sigma \in \varGamma$ and for any morphism $\varphi : B \to X$ in $\varPhi$ there exists a unique morphism $\varphi' : B \to X'$ in $\mathtt{K}$ such that

$$\begin{tikzcd} & X & \\ B \arrow[ur, "\forall\varphi\in\varPhi"] \arrow[rr, dashed, "\exists!\varphi'"] & & X' \arrow[ul, "\sigma\in\varGamma"'] \end{tikzcd} \tag{3.26}$$

- An enrichment $\rho : E \to X$ of $X \in \mathsf{Ob}(\mathtt{K})$ in $\varGamma$ by means of $\varPhi$ is called a *refinement of X in the class $\varGamma$ by means of $\varPhi$* if for any other enrichment $\sigma : X' \to X$ (of X in $\varGamma$ by means of $\varPhi$) there exists a unique morphism $\upsilon : E \to X'$ in $\mathtt{K}$ such that

$$\begin{tikzcd} & X & \\ E \arrow[ur, "\rho"] \arrow[rr, dashed, "\exists!\upsilon"] & & X' \arrow[ul, "\forall\sigma"'] \end{tikzcd} \tag{3.27}$$

For a refinement $\rho : E \to X$ we use the notation

$$\rho = \mathsf{ref}^{\varGamma}_{\varPhi} X. \tag{3.28}$$

The very object E is also called a *refinement* of X in $\varGamma$ by means of $\varPhi$, and is denoted by

$$E = \mathsf{Ref}^{\varGamma}_{\varPhi} X. \tag{3.29}$$

Remark 3.12. As in the case of envelopes, the refinement $\mathsf{Ref}^{\varGamma}_{\varPhi} X$ (if any) is defined up to isomorphism. The question when the correspondence $X \mapsto \mathsf{Ref}^{\varGamma}_{\varPhi} X$ can be defined as a functor is discussed below starting from p. 70.

Remark 3.13. If $\varGamma = \emptyset$, then, of course, neither enrichments nor refinements of the objects of $\mathtt{K}$ exist in $\varGamma$. So this construction is interesting only if $\varGamma$ is a non-empty class. The following two situations will be of special interest:

– $\varGamma = \mathsf{Mono}(\mathtt{K})$ (the class of all monomorphisms of $\mathtt{K}$); then we will use the notation

$$\mathsf{ref}^{\mathsf{Mono}}_{\varPhi} X := \mathsf{ref}^{\mathsf{Mono}(\mathtt{K})}_{\varPhi} X, \qquad \mathsf{Ref}^{\mathsf{Mono}}_{\varPhi} X := \mathsf{Ref}^{\mathsf{Mono}(\mathtt{K})}_{\varPhi} X. \tag{3.30}$$

– $\varGamma = \mathsf{Mor}(\mathtt{K})$ (the class of all morphisms of $\mathtt{K}$); in this case it is convenient to omit $\varGamma$ from the formulations and notation, so we will be speaking about *refinements of* $X \in \mathtt{K}$ *in* $\mathtt{K}$ *by means of* $\varPhi$, and the notation will be simplified to

$$\mathsf{ref}_{\varPhi} X := \mathsf{ref}^{\mathsf{Mor}(\mathtt{K})}_{\varPhi} X, \qquad \mathsf{Ref}_{\varPhi} X := \mathsf{Ref}^{\mathsf{Mor}(\mathtt{K})}_{\varPhi} X. \tag{3.31}$$

Remark 3.14. Another degenerate, but this time informative case is when $\varPhi = \emptyset$. It is essential that for a given object X, $\varPhi$ does not contain morphisms coming to X:

$$\varPhi_X = \{\varphi \in \varPhi : \mathsf{Ran}\,\varphi = X\} = \emptyset.$$

Then, obviously, any morphism $\sigma \in \varGamma$ coming to X, $\sigma : X \leftarrow X'$, is an enrichment of X (in $\varGamma$ by means of the class of morphisms $\emptyset$). If in addition $\varGamma = \mathsf{Mono}$, then the refinement will be the initial object of the category Mono_X (if it exists):

$$\mathsf{Ref}^{\varGamma}_{\emptyset} X = \min \mathsf{Mono}_X.$$

On the other hand, if $\mathtt{K}$ is a category with 0, and $\varGamma$ contains all morphisms going from 0, then the refinement in $\varGamma$ of each object by means of the empty class of morphisms is 0: $\mathsf{ref}^{\varGamma}_{\emptyset} X = 0$.

Remark 3.15. Another extreme situation is when $\varPhi = \mathsf{Mor}(\mathtt{K})$. For a given object X the essential thing here is that $\varPhi$ contains the local identity of X. Then for any enrichment σ the diagram

$$\begin{tikzcd} X & & X' \arrow[ll, "\sigma"'] \\ & B \arrow[ul, "1_X"] \arrow[ur, dashed] & \end{tikzcd}$$

implies that σ is a coretraction (moreover, the dashed arrow is unique). In the special case of $\Gamma \subseteq \mathsf{Mono}$ this is possible only if σ is an isomorphism. As a corollary, $\mathsf{ref}^{\Gamma}_{\mathsf{Mor}(\mathtt{K})} X$ coincides here with X (up to isomorphism).

PROPERTIES OF REFINEMENTS.

1° *Suppose $\Sigma \subseteq \Gamma$. Then for any object X and any class Φ of morphisms:*

(a) *every enrichment $\sigma : X \leftarrow X'$ in Σ by means of Φ is an enrichment in Γ by means of Φ;*

(b) *if there are refinements $\mathsf{ref}^{\Sigma}_{\Phi} X$ and $\mathsf{ref}^{\Gamma}_{\Phi} X$, then there is a unique morphism $\rho : \mathsf{Ref}^{\Sigma}_{\Phi} X \leftarrow \mathsf{Ref}^{\Gamma}_{\Phi} X$ such that*

$$\begin{tikzcd} & X & \\ \mathsf{Ref}^{\Sigma}_{\Phi} X \arrow[ur, "\mathsf{ref}^{\Sigma}_{\Phi} X"] & & \mathsf{Ref}^{\Gamma}_{\Phi} X \arrow[ul, "\mathsf{ref}^{\Gamma}_{\Phi} X"'] \arrow[ll, dashed, "\rho"'] \end{tikzcd} \tag{3.32}$$

(c) *if there is $\mathsf{ref}^{\Gamma}_{\Phi} X \in \Sigma$, then $\mathsf{ref}^{\Gamma}_{\Phi} X = \mathsf{ref}^{\Sigma}_{\Phi} X$.*

2° *Let Σ, Γ, Φ be classes of morphisms, and suppose for an object X that*

(a) *every enrichment $\sigma : X \leftarrow X'$ in Γ by means of Φ belongs to Σ.*

Then:

(b) *a refinement of X in Γ by means of Φ exists if and only if there exists a refinement of X in $\Gamma \cap \Sigma$ by means of Φ; in this case $\mathsf{ref}^{\Gamma}_{\Phi} = \mathsf{ref}^{\Gamma\cap\Sigma}_{\Phi}$;*

(c) *if $\Sigma \subseteq \Gamma$, then the existence of a refinement of X in Σ by means of Φ automatically implies the existence of a refinement of X in Γ by means of Φ and their coincidence.*

3° *Suppose $\Psi \subseteq \Phi$. Then for any object X and any class of Γ morphisms:*

(a) *every enrichment $\sigma : X \leftarrow X'$ of X in Γ by means of Φ is an enrichment of X in Γ by means of Ψ;*

(b) *if there are refinements $\mathsf{ref}^{\Gamma}_{\Psi} X$ and $\mathsf{ref}^{\Gamma}_{\Phi} X$, then there is a unique morphism $\alpha : \mathsf{ref}^{\Gamma}_{\Psi} X \to \mathsf{ref}^{\Gamma}_{\Phi} X$ such that*

$$\begin{tikzcd} & X & \\ \mathsf{Ref}^{\Gamma}_{\Psi} X \arrow[ur, "\mathsf{ref}^{\Gamma}_{\Psi} X"] \arrow[rr, dashed, "\alpha"] & & \mathsf{Ref}^{\Gamma}_{\Phi} X \arrow[ul, "\mathsf{ref}^{\Gamma}_{\Phi} X"'] \end{tikzcd} \tag{3.33}$$

4° *Suppose $\Phi \subseteq \Psi \circ \mathsf{Mor}(\mathtt{K})$. Then for any object X and for any class Γ of morphisms*:

(a) *if an enrichment $\sigma : X \leftarrow X'$ in Γ by means of Ψ is at the same time a monomorphism in $\mathtt{K}$, then it is an enrichment in Γ by means of Φ;*

(b) *if there are refinements $\mathsf{ref}^{\Gamma}_{\Psi} X$ and $\mathsf{ref}^{\Gamma}_{\Phi} X$, and $\mathsf{ref}^{\Gamma}_{\Psi} X$ is at the same time a monomorphism in $\mathtt{K}$, then there is a unique morphism $\beta : \mathsf{ref}^{\Gamma}_{\Psi} X \leftarrow \mathsf{ref}^{\Gamma}_{\Phi} X$ such that*

$$\begin{tikzcd} & X & \\ \mathsf{Ref}_{\Psi}^{\Gamma} X \arrow[ur, "\mathsf{ref}_{\Psi}^{\Gamma} X"] & & \mathsf{Ref}_{\Phi}^{\Gamma} X \arrow[ul, "\mathsf{ref}_{\Phi}^{\Gamma} X"'] \arrow[ll, dashed, "\beta"'] \end{tikzcd} \tag{3.34}$$

5° *Let Γ, Φ classes of morphisms and a monomorphism $\mu : X \leftarrow Y$ in $\mathtt{K}$ satisfy the following conditions:*

(a) *there is a refinement $\mathsf{Ref}_{\mu\circ\Phi}^{\Gamma} X$ (by means of $\mu \circ \Phi = \{\mu \circ \varphi;\ \varphi \in \Phi\}$);*
(b) *there is a refinement $\mathsf{Ref}_{\Phi}^{\Gamma} Y$;*
(c) *$\mu \circ \mathsf{ref}_{\Phi}^{\Gamma} Y \in \Gamma$.*

Then there is a unique morphism $\upsilon : \mathsf{Ref}_{\mu\circ\Phi}^{\Gamma} X \to \mathsf{Ref}_{\Phi}^{\Gamma} Y$ such that

$$\begin{tikzcd} X & & Y \arrow[ll, "\mu"'] \\ \mathsf{Ref}_{\mu\circ\Phi}^{\Gamma} X \arrow[u, "\mathsf{ref}_{\mu\circ\Phi}^{\Gamma} X"] \arrow[rr, dashed, "\upsilon"] & & \mathsf{Ref}_{\Phi}^{\Gamma} Y \arrow[u, "\mathsf{ref}_{\Phi}^{\Gamma} Y"'] \arrow[ull, "\mu \circ \mathsf{ref}_{\Phi}^{\Gamma} Y"'] \end{tikzcd} \tag{3.35}$$

- Let us say that in a category $\mathtt{K}$ *a class Φ of morphisms is generated on the outside by a class Ψ of morphisms* if

$$\Psi \subseteq \Phi \subseteq \Psi \circ \mathsf{Mor}(\mathtt{K}).$$

The following fact is dual to Theorem 3.5 and is proved by analogy:

THEOREM 3.16. *Suppose in a category $\mathtt{K}$ a class Φ of morphisms is generated on the outside by a class Ψ of morphisms. Then for any class Γ of monomorphisms (not necessarily all) and any object X the existence of $\mathsf{ref}_{\Psi}^{\Gamma} X$ is equivalent to the existence of $\mathsf{ref}_{\Phi}^{\Gamma} X$, and*

$$\mathsf{ref}_{\Psi}^{\Gamma} X = \mathsf{ref}_{\Phi}^{\Gamma} X. \tag{3.36}$$

- Let us say that a class Φ of morphisms in a category $\mathtt{K}$ *separates morphisms on the inside* if for any morphisms $\alpha \neq \beta : X \to Y$ there is a morphism $\varphi : M \to X$ in Φ such that $\alpha \circ \varphi \neq \beta \circ \varphi$.

The following result is dual to Theorem 3.6:

THEOREM 3.17. *If a class Φ of morphisms separates morphisms on the inside, then for any class Γ of morphisms:*

(i) *every enrichment in Γ by means of Φ is an epimorphism;*
(ii) *a refinement in Γ by means of Φ exists if and only if there exists a refinement in $\Gamma \cap \mathsf{Mono}$ by means of Φ; in that case $\mathsf{ref}_{\Phi}^{\Gamma} = \mathsf{ref}_{\Phi}^{\Gamma\cap\mathsf{Epi}}$;*
(iii) *if $\Gamma \supseteq \mathsf{Epi}$, then the existence of a refinement in Epi by means of Φ automatically implies the existence of a refinement in Γ by means of Φ, and their coincidence.*

- Let us recall that a class Φ of morphisms in a category $\mathtt{K}$ is called a *left ideal* if

$$\mathsf{Mor}(\mathtt{K}) \circ \Phi \subseteq \Phi.$$

The following is dual to Theorem 3.7:

THEOREM 3.18. *If a class Φ of morphisms separates morphisms on the inside and is a left ideal in a category* $\mathtt{K}$, *then for any class Γ of morphisms:*

(i) *every enrichment in Γ by means of Φ is a bimorphism;*
(ii) *a refinement in Γ by means of Φ exists if and only if there exists a refinement in $\Gamma \cap \mathsf{Bim}$ by means of Φ; in that case $\mathsf{ref}_{\Phi}^{\Gamma} = \mathsf{ref}_{\Phi}^{\Gamma\cap\mathsf{Bim}}$;*
(iii) *if Γ contains all bimorphisms, then a refinement in Γ by means of Φ exists if and only if there exists a refinement in Bim by means of Φ, and $\mathsf{ref}_{\Phi}^{\Gamma} = \mathsf{ref}_{\Phi}^{\mathsf{Bim}}$.*

3.2.2. Refinement in a class of objects by means of a class of objects. A special case is when Γ and/or Φ are classes of all morphisms from a given subclass of $\mathsf{Ob}(\mathtt{K})$. An exact formulation for the case when both Γ and Φ are defined in this way is the following. Suppose we have a category $\mathtt{K}$ and two subclasses $\mathtt{L}$ and $\mathtt{M}$ of $\mathsf{Ob}(\mathtt{K})$.

- A morphism $\sigma : X' \to X$ is called an *enrichment of the object $X \in \mathtt{K}$ in the class $\mathtt{L}$ by means of the class $\mathtt{M}$* if for any $B \in \mathtt{M}$ and any morphism $\varphi : B \to X$ there is a unique morphism $\varphi' : B \to X'$ such that

$$\begin{tikzcd} & X & \\ B \arrow[ur, "\forall\varphi"] \arrow[rr, dashed, "\exists!\varphi'"] & & X' \arrow[ul, "\sigma"'] \end{tikzcd} \qquad B \in \mathtt{M},\ X' \in \mathtt{L}$$

- An enrichment $\rho : E \to X$ of $X \in \mathtt{K}$ in $\mathtt{L}$ by means of $\mathtt{M}$ is called a *refinement of the object $X \in \mathtt{K}$ in the class $\mathtt{L}$ by means of the class* $\mathtt{M}$, in symbols

$$\rho = \mathsf{ref}_{\mathtt{M}}^{\mathtt{L}} X, \tag{3.37}$$

if for any other enrichment $\sigma : X' \to X$ (of $X \in \mathtt{K}$ in $\mathtt{L}$ by means of $\mathtt{M}$) there is a unique morphism $\upsilon : E \to X'$ such that

$$\begin{tikzcd} & X & \\ E \arrow[ur, "\rho"] \arrow[rr, dashed, "\exists!\upsilon"] & & X' \arrow[ul, "\forall\sigma"'] \end{tikzcd} \qquad E \in \mathtt{L},\ X' \in \mathtt{L} \tag{3.38}$$

The very object E is also called a *refinement* of $X \in \mathtt{K}$ in $\mathtt{L}$ by means of $\mathtt{M}$, and we write

$$E = \mathsf{Ref}_{\mathtt{M}}^{\mathtt{L}} X. \tag{3.39}$$

The following two extreme situations in the choice of $\mathtt{L}$ can occur:

– If $\mathtt{L} = \mathsf{Ob}(\mathtt{K})$, then we speak about a *refinement of the object $X \in \mathtt{K}$ in the category* $\mathtt{K}$ *by means of the class* $\mathtt{M}$, and the notation will be

$$\mathsf{ref}_{\mathtt{M}} X := \mathsf{ref}_{\mathtt{M}}^{\mathtt{K}} X, \qquad \mathsf{Ref}_{\mathtt{M}} X := \mathsf{Ref}_{\mathtt{M}}^{\mathtt{K}} X. \tag{3.40}$$

– If $\mathtt{L} = \mathtt{M}$, then the notions of enrichment and of refinement coincide: *every enrichment of $X \in \mathtt{K}$ in $\mathtt{L}$ by means of $\mathtt{L}$ is a refinement of X in $\mathtt{L}$ by means of $\mathtt{L}$* (since if $\rho : E \to X$

and $\sigma : X' \to X$ are two enrichments of X in $\mathtt{L}$ by means of $\mathtt{L}$, then in diagram (3.38) the morphism υ exists and is unique just because σ is an enrichment); for simplicity, in this case we will be speaking about a *refinement of* X *in* $\mathtt{L}$, and the notation will be

$$\mathsf{ref}^{\mathtt{L}}_{\mathtt{L}} X =: \mathsf{ref}^{\mathtt{L}} X, \qquad \mathsf{Ref}^{\mathtt{L}}_{\mathtt{L}} X =: \mathsf{Ref}^{\mathtt{L}} X. \tag{3.41}$$

- Let us say that a class $\mathtt{M}$ of objects in the category $\mathtt{K}$ *separates morphisms on the inside* if the class of all morphisms going from objects of $\mathtt{M}$ has this property, i.e. for any morphisms $\alpha \neq \beta : X \to Y$ there is a morphism $\varphi : M \to X$ such that $\alpha \circ \varphi \neq \beta \circ \varphi$.

Theorem 3.18 implies

THEOREM 3.19. *If a class* $\mathtt{M}$ *of objects separates morphisms on the inside, then for any class* $\mathtt{L}$ *of objects:*

(i) *every enrichment in* $\mathtt{L}$ *by means of* $\mathtt{M}$ *is a bimorphism,*
(ii) *a refinement in* $\mathtt{L}$ *by means of* $\mathtt{M}$ *exists if and only if there exists a refinement in the class of bimorphisms going from* $\mathtt{L}$ *by means of* $\mathtt{M}$*; in that case* $\mathsf{ref}^{\mathtt{L}}_{\mathtt{M}} = \mathsf{ref}^{\mathsf{Bim}(\mathtt{L},\mathtt{K})}_{\mathtt{M}}$.

3.2.3. Examples of refinements

EXAMPLE 3.20. A *simply connected covering* used in the theory of Lie groups is from the categorical point of view a refinement in the class of pointed simply connected coverings by means of the empty class of morphisms in the category of connected locally connected and semilocally simply connected topological spaces (see definitions in [33]).

EXAMPLE 3.21. The *bornologification* (see definition in [22]) X_{born} of a locally convex space X is a refinement of X in the category $\mathtt{LCS}$ of locally convex spaces by means of the subcategory $\mathtt{Norm}$ of normed spaces: $X_{\mathrm{born}} = \mathsf{Ref}^{\mathtt{LCS}}_{\mathtt{Norm}} X$.

Proof. This follows from the characterization of bornologification as the strongest locally convex topology on X for which all the imbeddings $X_B \to X$ are continuous, where B runs over the system of bounded absolutely convex subsets in X, and X_B is a normed space with unit ball B (see [22, Chapter I, Lemma 4.2]). ■

EXAMPLE 3.22. The *saturation* $X^{\blacktriangle}$ of a pseudocomplete locally convex space X is a refinement of X in the category $\mathtt{LCS}$ of locally convex spaces by means of the subcategory $\mathtt{Smi}$ of Smith spaces (see definitions in [2]): $X^{\blacktriangle} = \mathsf{Ref}^{\mathtt{LCS}}_{\mathtt{Smi}} X$.

3.3. Connection with factorizations and with nodal decomposition

3.3.1. Connection with projective and injective limits. The similarity between the notions of envelope and projective limit is formalized in the following

LEMMA 3.23. *The projective limit* $\rho = \varprojlim \rho^i : X \to \varprojlim X^i$ *of any projective cone* $\{\rho^i : X \to X^i;\ i \in I\}$ *from a given object* X *into a covariant* (*or contravariant*) *system* $\{X^i;\ \iota^j_i\}$ *is an envelope of* X *in an arbitrary class* Ω *containing* ρ *with respect to the system* $\{\rho^i;\ i \in I\}$:

$$\rho = \varprojlim \rho^i \in \Omega \ \Rightarrow\ \mathsf{Env}^{\Omega}_{\{\rho^i;\, i\in I\}} X = \varprojlim X^i. \tag{3.42}$$

In particular, this is always true for $\Omega = \mathsf{Mor}(\mathtt{K})$:

$$\mathsf{Env}^{\mathsf{Mor}(\mathtt{K})}_{\{\rho^i;\, i\in I\}} X = \varprojlim X^i. \tag{3.43}$$

Proof. First, ρ is an extension of X with respect to $\{\rho^i\}$, since the definition of projective limit guarantees that for any ρ^j there exists a unique continuation π^j on $\varprojlim X^i$:

$$\begin{tikzcd}
X \arrow[rr, "\rho"] \arrow[dr, "\rho^j"'] & & \varprojlim X^i \arrow[dl, dashed, "\pi^j"] \\
& X^j &
\end{tikzcd} \tag{3.44}$$

Suppose now that $\sigma : X \to X'$ is another extension. Then for any morphism $\rho^j : X \to X^j$ there is a unique morphism $\upsilon^j : X' \to X^j$ such that

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\rho^j"'] & & X' \arrow[dl, dashed, "\upsilon^j"] \\
& X^j &
\end{tikzcd} \tag{3.45}$$

For any indices $i \leq j$ in the diagram

$$\begin{tikzcd}
& X \arrow[d, "\sigma"] \arrow[ddl, bend right, "\rho^i"'] \arrow[ddr, bend left, "\rho^j"] & \\
& X' \arrow[dl, dashed, "\upsilon^i"] \arrow[dr, dashed, "\upsilon^j"'] & \\
X^i \arrow[rr, bend right, "\iota_i^j"] & & X^j
\end{tikzcd}$$

the following elements will be commutative: the two upper triangles (each has one dashed arrow) and the perimeter (without dashed arrows). This together with the uniqueness of υ^j in the upper right triangle implies that the lower triangle (with two dashed arrows) is commutative as well:

$$\begin{cases} (\iota_i^j \circ \upsilon^i) \circ \sigma = \iota_i^j \circ (\upsilon^i \circ \sigma) = \iota_i^j \circ \rho^i = \rho^j \\ \upsilon^j \circ \sigma = \rho^j \end{cases} \quad \Rightarrow \quad \iota_i^j \circ \upsilon^i = \upsilon^j.$$

The commutativity of the triangle with two dashed arrows means in turn that X' with the system of morphisms υ^i is a projective cone of the covariant system $\{X^i; \iota_i^j\}$. So there exists a unique morphism υ such that for any j in the diagram

$$\begin{tikzcd}
& X \arrow[d, "\sigma"] \arrow[ddl, bend right, "\rho"'] \arrow[ddr, bend left, "\rho^j"] & \\
& X' \arrow[dl, dashed, "\upsilon"] \arrow[dr, "\upsilon^j"'] & \\
\varprojlim X^i \arrow[rr, bend right, "\pi^j"] & & X^j
\end{tikzcd}$$

the lower triangle is commutative. On the other hand, the upper right triangle here is also commutative since this is diagram (3.45) turned around, and the perimeter is commutative since this is diagram (3.44) turned around. Together with the uniqueness of ρ in the system of all those perimeters with different j this implies that the upper left triangle is also commutative:

$$\left(\forall j \ \begin{cases} \pi^j \circ \upsilon \circ \sigma = \upsilon^j \circ \sigma = \rho^j \\ \pi^j \circ \rho = \rho^j \end{cases}\right) \ \Rightarrow \ \upsilon \circ \sigma = \rho.$$

We observe that there is a morphism υ such that diagram (3.4) is commutative (with $E = \varprojlim X^i$). It remains to verify that such a morphism is unique. Let υ' be another morphism with the same property: $\rho = \upsilon' \circ \sigma$. Consider the diagram

$$\begin{tikzcd}
& X \arrow[d, "\sigma"] \arrow[ddl, bend right, "\rho"'] \arrow[ddr, bend left, "\rho^j"] & \\
& X' \arrow[dl, dashed, "\upsilon'"] \arrow[dr, "\upsilon^j"'] & \\
\varprojlim X^i \arrow[rr, bend right, "\pi^j"] & & X^j
\end{tikzcd}$$

Here (besides the upper left triangle) the upper right triangle will be commutative (since this is diagram (3.45) turned around), and the perimeter as well (since this is diagram (3.44) turned around). Together with the uniqueness of the arrow υ^j in the upper right triangle, this implies that the lower triangle is also commutative:

$$\begin{cases} \pi^j \circ \upsilon' \circ \sigma = \pi^j \circ \rho = \rho^j \\ \upsilon^j \circ \sigma = \rho^j \end{cases} \ \Rightarrow \ \pi^j \circ \upsilon' = \upsilon^j.$$

This is true for each index j, so υ' must coincide with the morphism υ which we constructed before. ■

LEMMA 3.24. *Let $\varOmega$ be a monomorphically complemented class in a category K, $\{X^i;\ \iota_i^j\}$ a covariant (or contravariant) system, and $\{\rho^i : X \to X^i;\ i \in I\}$ a projective cone from a given object X into $\{X^i;\ \iota_i^j\}$. If $\rho = \varprojlim \rho^i : X \to \varprojlim X^i$ exists, then in its factorization*

$$\rho = \mu_\rho \circ \varepsilon_\rho, \qquad \mu_\rho \in {}^{\downarrow}\varOmega,\ \varepsilon_\rho \in \varOmega,$$

the epimorphism ε_ρ is an envelope of X with respect to the system $\{\rho^i;\ i \in I\}$ of morphisms in $\varOmega$:

$$\varepsilon_{\varprojlim \rho^i} = \varepsilon_\rho = \mathsf{env}^{\varOmega}_{\{\rho^i;\, i\in I\}} X, \qquad \mathsf{Ran}\,\varepsilon_{\varprojlim \rho^i} = \mathsf{Ran}\,\varepsilon_\rho = \mathsf{Env}^{\varOmega}_{\{\rho^i;\, i\in I\}} X. \tag{3.46}$$

Proof. By definition of projective limit every ρ^j has an extension π^j to $\varprojlim X^i$. The restriction of π^j to $\mathsf{Ran}\,\varepsilon_\rho$, i.e. the composition $\tau^j = \pi^j \circ \mu_\rho$, is an extension of ρ^j to $\mathsf{Ran}\,\varepsilon_\rho$ along ε_ρ:

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_\rho"'] \arrow[rr, bend left, "\rho"] \arrow[dr, bend right, "\rho^j"'] & \mathsf{Ran}\,\varepsilon_\rho \arrow[r, "\mu_\rho"'] \arrow[d, dashed, "\tau^j"] & \varprojlim X^i \arrow[dl, dashed, "\pi^j"] \\
& X^j &
\end{tikzcd} \tag{3.47}$$

Such an extension τ^j is unique since $\varepsilon_\rho \in \mathsf{Epi}$, and we can say that ε_ρ is an extension of X in $\varOmega$ with respect to the system $\{\rho^i\}$.

Now, let $\sigma : X \to X'$ be another extension of X in Ω with respect to $\{\rho^i\}$. As in the proof of Lemma 3.23, we find a morphism υ such that $\upsilon \circ \sigma = \rho$. We have

$$\upsilon \circ \sigma = \rho = \mu_\rho \circ \varepsilon_\rho,$$

and since $\sigma \in \Omega$, $\mu_\rho \in {}^{\downarrow}\Omega$, there exists a diagonal morphism δ such that

$$\delta \circ \sigma = \varepsilon_\rho.$$

This morphism is unique since $\sigma \in \Omega \subseteq \mathsf{Epi}$. ■

The dual results are as follows.

LEMMA 3.25. *The injective limit $\rho = \varinjlim \rho^i : X \leftarrow \varinjlim X^i$ of any injective cone $\{\rho^i : X \leftarrow X^i;\ i \in I\}$ into a given object X from a covariant (or contravariant) system $\{X^i;\ \iota_i^j\}$ is a refinement of X in an arbitrary class Γ of objects containing ρ by means of the system $\{\rho^i; i \in I\}$:*

$$\rho = \varinjlim \rho^i \in \Gamma \ \Rightarrow\ \mathsf{Ref}^{\Gamma}_{\{\rho^i;\, i\in I\}} X = \varinjlim X^i. \tag{3.48}$$

In particular, this is true for $\Gamma = \mathsf{Mor}(\mathtt{K})$:

$$\mathsf{Ref}^{\mathsf{Mor}(\mathtt{K})}_{\{\rho^i;\, i\in I\}} X = \varinjlim X^i. \tag{3.49}$$

LEMMA 3.26. *Let Γ be an epimorphically complementable class in a category $\mathtt{K}$, $\{X^i;\ \iota_i^j\}$ a covariant (or contravariant) system, and $\{\rho^i : X \leftarrow X^i;\ i \in I\}$ an injective cone from $\{X^i;\ \iota_i^j\}$ into a given object X. If $\rho = \varinjlim \rho^i : X \leftarrow \varinjlim X^i$ exists, then in its factorization*

$$\rho = \mu_\rho \circ \varepsilon_\rho, \qquad \mu_\rho \in \Gamma,\ \varepsilon_\rho \in \Gamma^{\downarrow},$$

the monomorphism μ_ρ is a refinement of X in Γ by means of the system $\{\rho^i; i \in I\}$:

$$\mathsf{ref}^{\Gamma}_{\{\rho^i;\, i\in I\}} X = \mu_\rho = \mu_{\varinjlim \rho^i}, \qquad \mathsf{Ref}^{\Gamma}_{\{\rho^i;\, i\in I\}} X = \mathsf{Dom}\, \mu_\rho = \mathsf{Dom}\, \mu_{\varinjlim \rho^i}. \tag{3.50}$$

3.3.2. Existence of envelopes and refinements for complementable classes

LEMMA 3.27. *Let Ω be a monomorphically complementable class in a category $\mathtt{K}$. Then for each object X and any class Φ of morphisms,*

$$\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Omega}_{\{\varepsilon_\varphi;\, \varphi\in\Phi\}} X \tag{3.51}$$

(*this means that if one of these envelopes exists then so does the other and they coincide*).

Proof. Let $\varphi = \mu_\varphi \circ \varepsilon_\varphi$ be a factorization with $\mu_\varphi \in {}^{\downarrow}\Omega$ and $\varepsilon_\varphi \in \Omega$. We need to verify that the extensions with respect to the classes Φ and $\{\varepsilon_\varphi;\ \varphi \in \Phi\}$ are the same. Let $\sigma : X \to X'$ be an extension of X in Ω with respect to $\{\varepsilon_\varphi;\ \varphi \in \Phi\}$. Then in the diagram

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon_\varphi"'] \arrow[ddr, bend right, "\varphi"'] & & X' \arrow[dl, dashed, "\varepsilon'"] \arrow[ddl, dashed, bend left, "\varphi'"] \\
& \mathsf{Ran}\,\varepsilon_\varphi \arrow[d, "\mu_\varphi"] & \\
& Y &
\end{tikzcd}$$

the existence of ε' for which the upper little triangle is commutative implies the existence of φ' for which the lower right triangle is commutative, and since the lower left triangle is commutative, we conclude that so is the perimeter. In addition, φ' is unique since σ is an epimorphism. Hence, $\sigma : X \to X'$ is an extension of X with respect to Φ.

Conversely, suppose that $\sigma : X \to X'$ is an extension of X with respect to Φ. Then for any $\varphi \in \Phi$ there exists a morphism φ' such that in the diagram

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon_\varphi"'] \arrow[ddr, bend right, "\varphi"'] & & X' \arrow[ddl, dashed, bend left, "\varphi'"] \\
& \mathsf{Ran}\,\varepsilon_\varphi \arrow[d, "\mu_\varphi"] & \\
& Y &
\end{tikzcd}$$

the perimeter is commutative. The lower left triangle is commutative as well due to (2.39), hence so is the quadrangle

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon_\varphi"'] & & X' \arrow[ddl, dashed, bend left, "\varphi'"] \\
& \mathsf{Ran}\,\varepsilon_\varphi \arrow[d, "\mu_\varphi"] & \\
& Y &
\end{tikzcd}$$

Here $\sigma \in \Omega$ and $\mu_\varphi \in \Omega^{\downarrow}$. Thus, there exists a diagonal ε':

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon_\varphi"'] & & X' \arrow[dl, dashed, "\varepsilon'"] \arrow[ddl, dashed, bend left, "\varphi'"] \\
& \mathsf{Ran}\,\varepsilon_\varphi \arrow[d, "\mu_\varphi"] & \\
& Y &
\end{tikzcd}$$

In particular, the upper triangle is commutative, and since this is true for any $\varphi \in \Phi$, $\sigma : X \to X'$ is an extension of X with respect to $\{\varepsilon_\varphi;\ \varphi \in \Phi\}$. ∎

Properties of envelopes in monomorphically complementable classes. *Let Ω be a monomorphically complementable class in a category $\mathtt{K}$.*

1° *For each morphism $\varphi : X \to Y$ in $\mathtt{K}$ the epimorphism ε_φ in the factorization $\varphi = \mu_\varphi \circ \varepsilon_\varphi$ (defined by the classes ${}^{\downarrow}\Omega$ and Ω) is an envelope of X in Ω with respect to φ:*

$$\mathsf{env}^{\Omega}_{\varphi} X = \varepsilon_\varphi, \qquad \mathsf{Env}^{\Omega}_{\varphi} X = \mathsf{Ran}\,\varepsilon_\varphi. \tag{3.52}$$

2° *If $\mathtt{K}$ is a category with finite products, then each object X in $\mathtt{K}$ has an envelope in Ω with respect to an arbitrary finite set Φ of morphisms going from X.*

3° *If $\mathtt{K}$ is a category with products* (1)*, then every object X in $\mathtt{K}$ has an envelope in Ω with respect to an arbitrary set Φ of morphisms going from X.*

4° *If $\mathtt{K}$ is a category with products, then every object X in $\mathtt{K}$ has an envelope in Ω with respect to an arbitrary class Φ of morphisms going from X and having a subset which generates Φ on the inside (see p. 47).*

5° *If $\mathtt{K}$ has products, and is co-well-powered in Ω, then every object X in $\mathtt{K}$ has an envelope in Ω with respect to an arbitrary class Φ of morphisms going from X.*

(1) In 3°–5° we assume that $\mathtt{K}$ has products over arbitrary index sets, not necessarily finite.

Proof. 1° The morphism ε_φ is an extension of X in Ω with respect to φ, as is seen from the diagram

$$\begin{tikzcd} X \arrow[rr, "\varepsilon_\varphi"] \arrow[dr, "\varphi"'] & & \mathsf{Ran}\,\varepsilon_\varphi \arrow[dl, dashed, "\mu_\varphi"] \\ & Y & \end{tikzcd} \tag{3.53}$$

Let $\sigma : X \to N$ be another extension of X in Ω with respect to φ:

$$\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & N \arrow[dl, dashed, "\exists!\nu"] \\ & Y & \end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\sigma"] \arrow[ddr, bend right, "\varepsilon_\varphi"'] & & Y \\ & N \arrow[ur, "\nu"] & \\ & \mathsf{Ran}\,\varepsilon_\varphi \arrow[uur, bend right, "\mu_\varphi"'] & \end{tikzcd}$$

Here $\sigma \in \Omega$ and $\mu_\varphi \in {}^{\downarrow}\Omega$, hence there exists a diagonal of the lower quadrangle:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\sigma"] \arrow[ddr, bend right, "\varepsilon_\varphi"'] & & Y \\ & N \arrow[ur, "\nu"] \arrow[d, dashed, "\upsilon"] & \\ & \mathsf{Ran}\,\varepsilon_\varphi \arrow[uur, bend right, "\mu_\varphi"'] & \end{tikzcd}$$

The morphism υ is the one in diagram (3.4) which connects the extension σ with the envelope ε_φ. Its uniqueness follows from the epimorphy of σ.

2° Let X be an object and Φ a finite set of morphisms. Clearly, it is sufficient to pick in Φ a subset $\Phi^X = \{\varphi : X \to Y_\varphi;\ \varphi \in \Phi^X\}$ of morphisms going from X. Then the envelope with respect to Φ is the same as the envelope with respect to Φ^X. Consider the product $\prod_{\varphi\in\Phi^X} Y_\varphi$ of objects and the product $\prod_{\varphi\in\Phi^X} \varphi : X \to \prod_{\varphi\in\Phi^X} Y_\varphi$ of morphisms. The envelope of X with respect to Φ^X is exactly the envelope of X with respect to one morphism, $\prod_{\varphi\in\Phi^X} \varphi$. We conclude by applying 1°.

3° Let $\mathtt{K}$ be a category with products over an arbitrary (not necessarily finite) index set. Then the above reasoning works in the case when Φ is a set (not necessarily finite) of morphisms.

4° Let $\Psi \subseteq \Phi$ be a subset (not a proper class) generating Φ on the inside. By 3°, every object X has an envelope with respect to Ψ. And by (3.14) this envelope coincides with the envelope with respect to Φ.

5° Let $\mathtt{K}$ be a category with products (over an arbitrary set of indices), A an object in $\mathtt{K}$, and Φ a class of morphisms (not necessarily a set). The idea of the proof is to replace the *class* Φ by a *set* M of morphisms such that the envelope will be the same. As in 2°, we can assume that Φ consists of morphisms going from X. Then for any $\varphi \in \Phi$ we consider the morphism ε_φ. By Lemma 3.27, we can replace Φ by the class $\{\varepsilon_\varphi;\ \varphi\in\Phi\}$:

$$\mathsf{env}^{\mathsf{Epi}}_{\Phi} X = \mathsf{env}^{\mathsf{Epi}}_{\{\varepsilon_\varphi;\,\varphi\in\Phi\}} X.$$

Next we recall that all ε_φ belong to Ω, and since our category is co-well-powered in the class Ω, we can choose among ε_φ a set M such that every ε_φ will be isomorphic to some $\varepsilon\in M$, i.e. $\varepsilon_\varphi=\iota\circ\varepsilon$ for some isomorphism ι. The set M now replaces the class $\{\varepsilon_\varphi;\ \varphi\in\Phi\}$ (and hence the class Φ), and so 3° works. ■

The dual results for refinements look as follows.

LEMMA 3.28. *Let Γ be an epimorphically complementable class in a category* $\mathtt{K}$. *Then for every object X and every class Φ of morphisms,*

$$\mathsf{ref}^{\Gamma}_{\Phi} X = \mathsf{ref}^{\Gamma}_{\{\mu_\varphi;\,\varphi\in\Phi\}} X \tag{3.54}$$

(*this means that if one of these refinements exists then so does the other and they coincide*).

PROPERTIES OF REFINEMENTS IN EPIMORPHICALLY COMPLEMENTABLE CLASSES. *Let Γ be an epimorphically complementable class of morphisms in a category* $\mathtt{K}$.

1° *For each morphism $\varphi : X \leftarrow Y$ in* $\mathtt{K}$ *the monomorphism μ_φ in the factorization $\varphi=\mu_\varphi\circ\varepsilon_\varphi$ (defined by the classes Γ and $\Gamma^{\downarrow}$) is a refinement of X in Γ by means of φ:*

$$\mathsf{ref}^{\Gamma}_{\varphi} X = \mu_\varphi, \qquad \mathsf{Ref}^{\Gamma}_{\varphi} X = \mathsf{Dom}\,\mu_\varphi. \tag{3.55}$$

2° *If* $\mathtt{K}$ *is a category with finite coproducts, then every object X in* $\mathtt{K}$ *has a refinement in Γ by means of an arbitrary finite set Φ of morphisms going to X.*

3° *If* $\mathtt{K}$ *is a category with coproducts* (2), *then every object X in* $\mathtt{K}$ *has a refinement in Γ by means of some set Φ of morphisms going to X.*

4° *If* $\mathtt{K}$ *is a category with coproducts, then every object X in* $\mathtt{K}$ *has a refinement in Γ by means of an arbitrary set Φ of morphisms going to X such that there is a set that generates Φ on the inside.*

5° *If* $\mathtt{K}$ *has coproducts and is well-powered in Γ, then every object X in* $\mathtt{K}$ *has a refinement in Γ by means of an arbitrary class Φ of morphisms going to X.*

3.3.3. Existence of envelopes and refinements in categories with nodal decomposition. The general properties on p. 61, when applied to $\Omega=\mathsf{Epi}$ and $\Omega=\mathsf{SEpi}$, give the following:

PROPERTIES OF ENVELOPES IN Epi AND IN SEpi IN A CATEGORY WITH NODAL DECOMPOSITION. *Let* $\mathtt{K}$ *be a category with nodal decomposition.*

1° *For each morphism $\varphi : X\to Y$ in* $\mathtt{K}$:

- *the epimorphism $\mathsf{red}_\infty\varphi\circ\mathsf{coim}_\infty\varphi$ in the nodal decomposition of φ is an envelope of X in the class* Epi *of all epimorphisms with respect to φ:*

$$\mathsf{env}^{\mathsf{Epi}}_{\varphi} X = \mathsf{red}_\infty\varphi\circ\mathsf{coim}_\infty\varphi, \qquad \mathsf{Env}^{\mathsf{Epi}}_{\varphi} X = \mathsf{Im}_\infty\varphi; \tag{3.56}$$

(2) In 3°–5° we assume that $\mathtt{K}$ has coproducts over arbitrary index sets, not necessarily finite.

– *the epimorphism* $\mathsf{coim}_\infty \varphi$ *in the nodal decomposition of* φ *is an envelope of* X *in the class* SEpi *of strong epimorphisms with respect to* φ*:*

$$\mathsf{env}_\varphi^{\mathsf{SEpi}} X = \mathsf{coim}_\infty \varphi, \qquad \mathsf{Env}_\varphi^{\mathsf{SEpi}} X = \mathsf{Coim}_\infty \varphi. \tag{3.57}$$

2° *If* $\mathtt{K}$ *has finite products, then every object* X *in* $\mathtt{K}$ *has envelopes in* Epi *and* SEpi *with respect to an arbitrary finite set* Φ *of morphisms going from* X*.*

3° *If* $\mathtt{K}$ *is a category with products* (3)*, then every object* X *in* $\mathtt{K}$ *has envelopes in* Epi *and in* SEpi *with respect to an arbitrary set* Φ *of morphisms going from* X*.*

4° *If* $\mathtt{K}$ *is a category with products, then every object* X *in* $\mathtt{K}$ *has envelopes in* Epi *and in* SEpi *with respect to an arbitrary class* Φ *of morphisms going from* X *such that there is a set that generates* Φ *on the inside.*

5° *If* $\mathtt{K}$ *is a category with products, co-well-powered in* Epi (*respectively, in* SEpi)*, then every object* X *in* $\mathtt{K}$ *has an envelope in* Epi (*respectively, in* SEpi) *with respect to an arbitrary class* Φ *of morphisms going from* X*.*

Proposition 3.29. *If* $\mathtt{K}$ *is a category with products, with nodal decomposition, and co-well-powered in* Epi*, then every object* X *in* $\mathtt{K}$ *has an envelope in each class* $\Omega \supseteq \mathsf{Bim}$ *with respect to an arbitrary right ideal* Φ *of morphisms going from* X *which separates morphisms on the outside* (4)*, and*

$$\mathsf{env}_\Phi^\Omega X = \mathsf{env}_\Phi^{\mathsf{Bim}} X = \mathsf{env}_\Phi^{\mathsf{Epi}} X.$$

Proof. By property 5°, there exists an envelope $\mathsf{env}_\Phi^{\mathsf{Epi}} X$. By Theorem 3.6(i) this envelope is a monomorphism, and hence a bimorphism. Then by property 1°(c) on p. 44, $\mathsf{env}_\Phi^{\mathsf{Epi}} X = \mathsf{env}_\Phi^{\mathsf{Bim}} X$. Now by Theorem 3.7, $\mathsf{env}_\Phi^{\mathsf{Bim}} X = \mathsf{env}_\Phi^\Omega X$. ∎

The dual results for refinements look as follows.

Properties of refinements in Mono and SMono in a category with nodal decomposition. *Let* $\mathtt{K}$ *be a category with nodal decomposition.*

1° *For each morphism* $\varphi : X \leftarrow Y$ *in* $\mathtt{K}$*:*

– *the monomorphism* $\mathsf{im}_\infty \varphi \circ \mathsf{red}_\infty \varphi$ *in the nodal decomposition of* φ *is a refinement in the class* Mono *of all monomorphisms in* X *by means of* φ*:*

$$\mathsf{ref}_\varphi^{\mathsf{Mono}} X = \mathsf{im}_\infty \varphi \circ \mathsf{red}_\infty \varphi, \qquad \mathsf{Ref}_\varphi^{\mathsf{Mono}} X = \mathsf{Coim}_\infty \varphi; \tag{3.58}$$

– *the monomorphism* $\mathsf{im}_\infty \varphi$ *in the nodal decomposition of* φ *is a refinement in the class* SMono *of strong monomorphisms in* X *by means of* φ*:*

$$\mathsf{ref}_\varphi^{\mathsf{SMono}} X = \mathsf{im}_\infty \varphi, \qquad \mathsf{Ref}_\varphi^{\mathsf{SMono}} X = \mathsf{Im}_\infty \varphi. \tag{3.59}$$

2° *If* $\mathtt{K}$ *is a category with finite coproducts, then every object* X *in* $\mathtt{K}$ *has refinements in* Mono *and in* SMono *by means of an arbitrary finite set* Φ *of morphisms going to* X*.*

3° *If* $\mathtt{K}$ *is a category with coproducts* (5)*, then every object* X *in* $\mathtt{K}$ *has refinements in* Mono *and in* SMono *by means of an arbitrary set* Φ *of morphisms going to* X*.*

---

(3) Similarly to footnote (2).

(4) See definition on p. 47.

(5) In 3°–5° we assume that $\mathtt{K}$ has coproducts over arbitrary sets of indices, not necessarily finite.

4° *If* $\mathtt{K}$ *is a category with coproducts, then every object* X *in* $\mathtt{K}$ *has refinements in* Mono *and in* SMono *by means of an arbitrary class* $\varPhi$ *of morphisms coming to* X *such that there is a set which generates* $\varPhi$ *on the outside.*

5° *If* $\mathtt{K}$ *is a category with coproducts, and well-powered in* Mono (*respectively, in* SMono), *then each object* X *in* $\mathtt{K}$ *has a refinement in* Mono (*respectively, in* SMono) *by means of an arbitrary class* $\varPhi$ *of morphisms going to* X.

PROPOSITION 3.30. *If* $\mathtt{K}$ *is a category with coproducts, with nodal decomposition, and well-powered in* Mono, *then every object* X *in* $\mathtt{K}$ *has a refinement in an arbitrary class* $\varGamma \supseteq \mathsf{Bim}$ *by means of an arbitrary left ideal* $\varPhi$ *of morphisms going to* X *which separates morphisms on the inside* (6), *and*

$$\mathsf{ref}_{\varPhi}^{\varGamma} X = \mathsf{ref}_{\varPhi}^{\mathsf{Bim}} X = \mathsf{ref}_{\varPhi}^{\mathsf{Mono}} X.$$

3.3.4. Existence of nodal decomposition in categories with envelopes and refinements. By analogy with definitions on p. 31 we will say that in a category $\mathtt{K}$:

– *epimorphisms discern monomorphisms* if from the fact that a morphism μ is not a monomorphism it follows that μ can be represented as $\mu = \mu' \circ \varepsilon$ where ε is an epimorphism which is not an isomorphism;
– *monomorphisms discern epimorphisms* if from the fact that a morphism ε is not an epimorphism it follows that ε can be represented as $\varepsilon = \mu \circ \varepsilon'$ where μ is a monomorphism which is not an isomorphism.

THEOREM 3.31. *Suppose that in a category* $\mathtt{K}$*:*

(a) *epimorphisms discern monomorphisms, and dually, monomorphisms discern epimorphisms;*
(b) *every immediate monomorphism is a strong monomorphism, and dually, every immediate epimorphism is a strong epimorphism;*
(c) *every object* X *has an envelope in* Epi *with respect to any morphism starting from* X, *and dually, in every object* X *there is a refinement in* Mono *by means of any morphism coming to* X.

Then $\mathtt{K}$ *is a category with nodal decomposition.*

Proof. Consider a morphism $\varphi : X \to Y$.

Suppose $\varepsilon : X \to N$ is an envelope of X in Epi with respect to φ, and denote by β the dashed arrow in (3.3): $\varphi = \beta \circ \varepsilon$. Note first that β is a monomorphism. Indeed, if not, then by (a), there exists a decomposition $\beta = \beta' \circ \pi$ where π is an epimorphism, but not an isomorphism. If we denote by N' the range of π, then we get a diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] \arrow[dr, "\varepsilon'"] & Y \\ N \arrow[r, "\pi"'] \arrow[ur, "\beta"] & N' \arrow[u, "\beta'"'] \end{tikzcd} \tag{3.60}$$

(6) See definition on p. 55.

where by definition $\varepsilon' = \pi \circ \varepsilon$, and this is an epimorphism, as a composition of two epimorphisms. Thus, ε' is another extension of X with respect to φ. Hence, there exists a unique morphism υ such that

$$\begin{tikzcd} & X \arrow[dl, "\varepsilon"'] \arrow[dr, "\varepsilon'"] & \\ N & & N' \arrow[ll, dashed, "\upsilon"'] \end{tikzcd}$$

Here

$$\begin{aligned} &\pi \circ \varepsilon = \varepsilon' \ \Rightarrow\ \upsilon \circ \pi \circ \varepsilon = \upsilon \circ \varepsilon' = \varepsilon = 1_N \circ \varepsilon \ \Rightarrow\ \upsilon \circ \pi = 1_N, \\ &\upsilon \circ \varepsilon' = \varepsilon \ \Rightarrow\ \pi \circ \upsilon \circ \varepsilon' = \pi \circ \varepsilon = \varepsilon' = 1_{N'} \circ \varepsilon' \ \Rightarrow\ \pi \circ \upsilon = 1_{N'}. \end{aligned}$$

That is, π is an isomorphism, contrary to our assumption.

Similarly one can prove that β is an immediate monomorphism. Indeed, any factorization $\beta = \beta' \circ \pi$ leads again to diagram (3.60), and the same reasoning shows that π is an isomorphism.

The fact that β is an immediate monomorphism together with condition (b) implies that β is a strong monomorphism.

Denote by $\mu : M \to Y$ the refinement of Y in Mono by means of φ, and by α the dashed arrow in the corresponding diagram (3.26), i.e. $\varphi = \mu \circ \alpha$. Using the dual reasoning to the one used when proving that β is a strong monomorphism, we can show that α is a strong epimorphism.

Consider now a diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, "\varepsilon" near end] & Y \\ M \arrow[ur, "\mu" near end] & N \arrow[u, "\beta"'] \end{tikzcd}$$

As we already observed, here α is an epimorphism, hence α is an extension of X in Epi with respect to φ. At the same time ε is an envelope of X in Epi with respect to φ. Hence there exists a morphism υ such that

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, "\varepsilon"] & \\ M \arrow[r, "\upsilon"] & N \end{tikzcd}$$

As a corollary, the following diagram is commutative as well:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, "\varepsilon"] & Y \\ M \arrow[r, "\upsilon"] & N \arrow[u, "\beta"'] \end{tikzcd} \tag{3.61}$$

Similarly, β is a monomorphism, so it is an enrichment of Y in Mono by means of φ. At the same time, μ is a refinement of Y in Mono by means of φ. Hence, there exists a morphism υ' such that

$$\begin{tikzcd} & Y \\ M \arrow[ur, "\mu"] \arrow[r, "\upsilon'"] & N \arrow[u, "\beta"'] \end{tikzcd}$$

As a corollary, we get

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & Y \\ M \arrow[ur, "\mu"] \arrow[r, "\upsilon'"] & N \arrow[u, "\beta"'] \end{tikzcd} \tag{3.62}$$

From (3.61) and (3.62) we have

$$\underset{\mathsf{Mono}}{\underset{\Cap}{\beta}} \circ \upsilon \circ \underset{\mathsf{Epi}}{\underset{\Cap}{\alpha}} = \varphi = \underset{\mathsf{Mono}}{\underset{\Cap}{\beta}} \circ \upsilon' \circ \underset{\mathsf{Epi}}{\underset{\Cap}{\alpha}} \Rightarrow \upsilon = \upsilon',$$

that is, the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, "\varepsilon"] & Y \\ M \arrow[ur, "\mu"] \arrow[r, "\upsilon"] & N \arrow[u, "\beta"'] \end{tikzcd}$$

Here $\varepsilon = \upsilon \circ \alpha$ is an epimorphism, hence so is υ. On the other hand, $\mu = \beta \circ \upsilon$ is a monomorphism, hence so is υ. Thus, υ is a bimorphism, and $\varphi = \beta \circ \upsilon \circ \alpha$ is a nodal decomposition of φ. ∎

THEOREM 3.32. *Suppose that in a category* $\mathtt{K}$*:*

(a) *strong epimorphisms discern monomorphisms and strong monomorphisms discern epimorphisms* (7)*;*
(b) *each object X has an envelope in the class* SEpi *of all strong epimorphisms with respect to an arbitrary morphism that goes from X, and dually, in each object X there is a refinement in the class* SMono *of all strong monomorphisms by means of an arbitrary morphism that comes to X.*

Then $\mathtt{K}$ *is a category with nodal decomposition.*

Proof. Take a morphism $\varphi : X \to Y$.

By (b), there is an envelope $\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X : X \to \mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X$. Denote by α the morphism that extends φ onto $\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X$:

$$\begin{tikzcd} X \arrow[d, "\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"'] \arrow[dr, "\varphi"] & \\ \mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X \arrow[r, dashed, "\alpha"] & Y \end{tikzcd}$$

(7) See definitions on p. 31.

Similarly, by (b) there is a refinement $\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y : \mathsf{Ref}^{\mathsf{SMono}}_{\varphi} Y \to Y$. Denote by β the morphism that lifts φ to $\mathsf{ref}^{\mathsf{SMono}}_{\varphi} X$:

$$\begin{tikzcd}
X \arrow[r, dashed, "\beta"] \arrow[dr, "\varphi"'] & \mathsf{Ref}^{\mathsf{SEpi}}_{\varphi} Y \arrow[d, "\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y"] \\
& Y
\end{tikzcd}$$

Pasting these triangles together along the common side φ, and throwing away this side, we obtain a quadrangle:

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[d, "\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"'] & \mathsf{Ref}^{\mathsf{SEpi}}_{\varphi} Y \arrow[d, "\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y"] \\
\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

Here $\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X$ is a strong epimorphism, and $\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y$ a monomorphism, so there is a diagonal δ:

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[d, "\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"'] & \mathsf{Ref}^{\mathsf{SEpi}}_{\varphi} Y \arrow[d, "\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y"] \\
\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X \arrow[r, "\alpha"'] \arrow[ur, dashed, "\delta"'] & Y
\end{tikzcd} \tag{3.63}$$

Let us show that δ is a bimorphism.

Suppose first that δ is not a monomorphism. Then, since strong epimorphisms discern monomorphisms (by (a)), there is a decomposition $\delta = \delta' \circ \varepsilon$ where ε is a strong epimorphism which is not an isomorphism. As a corollary, the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"'] \arrow[dr, "\beta"] & Y \\
\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X \arrow[r, "\delta"'] \arrow[d, dashed, "\varepsilon"'] & \mathsf{Ref}^{\mathsf{SEpi}}_{\varphi} Y \arrow[u, "\mathsf{ref}^{\mathsf{SMono}}_{\varphi} Y"'] \\
M \arrow[ur, dashed, bend right, "\delta'"'] &
\end{tikzcd}$$

We see that $\mathsf{imp}^{\mathsf{SMono}}_{\varphi} Y \circ \delta'$ is a continuation of φ along $\varepsilon \circ \mathsf{env}^{\mathsf{SEpi}}_{\varphi} X$, which in turn is a strong epimorphism (as a composition of two strong epimorphisms). This means that $\varepsilon \circ \mathsf{env}^{\mathsf{SEpi}}_{\varphi} X$ is an extension of X in SEpi with respect to φ. Hence, there is a morphism υ from M to $\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X$ such that diagram (3.4) is commutative:

$$\begin{tikzcd}
& X \arrow[dl, bend right, "\mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"'] \arrow[dr, bend left, "\varepsilon \circ \mathsf{env}^{\mathsf{SEpi}}_{\varphi} X"] & \\
\mathsf{Env}^{\mathsf{SEpi}}_{\varphi} X & & M \arrow[ll, dashed, "\upsilon"']
\end{tikzcd}$$

We now have $\upsilon \circ \varepsilon \circ \mathsf{env}_{\varphi}^{\mathsf{SEpi}} X = \mathsf{env}_{\varphi}^{\mathsf{SEpi}} X = 1_M \circ \mathsf{env}_{\varphi}^{\mathsf{SEpi}} X$, and since $\mathsf{env}_{\varphi}^{\mathsf{SEpi}} X$ is an epimorphism, this implies $\upsilon \circ \varepsilon = 1_M$, which means that ε is a coretraction. On the other hand, ε is an epimorphism, and hence an isomorphism. This contradicts the choice of ε.

Thus, δ must be a monomorphism. By analogy we prove that it is an epimorphism. Let us now add φ to diagram (3.63) and twist it as follows:

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\mathsf{env}_{\varphi}^{\mathsf{SEpi}} X"'] \arrow[dr, "\beta"] & Y \\
\mathsf{Env}_{\varphi}^{\mathsf{SEpi}} X \arrow[r, "\delta"] \arrow[ur, "\alpha"] & \mathsf{Ref}_{\varphi}^{\mathsf{SEpi}} Y \arrow[u, "\mathsf{ref}_{\varphi}^{\mathsf{SMono}} Y"']
\end{tikzcd}$$

We then see that $\varphi = \mathsf{ref}_{\varphi}^{\mathsf{SMono}} Y \circ \delta \circ \mathsf{env}_{\varphi}^{\mathsf{SEpi}} X$ is a nodal decomposition of φ. ∎

3.4. Nets and functoriality. In general, the operations of taking envelopes and refinements are not functors. But under some assumptions they are, and in the last part of this section we discuss this. Let us make the following definition. Suppose $\varOmega$, $\varPhi$, $\varGamma$ are classes of morphisms in a category $\mathtt{K}$.

- Let us say that *the envelope* $\mathsf{Env}_{\varPhi}^{\varOmega}$ *can be defined as a functor* if there exist

  E.1. a map $X \mapsto (E(X), e_X)$ that to each object X in $\mathtt{K}$ assigns a morphism $e_X : X \to E(X)$ in $\mathtt{K}$ which is an envelope in $\varOmega$ with respect to $\varPhi$:
  $$E(X) = \mathsf{Env}_{\varPhi}^{\varOmega} X, \qquad e_X = \mathsf{env}_{\varPhi}^{\varOmega} X,$$

  E.2. a map $\alpha \mapsto E(\alpha)$ that turns each morphism $\alpha : X \to Y$ in $\mathtt{K}$ into a morphism $E(\alpha) : E(X) \to E(Y)$ in $\mathtt{K}$ in such a way that
  $$\begin{tikzcd}
  X \arrow[r, "e_X"] \arrow[d, "\alpha"'] & E(X) \arrow[d, dashed, "E(\alpha)"] \\
  Y \arrow[r, "e_Y"] & E(Y)
  \end{tikzcd} \tag{3.64}$$

  and the following identities hold:
  $$E(1_X) = 1_{E(X)}, \qquad E(\beta \circ \alpha) = E(\beta) \circ E(\alpha). \tag{3.65}$$

  Clearly, in this case the map $(X, \alpha) \mapsto (E(X), E(\alpha))$ is a covariant functor from $\mathtt{K}$ into $\mathtt{K}$, and $X \mapsto e_X$ is a natural transformation of the identity functor $(X, \alpha) \mapsto (X, \alpha)$ into the functor $(X, \alpha) \mapsto (E(X), E(\alpha))$.
- Let us say that *the envelope* $\mathsf{Env}_{\varPhi}^{\varOmega}$ *can be defined as an idempotent functor* if in addition to E.1 and E.2 one can ensure the condition

  E.3. for each $X \in \mathsf{Ob}(\mathtt{K})$ the morphism $e_{E(X)} : E(X) \to E(E(X))$ is the local identity:
  $$E(E(X)) = E(X), \quad e_{E(X)} = 1_{E(X)}, \qquad X \in \mathsf{Ob}(\mathtt{K}). \tag{3.66}$$

REMARK 3.33. If $\varOmega \subseteq \mathsf{Epi}$, then (3.66) implies
$$E(e_X) = 1_{E(X)}, \qquad X \in \mathsf{Ob}(\mathtt{K}). \tag{3.67}$$

Indeed, if we insert $\alpha = e_X$ into (3.64), we obtain

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "e_X"'] & E(X) \arrow[d, "E(e_X)"] \\ E(X) \arrow[r, "e_{E(X)}=1_{E(X)}"] & E(E(X)) = E(X) \end{tikzcd}$$

i.e. $E(e_X) \circ e_X = 1_{E(X)} \circ e_X$, and since $e_X \in \Omega \subseteq \mathsf{Epi}$, we can cancel it: $E(e_X) = 1_{E(X)}$.

- Let us say that *the refinement* $\mathsf{Ref}^{\Gamma}_{\Phi}$ *can be defined as a functor* if there exist

R.1. a map $X \mapsto (I(X), i_X)$ that to each object X in $\mathtt{K}$ assigns a morphism $i_X : I(X) \to X$ in $\mathtt{K}$, which is a refinement in Γ by means of Φ:

$$I(X) = \mathsf{Ref}^{\Gamma}_{\Phi} X, \quad i_X = \mathsf{ref}^{\Gamma}_{\Phi} X,$$

R.2. a map $\alpha \mapsto I(\alpha)$ that turns each morphism $\alpha : X \leftarrow Y$ in $\mathtt{K}$ into a morphism $I(\alpha) : I(X) \leftarrow I(Y)$ in $\mathtt{K}$ in such a way that

$$\begin{tikzcd} X & I(X) \arrow[l, "i_X"'] \\ Y \arrow[u, "\alpha"] & I(Y) \arrow[l, "i_Y"'] \arrow[u, dashed, "I(\alpha)"'] \end{tikzcd} \tag{3.68}$$

and the following identities hold:

$$I(1_X) = 1_{I(X)}, \quad I(\beta \circ \alpha) = I(\beta) \circ I(\alpha). \tag{3.69}$$

In this case $(X, \alpha) \mapsto (I(X), I(\alpha))$ is a covariant functor from $\mathtt{K}$ into $\mathtt{K}$, and $X \mapsto i_X$ is a natural transformation of the identity functor $(X, \alpha) \mapsto (X, \alpha)$ into the functor $(X, \alpha) \mapsto (I(X), I(\alpha))$.

- Let us say that *the refinement* $\mathsf{Ref}^{\Gamma}_{\Phi}$ *can be defined as an idempotent functor* if in addition to R.1 and R.2 one can ensure the condition

R.3. for each $X \in \mathsf{Ob}(\mathtt{K})$ the morphism $i_{I(X)} : I(X) \leftarrow I(I(X))$ is the local identity:

$$I(I(X)) = I(X), \quad i_{I(X)} = 1_{I(X)}, \quad X \in \mathsf{Ob}(\mathtt{K}). \tag{3.70}$$

Remark 3.34. If $\Gamma \subseteq \mathsf{Mono}$, then (3.70) implies

$$I(i_X) = 1_{I(X)}, \quad X \in \mathsf{Ob}(\mathtt{K}). \tag{3.71}$$

Remark 3.35. For envelopes, in the most important case when $\Omega \subseteq \mathsf{Epi}$, the identities (3.65) automatically follow from E.1 and E.2. Dually, for refinements, when $\Gamma \subseteq \mathsf{Mono}$, the identities (3.69) automatically follow from R.1 and R.2.

3.4.1. Nets of epimorphisms. Suppose that to each object X in a category $\mathtt{K}$ there is assigned a subset $\mathcal{N}^X$ in the class Epi^X of all epimorphisms of $\mathtt{K}$ going from X, and the following three requirements are fulfilled:

(a) for each X the set $\mathcal{N}^X$ is non-empty and is directed to the left with respect to the pre-order (2.17) inherited from Epi^X:

$$\forall \sigma, \sigma' \in \mathcal{N}^X \ \exists \rho \in \mathcal{N}^X \quad \rho \to \sigma \ \& \ \rho \to \sigma';$$

(b) for each X the covariant system of morphisms generated by $\mathcal{N}^X$ given by

$$\mathsf{Bind}(\mathcal{N}^X) := \{\iota_\rho^\sigma;\ \rho, \sigma \in \mathcal{N}^X,\ \rho \to \sigma\} \tag{3.72}$$

(the morphisms ι_ρ^σ were defined in (2.18); by (2.19) this system is a covariant functor from $\mathcal{N}^X$ considered as a full subcategory in Epi^X into $\mathtt{K}$) has a projective limit in $\mathtt{K}$;

(c) for each morphism $\alpha : X \to Y$ and each $\tau \in \mathcal{N}^Y$ there is $\sigma \in \mathcal{N}^X$ and a morphism $\alpha_\sigma^\tau : \mathsf{Ran}\,\sigma \to \mathsf{Ran}\,\tau$ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, dashed, "\sigma"'] & Y \arrow[d, "\tau"] \\ \mathsf{Ran}\,\sigma \arrow[r, dashed, "\alpha_\sigma^\tau"] & \mathsf{Ran}\,\tau \end{tikzcd} \tag{3.73}$$

(for given α, σ and τ the morphism α_σ^τ, if it exists, is unique, since σ is an epimorphism).

Then:

- We call the family $\mathcal{N} = \{\mathcal{N}^X;\ X \in \mathsf{Ob}(\mathtt{K})\}$ a *net of epimorphisms* in $\mathtt{K}$, and the elements of the sets $\mathcal{N}^X$ *elements of the net* $\mathcal{N}$.
- For each X the system $\mathsf{Bind}(\mathcal{N}^X)$ defined by (3.72) will be called the *system of binding morphisms of the net* $\mathcal{N}$ *over the vertex* X. Its projective limit (which exists by (b)) is a projective cone whose vertex will be denoted by $X_\mathcal{N}$, and the morphisms going from it by $\sigma_\mathcal{N} = \varprojlim_{\rho \in \mathcal{N}^X} \iota_\rho^\sigma : X_\mathcal{N} \to \mathsf{Ran}\,\sigma$:

$$\begin{tikzcd} & X_\mathcal{N} \arrow[dl, "\rho_\mathcal{N}"'] \arrow[dr, "\sigma_\mathcal{N}"] & \\ \mathsf{Ran}\,\rho \arrow[rr, "\iota_\rho^\sigma"] & & \mathsf{Ran}\,\sigma \end{tikzcd} \qquad (\rho \to \sigma) \tag{3.74}$$

In addition, by (2.18), the system $\mathcal{N}^X$ is also a projective cone of $\mathsf{Bind}(\mathcal{N}^X)$:

$$\begin{tikzcd} & X \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\ \mathsf{Ran}\,\rho \arrow[rr, "\iota_\rho^\sigma"] & & \mathsf{Ran}\,\sigma \end{tikzcd} \qquad (\rho \to \sigma) \tag{3.75}$$

so there exists a natural morphism from X into the vertex $X_\mathcal{N}$ of the projective limit of the system $\mathsf{Bind}(\mathcal{N}^X)$. We denote this morphism by $\varprojlim \mathcal{N}^X$ and call it the *local limit of the net* $\mathcal{N}$ *at the object* X:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varprojlim \mathcal{N}^X"] \arrow[dr, "\sigma"'] & & X_\mathcal{N} \arrow[dl, "\sigma_\mathcal{N}"] \\ & \mathsf{Ran}\,\sigma & \end{tikzcd} \qquad (\sigma \in \mathcal{N}^X) \tag{3.76}$$

- The element σ of the net in diagram (3.73) will be called a *counterfort* of the element τ of the net.

Examples of nets of epimorphisms will be given in Sections 5.4 and 5.5.

THEOREM 3.36. *Let $\mathcal{N}$ be a net of epimorphisms in a category $\mathtt{K}$. Then:*

(i) *for each object X in $\mathtt{K}$ the local limit $\varprojlim \mathcal{N}^X : X \to X_{\mathcal{N}}$ is an envelope $\mathsf{env}_{\mathcal{N}} X$ in $\mathtt{K}$ with respect to the class $\mathcal{N}$:*

$$\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}} X; \tag{3.77}$$

(ii) *for each morphism $\alpha : X \to Y$ in $\mathtt{K}$ and any choice of local limits $\varprojlim \mathcal{N}^X$ and $\varprojlim \mathcal{N}^Y$ the formula*

$$\alpha_{\mathcal{N}} = \varprojlim_{\tau \in \mathcal{N}_Y} \varprojlim_{\sigma \in \mathcal{N}^X} \alpha_\sigma^\tau \circ \sigma_{\mathcal{N}} \tag{3.78}$$

defines a morphism $\alpha_{\mathcal{N}} : X_{\mathcal{N}} \to Y_{\mathcal{N}}$ such that

$$\begin{tikzcd} X \arrow[rr, "\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}} X"] \arrow[d, "\alpha"'] && X_{\mathcal{N}} = \mathsf{Env}_{\mathcal{N}} X \arrow[d, dashed, "\alpha_{\mathcal{N}}"] \\ Y \arrow[rr, "\varprojlim \mathcal{N}^Y = \mathsf{env}_{\mathcal{N}} Y"] && Y_{\mathcal{N}} = \mathsf{Env}_{\mathcal{N}} Y \end{tikzcd} \tag{3.79}$$

(iii) *the envelope $\mathsf{Env}_{\mathcal{N}}$ can be defined as a functor.*

Proof. (i) By Lemma 3.23, $\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}^X} X$. Here one can replace $\mathcal{N}^X$ by $\mathcal{N}$, since $\mathcal{N}^X$ is exactly the subclass in $\mathcal{N}$ consisting of morphisms with X as domain: $\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}^X} X = \mathsf{env}_{\mathcal{N}} X$.

(ii) Let us first explain the meaning of (3.78). Take a morphism $\alpha : X \to Y$. For each $\tau \in \mathcal{N}^Y$ denote

$$\alpha^\tau = \tau \circ \alpha. \tag{3.80}$$

Clearly, the family $\{\alpha^\tau : X \to \mathsf{Ran}\,\tau;\ \tau \in \mathcal{N}^Y\}$ is a projective cone of the system $\mathsf{Bind}(\mathcal{N}^Y)$ of binding morphisms:

$$\begin{tikzcd} & X \arrow[dl, "\alpha^\tau"'] \arrow[dr, "\alpha^\upsilon"] & \\ \mathsf{Ran}\,\tau \arrow[rr, "\iota_\tau^\upsilon"] && \mathsf{Ran}\,\upsilon \end{tikzcd} \qquad (\tau \to \upsilon) \tag{3.81}$$

By (c) for each $\tau \in \mathcal{N}^Y$ there are $\sigma \in \mathcal{N}^X$ and a morphism $\alpha_\sigma^\tau : \mathsf{Ran}\,\sigma \to \mathsf{Ran}\,\tau$ such that diagram (3.73) is commutative, and we have already denoted by α^τ the diagonal there:

$$\alpha^\tau = \tau \circ \alpha = \alpha_\sigma^\tau \circ \sigma. \tag{3.82}$$

Let

$$\alpha_{\mathcal{N}}^\tau = \alpha_\sigma^\tau \circ \sigma_{\mathcal{N}}. \tag{3.83}$$

Then we obtain a diagram

$$\begin{tikzcd} X \arrow[rr, "\varprojlim \mathcal{N}^X"] \arrow[dr, "\sigma"'] \arrow[ddr, bend right, "\alpha^\tau"'] && X_{\mathcal{N}} \arrow[dl, "\sigma_{\mathcal{N}}"] \arrow[ddl, dashed, bend left, "\alpha_{\mathcal{N}}^\tau"] \\ & \mathsf{Ran}\,\sigma \arrow[d, "\alpha_\sigma^\tau"] & \\ & \mathsf{Ran}\,\tau & \end{tikzcd} \qquad (\sigma \in \mathcal{N}^X) \tag{3.84}$$

Note that for any other $\rho \in \mathcal{N}^X$ such that $\rho \to \sigma$ the following equality analogous to (3.83) is true:

$$\alpha^\tau_{\mathcal{N}} = \alpha^\tau_\rho \circ \rho_{\mathcal{N}}, \qquad \rho \to \sigma. \tag{3.85}$$

Indeed, for $\rho \to \sigma$ diagram (3.73) can be added to the diagram

$$\begin{tikzcd}
 & X \arrow[r, "\alpha"] \arrow[d, "\sigma"'] \arrow[dl, bend right, "\rho"'] & Y \arrow[d, "\tau"] \\
 & \operatorname{Ran}\sigma \arrow[r, "\alpha^\tau_\sigma"] & \operatorname{Ran}\tau \\
\operatorname{Ran}\rho \arrow[ur, "\iota^\sigma_\rho"] \arrow[urr, dashed, "\alpha^\tau_\rho"'] & &
\end{tikzcd} \tag{3.86}$$

(here the dashed arrow is initially defined as $\alpha^\tau_\sigma \circ \iota^\sigma_\rho$; since such an arrow, if it exists, is unique, we deduce that this is the morphism α^τ_ρ). So we have

$$\alpha^\tau_{\mathcal{N}} = \alpha^\tau_\sigma \circ \sigma_{\mathcal{N}} \overset{(3.74)}{=} \alpha^\tau_\sigma \circ \iota^\sigma_\rho \circ \rho_{\mathcal{N}} \overset{(3.86)}{=} \alpha^\tau_\rho \circ \rho_{\mathcal{N}}.$$

From (3.85) it follows that the definition of $\alpha^\tau_{\mathcal{N}}$ in (3.83) does not depend on the choice of $\sigma \in \mathcal{N}^X$, since if $\sigma' \in \mathcal{N}^X$ is another element such that there exists a morphism $\alpha^\tau_{\sigma'} : \mathsf{Ran}\,\sigma' \to \mathsf{Ran}\,\tau$ for which diagram (3.73) is commutative (with σ replaced by σ'), then we can take $\rho \in \mathcal{N}^X$ standing to the left of σ and σ', in symbols $\rho \to \sigma$ and $\rho \to \sigma'$ (at this moment we use property (a) of a net of epimorphisms), and get

$$\alpha^\tau_{\mathcal{N}} = \alpha^\tau_\sigma \circ \sigma_{\mathcal{N}} \overset{(3.85)}{=} \alpha^\tau_\rho \circ \rho_{\mathcal{N}} \overset{(3.85)}{=} \alpha^\tau_{\sigma'} \circ \sigma'_{\mathcal{N}}.$$

We can deduce that now formula (3.83) correctly defines a map $\tau \in \mathcal{N}^Y \mapsto \alpha^\tau_{\mathcal{N}}$. Let us show that $\{\alpha^\tau_{\mathcal{N}} : X_{\mathcal{N}} \to \mathsf{Ran}\,\tau;\ \tau \in \mathcal{N}^Y\}$ is a projective cone $\mathsf{Bind}(\mathcal{N}^Y)$. We have

$$\begin{tikzcd}
 & X_{\mathcal{N}} \arrow[dl, "\alpha^\tau_{\mathcal{N}}"'] \arrow[dr, "\alpha^\upsilon_{\mathcal{N}}"] & \\
\operatorname{Ran}\tau \arrow[rr, "\iota^\upsilon_\tau"] & & \operatorname{Ran}\upsilon
\end{tikzcd} \qquad (\tau \to \upsilon \in \mathcal{N}^Y) \tag{3.87}$$

For $\tau \to \upsilon$ diagram (3.73) can be added to the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"] \arrow[dr, bend left, "\upsilon"] & \\
\operatorname{Ran}\sigma \arrow[r, "\alpha^\tau_\sigma"] \arrow[drr, dashed, "\alpha^\upsilon_\sigma"'] & \operatorname{Ran}\tau \arrow[dr, "\iota^\upsilon_\tau"] & \\
 & & \operatorname{Ran}\upsilon
\end{tikzcd} \tag{3.88}$$

(where the dashed arrow is initially defined as $\iota^\upsilon_\tau \circ \alpha^\tau_\sigma$; since such an arrow, if it exists, is unique, we deduce that this is the morphism α^υ_σ). Using this diagram we have

$$\iota^\upsilon_\tau \circ \alpha^\tau_{\mathcal{N}} \overset{(3.83)}{=} \iota^\upsilon_\tau \circ \alpha^\tau_\sigma \circ \sigma_{\mathcal{N}} \overset{(3.88)}{=} \alpha^\upsilon_\sigma \circ \sigma_{\mathcal{N}} \overset{(3.83)}{=} \alpha^\upsilon_{\mathcal{N}}.$$

From diagram (3.87) it now follows that there exists a natural morphism $\alpha_{\mathcal{N}}$ from $X_{\mathcal{N}}$

into the projective limit $Y_{\mathcal{N}}$ of $\mathsf{Bind}(\mathcal{N}^Y)$:

$$\begin{tikzcd} X_{\mathcal{N}} \arrow[rr, dashed, "\alpha_{\mathcal{N}}"] \arrow[dr, "\alpha^{\tau}_{\mathcal{N}}"'] & & Y_{\mathcal{N}} \arrow[dl, "\tau_{\mathcal{N}}"] \\ & \mathsf{Ran}\,\tau & \end{tikzcd} \qquad (\tau \in \mathcal{N}^Y) \tag{3.89}$$

Recall now that by property (b) of nets the passage from X to $\varprojlim \mathsf{Bind}(\mathcal{N}^X)$ can be understood as a map. The further steps of building $\alpha_{\mathcal{N}}$ (the choice of the vertex $X_{\mathcal{N}}$ of the cone $\varprojlim \mathsf{Bind}(\mathcal{N}^X)$, and then the choice of the arrow $\alpha_{\mathcal{N}}$ such that all diagrams (3.89) are commutative) are also unambiguous, so the correspondence $\alpha \mapsto \alpha_{\mathcal{N}}$ can also be treated as a map.

Note further that for the morphisms $\alpha_{\mathcal{N}}$ the diagrams of the form (3.79) are commutative. In the diagram

$$\begin{tikzcd} X \arrow[rr, "\varprojlim \mathcal{N}^X"] \arrow[dd, "\alpha"'] \arrow[dr, "\alpha^{\tau}"] & & X_{\mathcal{N}} \arrow[dl, "\alpha^{\tau}_{\mathcal{N}}"'] \arrow[dd, dashed, "\alpha_{\mathcal{N}}"] \\ & \mathsf{Ran}\,\tau & \\ Y \arrow[rr, "\varprojlim \mathcal{N}^Y"'] \arrow[ur, "\tau"] & & Y_{\mathcal{N}} \arrow[ul, "\tau_{\mathcal{N}}"'] \end{tikzcd}$$

all the (small) triangles are commutative: the upper triangle is so because it is the perimeter of (3.84), the left triangle because this is a variant of formula (3.80), the lower triangle because up to notation it is diagram (3.76), and the right triangle because it is a rotated diagram (3.89). Therefore

$$\tau_{\mathcal{N}} \circ \varprojlim \mathcal{N}_Y \circ \alpha = \alpha^{\tau} = \tau_{\mathcal{N}} \circ \alpha_{\mathcal{N}} \circ \varprojlim \mathcal{N}^X \qquad (\tau \in \mathcal{N}^Y).$$

One can interpret this as follows: each of the morphisms $\varprojlim \mathcal{N}^Y \circ \alpha$ and $\alpha_{\mathcal{N}} \circ \varprojlim \mathcal{N}^X$ is a lifting of the projective cone $\{\alpha^{\tau} : X \to \mathsf{Ran}\,\tau;\ \tau \in \mathcal{N}^Y\}$ for the system of binding morphisms $\mathsf{Bind}(\mathcal{N}_Y)$ which we were talking about in diagram (3.81) to the projective limit of this system. That is, $\varprojlim \mathcal{N}^Y \circ \alpha$ and $\alpha_{\mathcal{N}} \circ \varprojlim \mathcal{N}^X$ are the same dashed arrow in the definition of projective limit, for which all the diagrams of the form

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\alpha^{\tau}"'] & & Y_{\mathcal{N}} \arrow[dl, "\tau_{\mathcal{N}}"] \\ & \mathsf{Ran}\,\tau & \end{tikzcd} \qquad (\tau \in \mathcal{N}^Y)$$

are commutative. But such an arrow is unique, so

$$\varprojlim \mathcal{N}^Y \circ \alpha = \alpha_{\mathcal{N}} \circ \varprojlim \mathcal{N}_X.$$

This gives diagram (3.79).

(iii) The theorem on well-ordering of the class of all sets [25, V, 4.1] allows us to define the operation of taking local limit as a map:

$$X \mapsto \varprojlim \mathsf{Bind}(\mathcal{N}^X)$$

(i.e. there is a map that assigns to each $X \in \mathsf{Ob}(\mathtt{K})$ a concrete projective limit of the subcategory $\mathsf{Bind}(\mathcal{N}^X)$ among all its projective limits in $\mathtt{K}$). Let us show that in this case the resulting map $(X,\alpha)\mapsto(X_{\mathcal N},\alpha_{\mathcal N})$ is a functor, that is,

$$(1_X)_{\mathcal N}=1_{X_{\mathcal N}},\qquad(\beta\circ\alpha)_{\mathcal N}=\beta_{\mathcal N}\circ\alpha_{\mathcal N}.\tag{3.90}$$

Suppose first that $\alpha=1_X:X\to X$. Then

$$\alpha^\tau\overset{(3.80)}{=}\tau\circ\alpha=\tau\circ 1_X=\tau\ \Rightarrow\ \alpha^\tau_\sigma\circ\sigma\overset{(3.82)}{=}\alpha^\tau=\tau\overset{(2.18)}{=}\iota^\tau_\sigma\circ\sigma$$
$$\Rightarrow\ \alpha^\tau_\sigma=\iota^\tau_\sigma\ \Rightarrow\ \alpha^\tau_{\mathcal N}=\iota^\tau_\sigma\circ\sigma_{\mathcal N}=\tau_{\mathcal N}.$$

So in diagrams (3.89) we can replace $\alpha^\tau_{\mathcal N}$ by $\tau_{\mathcal N}$:

$$\begin{tikzcd} X_{\mathcal N} \arrow[rr, dashed, "\alpha_{\mathcal N}"] \arrow[dr, "\tau_{\mathcal N}"'] & & X_{\mathcal N} \arrow[dl, "\tau_{\mathcal N}"] \\ & \mathsf{Ran}\,\tau & \end{tikzcd}\qquad(\tau\in\mathcal N^X)$$

These diagrams are commutative for all $\tau\in\mathcal N^X$, and the dashed arrow $\alpha_{\mathcal N}$ is defined as the lifting of the projective cone $\{\alpha^\tau_{\mathcal N}=\tau_{\mathcal N}:X_{\mathcal N}\to\mathsf{Ran}\,\tau\}$ to the projective limit $\{\tau_{\mathcal N}:X_{\mathcal N}\to\mathsf{Ran}\,\tau\}$. Such an arrow is unique, so it must coincide with the morphism $1_{X_{\mathcal N}}$, for which all these diagrams are trivially commutative.

Let us now prove the second identity in (3.90). Consider morphisms $X\xrightarrow{\alpha}Y\xrightarrow{\beta}Z$. Take $\upsilon\in\mathcal N^Z$ and, using property (c), choose first $\tau\in\mathcal N^Y$ and a morphism β^υ_τ such that $\upsilon\circ\beta=\beta^\upsilon_\tau\circ\tau$, and then, again using (c), choose $\sigma\in\mathcal N^X$ and a morphism α^τ_σ such that $\tau\circ\alpha=\alpha^\tau_\sigma\circ\sigma$. We get

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & Y \arrow[r, "\beta"] \arrow[d, "\tau"] & Z \arrow[d, "\upsilon"] \\ \mathsf{Ran}\,\sigma \arrow[r, "\alpha^\tau_\sigma"] & \mathsf{Ran}\,\tau \arrow[r, "\beta^\upsilon_\tau"] & \mathsf{Ran}\,\upsilon \end{tikzcd}$$

If we remove the middle arrow, then we obtain

$$\begin{tikzcd} X \arrow[r, "\beta\circ\alpha"] \arrow[d, "\sigma"'] & Z \arrow[d, "\upsilon"] \\ \mathsf{Ran}\,\sigma \arrow[r, "\beta^\upsilon_\tau\circ\alpha^\tau_\sigma"] & \mathsf{Ran}\,\upsilon \end{tikzcd}$$

which can be understood as follows: the morphism $\beta^\upsilon_\tau\circ\alpha^\tau_\sigma$ is exactly the composition of the dashed arrows from diagram (3.73), but the difference is that Y is replaced here by Z, α by $\beta\circ\alpha$, and τ by υ. Hence there exists a morphism $(\beta\circ\alpha)^\upsilon_\sigma$ such that

$$\beta^\upsilon_\tau\circ\alpha^\tau_\sigma=(\beta\circ\alpha)^\upsilon_\sigma.\tag{3.91}$$

This equality is used in the following chain:

$$\underbrace{\upsilon_{\mathcal N}\circ\beta_{\mathcal N}}_{\substack{\|\ (3.89)\\ \beta^\upsilon_{\mathcal N}\\ \|\ (3.83)\\ \beta^\upsilon_\tau\circ\tau_{\mathcal N}}}\circ\,\alpha_{\mathcal N}=\beta^\upsilon_\tau\circ\underbrace{\tau_{\mathcal N}\circ\alpha_{\mathcal N}}_{\substack{\|\ (3.89)\\ \alpha^\tau_{\mathcal N}\\ \|\ (3.83)\\ \alpha^\tau_\sigma\circ\sigma_{\mathcal N}}}=\underbrace{\beta^\upsilon_\tau\circ\alpha^\tau_\sigma}_{\substack{\|\ (3.91)\\ (\beta\circ\alpha)^\upsilon_\sigma}}\circ\,\sigma_{\mathcal N}=\underbrace{(\beta\circ\alpha)^\upsilon_\sigma\circ\sigma_{\mathcal N}}_{\substack{\|\ (3.83)\\ (\beta\circ\alpha)^\upsilon_{\mathcal N}}}$$
$$=(\beta\circ\alpha)^\upsilon_{\mathcal N}\overset{(3.89)}{=}\upsilon_{\mathcal N}\circ(\beta\circ\alpha)_{\mathcal N}.$$

If we omit the intermediate calculations, we arrive at

$$\upsilon_{\mathcal{N}} \circ (\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}) = (\beta \circ \alpha)^{\upsilon}_{\mathcal{N}} = \upsilon_{\mathcal{N}} \circ (\beta \circ \alpha)_{\mathcal{N}}.$$

This is true for each $\upsilon \in \mathcal{N}^Z$. So this can be treated as if both $\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}$ and $(\beta \circ \tau)_{\mathcal{N}}$ were liftings of the projective cone $\{(\beta\circ\alpha)^{\upsilon}_{\mathcal{N}} : X_{\mathcal{N}} \to \mathsf{Ran}\,\upsilon;\ \upsilon \in \mathcal{N}^Z\}$ for $\mathsf{Bind}(\mathcal{N}^Z)$ (and this family is indeed a projective cone due to diagram (3.88) where one should replace Y by Z, and α by $\beta \circ \alpha$) to the projective limit of this system. Thus, both $\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}$ and $(\beta \circ \tau)_{\mathcal{N}}$ are exactly the dashed arrow in the definition of projective limit, for which all the diagrams

$$\begin{tikzcd} X_{\mathcal{N}} \arrow[rr, dashed] \arrow[dr, "(\beta\circ\alpha)^{\upsilon}_{\mathcal{N}}"'] & & Z_{\mathcal{N}} \arrow[dl, "\upsilon_{\mathcal{N}}"] \\ & \mathsf{Ran}\,\upsilon & \end{tikzcd} \qquad (\upsilon \in \mathcal{N}^Z)$$

are commutative. But this dashed arrow is unique, so

$$\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}} = (\beta \circ \tau)_{\mathcal{N}}.$$

This is the identity (3.90). ∎

Theorem 3.37. *Let $\mathcal{N}$ be a net of epimorphisms in a category $\mathtt{K}$ that generates a class $\varPhi$ of morphisms on the inside: $\mathcal{N} \subseteq \varPhi \subseteq \mathsf{Mor}(\mathtt{K}) \circ \mathcal{N}$. Then for any class $\varOmega$ of epimorphisms in $\mathtt{K}$ with*

$$\{\varprojlim \mathcal{N}^X;\ X \in \mathsf{Ob}(\mathtt{K})\} \subseteq \varOmega \subseteq \mathsf{Epi}(\mathtt{K}), \tag{3.92}$$

the following hold:

(a) *for each object X in $\mathtt{K}$,*

$$\varprojlim \mathcal{N}^X = \mathsf{env}^{\varOmega}_{\varPhi} X; \tag{3.93}$$

(b) *the envelope $\mathsf{Env}^{\varOmega}_{\varPhi}$ can be defined as a functor.*

Proof. By Theorem 3.36,

$$\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}} X := \mathsf{env}^{\mathsf{Mor}(\mathtt{K})}_{\mathcal{N}} X.$$

On the other hand, by (i), $\varprojlim \mathcal{N}^X$ belongs to a narrower class $\varOmega$, so by 1°(c) on p. 44,

$$\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}} X = \mathsf{env}^{\mathsf{Mor}(\mathtt{K})}_{\mathcal{N}} X = \mathsf{env}^{\varOmega}_{\mathcal{N}} X.$$

Further, since $\mathcal{N}$ generates $\varPhi$ on the inside, and $\varOmega$ consists of epimorphisms, by (3.14) we have

$$\varprojlim \mathcal{N}^X = \mathsf{env}_{\mathcal{N}} X = \mathsf{env}^{\mathsf{Mor}(\mathtt{K})}_{\mathcal{N}} X = \mathsf{env}^{\varOmega}_{\mathcal{N}} X = \mathsf{env}^{\varOmega}_{\varPhi} X.$$

This proves (3.93). Part (b) follows from Theorem 3.36(iii). ∎

One can get rid of the left side of (3.92) if the class $\varOmega$ is monomorphically complementable:

Theorem 3.38. *Let $\mathcal{N}$ be a net of epimorphisms in a category $\mathtt{K}$ that generates a class $\varPhi$ of morphisms on the inside: $\mathcal{N} \subseteq \varPhi \subseteq \mathsf{Mor}(\mathtt{K}) \circ \mathcal{N}$. Then for each monomorphically complementable* ([8]) *class $\varOmega$ of epimorphisms the following hold:*

([8]) See definition on p. 41.

(a) *for each object X in $\mathtt{K}$ the morphism $\varepsilon_{\varprojlim \mathcal{N}^X}$ in the factorization (2.39) defined by the classes ${}^{\downarrow}\Omega$ and Ω is an envelope $\mathsf{env}^{\Omega}_{\Phi} X$ in Ω with respect to Φ:*

$$\varepsilon_{\varprojlim \mathcal{N}^X} = \mathsf{env}^{\Omega}_{\Phi} X; \tag{3.94}$$

(b) *for each morphism $\alpha : X \to Y$ in $\mathtt{K}$ and any choice of $\mathsf{env}^{\Omega}_{\Phi} X$ and $\mathsf{env}^{\Omega}_{\Phi} Y$ there exists a unique morphism $\mathsf{Env}^{\Omega}_{\Phi}\,\alpha : \mathsf{Env}^{\Omega}_{\Phi} X \to \mathsf{Env}^{\Omega}_{\Phi} Y$ in $\mathtt{K}$ such that*

$$\begin{tikzcd}
X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[d, "\alpha"'] & \mathsf{Env}^{\Omega}_{\Phi} X \arrow[d, dashed, "\mathsf{Env}^{\Omega}_{\Phi}\,\alpha"] \\
Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y
\end{tikzcd} \tag{3.95}$$

(c) *if $\mathtt{K}$ is co-well-powered in Ω, then $\mathsf{Env}^{\Omega}_{\Phi}$ can be defined as a functor.*

Proof. (a) Since $\mathcal{N}$ generates Φ, and Ω consists of epimorphisms, by (3.14) we have $\mathsf{env}^{\Omega}_{\mathcal{N}} X = \mathsf{env}^{\Omega}_{\Phi} X$. Hence (3.46) implies (3.94):

$$\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Omega}_{\mathcal{N}} X = \varepsilon_{\varprojlim \mathcal{N}_X}.$$

(b) The property (3.95) is proved as follows. First we add diagram (3.79) by decomposing $\varprojlim \mathcal{N}_X$ and $\varprojlim \mathcal{N}^Y$:

$$\begin{tikzcd}
X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"'] \arrow[d, "\alpha"'] \arrow[rr, bend left, "\varprojlim \mathcal{N}^X"] & \mathsf{Env}^{\Omega}_{\Phi} X = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^X} \arrow[r, "\mu_{\varprojlim \mathcal{N}^X}"'] & X_{\mathcal{N}} \arrow[d, "\alpha_{\mathcal{N}}"] \\
Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] \arrow[rr, bend right, "\varprojlim \mathcal{N}^Y"'] & \mathsf{Env}^{\Omega}_{\Phi} Y = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^Y} \arrow[r, "\mu_{\varprojlim \mathcal{N}^Y}"] & Y_{\mathcal{N}}
\end{tikzcd}$$

Then we represent the inner quadrangle as

$$\begin{tikzcd}
X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[d, "\alpha"'] & \mathsf{Env}^{\Omega}_{\Phi} X = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^X} \arrow[dr, "\alpha_{\mathcal{N}} \circ \mu_{\varprojlim \mathcal{N}^X}"] & \\
Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^Y} \arrow[r, "\mu_{\varprojlim \mathcal{N}^Y}"] & Y_{\mathcal{N}}
\end{tikzcd}$$

Here the upper horizontal arrow, $\mathsf{env}^{\Omega}_{\Phi} X$, belongs to Ω, and the second lower horizontal arrow, $\mu_{\varprojlim \mathcal{N}^Y}$, belongs to $\Gamma = \Omega^{\downarrow}$. Hence there exists a morphism ξ such that

$$\begin{tikzcd}
X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[d, "\alpha"'] & \mathsf{Env}^{\Omega}_{\Phi} X = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^X} \arrow[d, dashed, "\xi"] \arrow[dr, "\alpha_{\mathcal{N}} \circ \mu_{\varprojlim \mathcal{N}^X}"] & \\
Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y = \mathsf{Dom}\,\mu_{\varprojlim \mathcal{N}^Y} \arrow[r, "\mu_{\varprojlim \mathcal{N}^Y}"] & Y_{\mathcal{N}}
\end{tikzcd}$$

This ξ will be the vertical arrow in (3.95) that we need.

(c) Let K be co-well-powered in Ω, i.e. for each object X the category $\Omega^X = \Omega \cap \mathsf{Epi}^X$ is skeletally small. Let S_X be its skeleton, which is a set. Using Theorem 2.31, we can fix a map $X \mapsto S_X$ which assigns to each object X a skeleton S_X in Ω^X. To define the envelope $\mathsf{Env}^{\Omega}_{\Phi}$ as a functor, we now define (by the axiom of choice) a map $X \in \mathsf{Ob}(\mathsf{K}) \mapsto \mathsf{env}^{\Omega}_{\Phi} X \in S_X$. Then the object $\mathsf{Env}^{\Omega}_{\Phi} X$ is defined as the domain of the morphism $\mathsf{env}^{\Omega}_{\Phi} X$, and the morphism $\mathsf{Env}^{\Omega}_{\Phi} \alpha$ in (3.95) arises automatically (as the unique possible morphism). ∎

3.4.2. Nets of monomorphisms. Suppose that to each object X in a category K there is assigned a subset $\mathcal{N}_X$ in the class Mono_X of all monomorphisms of K coming to X, and the following three requirements are fulfilled:

(a) for each object X the set $\mathcal{N}_X$ is non-empty and is directed to the right with respect to pre-order (2.12) inherited from Mono_X:

$$\forall \rho, \rho' \in \mathcal{N}_X \ \exists \sigma \in \mathcal{N}_X \quad \rho \to \sigma \ \& \ \rho' \to \sigma;$$

(b) for each object X the covariant system of morphisms generated by the set $\mathcal{N}_X$ given by

$$\mathsf{Bind}(\mathcal{N}_X) := \{\varkappa^{\sigma}_{\rho};\ \rho, \sigma \in \mathcal{N}_X,\ \rho \to \sigma\} \tag{3.96}$$

(the morphisms $\varkappa^{\sigma}_{\rho}$ were defined in (2.13); according to (2.14), this system is a covariant functor from $\mathcal{N}_X$ considered as a full subcategory in Mono_X into K) has an injective limit in K;

(c) for each morphism $\alpha : X \to Y$ and each $\sigma \in \mathcal{N}_X$ there is $\tau \in \mathcal{N}_Y$ and a morphism $\alpha^{\tau}_{\sigma} : \mathsf{Dom}\,\sigma \to \mathsf{Dom}\,\tau$ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \\ \mathsf{Dom}\,\sigma \arrow[u, "\sigma"] \arrow[r, dashed, "\alpha^{\tau}_{\sigma}"] & \mathsf{Dom}\,\tau \arrow[u, dashed, "\tau"'] \end{tikzcd} \tag{3.97}$$

(for given α, σ and τ the morphism α^{τ}_{σ}, if it exists, is unique, since τ is a monomorphism).

Then:

- We call the family $\mathcal{N} = \{\mathcal{N}_X;\ X \in \mathsf{Ob}(\mathsf{K})\}$ of sets a *net of monomorphisms* in K, and the elements of the sets $\mathcal{N}_X$ *elements of the net* $\mathcal{N}$.
- For each X the system $\mathsf{Bind}(\mathcal{N}_X)$ defined by (3.96) will be called a *system of binding morphisms of the net* $\mathcal{N}$ *over the vertex* X. Its injective limit (which exists by (b)) is an injective cone whose vertex will be denoted by $X_{\mathcal{N}}$, and the morphisms coming to it by $\rho_{\mathcal{N}} = \varinjlim_{\sigma \in \mathcal{N}_X} \varkappa^{\sigma}_{\rho} : X_{\mathcal{N}} \leftarrow \mathsf{Ran}\,\sigma$:

$$\begin{tikzcd} & X_{\mathcal{N}} & \\ \mathsf{Dom}\,\rho \arrow[ur, "\rho_{\mathcal{N}}"] \arrow[rr, "\varkappa^{\sigma}_{\rho}"] & & \mathsf{Dom}\,\sigma \arrow[ul, "\sigma_{\mathcal{N}}"'] \end{tikzcd} \qquad (\rho \to \sigma) \tag{3.98}$$

In addition, by (2.13), the system $\mathcal{N}_X$ is also an injective cone of $\mathsf{Bind}(\mathcal{N}_X)$:

$$\begin{tikzcd} & X & \\ \mathsf{Dom}\,\rho \arrow[ur, "\rho"] \arrow[rr, "\varkappa_\rho^\sigma"] & & \mathsf{Dom}\,\sigma \arrow[ul, "\sigma"'] \end{tikzcd} \qquad (\rho \to \sigma) \tag{3.99}$$

so there exists a natural morphism into X from the vertex $X_\mathcal{N}$ of the injective limit of $\mathsf{Bind}(\mathcal{N}_X)$. This morphism will be denoted by $\varinjlim \mathcal{N}_X$ and called a *local limit of the net $\mathcal{N}$ at the object X*:

$$\begin{tikzcd} X_\mathcal{N} \arrow[rr, dashed, "\varinjlim \mathcal{N}_X"] & & X \\ & \mathsf{Dom}\,\sigma \arrow[ul, "\sigma_\mathcal{N}"] \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad (\sigma \in \mathcal{N}_X) \tag{3.100}$$

– The element τ of the net in diagram (3.97) will be called a *shed* for the element σ of the net.

The following results are dual to Theorems 3.36–3.38.

THEOREM 3.39. *Let $\mathcal{N}$ be a net of monomorphisms in a category* $\mathtt{K}$. *Then:*

(i) *for each object X in $\mathtt{K}$ the local limit $\varinjlim \mathcal{N}_X : X_\mathcal{N} \to X$ is a refinement $\mathsf{ref}_\mathcal{N} X$ of X in $\mathtt{K}$ by means of the class $\mathcal{N}$:*

$$\varinjlim \mathcal{N}_X = \mathsf{ref}_\mathcal{N} X; \tag{3.101}$$

(ii) *for each morphism $\alpha : X \to Y$ in $\mathtt{K}$ and for choice of local limits $\varinjlim \mathcal{N}_X$ and $\varinjlim \mathcal{N}_Y$ the formula*

$$\alpha_\mathcal{N} = \varinjlim_{\sigma \in \mathcal{N}_X} \varinjlim_{\tau \in \mathcal{N}_Y} \tau_\mathcal{N} \circ \alpha_\sigma^\tau \tag{3.102}$$

defines a morphism $\alpha_\mathcal{N} : X_\mathcal{N} \to Y_\mathcal{N}$ such that

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & X_\mathcal{N} = \mathsf{ref}_\mathcal{N} X \arrow[l, "\varinjlim \mathcal{N}_X = \mathsf{ref}_\mathcal{N} X"'] \arrow[d, dashed, "\alpha_\mathcal{N}"] \\ Y & Y_\mathcal{N} = \mathsf{ref}_\mathcal{N} Y \arrow[l, "\varinjlim \mathcal{N}_Y = \mathsf{ref}_\mathcal{N} Y"'] \end{tikzcd} \tag{3.103}$$

(iii) *the refinement $\mathsf{Ref}_\mathcal{N}$ can be defined as a functor.*

THEOREM 3.40. *Let $\mathcal{N}$ be a net of monomorphisms in a category $\mathtt{K}$ that generates a class $\varPhi$ of morphisms on the outside: $\mathcal{N} \subseteq \varPhi \subseteq \mathcal{N} \circ \mathsf{Mor}(\mathtt{K})$. Then for every class $\varGamma$ of monomorphisms in $\mathtt{K}$ that contains all local limits, the following hold:*

(a) *for each object X in* $\mathtt{K}$,

$$\varinjlim \mathcal{N}_X = \mathsf{ref}_\varPhi^\varGamma X; \tag{3.104}$$

(b) *the refinement $\mathsf{Ref}_\varPhi^\varGamma$ can be defined as a functor.*

THEOREM 3.41. *Let $\mathcal{N}$ be a net of monomorphisms in a category* $\mathtt{K}$ *that generates the class Φ on the outside:* $\mathcal{N} \subseteq \Phi \subseteq \mathcal{N} \circ \mathsf{Mor}(\mathtt{K})$. *Then for every epimorphically complementable* ([9]) *class Γ of monomorphisms the following hold:*

(a) *in* $\mathtt{K}$ *there exists a net $\mathcal{N}$ of monomorphisms such that for any object X in* $\mathtt{K}$ *the morphism $\mu_{\underrightarrow{\lim}\,\mathcal{N}_X}$ in the factorization* (2.39) *is a refinement $\mathsf{ref}^{\Gamma}_{\Phi} X$ in Γ by means of Φ:*

$$\mu_{\underrightarrow{\lim}\,\mathcal{N}_X} = \mathsf{ref}^{\Gamma}_{\Phi} X, \tag{3.105}$$

(b) *for each morphism $\alpha : X \to Y$ in* $\mathtt{K}$ *and any choice of $\mathsf{ref}^{\Gamma}_{\Phi} X$ and $\mathsf{ref}^{\Gamma}_{\Phi} Y$ there is a unique morphism $\mathsf{Ref}^{\Gamma}_{\Phi}\,\alpha : \mathsf{Ref}^{\Gamma}_{\Phi} X \to \mathsf{Ref}^{\Gamma}_{\Phi} Y$ in* $\mathtt{K}$ *such that*

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & \mathsf{Ref}^{\Gamma}_{\Phi} X \arrow[l, "\mathsf{ref}^{\Gamma}_{\Phi} X"'] \arrow[d, dashed, "\mathsf{Ref}^{\Gamma}_{\Phi}\,\alpha"] \\ Y & \mathsf{Ref}^{\Gamma}_{\Phi} Y \arrow[l, "\mathsf{ref}^{\Gamma}_{\Phi} Y"'] \end{tikzcd} \tag{3.106}$$

(c) *if* $\mathtt{K}$ *is well-powered in Γ, then $\mathsf{Ref}^{\Gamma}_{\Phi}$ can be defined as a functor.*

3.4.3. Existence of nets of epimorphisms and semiregular envelopes

THEOREM 3.42. *Suppose a category* $\mathtt{K}$ *and classes Ω and Φ of morphisms in it satisfy the following conditions:*

RE.1. $\mathtt{K}$ *is projectively complete;*
RE.2. Ω *is monomorphically complementable;*
RE.3. $\mathtt{K}$ *is co-well-powered in Ω;*
RE.4. Φ *goes from* ([10]) $\mathsf{Ob}(\mathtt{K})$ *and is a right ideal in* $\mathtt{K}$.

Then:

(a) *there is a net $\mathcal{N}$ of epimorphisms in* $\mathtt{K}$ *such that for each object X in* $\mathtt{K}$ *the morphism $\varepsilon_{\underleftarrow{\lim}\,\mathcal{N}^X}$ in the factorization* (2.39) *is an envelope $\mathsf{env}^{\Omega}_{\Phi} X$ in Ω with respect to Φ:*

$$\varepsilon_{\underleftarrow{\lim}\,\mathcal{N}^X} = \mathsf{env}^{\Omega}_{\Phi} X; \tag{3.107}$$

(b) *for each morphism $\alpha : X \to Y$ in* $\mathtt{K}$ *and any choice of $\mathsf{env}^{\Omega}_{\Phi} X$ and $\mathsf{env}^{\Omega}_{\Phi} Y$ there exists a unique morphism $\mathsf{Env}^{\Omega}_{\Phi}\,\alpha : \mathsf{Env}^{\Omega}_{\Phi} X \to \mathsf{Env}^{\Omega}_{\Phi} Y$ in* $\mathtt{K}$ *such that*

$$\begin{tikzcd} X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[d, "\alpha"'] & \mathsf{Env}^{\Omega}_{\Phi} X \arrow[d, dashed, "\mathsf{Env}^{\Omega}_{\Phi}\,\alpha"] \\ Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y \end{tikzcd} \tag{3.108}$$

(c) *the envelope $\mathsf{Env}^{\Omega}_{\Phi}$ can be defined as a functor.*

- If RE.1–RE.4 are fulfilled, then we say that *the classes Ω and Φ define a semiregular envelope in* $\mathtt{K}$, or *the envelope $\mathsf{Env}^{\Omega}_{\Phi}$ is semiregular.*

([9]) See definition on p. 41.
([10]) In the sense of the definition on p. 9.

Proof of Theorem 3.42. (a) By RE.3, for each X the category $\Omega^X = \Omega \cap \mathsf{Epi}^X$ is skeletally small. Let S_X be its skeleton (which is a set). Using Theorem 2.31, we can choose a map $X \mapsto S_X$ that to each object X assigns a skeleton S_X in Ω^X.

For every object X in $\mathtt{K}$ set $\Phi^X = \{\varphi \in \Phi;\ \mathsf{Dom}\,\varphi = X\}$ (from RE.4 it follows that $\Phi^X \neq \emptyset$) and denote by 2_{Φ^X} the class of finite subsets in Φ^X. To each object X in $\mathtt{K}$ and each morphism $\Psi \in 2_{\Phi^X}$ we assign a morphism

$$\overline{\Psi} = \prod_{\psi \in \Psi} \psi : X \to \prod_{\psi \in \Psi} \mathsf{Ran}\,\psi,$$

and morphisms $\mu_\Psi \in \Gamma$ and $\varepsilon_\Psi \in S_X$ such that

$$\overline{\Psi} = \mu_\Psi \circ \varepsilon_\Psi \tag{3.109}$$

(since S_X is a skeleton in Ω^X, such morphisms are unique). Let $\mathcal{N}^X = \{\varepsilon_\Psi;\ \Psi \in 2_{\Phi^X}\}$. Since $\mathcal{N}^X \subseteq S_X$, this is a set, and since the correspondence $X \mapsto S_X$ is a map, we obtain a map $X \mapsto \mathcal{N}^X$.

(b) Let us check that $\mathcal{N}$ satisfies the axioms of a net of epimorphisms (p. 70). First, we show that $\mathcal{N}^X$ is directed to the left with respect to the pre-order (2.17) inherited from Epi^X. For any $\Psi, \Psi' \in 2_{\Phi^X}$ consider the diagram

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\overline{\Psi}"'] \arrow[d, "\overline{\Psi\cup\Psi'}"] \arrow[dr, bend left, "\overline{\Psi'}"] & \\
\mathsf{Ran}\,\overline{\Psi} \arrow[d, equal] & \mathsf{Ran}\,\overline{\Psi\cup\Psi'} \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, equal] & \mathsf{Ran}\,\overline{\Psi'} \arrow[d, equal] \\
\prod_{\psi\in\Psi} \mathsf{Ran}\,\psi & \prod_{\psi\in\Psi\cup\Psi'} \mathsf{Ran}\,\psi & \prod_{\psi\in\Psi'} \mathsf{Ran}\,\psi
\end{tikzcd}$$

where π and π' are natural projections. Let us decompose the arrows going from X by using (3.109):

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\varepsilon_\Psi"'] \arrow[d, "\varepsilon_{\Psi\cup\Psi'}"] \arrow[dr, bend left, "\varepsilon_{\Psi'}"] & \\
\mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"'] & \mathsf{Ran}\,\varepsilon_{\Psi\cup\Psi'} \arrow[d, "\mu_{\Psi\cup\Psi'}"] & \mathsf{Ran}\,\varepsilon_{\Psi'} \arrow[d, "\mu_{\Psi'}"] \\
\mathsf{Ran}\,\overline{\Psi} & \mathsf{Ran}\,\overline{\Psi\cup\Psi'} \arrow[l, "\pi"'] \arrow[r, "\pi'"] & \mathsf{Ran}\,\overline{\Psi'}
\end{tikzcd} \tag{3.110}$$

We represent the left side as a quadrangle:

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\varepsilon_\Psi"'] \arrow[d, "\varepsilon_{\Psi\cup\Psi'}"] \\
\mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"'] & \mathsf{Ran}\,\varepsilon_{\Psi\cup\Psi'} \arrow[dl, bend left, "\pi\circ\mu_{\Psi\cup\Psi'}"] \\
\mathsf{Ran}\,\overline{\Psi} &
\end{tikzcd}$$

Here $\varepsilon_{\Psi\cup\Psi'}$ is an epimorphism, and μ_Ψ a strong monomorphism, hence there exists a leftward horizontal arrow:

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\varepsilon_\Psi"'] \arrow[d, "\varepsilon_{\Psi\cup\Psi'}"] \\
\mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"'] & \mathsf{Ran}\,\varepsilon_{\Psi\cup\Psi'} \arrow[l, dashed, "\delta"'] \arrow[dl, bend left, "\pi\circ\mu_{\Psi\cup\Psi'}"] \\
\mathsf{Ran}\,\overline{\Psi} &
\end{tikzcd}$$

For the same reason there is a rightward arrow in (3.110), and we obtain a diagram

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\varepsilon_\Psi"'] \arrow[d, "\varepsilon_{\Psi\cup\Psi'}"] \arrow[dr, bend left, "\varepsilon_{\Psi'}"] & \\
\mathsf{Ran}\,\varepsilon_\Psi & \mathsf{Ran}\,\varepsilon_{\Psi\cup\Psi'} \arrow[l, dashed, "\delta"'] \arrow[r, dashed, "\delta'"] & \mathsf{Ran}\,\varepsilon_{\Psi'}
\end{tikzcd}$$

This means that in the category Ω^X the morphism $\varepsilon_{\Psi\cup\Psi'}$ majorizes ε_Ψ and $\varepsilon_{\Psi'}$:

$$\varepsilon_{\Psi\cup\Psi'} \to \varepsilon_\Psi, \qquad \varepsilon_{\Psi\cup\Psi'} \to \varepsilon_{\Psi'}.$$

The second condition in the definition of the net of epimorphisms is fulfilled automatically: since K is projectively complete, the system $\mathsf{Bind}(\mathcal{N}^X)$, defined in (3.72), always has a projective limit.

Let us check the third condition. Let $\alpha : X \to Y$ be a morphism in $\Psi \in 2_{\Phi_Y}$. By RE.4, Φ is a right ideal, hence for each $\psi \in \Psi \subseteq \Phi$ the composition $\psi \circ \alpha$ belongs to Φ, and we can consider the set $\Psi \circ \alpha \in 2_{\Phi X}$. We obtain the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, bend right, "\prod_{\psi\in\Psi}(\psi\circ\alpha)=\overline{\Psi\circ\alpha}"'] & Y \arrow[d, "\overline{\Psi}=\prod_{\psi\in\Psi}\psi"] & \\
 & \mathsf{Ran}\,\overline{\Psi} \arrow[r, equal] & \prod_{\psi\in\Psi}\mathsf{Ran}\,\psi
\end{tikzcd}$$

Let us represent the morphisms coming to $\mathsf{Ran}\,\overline{\Psi}$ as their factorizations (3.109):

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\varepsilon_{\Psi\circ\alpha}"'] & Y \arrow[d, "\varepsilon_\Psi"] \\
\mathsf{Ran}\,\varepsilon_{\Psi\circ\alpha} \arrow[dr, bend right, "\mu_{\Psi\circ\alpha}"'] & \mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"] \\
 & \mathsf{Ran}\,\overline{\Psi}
\end{tikzcd}$$

This diagram can be represented as a quadrangle

$$\begin{tikzcd}
X \arrow[dr, bend left, "\varepsilon_\Psi\circ\alpha"] \arrow[d, "\varepsilon_{\Psi\circ\alpha}"'] & \\
\mathsf{Ran}\,\varepsilon_{\Psi\circ\alpha} \arrow[dr, bend right, "\mu_{\Psi\circ\alpha}"'] & \mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"] \\
 & \mathsf{Ran}\,\overline{\Psi}
\end{tikzcd}$$

where $\varepsilon_{\Psi\circ\alpha} \in \Omega$ and $\mu_\Psi \in \Gamma = \Omega^\downarrow$. So there must exist a rightward horizontal arrow:

$$\begin{tikzcd}
X \arrow[d, "\varepsilon_{\Psi\circ\alpha}"'] \arrow[dr, bend left, "\varepsilon_\Psi\circ\alpha"] & \\
\mathsf{Ran}\,\varepsilon_{\Psi\circ\alpha} \arrow[r, dashed, "\delta"] \arrow[dr, bend right, "\mu_{\Psi\circ\alpha}"'] & \mathsf{Ran}\,\varepsilon_\Psi \arrow[d, "\mu_\Psi"] \\
& \mathsf{Ran}\,\overline{\Psi}
\end{tikzcd}$$

This will be the horizontal arrow that we need in (3.73):

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\varepsilon_{\Psi\circ\alpha}"'] & Y \arrow[d, "\varepsilon_\Psi"] \\
\mathsf{Ran}\,\varepsilon_{\Psi\circ\alpha} \arrow[r, dashed, "\delta"] & \mathsf{Ran}\,\varepsilon_\Psi
\end{tikzcd}$$

(c) Note further that

$$\mathsf{env}^{\Omega}_{\Phi} X = \mathsf{env}^{\Omega}_{2_\Phi} X \overset{(3.51)}{=} \mathsf{env}^{\Omega}_{\{\varepsilon_\Psi\,;\,\Psi\in 2_\Phi\}} X = \mathsf{env}^{\Omega}_{\mathcal{N}} X.$$

Now the proof of Theorem 3.38 can be applied. ∎

3.4.4. Existence of nets of monomophisms and semiregular refinements. The dual result for refinements look as follows:

THEOREM 3.43. *Suppose a category* $\mathtt{K}$ *and classes* Γ *and* Φ *of morphisms satisfy the following conditions:*

RR.1. $\mathtt{K}$ *is injectively complete;*
RR.2. Γ *is epimorphically complementable in* $\mathtt{K}$*;*
RR.3. $\mathtt{K}$ *is well-powered in* Γ*;*
RR.4. Φ *goes to* ([11]) $\mathsf{Ob}(\mathtt{K})$ *and is a left ideal in* $\mathtt{K}$.

Then:

(a) *there exists a net* $\mathcal{N}$ *of monomorphisms in* $\mathtt{K}$ *such that for each object* X *in* $\mathtt{K}$ *the morphism* $\mu_{\varinjlim \mathcal{N}_X}$ *in the factorization* (2.39) *is a refinement* $\mathsf{ref}^{\Gamma}_{\Phi} X$ *in* Γ *by means of* Φ*:*

$$\mu_{\varinjlim \mathcal{N}_X} = \mathsf{ref}^{\Gamma}_{\Phi} X; \tag{3.111}$$

(b) *for each morphism* $\alpha : X \to Y$ *in* $\mathtt{K}$ *and any choice of* $\mathsf{ref}^{\Gamma}_{\Phi} X$ *and* $\mathsf{ref}^{\Gamma}_{\Phi} Y$ *there is a unique morphism* $\mathsf{Ref}^{\Gamma}_{\Phi}\,\alpha : \mathsf{Ref}^{\Gamma}_{\Phi} X \to \mathsf{Ref}^{\Gamma}_{\Phi} Y$ *in* $\mathtt{K}$ *such that*

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] & \mathsf{Ref}^{\Gamma}_{\Phi} X \arrow[l, "\mathsf{ref}^{\Gamma}_{\Phi} X"'] \arrow[d, dashed, "\mathsf{Ref}^{\Gamma}_{\Phi}\,\alpha"] \\
Y & \mathsf{Ref}^{\Gamma}_{\Phi} Y \arrow[l, "\mathsf{ref}^{\Gamma}_{\Phi} Y"']
\end{tikzcd} \tag{3.112}$$

(c) *the refinement* $\mathsf{Ref}^{\Gamma}_{\Phi}$ *can be defined as a functor.*

- If RR.1–RR.4 are fulfilled, then we say that *the classes* Γ *and* Φ *define a semiregular refinement* $\mathsf{Ref}^{\Gamma}_{\Phi}$ *in* $\mathtt{K}$, or *the refinement* $\mathsf{Ref}^{\Gamma}_{\Phi}$ *is semiregular.*

([11]) In the sense of the definition on p. 9.

3.4.5. Pushing, regular envelope and complete objects

• Let us say that *a class Ω of morphisms pushes a class Φ of morphisms* if

$$\forall \psi \in \mathsf{Mor}(\mathtt{K})\ \forall \sigma \in \Omega \quad (\psi \circ \sigma \in \Phi \Rightarrow \psi \in \Phi). \tag{3.113}$$

Remark 3.44. Obviously, (3.113) holds if $\Phi = \{\varphi \in \mathsf{Mor}(\mathtt{K});\ \mathsf{Ran}\,\varphi \in \mathtt{M}\}$ for some class $\mathtt{M}$ of objects in $\mathtt{K}$.

Lemma 3.45. *If Ω pushes Φ, then the composition $\sigma \circ \rho : X \to X''$ of any two extensions $\rho : X \to X'$ and $\sigma : X' \to X''$ (in Ω with respect to Φ) is an extension (in Ω with respect to Φ).*

Proof. This is seen from the diagram

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, bend right, "\varphi"'] & X' \arrow[r, "\sigma"] \arrow[d, dashed, "\varphi'"] & X'' \arrow[dl, dashed, "\varphi''"] \\ & \mathtt{M} & \end{tikzcd}$$

Since ρ is an extension, for any $\varphi \in \Phi$ there exists φ', and since Ω pushes Φ, we have $\varphi' \in \Phi$. Then since σ is an extension, there exists φ''. This way every next arrow is uniquely defined by the previous one. ∎

Proposition 3.46. *Suppose $\Omega \subseteq \mathsf{Epi}$. Then for each $A \in \mathsf{Ob}(\mathtt{K})$ the following conditions are equivalent:*

(i) *each extension $\sigma : A \to A'$ in Ω with respect to Φ is an isomorphism;*
(ii) *the local identity $1_A : A \to A$ is an envelope of A in Ω with respect to Φ;*
(iii) *there exists $\mathsf{env}^{\Omega}_{\Phi} A \in \mathsf{Iso}$.*

If in addition Ω pushes Φ, then these conditions are equivalent to:

(iv) *$A \cong \mathsf{Env}^{\Omega}_{\Phi} X$ for some $X \in \mathsf{Ob}(\mathtt{K})$.*

• We will say that an object A in $\mathtt{K}$ is *complete* in a class $\Omega \subseteq \mathsf{Epi}$ with respect to a class Φ if it satisfies the above properties (i)–(iii).

Proof of Proposition 3.46. (i)⇒(ii). It follows from (i) that for the local identity $1_A : A \to A$ (which is also an extension) we have the diagram

$$\begin{tikzcd} & A \arrow[dl, "\sigma"'] \arrow[dr, "1_A"] & \\ A' \arrow[rr, "\sigma^{-1}"] & & A \end{tikzcd}$$

which can be considered as the special case of (3.4), and this means that 1_A is an envelope.

(ii)⇒(iii) is obvious.

(iii)⇒(i). Let $\rho : A \to E$ be an envelope and at the same time an isomorphism. Then for any extension $\sigma : A \to A'$ we can take a morphism υ in (3.4) and we get $\upsilon \circ \sigma = \rho \in \mathsf{Iso}$, hence σ is a coretraction. On the other hand, $\sigma \in \Omega \subseteq \mathsf{Epi}$, hence $\sigma \in \mathsf{Iso}$.

(iii)⇒(iv) is also obvious: if $\mathsf{env}^{\Omega}_{\Phi} A \in \mathsf{Iso}$, then $A \cong \mathsf{Env}^{\Omega}_{\Phi} A$.

It is sufficient to prove (iv)⇒(i) in the case when Ω pushes Φ. Suppose that $A \cong \mathsf{Env}^{\Omega}_{\Phi} X$ for some $X \in \mathsf{Ob}(\mathtt{K})$. Then A can be considered as an envelope of X, i.e. there exists a morphism $\rho : X \to A$ which is an envelope. Take any extension $\sigma : A \to A'$ of A. By Lemma 3.45, the composition $\sigma \circ \rho : X \to A'$ is an extension of X, so there is a morphism υ such that (3.4) is commutative:

$$\begin{tikzcd} & X \arrow[dl, "\sigma\circ\rho"'] \arrow[dr, "\rho"] & \\ A' \arrow[rr, dashed, "\upsilon"] & & A \end{tikzcd}$$

Now we have

$$\upsilon \circ \sigma \circ \rho = \rho = 1_A \circ \underset{\substack{\Cap \\ \mathsf{Epi}}}{\rho} \ \Rightarrow\ \upsilon \circ \sigma = 1_A.$$

In the last equality, υ is unique, since σ is an epimorphism. We observe that the extension σ is subordinated to the extension 1_A, and since this is true for each σ, the morphism ρ is an envelope of A. ■

Let us denote by $\mathtt{L}$ the class of complete objects in $\mathtt{K}$ (in Ω with respect to Φ). We consider $\mathtt{L}$ as a full subcategory in $\mathtt{K}$.

Proposition 3.47. *Under the conditions of Theorem* 3.42 *the functor of envelope* $(X, \alpha) \mapsto (E(X), E(\alpha))$ *on* $\mathtt{L} \subseteq \mathtt{K}$ *is isomorphic to the identity functor:*

$$\forall A \in \mathtt{L} \quad E(A) \cong A, \qquad \forall \alpha : \underset{\substack{\Cap \\ \mathtt{L}}}{A} \to \underset{\substack{\Cap \\ \mathtt{L}}}{A'} \quad E(\alpha) = e_{A'} \circ \alpha \circ e_A^{-1}. \tag{3.114}$$

Proof. Take an arbitrary morphism $\alpha : A \to A'$ in $\mathtt{L}$, i.e. a morphism in $\mathtt{K}$ whose domain and range belong to $\mathtt{L}$. Then in diagram (3.64) the horizontal arrows are isomorphisms, so

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & A \arrow[l, "e_A^{-1}"'] \arrow[d, "E(\alpha)"] \\ A' \arrow[r, "e_{A'}"] & A' \end{tikzcd}$$

■

- We say that *classes* Ω *and* Φ *define a regular envelope in a category* $\mathtt{K}$, or the *envelope* Env_Φ^Ω *is regular*, if in addition to conditions RE.1–RE.4 of Theorem 3.42 the class Ω pushes Φ.

Theorem 3.48. *If* Ω *and* Φ *define a regular envelope in* $\mathtt{K}$*, then* Env_Φ^Ω *can be defined as an idempotent functor.*

Proof. Consider the functor of envelope E built in Theorem 3.42, and denote by $\mathtt{L}_0$ the class of all objects which are values of the map $X \mapsto E(X)$:

$$A \in \mathtt{L}_0 \ \Leftrightarrow\ \exists X \in \mathsf{Ob}(\mathtt{K}) \ \ A = E(X). \tag{3.115}$$

Define a system of isomorphisms

$$\forall X \in \mathsf{Ob}(\mathtt{K}) \quad \zeta_X = \begin{cases} 1_X, & X \notin \mathtt{L}_0, \\ e_X^{-1}, & X \in \mathtt{L}_0 \end{cases}$$

(this definition is correct by Proposition 3.46). Now consider the maps $X \mapsto F(X)$, $X \mapsto f_X$, $\alpha \mapsto F(\alpha)$, defined by

$$\forall X \in \mathsf{Ob}(\mathtt{K}) \quad F(X) = \begin{cases} E(X), & X \notin \mathtt{L}_0, \\ X, & X \in \mathtt{L}_0, \end{cases} \qquad f_X = \begin{cases} e_X, & X \notin \mathtt{L}_0, \\ 1_X, & X \in \mathtt{L}_0, \end{cases}$$

$$\forall \alpha \in \mathsf{Mor}(\mathtt{K}) \quad F(\alpha) = \zeta_{\mathsf{Ran}\, E(\alpha)} \circ E(\alpha) \circ \zeta^{-1}_{\mathsf{Dom}\, E(\alpha)}.$$

The connection with the functor E is reflected in the diagram

$$\begin{tikzcd}
 & & F(X) \arrow[ddd, "F(\alpha)"] \\
X \arrow[r, "e_X"] \arrow[d, "\alpha"'] \arrow[urr, bend left, "f_X"] & E(X) \arrow[ur, "\zeta_X"'] \arrow[d, "E(\alpha)"] & \\
Y \arrow[r, "e_Y"] \arrow[drr, bend right, "f_Y"'] & E(Y) \arrow[dr, "\zeta_Y"] & \\
 & & F(Y)
\end{tikzcd} \tag{3.116}$$

For any X the morphism $f_X : X \to F(X)$ is an envelope of X, since f_X and e_X are connected by the isomorphism ζ_X. The map $(X, \alpha) \mapsto (F(X), F(\alpha))$ is a functor, since, first,

$$\begin{aligned} F(\beta \circ \alpha) &= \zeta_{\mathsf{Ran}\,\beta} \circ E(\beta \circ \alpha) \circ \zeta^{-1}_{\mathsf{Dom}\,\alpha} = \zeta_{\mathsf{Ran}\,\beta} \circ E(\beta) \circ E(\alpha) \circ \zeta^{-1}_{\mathsf{Dom}\,\alpha} \\ &= \zeta_{\mathsf{Ran}\,\beta} \circ E(\beta) \circ \zeta^{-1}_{\mathsf{Dom}\,\beta} \circ \zeta_{\mathsf{Ran}\,\alpha} \circ E(\alpha) \circ \zeta^{-1}_{\mathsf{Dom}\,\alpha} = F(\beta) \circ F(\alpha), \end{aligned}$$

and, second, for $X \notin \mathtt{L}_0$ diagram (3.116) has the form

$$\begin{tikzcd}
 & & E(X) \arrow[ddd, "E(1_X)"] \\
X \arrow[r, "e_X"] \arrow[d, "\alpha"'] \arrow[urr, bend left, "e_X"] & E(X) \arrow[ur, "1_{E(X)}"'] \arrow[d, "1_{E(X)}"] & \\
X \arrow[r, "e_X"] \arrow[drr, bend right, "e_X"'] & E(X) \arrow[dr, "1_{E(X)}"] & \\
 & & E(X)
\end{tikzcd}$$

hence

$$F(1_X) = \zeta_X \circ E(1_X) \circ \zeta_X^{-1} = 1^{-1}_{E(X)} \circ 1_{E(X)} \circ 1_{E(X)} = 1_{E(X)} = 1_{F(X)},$$

and for $X \in \mathsf{L}_0$ diagram (3.116) turns into

$$
\begin{tikzcd}
 & & X \arrow[ddd, "F(1_X)"] \\
X \arrow[r, "e_X"] \arrow[urr, bend left, "1_X"] \arrow[d, "1_X"'] & E(X) \arrow[ur, "e_X^{-1}"'] \arrow[d, "1_{E(X)}"] & \\
X \arrow[r, "e_X"] \arrow[drr, bend right, "1_X"'] & E(X) \arrow[dr, "e_X^{-1}"] & \\
 & & X
\end{tikzcd}
$$

If we replace $F(1_X)$ by 1_X, then the perimeter will still be a commutative diagram. Since this arrow is unique we have

$$F(1_X) = 1_X = 1_{F(X)}.$$

Condition (3.66) holds for F by definition: since always $F(X) \in \mathsf{L}_0$, we have $f_{F(X)} = 1_{F(X)}$. ∎

THEOREM 3.49 (description of envelope in terms of complete objects). *Suppose that Ω and Φ define a regular envelope in* K. *Then a given morphism $\rho : X \to A$ is an envelope (in Ω with respect to Φ) if and only if the following conditions are fulfilled:*

(i) *$\rho : X \to A$ is an epimorphism;*
(ii) *A is a complete object (in Ω with respect to Φ);*
(iii) *for any complete object B (in Ω with respect to Φ) and for any morphism $\xi : X \to B$ there is a unique morphism $\xi' : A \to B$ such that*

$$
\begin{tikzcd}
X \arrow[rr, "\rho"] \arrow[dr, "\xi"'] & & A \arrow[dl, dashed, "\xi'"] \\
 & B &
\end{tikzcd}
\tag{3.117}
$$

Proof. Let $\rho : X \to A$ be an envelope. Then, first, this is an epimorphism, since $\Omega \subseteq \mathsf{Epi}$. Second, by Proposition 3.46, $A \cong \mathsf{Env}_\Phi^\Omega X$ is a complete object. Third, if $\xi : X \to B$ is a morphism to a complete object B, then we can consider diagram (3.108) which in this situation has the form

$$
\begin{tikzcd}
X \arrow[r, "\rho = \mathsf{env}_\Phi^\Omega X"] \arrow[d, "\xi"'] & A = \mathsf{Env}_\Phi^\Omega X \arrow[d, "\mathsf{Env}_\Phi^\Omega \xi"] \\
B \arrow[r, "\mathsf{env}_\Phi^\Omega B"] & \mathsf{Env}_\Phi^\Omega B
\end{tikzcd}
$$

In this case $\mathsf{env}_\Phi^\Omega B$ is an isomorphism, and as a corollary, there exists a morphism

$$\xi' = (\mathsf{env}_\Phi^\Omega B)^{-1} \circ \mathsf{Env}_\Phi^\Omega \xi.$$

It is the dashed arrow in (3.117).

Conversely, suppose (i)–(iii) hold. In our circumstances Theorem 3.42 applies, so we can consider diagram (3.108), now in the form

$$\begin{tikzcd} X \arrow[r, "\mathsf{env}^{\varOmega}_{\varPhi} X"] \arrow[d, "\rho"'] & \mathsf{Env}^{\varOmega}_{\varPhi} X \arrow[d, "\mathsf{Env}^{\varOmega}_{\varPhi} \rho"] \\ A \arrow[r, "\mathsf{env}^{\varOmega}_{\varPhi} A"] & \mathsf{Env}^{\varOmega}_{\varPhi} A \end{tikzcd}$$

Here $\mathsf{env}^{\varOmega}_{\varPhi} A$ is an isomorphism (since A is a complete object). Hence if we take $\zeta = \mathsf{env}^{\varOmega}_{\varPhi} A^{-1} \circ \mathsf{Env}^{\varOmega}_{\varPhi}(\rho)$, we obtain

$$\begin{tikzcd} X \arrow[r, "\mathsf{env}^{\varOmega}_{\varPhi} X"] \arrow[d, "\rho"'] & \mathsf{Env}^{\varOmega}_{\varPhi} X \arrow[dl, dashed, "\zeta"] \\ A & \end{tikzcd}$$

On the other hand, by Proposition 3.46, $\mathsf{Env}^{\varOmega}_{\varPhi} X$ is a complete object, so by (iii), there exists a morphism η such that

$$\begin{tikzcd} X \arrow[r, "\mathsf{env}^{\varOmega}_{\varPhi} X"] \arrow[d, "\rho"'] & \mathsf{Env}^{\varOmega}_{\varPhi} X \\ A \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

In these diagrams both ρ and $\mathsf{env}^{\varOmega}_{\varPhi} X$ are epimorphisms, so ζ and η are mutually inverse morphisms. Thus, $\rho = \zeta \circ \mathsf{env}^{\varOmega}_{\varPhi} X$, where $\zeta \in \mathsf{Iso}$. By (2.41), we see that $\rho \in \varOmega$, and thus it is an envelope. ■

3.4.6. Pulling, regular refinement and saturated objects

• Let us say that *a class $\varGamma$ of morphisms pulls a class $\varPhi$ of morphisms* if

$$\forall \psi \in \mathsf{Mor}(\mathtt{K})\ \forall \sigma \in \varGamma \quad (\sigma \circ \psi \in \varPhi \Rightarrow \psi \in \varPhi). \tag{3.118}$$

REMARK 3.50. Obviously, (3.118) holds if $\varPhi$ is the class of morphisms with domains in a subclass $\mathtt{M}$ of objects in $\mathtt{K}$.

LEMMA 3.51. *If $\varGamma$ pulls $\varPhi$, then the composition $\sigma \circ \rho : X \leftarrow X''$ of any two enrichments $\sigma : X \leftarrow X'$ and $\rho : X' \leftarrow X''$ (in $\varGamma$ by means of $\varPhi$) is an enrichment (in $\varGamma$ by means of $\varPhi$).*

Proof. This is seen from the diagram

$$\begin{tikzcd} X & X' \arrow[l, "\sigma"'] & X'' \arrow[l, "\rho"'] \\ & \mathtt{M} \arrow[ul, bend left, "\varphi"] \arrow[u, dashed, "\varphi'"'] \arrow[ur, dashed, bend right, "\varphi''"'] & \end{tikzcd}$$

■

Proposition 3.52. *Suppose $\Gamma \subseteq \mathsf{Mono}$. Then for $A \in \mathsf{Ob}(\mathtt{K})$ the following conditions are equivalent:*

(i) *every enrichment $\sigma : A \leftarrow A'$ in Γ by means of Φ is an isomorphism;*
(ii) *$1_A : A \to A$ is a refinement of A in Γ by means of Φ;*
(iii) *there exists $\mathsf{ref}^{\Gamma}_{\Phi} A \in \mathsf{Iso}$.*

If in addition Γ pulls Φ, then these conditions are equivalent to:

(iv) *A is isomorphic to a refinement of some $X \in \mathsf{Ob}(\mathtt{K})$.*

- We say that an object A in $\mathtt{K}$ is *saturated* in $\Gamma \subseteq \mathsf{Mono}$ by means of Φ if it satisfies the above conditions (i)–(iii).

Denote by $\mathtt{L}$ the class of all saturated objects in $\mathtt{K}$ (in $\Gamma \subseteq \mathsf{Mono}$ by means of Φ). We consider $\mathtt{L}$ as a full subcategory in $\mathtt{K}$.

Proposition 3.53. *Under the conditions of Theorem 3.43 the functor of refinement $(X, \alpha) \mapsto (I(X), I(\alpha))$ on $\mathtt{L} \subseteq \mathtt{K}$ is isomorphic to the identity functor:*

$$\forall A \in \mathtt{L} \quad I(A) \cong A, \qquad \forall \alpha : \underset{\underset{\mathtt{L}}{\cap}}{A} \leftarrow \underset{\underset{\mathtt{L}}{\cap}}{A'} \quad E(\alpha) = i_A^{-1} \circ \alpha \circ i_{A'}. \tag{3.119}$$

- We say that *classes Γ and Φ define a regular refinement in* $\mathtt{K}$, or *the refinement $\mathsf{Ref}^{\Gamma}_{\Phi}$ is regular*, if in addition to conditions RR.1–RR.4 of Theorem 3.43 the class Γ pulls Φ.

Theorem 3.54. *If Γ and Φ define a regular refinement in $\mathtt{K}$, then $\mathsf{Ref}^{\Gamma}_{\Phi}$ can be defined as an idempotent functor.*

Theorem 3.55 (description of refinement in terms of saturated objects). *Suppose Γ and Φ define a regular refinement in $\mathtt{K}$. Then a given morphism $\rho : X \leftarrow A$ is a refinement (in Γ by means of Φ) if and only if the following conditions hold:*

(i) *$\rho : X \leftarrow A$ is a monomorphism;*
(ii) *A is a saturated object (in Γ by means of Φ);*
(iii) *for any saturated object B (in Γ by means of Φ) and for any morphism $\xi : X \leftarrow B$ there is a unique morphism $\xi' : A \leftarrow B$ such that*

$$\begin{tikzcd} X & & A \arrow[ll, "\rho"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, dashed, "\xi'"'] & \end{tikzcd} \tag{3.120}$$

3.4.7. Functoriality on epimorphisms and monomorphisms. Denote by $\mathtt{K}^{\mathsf{Epi}}$ the subcategory in $\mathtt{K}$ with the same class of objects as in $\mathtt{K}$, but with epimorphisms from $\mathtt{K}$ as morphisms: $\mathsf{Ob}(\mathtt{K}^{\mathsf{Epi}}) = \mathsf{Ob}(\mathtt{K})$, $\mathsf{Mor}(\mathtt{K}^{\mathsf{Epi}}) = \mathsf{Epi}(\mathtt{K})$.

Theorem 3.56. *Let $\mathtt{K}$ be a category with products (over arbitrary index sets), and suppose classes Ω and Φ of morphisms in $\mathtt{K}$ satisfy the following conditions:*

– *Ω is monomorphically complementable in* $\mathtt{K}$*;*
– $\mathtt{K}$ *is co-well-powered in the class Ω;*
– *Φ goes from* ([12]) $\mathtt{K}$*;*
– *$\Phi \circ \Omega \subseteq \Phi$.*

Then:

(a) *each object X in* $\mathtt{K}$ *has an envelope $\mathsf{Env}^{\Omega}_{\Phi} X$ in Ω with respect to Φ;*
(b) *for each epimorphism $\pi : X \to Y$ there is a unique epimorphism*

$$\mathsf{Env}^{\Omega}_{\Phi}\, \pi : \mathsf{Env}^{\Omega}_{\Phi} X \to \mathsf{Env}^{\Omega}_{\Phi} Y$$

such that

$$\begin{tikzcd} X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[d, "\pi"'] & \mathsf{Env}^{\Omega}_{\Phi} X \arrow[d, dashed, "\mathsf{Env}^{\Omega}_{\Phi}\, \pi"] \\ Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y \end{tikzcd} \tag{3.121}$$

(c) *the envelope $\mathsf{Env}^{\Omega}_{\Phi}$ can be defined as a functor from $\mathtt{K}^{\mathsf{Epi}}$ into $\mathtt{K}^{\mathsf{Epi}}$.*

We will need

LEMMA 3.57. *If* $\mathtt{K}$ *is a category with products* (*over arbitrary index sets*), *co-well-powered in Ω, and Ω is monomorphically complementable in* $\mathtt{K}$, *then for any class Φ of morphisms and any epimorphism $\pi : X \to Y$ we have*

$$\mathsf{Env}^{\Omega}_{\Phi\circ\pi} X = \mathsf{Env}^{\Omega}_{\Phi} Y. \tag{3.122}$$

Proof. Note first that the existence of the envelopes in (3.122) is guaranteed by property 5° on p. 61. In addition, by 5° on p. 45, there exists a morphism υ such that (3.12) is commutative:

$$\begin{tikzcd} & X \arrow[dl, "\mathsf{env}^{\Omega}_{\Phi} Y \circ \pi"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi\circ\pi} X"] & \\ \mathsf{Env}^{\Omega}_{\Phi} Y \arrow[rr, dashed, "\upsilon"] & & \mathsf{Env}^{\Omega}_{\Phi\circ\pi} X \end{tikzcd}$$

Let us show that there is an inverse morphism. Consider the envelope $\mathsf{env}^{\Omega}_{\Phi} Y : Y \to \mathsf{Env}^{\Omega}_{\Phi} Y$ and represent it as an envelope with respect to a set M of morphisms, as in the proof of property 5° on p. 61. Then, as in the proof of 3° on p. 61, replace M by a unique morphism $\psi = \prod_{\chi\in M} \chi$. By property 1° on p. 61, the envelope with respect to ψ will be described as an epimorphism ε_{ψ} in the factorization of ψ:

$$\mathsf{env}^{\Omega}_{\Phi} Y = \mathsf{env}^{\Omega}_{M} Y = \mathsf{env}^{\Omega}_{\psi} Y = \varepsilon_{\psi}.$$

([12]) In the sense of definition on p. 9.

We obtain a diagram

$$
\begin{tikzcd}
X \arrow[rrr, bend left=10, "\mathsf{env}^{\Omega}_{\Phi\circ\pi} X"] \arrow[dr, bend left, "\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi=\varepsilon_{\psi}\circ\pi"] \arrow[dd, "\pi"'] & & & \mathsf{Env}^{\Omega}_{\Phi\circ\pi} X \arrow[dl, dashed, bend right, "\delta"'] \arrow[dd, "(\psi\circ\pi)'"] \\
& \mathsf{Env}^{\Omega}_{\Phi} Y \arrow[r, equal] & \mathsf{Ran}\,\varepsilon_{\psi} \arrow[dr, bend right, "\mu_{\psi}"] & \\
Y \arrow[ur, bend right, "\mathsf{env}^{\Omega}_{\Phi} Y=\varepsilon_{\psi}"'] \arrow[rrr, bend right=10, "\psi"'] & & & B
\end{tikzcd}
$$

where $(\psi\circ\pi)'$ is an extension of $\psi\circ\pi\in\Phi\circ\pi$ along the envelope $\mathsf{env}^{\Omega}_{\Phi\circ\pi} X$. Here the existence of the morphism δ follows from the fact that $\mathsf{env}^{\Omega}_{\Phi\circ\pi} X\in\Omega$, and $\mu_{\psi}\in{}^{\downarrow}\Omega$. We now have the diagram

$$
\begin{tikzcd}
& X \arrow[dl, "\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi"'] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi\circ\pi} X"] & \\
\mathsf{Env}^{\Omega}_{\Phi} Y & & \mathsf{Env}^{\Omega}_{\Phi\circ\pi} X \arrow[ll, dashed, "\delta"']
\end{tikzcd}
$$

It remains to verify that υ and δ are mutually inverse. First,

$$
\delta\circ\upsilon\circ\underbrace{\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi}_{\in\,\mathsf{Epi}}=\delta\circ\mathsf{env}^{\Omega}_{\Phi\circ\pi} X=\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi=1_{\mathsf{Env}^{\Omega}_{\Phi} Y}\circ\underbrace{\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi}_{\in\,\mathsf{Epi}}\ \Rightarrow\ \delta\circ\upsilon=1_{\mathsf{Env}^{\Omega}_{\Phi} Y}.
$$

And second,

$$
\upsilon\circ\delta\circ\underbrace{\mathsf{env}^{\Omega}_{\Phi\circ\pi} X}_{\in\,\mathsf{Epi}}=\upsilon\circ\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi=\mathsf{env}^{\Omega}_{\Phi\circ\pi} X=1_{\mathsf{Env}^{\Omega}_{\Phi\circ\pi} X}\circ\underbrace{\mathsf{env}^{\Omega}_{\Phi\circ\pi} X}_{\in\,\mathsf{Epi}}\ \Rightarrow\ \upsilon\circ\delta=1_{\mathsf{Env}^{\Omega}_{\Phi\circ\pi} X}.\ \blacksquare
$$

Proof of Theorem 3.56. Part (a) follows from property 5° on p. 61. Let us prove (b). By Lemma 3.57, $\mathsf{Env}^{\Omega}_{\Phi} Y=\mathsf{Env}^{\Omega}_{\Phi\circ\pi} X$, and by property 3° on p. 44, when we pass to a narrower class of morphisms $\Phi\circ\pi\subseteq\Phi$, a dashed arrow arises in the upper triangle of the diagram

$$
\begin{tikzcd}
X \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} X"] \arrow[dr, "\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi"'] \arrow[dd, "\pi"'] & \mathsf{Env}^{\Omega}_{\Phi} X \arrow[d, dashed, "\mathsf{Env}^{\Omega}_{\Phi}\pi"] \\
& \mathsf{Env}^{\Omega}_{\Phi\circ\pi} X \arrow[d, equal] \\
Y \arrow[r, "\mathsf{env}^{\Omega}_{\Phi} Y"] & \mathsf{Env}^{\Omega}_{\Phi} Y
\end{tikzcd}
$$

It will be the dashed arrow in (3.121), but we need to verify that it is an epimorphism (so that it will be a morphism in $\mathsf{K}^{\mathsf{Epi}}$). This follows from property 3° on p. 15: since $\mathsf{Env}^{\Omega}_{\Phi}\pi\circ\mathsf{env}^{\Omega}_{\Phi} X=\mathsf{env}^{\Omega}_{\Phi} Y\circ\pi\in\mathsf{Epi}$, we have $\mathsf{Env}^{\Omega}_{\Phi}\pi\in\mathsf{Epi}$.

As (a) and (b) are proven, (c) becomes a corollary due to Theorem 2.24: $\mathtt{K}$ is co-well-powered in Ω, hence we can choose a map $X \mapsto S_X$, which assigns to each object a skeleton S_X in the category $\Omega \cap \mathsf{Epi}^X$. Then it becomes possible to choose a map $X \mapsto \mathsf{env}_\Phi^\Omega X$, and for any epimorphism $\pi : X \to Y$ the arrow $\mathsf{Env}_\Phi^\Omega \pi$ automatically appears from diagram (3.121). ∎

The dual results for refinements look as follows. Denote by $\mathtt{K}^{\mathsf{Mono}}$ the subcategory in $\mathtt{K}$ with the same class of objects as in $\mathtt{K}$, but with monomorphisms from $\mathtt{K}$ as morphisms.

Theorem 3.58. *Let $\mathtt{K}$ be a category with coproducts (over arbitrary index sets), and suppose classes Γ and Φ of morphisms in $\mathtt{K}$ satisfy the following conditions:*

– *Γ is epimorphically complementable in $\mathtt{K}$;*
– *$\mathtt{K}$ is well-powered in the class Γ;*
– *Φ goes to* [13] *$\mathtt{K}$;*
– *$\Gamma \circ \Phi \subseteq \Phi$.*

Then:

(a) *each object X in $\mathtt{K}$ has a refinement $\mathsf{Ref}_\Phi^\Gamma X$ in Γ by means of Φ;*
(b) *for each monomorphism $\pi : X \to Y$ there exists a unique monomorphism $\mathsf{Ref}_\Phi^\Gamma \pi : \mathsf{Ref}_\Phi^\Gamma X \to \mathsf{Ref}_\Phi^\Gamma Y$ such that*

$$\begin{tikzcd} X & \mathsf{Ref}_\Phi^\Gamma X \arrow[l, "\mathsf{Ref}_\Phi^\Gamma X"'] \arrow[d, dashed, "\mathsf{Ref}_\Phi^\Gamma \pi"] \\ Y \arrow[from=u, "\pi"'] & \mathsf{Ref}_\Phi^\Gamma Y \arrow[l, "\mathsf{ref}_\Phi^\Gamma Y"] \end{tikzcd} \tag{3.123}$$

(c) *the refinement Ref_Φ^Γ can be defined as a functor from $\mathtt{K}^{\mathsf{Mono}}$ into $\mathtt{K}^{\mathsf{Mono}}$.*

The following lemma is used in the proof:

Lemma 3.59. *If $\mathtt{K}$ is a category with coproducts, well-powered in Γ, and Γ is epimorphically complemented in $\mathtt{K}$, then for each class Φ of morphisms and each monomorphism $\pi : X \leftarrow Y$ we have*

$$\mathsf{Ref}_\Phi^\Gamma X = \mathsf{Ref}_{\pi\circ\Phi}^\Gamma Y \tag{3.124}$$

(here $\pi \circ \Phi = \{\pi \circ \varphi;\ \varphi \in \Phi\}$).

3.4.8. The case of $\mathsf{Env}_{\mathtt{L}}^{\mathtt{L}}$ and $\mathsf{Ref}_{\mathtt{L}}^{\mathtt{L}}$. Theorem 3.42 has important corollaries in the case when the classes of test morphisms and realizing morphisms coincide, i.e. $\Phi = \Omega$, and are the class of all morphisms with ranges in a given class $\mathtt{L}$ of objects (this is the special case of the situation described on p. 49, when $\mathtt{L} = \mathtt{M}$).

Theorem 3.60. *Suppose a category $\mathtt{K}$ and a class $\mathtt{L}$ of objects have the following properties:*

(i) *$\mathtt{K}$ is projectively complete;*
(ii) *$\mathtt{K}$ has nodal decomposition;*

[13] In the sense of the definition on p. 9.

(iii) $\mathtt{K}$ *is co-well-powered in* Epi;
(iv) $\mathsf{Mor}(\mathtt{K},\mathtt{L})$ *goes from* $\mathtt{K}$;
(v) $\mathtt{L}$ *separates morphisms on the outside;*
(vi) $\mathtt{L}$ *is closed with respect to passage to projective limits;*
(vii) $\mathtt{L}$ *is closed with respect to passage from the range of a morphism to its nodal image: if* $\mathsf{Ran}\,\alpha \in \mathtt{L}$, *then* $\mathsf{Im}_\infty\,\alpha \in \mathtt{L}$.

Then:

(a) *each object X has an envelope* $\mathsf{env}^{\mathtt{L}}_{\mathtt{L}} X$;
(b) *each envelope* $\mathsf{env}^{\mathtt{L}}_{\mathtt{L}} X$ *is a bimorphism;*
(c) *the envelope* $\mathsf{Env}^{\mathtt{L}}_{\mathtt{L}}$ *can be defined as a functor.*

Proof. Conditions (i)–(v) mean that the classes Epi and $\varPhi = \mathsf{Mor}(\mathtt{K},\mathtt{L})$ satisfy the premises of Theorem 3.42, i.e. define a semiregular envelope $\mathsf{Env}^{\mathsf{Epi}}_{\varPhi} = \mathsf{Env}^{\mathsf{Epi}}_{\mathtt{L}}$. In the proof of Theorem 3.42 this envelope is constructed by passing from the spaces $\mathsf{Ran}\,\varphi \in \mathtt{L}$ ($\varphi \in \varPhi$) to their projective limits, which belong to $\mathtt{K}$ by (vi), and then to the nodal images, which belong to $\mathtt{L}$ by (vii). Therefore, $\mathsf{Env}^{\mathsf{Epi}}_{\mathtt{L}} \in \mathtt{L}$, hence by property 1° on p. 44,

$$\mathsf{Env}^{\mathsf{Epi}}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Epi}(\mathtt{K},\mathtt{L})}_{\mathtt{L}}.$$

By construction, the class $\varPhi$ is a right ideal, and by (v), $\varPhi$ separates morphisms on the outside. So by Theorem 3.7,

$$\mathsf{Env}^{\mathsf{Epi}(\mathtt{K},\mathtt{L})}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Bim}(\mathtt{K},\mathtt{L})}_{\mathtt{L}}.$$

Still by Theorem 3.7 an envelope in the class $\mathsf{Bim}(\mathtt{K},\mathtt{L}) = \mathsf{Mor}(\mathtt{K},\mathtt{L}) \cap \mathsf{Bim}$ exists if and only if there exist the envelope in the class $\mathsf{Mor}(\mathtt{K},\mathtt{L})$, and $\mathsf{Env}^{\mathsf{Bim}(\mathtt{K},\mathtt{L})}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Mor}(\mathtt{K},\mathtt{L})}_{\mathtt{L}}$. We obtain the following logical chain:

$$\mathsf{Env}^{\mathsf{Epi}}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Epi}(\mathtt{K},\mathtt{L})}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Bim}(\mathtt{K},\mathtt{L})}_{\mathtt{L}} = \mathsf{Env}^{\mathsf{Mor}(\mathtt{K},\mathtt{L})}_{\mathtt{L}} = \mathsf{Env}^{\mathtt{L}}_{\mathtt{L}}.$$

This proves (a) and (c), and incidentally (b). ∎

The dual result is as follows:

THEOREM 3.61. *Suppose a category* $\mathtt{K}$ *and a class* $\mathtt{L}$ *of objects satisfy the following conditions:*

(i) $\mathtt{K}$ *is injectively complete;*
(ii) $\mathtt{K}$ *has nodal decomposition;*
(iii) $\mathtt{K}$ *is well-powered in* Mono;
(iv) $\mathsf{Mor}(\mathtt{L},\mathtt{K})$ *goes to* $\mathtt{K}$;
(v) $\mathtt{L}$ *separates morphisms on the inside;*
(vi) $\mathtt{L}$ *is closed with respect to the operation of taking injective limits;*
(vii) $\mathtt{L}$ *is closed with respect to passage from the domain of a morphism to its nodal coimage: if* $\mathsf{Dom}\,\alpha \in \mathtt{L}$, *then* $\mathsf{Coim}_\infty\,\alpha \in \mathtt{L}$.

Then:

(a) *each object X has a refinement* $\mathsf{ref}^{\mathtt{L}}_{\mathtt{L}} X$;
(b) *each refinement* $\mathsf{ref}^{\mathtt{L}}_{\mathtt{L}} X$ *is a bimorphism;*
(c) *the refinement* $\mathsf{Ref}^{\mathtt{L}}_{\mathtt{L}}$ *can be defined as a functor.*

3.5. Envelopes in monoidal categories

3.5.1. Envelopes coherent with tensor product. Let $\mathtt{K}$ be a monoidal category [26] with tensor product $\otimes$ and unit object I.

- Let us say that *the envelope* $\mathsf{Env}_{\Phi}^{\Omega}$ *is coherent with the tensor product* $\otimes$ *in* $\mathtt{K}$ if the following conditions are fulfilled:

  T.1. The tensor product $\rho \otimes \sigma : X \otimes Y \to X' \otimes Y'$ of any two extensions $\rho : X \to X'$ and $\sigma : Y \to Y'$ (in Ω with respect to Φ) is an extension (in Ω with respect to Φ).

  T.2. The local identity $1_I : I \to I$ is an envelope (in Ω with respect to Φ):
$$\mathsf{env}_{\Phi}^{\Omega} I = 1_I. \tag{3.125}$$

In this section we consider the case when Ω and Φ define a regular envelope in $\mathtt{K}$. By Theorem 3.48 this means that $\mathsf{Env}_{\Phi}^{\Omega}$ can be defined as an idempotent functor. We denote it by $E : \mathtt{K} \to \mathtt{K}$, and the natural transformation of the identity functor into E is denoted by e:
$$E(X) := \mathsf{Env}_{\Phi}^{\Omega} X, \qquad E(\varphi) := \mathsf{Env}_{\Phi}^{\Omega} \varphi, \qquad e_X := \mathsf{env}_{\Phi}^{\Omega} X.$$
The class of all complete objects in $\mathtt{K}$ (in Ω with respect to Φ) is denoted by $\mathtt{L}$.

LEMMA 3.62. *Let $\mathsf{Env}_{\Phi}^{\Omega}$ be a regular envelope coherent with the tensor product in $\mathtt{K}$. Then:*

(i) *For any objects $A \in \mathtt{L}$ and $X \in \mathsf{Ob}(\mathtt{K})$ the envelope $E(1_A \otimes e_X)$ of the morphism $1_A \otimes e_X : A \otimes X \to A \otimes E(X)$ is an isomorphism (in $\mathtt{K}$ and in $\mathtt{L}$):*
$$E(1_A \otimes e_X) \in \mathsf{Iso}. \tag{3.126}$$

(ii) *For any $X, Y \in \mathsf{Ob}(\mathtt{K})$ the envelope $E(e_X \otimes e_Y)$ of the morphism $e_X \otimes e_Y : X \otimes Y \to E(X) \otimes E(Y)$ is an isomorphism (in $\mathtt{K}$ and in $\mathtt{L}$):*
$$E(e_X \otimes e_Y) \in \mathsf{Iso}. \tag{3.127}$$

Proof. (i) Take $A \in \mathtt{L}$ and $X \in \mathsf{Ob}(\mathtt{K})$. The product of the morphisms $1_A : A \to A$ and $e_X : X \to E(X)$ is $1_A \otimes e_X : A \otimes X \to A \otimes E(X)$. If we insert it instead of α into (3.64), we obtain
$$\begin{tikzcd} A \otimes X \arrow[r, "e_{A\otimes X}"] \arrow[d, "1_A \otimes e_X"'] & E(A \otimes X) \arrow[d, "E(1_A \otimes e_X)"] \\ A \otimes E(X) \arrow[r, "e_{A\otimes E(X)}"] & E(A \otimes E(X)) \end{tikzcd} \tag{3.128}$$
From the diagram
$$\begin{tikzcd} A \otimes X \arrow[r, "1_A \otimes e_X"] \arrow[dr, bend right, "\varphi"'] & A \otimes E(X) \arrow[r, "e_{A\otimes E(X)}"] \arrow[d, dashed, "\varphi'"] & E(A \otimes E(X)) \arrow[dl, dashed, bend left, "\varphi''"] \\ & B & \end{tikzcd}$$

it is seen that $e_{A\otimes E(X)} \circ 1_A \otimes e_X$ is an extension of $A \otimes X$ (here in the left triangle we use T.1). Hence $e_{A\otimes E(X)} \circ 1_A \otimes e_X$ is subordinated to the envelope of $A \otimes X$:

$$\begin{tikzcd}
A \otimes X \arrow[r, "e_{A\otimes X}"] \arrow[d, "1_A \otimes e_X"'] & E(A \otimes X) \\
A \otimes E(X) \arrow[r, "e_{A\otimes E(X)}"] & E(A \otimes E(X)) \arrow[u, dashed, "\upsilon"']
\end{tikzcd} \tag{3.129}$$

for some (unique) υ. In addition, $\Omega \subseteq \mathsf{Epi}$, hence $e_{A\otimes E(X)} \circ 1_A \otimes e_X$ and $e_{A\otimes X}$, being extensions, are epimorphisms. As a corollary, (3.128) and (3.129) together give

$$\upsilon = E(1_A \otimes e_X)^{-1}.$$

(ii) For any two objects X and Y the product of $e_X : X \to E(X)$ and $e_Y : Y \to E(Y)$ is $e_X \otimes e_Y : X \otimes Y \to E(X) \otimes E(Y)$. If we insert it instead of α into (3.64), we get

$$\begin{tikzcd}
X \otimes Y \arrow[r, "e_{X\otimes Y}"] \arrow[d, "e_X \otimes e_Y"'] & E(X \otimes Y) \arrow[d, "E(e_X \otimes e_Y)"] \\
E(X) \otimes E(Y) \arrow[r, "e_{E(X)\otimes E(Y)}"] & E(E(X) \otimes E(Y))
\end{tikzcd} \tag{3.130}$$

From the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "e_X \otimes e_Y"] \arrow[dr, bend right, "\varphi"'] & E(X) \otimes E(Y) \arrow[r, "e_{E(X)\otimes E(Y)}"] \arrow[d, dashed, "\varphi'"] & E(E(X) \otimes E(Y)) \arrow[dl, dashed, bend left, "\varphi''"] \\
& B &
\end{tikzcd}$$

we see that $e_{E(X)\otimes E(Y)} \circ e_X \otimes e_Y$ is an extension of $X \otimes Y$ (again in the left triangle we use T.1). Hence $e_{E(X)\otimes E(Y)} \circ e_X \otimes e_Y$ is subordinated to $X \otimes Y$:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "e_{X\otimes Y}"] \arrow[d, "e_X \otimes e_Y"'] & E(X \otimes Y) \\
E(X) \otimes E(Y) \arrow[r, "e_{E(X)\otimes E(Y)}"] & E(E(X) \otimes E(Y)) \arrow[u, dashed, "\upsilon"']
\end{tikzcd} \tag{3.131}$$

for some (unique) υ. And as in the previous case, $e_{E(X)\otimes E(Y)} \circ e_X \otimes e_Y$ and $e_{X\otimes Y}$, being extensions, are epimorphisms, so (3.130) and (3.131) together give

$$\upsilon = E(e_X \otimes e_Y)^{-1}. \ \blacksquare$$

3.5.2. Monoidal structure on the class of complete objects. Let Env_Φ^Ω be a regular envelope coherent with the tensor product in $\mathtt{K}$, $E = \mathsf{Env}_\Phi^\Omega$ the idempotent functor built in Theorem 3.48, and $\mathtt{L}$ the (full) subcategory of complete objects in $\mathtt{K}$. For any objects $A, B \in \mathtt{L}$ and any morphisms $\varphi, \psi \in \mathtt{L}$ we define

$$A \overset{E}{\otimes} B := E(A \otimes B), \qquad \varphi \overset{E}{\otimes} \psi := E(\varphi \otimes \psi). \tag{3.132}$$

Notice the identity

$$E(X) \overset{E}{\otimes} E(Y) = E(E(X) \otimes E(Y)), \qquad X, Y \in \mathsf{Ob}(\mathsf{K}) \tag{3.133}$$

(this is an equality of objects, since by Proposition 3.46, always $E(X), E(Y) \in \mathsf{L}$).

THEOREM 3.63. *Suppose* $\mathsf{Env}_{\Phi}^{\Omega}$ *is a regular envelope coherent with the tensor product in* K. *Then the formulas* (3.132) *define a structure of monoidal category on* L (*with* $\overset{E}{\otimes}$ *as tensor product and* I *as unit object*).

Proof. The tensor product of local identities is a local identity. Insert $1_{A\otimes B}$ instead of α into (3.64):

$$\begin{tikzcd} A \otimes B \arrow[r, "e(A\otimes B)"] \arrow[d, "1_{A\otimes B}"'] & A \overset{E}{\otimes} B \arrow[d, "E(1_{A\otimes B})"] \\ A \otimes B \arrow[r, "e(A\otimes B)"] & A \overset{E}{\otimes} B \end{tikzcd}$$

If we replace here $E(1_{A\otimes B})$ by $1_{A \overset{E}{\otimes} B}$, then the diagram will remain commutative. But this arrow is unique (since $e(A \otimes B)$ is an epimorphism), so these arrows must coincide, and this is used in the last equality of the following chain:

$$1_A \overset{E}{\otimes} 1_B \overset{(3.132)}{=} E(1_A \otimes 1_B) = E(1_{A\otimes B}) = 1_{A \overset{E}{\otimes} B}.$$

The tensor product of commutative diagrams is a commutative diagram. Suppose we have two commutative diagrams in L:

$$\begin{tikzcd} & B \arrow[dr, "\chi"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\psi"] & & C \end{tikzcd} \qquad \begin{tikzcd} & B' \arrow[dr, "\chi'"] & \\ A' \arrow[ur, "\varphi'"] \arrow[rr, "\psi'"] & & C' \end{tikzcd}$$

If we multiply them in K, we obtain a commutative diagram

$$\begin{tikzcd} & B \otimes B' \arrow[dr, "\chi\otimes\chi'"] & \\ A \otimes A' \arrow[ur, "\varphi\otimes\varphi'"] \arrow[rr, "\psi\otimes\psi'"] & & C \otimes C' \end{tikzcd}$$

Then we apply the functor E and again obtain a commutative diagram:

$$\begin{tikzcd} & E(B \otimes B') \arrow[dr, "E(\chi\otimes\chi')"] & \\ E(A \otimes A') \arrow[ur, "E(\varphi\otimes\varphi')"] \arrow[rr, "E(\psi\otimes\psi')"] & & E(C \otimes C') \end{tikzcd}$$

By (3.132) this is the diagram that we need:

$$\begin{tikzcd} & B \overset{E}{\otimes} B' \arrow[dr, "\chi \overset{E}{\otimes} \chi'"] & \\ A \overset{E}{\otimes} A' \arrow[ur, "\varphi \overset{E}{\otimes} \varphi'"] \arrow[rr, "\psi \overset{E}{\otimes} \psi'"] & & C \overset{E}{\otimes} C' \end{tikzcd}$$

Notice that from what we already proved it follows that the tensor product of isomorphisms in L is also an isomorphism:

$$\varphi, \psi \in \mathsf{Iso} \ \Rightarrow\ \varphi \overset{E}{\otimes} \psi := E(\varphi \otimes \psi) \in \mathsf{Iso}. \tag{3.134}$$

Indeed,

$$(\varphi \otimes \psi) \circ (\varphi^{-1} \otimes \psi^{-1}) = (\varphi \circ \varphi^{-1}) \otimes (\psi \circ \psi^{-1}) = 1 \otimes 1 = 1,$$

so

$$\begin{aligned}(\varphi \overset{E}{\otimes} \psi) \circ (\varphi^{-1} \overset{E}{\otimes} \psi^{-1}) &= E(\varphi \otimes \psi) \circ E(\varphi^{-1} \otimes \psi^{-1}) \\ &= E((\varphi \otimes \psi) \circ (\varphi^{-1} \otimes \psi^{-1})) = E(1) = 1.\end{aligned}$$

And similarly,

$$(\varphi^{-1} \overset{E}{\otimes} \psi^{-1}) \circ (\varphi \overset{E}{\otimes} \psi) = 1.$$

If $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is the associativity transform in K, then the associativity transform $\alpha^E_{A,B,C} : (A \overset{E}{\otimes} B) \overset{E}{\otimes} C \to A \overset{E}{\otimes} (B \overset{E}{\otimes} C)$ in L is defined by the diagram

$$\begin{tikzcd} E((A \otimes B) \otimes C) \arrow[rr, "E(\alpha_{A,B,C})"] & & E(A \otimes (B \otimes C)) \arrow[d, "E(1_A \otimes e_{B \otimes C})"] \\ E(E(A \otimes B) \otimes C) \arrow[u, "E(e_{A\otimes B} \otimes 1_C)^{-1}"] \arrow[d, equal] & & E(A \otimes E(B \otimes C)) \arrow[d, equal] \\ E(A \otimes B) \overset{E}{\otimes} C \arrow[d, equal] & & A \overset{E}{\otimes} E(B \otimes C) \arrow[d, equal] \\ (A \overset{E}{\otimes} B) \overset{E}{\otimes} C \arrow[rr, "\alpha^E_{A,B,C}"] & & A \overset{E}{\otimes} (B \overset{E}{\otimes} C) \end{tikzcd} \tag{3.135}$$

(here we use (3.133) and Lemma 3.62, which implies that the morphism $E(e_{A\otimes B} \otimes 1_C)$ is invertible).

Let us show that the transform α^E is natural with respect to the tensor product:

$$\alpha^E : \big((A, B, C) \mapsto (A \overset{E}{\otimes} B) \overset{E}{\otimes} C\big) \rightarrowtail \big((A, B, C) \mapsto A \overset{E}{\otimes} (B \overset{E}{\otimes} C)\big).$$

Take morphisms $\varphi : A \to A'$, $\chi : B \to B'$, $\psi : C \to C'$ in L, and consider the diagram of naturality for α:

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "(\varphi\otimes\chi)\otimes\psi"'] & A\otimes(B\otimes C) \arrow[d, "\varphi\otimes(\chi\otimes\psi)"] \\
(A'\otimes B')\otimes C' \arrow[r, "\alpha_{A',B',C'}"] & A'\otimes(B'\otimes C')
\end{tikzcd} \tag{3.136}$$

After applying the functor E we have

$$\begin{tikzcd}
E((A\otimes B)\otimes C) \arrow[r, "E(\alpha_{A,B,C})"] \arrow[d, "E((\varphi\otimes\chi)\otimes\psi)"'] & E(A\otimes(B\otimes C)) \arrow[d, "E(\varphi\otimes(\chi\otimes\psi))"] \\
E((A'\otimes B')\otimes C') \arrow[r, "E(\alpha_{A',B',C'})"] & E(A'\otimes(B'\otimes C'))
\end{tikzcd}$$

Let us extend this diagram as follows:

$$\begin{tikzcd}
(A\overset{E}{\otimes} B)\overset{E}{\otimes} C \arrow[r, dashed, "E(e_{A\otimes B}\otimes 1_C)^{-1}"'] \arrow[rrr, dashed, bend left, "\alpha^E_{A,B,C}"] \arrow[d, dashed, "(\varphi\overset{E}{\otimes}\chi)\overset{E}{\otimes}\psi"] & E((A\otimes B)\otimes C) \arrow[r, "E(\alpha_{A,B,C})"] \arrow[d, "E((\varphi\otimes\chi)\otimes\psi)"'] & E(A\otimes(B\otimes C)) \arrow[r, dashed, "E(1_A\otimes e_{B\otimes C})"'] \arrow[d, "E(\varphi\otimes(\chi\otimes\psi))"'] & A\overset{E}{\otimes}(B\overset{E}{\otimes} C) \arrow[d, dashed, "\varphi\overset{E}{\otimes}(\chi\overset{E}{\otimes}\psi)"'] \\
(A'\overset{E}{\otimes} B')\overset{E}{\otimes} C' \arrow[r, dashed, "E(e_{A'\otimes B'}\otimes 1_{C'})^{-1}"] \arrow[rrr, dashed, bend right, "\alpha^E_{A',B',C'}"'] & E((A'\otimes B')\otimes C') \arrow[r, "E(\alpha_{A',B',C'})"'] & E(A'\otimes(B'\otimes C')) \arrow[r, dashed, "E(1_{A'}\otimes e_{B'\otimes C'})"] & A'\overset{E}{\otimes}(B'\overset{E}{\otimes} C')
\end{tikzcd}$$

If we remove the inner vertices, we obtain the diagram of naturality for α^E:

$$\begin{tikzcd}
(A\overset{E}{\otimes} B)\overset{E}{\otimes} C \arrow[r, "\alpha^E_{A,B,C}"] \arrow[d, "(\varphi\overset{E}{\otimes}\chi)\overset{E}{\otimes}\psi"'] & A\overset{E}{\otimes}(B\overset{E}{\otimes} C) \arrow[d, "\varphi\overset{E}{\otimes}(\chi\overset{E}{\otimes}\psi)"] \\
(A'\overset{E}{\otimes} B')\overset{E}{\otimes} C' \arrow[r, "\alpha^E_{A',B',C'}"] & A'\overset{E}{\otimes}(B'\overset{E}{\otimes} C')
\end{tikzcd}$$

Let us show that α^E satisfies the associativity conditions. For α they look as the pentagon

$$\begin{tikzcd}
 & (A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A\otimes((B\otimes C)\otimes D) \arrow[dr, "1_A\otimes\alpha_{B,C,D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "\alpha_{A,B,C}\otimes 1_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & & A\otimes(B\otimes(C\otimes D)) \\
 & & (A\otimes B)\otimes(C\otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd} \tag{3.137}$$

Let us apply E and add the diagram to the following prism:

$$
\begin{tikzcd}[column sep=small]
& E((A\otimes(B\otimes C))\otimes D) \arrow[rr, "E(\alpha_{A,B\otimes C,D})"] \arrow[d, "E(e_{A\otimes(B\otimes C)}\otimes 1_D)"] & & E(A\otimes((B\otimes C)\otimes D)) \arrow[d, "E(1_A\otimes e_{(B\otimes C)\otimes D})"] \arrow[ddr, bend left, "E(1_A\otimes \alpha_{B,C,D})"] & \\
& E(E(A\otimes(B\otimes C))\otimes D) \arrow[d, equal] & & E(A\otimes E((B\otimes C)\otimes D)) \arrow[d, equal] & \\
E(((A\otimes B)\otimes C)\otimes D) \arrow[uur, bend left, "E(\alpha_{A,B,C}\otimes 1_D)"] \arrow[d, "E(e_{(A\otimes B)\otimes C}\otimes 1_D)"] \arrow[ddrr, "E(\alpha_{A\otimes B,C,D})"] & E(A\otimes(B\otimes C))\overset{E}{\otimes} D \arrow[dd, "E(1_A\otimes e_{B\otimes C})\overset{E}{\otimes} 1_D"] & & A\overset{E}{\otimes} E((B\otimes C)\otimes D) \arrow[dd, "1_A\overset{E}{\otimes} E(e_{B\otimes C}\otimes 1_D)"] & E(A\otimes(B\otimes(C\otimes D))) \arrow[d, "E(1_A\otimes e_{B\otimes(C\otimes D)})"] \\
E(E((A\otimes B)\otimes C)\otimes D) \arrow[d, equal] & & & & E(A\otimes E(B\otimes(C\otimes D))) \arrow[d, equal] \\
E((A\otimes B)\otimes C)\overset{E}{\otimes} D \arrow[d, "E(e_{A\otimes B}\otimes 1_C)\overset{E}{\otimes} 1_D"] & E(A\otimes E(B\otimes C))\overset{E}{\otimes} D \arrow[d, equal] & E((A\otimes B)\otimes(C\otimes D)) \arrow[uurr, "E(\alpha_{A,B,C\otimes D})"] \arrow[dd, "E(e_{A\otimes B}\otimes e_{C\otimes D})"] & A\overset{E}{\otimes} E(E(B\otimes C)\otimes D) \arrow[d, equal] & A\overset{E}{\otimes} E(B\otimes(C\otimes D)) \arrow[d, "1_A\overset{E}{\otimes} E(1_B\otimes e_{C\otimes D})"] \\
E(E(A\otimes B)\otimes C)\overset{E}{\otimes} D \arrow[d, equal] & (A\overset{E}{\otimes}(B\overset{E}{\otimes} C))\overset{E}{\otimes} D \arrow[rr, "\alpha^E_{A,B\overset{E}{\otimes}C,D}"] & & A\overset{E}{\otimes}((B\overset{E}{\otimes} C)\overset{E}{\otimes} D) \arrow[dr, "1_A\overset{E}{\otimes}\alpha^E_{B,C,D}"] & A\overset{E}{\otimes} E(B\otimes E(C\otimes D)) \arrow[d, equal] \\
((A\overset{E}{\otimes} B)\overset{E}{\otimes} C)\overset{E}{\otimes} D \arrow[ur, "\alpha^E_{A,B,C}\overset{E}{\otimes} 1_D"] \arrow[drr, "\alpha^E_{A\overset{E}{\otimes}B,C,D}"] & & E(E(A\otimes B)\otimes E(C\otimes D)) \arrow[d, equal] & & A\overset{E}{\otimes}(B\overset{E}{\otimes}(C\overset{E}{\otimes} D)) \\
& & (A\overset{E}{\otimes} B)\overset{E}{\otimes}(C\overset{E}{\otimes} D) \arrow[urr, "\alpha^E_{A,B,C\overset{E}{\otimes}D}"] & &
\end{tikzcd}
\tag{3.138}
$$

The upper base of this prism is commutative, since this is the action of the functor E on the diagram (3.137), and the commutativity of the lateral sides can be verified by changing the vertical arrows in an equivalent way.

For example, the commutativity of the near side becomes obvious if we represent it as the perimeter of the following diagram:

$$
\begin{tikzcd}
 & E(((A \otimes B) \otimes C) \otimes D) \arrow[r, "E(\alpha_{A \otimes B, C, D})"] \arrow[dl, bend right, "E(e_{(A \otimes B) \otimes C} \otimes 1_D)"'] \arrow[d, "E((e_{A \otimes B} \otimes 1_C) \otimes 1_D)"'] & E((A \otimes B) \otimes (C \otimes D)) \arrow[d, "E(e_{A \otimes B} \otimes (1_C \otimes 1_D))"'] \arrow[ddrd, bend left, "E(e_{A \otimes B} \otimes e_{C \otimes D})"] & \\
E(E((A \otimes B) \otimes C) \otimes D) \arrow[d, equal] & E((E(A \otimes B) \otimes C) \otimes D) \arrow[d, equal] & E(E(A \otimes B) \otimes (C \otimes D)) \arrow[d, equal] & \\
E((A \otimes B) \otimes C) \overset{E}{\otimes} D \arrow[d, "E(e_{A \otimes B} \otimes 1_C) \overset{E}{\otimes} 1_D"] & E(((A \overset{E}{\otimes} B) \otimes C) \otimes D) \arrow[r, "E(\alpha_{A \overset{E}{\otimes} B, C, D})"] \arrow[d, "E(e_{(A \overset{E}{\otimes} B) \otimes C} \otimes 1_D)"] & E((A \overset{E}{\otimes} B) \otimes (C \otimes D)) \arrow[d, "E(1_{A \overset{E}{\otimes} B} \otimes e_{C \otimes D})"] & \\
E(E(A \otimes B) \otimes C) \overset{E}{\otimes} D \arrow[dr, bend right, equal] & E(E((A \overset{E}{\otimes} B) \otimes C) \otimes D) \arrow[d, equal] & E((A \overset{E}{\otimes} B) \otimes E(C \otimes D)) \arrow[r, equal] \arrow[d, equal] & E(E(A \otimes B) \otimes E(C \otimes D)) \arrow[dl, bend left, equal] \\
 & ((A \overset{E}{\otimes} B) \overset{E}{\otimes} C) \overset{E}{\otimes} D \arrow[r, "\alpha^E_{A \overset{E}{\otimes} B, C, D}"] & (A \overset{E}{\otimes} B) \overset{E}{\otimes} (C \overset{E}{\otimes} D) &
\end{tikzcd}
$$

Here the upper inner triangle (or quadrangle) is the result of applying E to the diagram

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "(e_{A\otimes B}\otimes 1_C)\otimes 1_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[d, "e_{A\otimes B}\otimes(1_C\otimes 1_D)"] \\
(E(A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{E(A\otimes B),C,D}"] & E(A \otimes B) \otimes (C \otimes D)
\end{tikzcd}$$

(this is a corollary of (3.136)). The lower inner hexagon is diagram (3.135) for α^E on the components $A \overset{E}{\otimes} B$, C, D. The big octagon can be represented as a rhombus

$$\begin{tikzcd}
& E(((A \otimes B) \otimes C) \otimes D) \arrow[dl, "E(e_{(A\otimes B)\otimes C}\otimes 1_D)"'] \arrow[dr, "E((e_{A\otimes B}\otimes 1_C)\otimes 1_D)"] & \\
E(E((A \otimes B) \otimes C) \otimes D) \arrow[dr, "E(E(e_{A\otimes B}\otimes 1_C)\otimes 1_D)"'] & & E((E(A \otimes B) \otimes C) \otimes D) \arrow[dl, "E(e_{E(A\otimes B)\otimes C}\otimes 1_D)"] \\
& E(E(E(A \otimes B) \otimes C) \otimes D) &
\end{tikzcd}$$

which is a result of applying E to the rhombus

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl, "e_{(A\otimes B)\otimes C}\otimes 1_D"'] \arrow[dr, "(e_{A\otimes B}\otimes 1_C)\otimes 1_D"] & \\
E((A \otimes B) \otimes C) \otimes D \arrow[dr, "E(e_{A\otimes B}\otimes 1_C)\otimes 1_D"'] & & (E(A \otimes B) \otimes C) \otimes D \arrow[dl, "e_{E(A\otimes B)\otimes C}\otimes 1_D"] \\
& E(E(A \otimes B) \otimes C) \otimes D &
\end{tikzcd}$$

which in turn results from multiplying by D on the right the diagram

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[dl, "e_{(A\otimes B)\otimes C}"'] \arrow[dr, "e_{A\otimes B}\otimes 1_C"] & \\
E((A \otimes B) \otimes C) \arrow[dr, "E(e_{A\otimes B}\otimes 1_C)"'] & & E(A \otimes B) \otimes C \arrow[dl, "e_{E(A\otimes B)\otimes C}"] \\
& E(E(A \otimes B) \otimes C) &
\end{tikzcd}$$

and it can be viewed as diagram (3.64) where α is replaced by $e_{A\otimes B} \otimes 1_C$. Finally, the upper right pentagon is the triangle

$$\begin{tikzcd}
E((A \otimes B) \otimes (C \otimes D)) \arrow[d, "E(e_{A\otimes B}\otimes(1_C\otimes 1_D))"'] \arrow[dr, "E(e_{A\otimes B}\otimes e_{C\otimes D})"] & \\
E(E(A \otimes B) \otimes (C \otimes D)) \arrow[r, "E(1_{A\otimes B}\otimes e_{C\otimes D})"] & E(E(A \otimes B) \otimes E(C \otimes D))
\end{tikzcd}$$

which is a result of applying E to the triangle

$$\begin{tikzcd}
(A\otimes B)\otimes(C\otimes D) \arrow[d, "e_{A\otimes B}\otimes(1_C\otimes 1_D)"'] \arrow[dr, "e_{A\otimes B}\otimes e_{C\otimes D}"] & \\
E(A\otimes B)\otimes(C\otimes D) \arrow[r, "1_{A\otimes B}\otimes e_{C\otimes D}"] & E(A\otimes B)\otimes E(C\otimes D)
\end{tikzcd}$$

The same reasoning works for the other vertical sides of (3.138). In addition, the vertical arrows are isomorphisms (by Lemma 3.62 and property (3.134)), so we deduce that the lower base of this prism is commutative as well, and this is the diagram that we need for α^E.

Let $\lambda_X : I\otimes X \to X$ be the left identity in the monoidal category $\mathtt{K}$, and $\rho_X : X\otimes I \to X$ the right identity. For any $A \in \mathtt{Ob}(\mathtt{L})$ we set

$$\begin{aligned}
\lambda^E_A &= E(\lambda_A) : I \overset{E}{\otimes} A = E(I\otimes A) \to E(A) = A,\\
\rho^E_A &= E(\rho_A) : A \overset{E}{\otimes} I = E(A\otimes I) \to E(A) = A,
\end{aligned} \tag{3.139}$$

and these will be the left and the right identities for $\mathtt{L}$. Indeed, for any morphism $\varphi : A \to A'$ in $\mathtt{L}$ the diagrams

$$\begin{tikzcd}
I\otimes A \arrow[r, "\lambda_A"] \arrow[d, "1_I\otimes\varphi"'] & A \arrow[d, "\varphi"] \\
I\otimes A' \arrow[r, "\lambda_{A'}"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[r, "\rho_A"] \arrow[d, "\varphi\otimes 1_I"'] & A \arrow[d, "\varphi"] \\
A'\otimes I \arrow[r, "\rho_{A'}"] & A'
\end{tikzcd} \tag{3.140}$$

give

$$\begin{tikzcd}
I \overset{E}{\otimes} A \arrow[r, equal] & E(I\otimes A) \arrow[r, "\lambda^E_A = E(\lambda_A)"] \arrow[d, "1_I\overset{E}{\otimes}\varphi = E(1_I\otimes\varphi)"'] & A \arrow[d, "\varphi"] \\
I \overset{E}{\otimes} A' \arrow[r, equal] & E(I\otimes A') \arrow[r, "\lambda^E_{A'} = E(\lambda_{A'})"] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
A \overset{E}{\otimes} I \arrow[r, equal] & E(A\otimes I) \arrow[r, "\rho^E_A = E(\rho_A)"] \arrow[d, "\varphi\overset{E}{\otimes}1_I = E(\varphi\otimes 1_I)"'] & A \arrow[d, "\varphi"] \\
A' \overset{E}{\otimes} I \arrow[r, equal] & E(A'\otimes I) \arrow[r, "\rho^E_{A'} = E(\rho_{A'})"] & A'
\end{tikzcd}$$

Moreover, the identity $\lambda_I = \rho_I$ implies $\lambda^E_I = E(\lambda_I) = E(\rho_I) = \rho^E_I$, and the diagram

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A\otimes 1_B"'] & & A\otimes(I\otimes B) \arrow[dl, "1_A\otimes\lambda_B"] \\
& A\otimes B &
\end{tikzcd}$$

gives the upper base of the prism

$$\begin{tikzcd}
E((A\otimes I)\otimes B) \arrow[rr, "E(\alpha_{A,I,B})"] \arrow[d, "E(e_{A\otimes I}\otimes 1_B)"'] \arrow[dr, "E(\rho_A\otimes 1_B)"] & & E(A\otimes (I\otimes B)) \arrow[d, "E(1_A\otimes e_{I\otimes B})"] \arrow[dl, "E(1_A\otimes\lambda_B)"'] \\
E(E(A\otimes I)\otimes B) \arrow[r, "E(\rho^E_A\otimes 1_B)"'] \arrow[d, equal] & E(A\otimes B) \arrow[dd, equal] & E(A\otimes E(I\otimes B)) \arrow[l, "E(1_A\otimes\lambda^E_B)"] \arrow[d, equal] \\
(A\overset{E}{\otimes} I)\overset{E}{\otimes} B \arrow[rr, "\alpha^E_{A,I,B}"] \arrow[dr, "\rho^E_A\overset{E}{\otimes}1_B"'] & & A\overset{E}{\otimes}(I\overset{E}{\otimes} B) \arrow[dl, "1_A\overset{E}{\otimes}\lambda^E_B"] \\
& A\overset{E}{\otimes} B &
\end{tikzcd}$$

The commutativity of its lateral sides is obvious, and the vertical arrows are isomorphisms, so the lower base is also commutative. ■

3.5.3. Envelope as a monoidal functor

THEOREM 3.64. *Let* $\mathsf{Env}^{\Omega}_{\Phi}$ *be a regular envelope coherent with the tensor product in* $\mathtt{K}$. *Then the functor of envelope* $E:\mathtt{K}\to\mathtt{L}$ *built in Theorem* 3.48 *is monoidal.*

Proof. To be monoidal the functor $E:\mathtt{K}\to\mathtt{L}$ must define a morphism of bifunctors

$$\big((X,Y)\mapsto E(X)\overset{E}{\otimes}E(Y)\big)\overset{E^{\otimes}}{\rightarrowtail}\big((X,Y)\mapsto E(X\otimes Y)\big).$$

In this case this is a family of morphisms

$$E^{\otimes}_{X,Y}=E(e_X\otimes e_Y)^{-1}:E(X)\overset{E}{\otimes}E(Y)=E(E(X)\otimes E(Y))\to E(X\otimes Y)$$

(by Lemma 3.62 all morphisms $E(e_X\otimes e_Y)$ are isomorphisms, so $E(e_X\otimes e_Y)^{-1}$ exists) and a morphism E^I in $\mathtt{L}$ that turns the identity object I of $\mathtt{L}$ into the image $E(I)$ in $\mathtt{K}$; in this situation this will be the local identity:

$$E^I=1_I:I\to I\overset{(3.125)}{=}E(I).$$

Let us check the axioms of monoidal functor for these components. The diagram of coherence with associativity:

$$\begin{tikzcd}
E\big((X\otimes Y)\otimes Z\big) \arrow[r, "E(\alpha_{X,Y,Z})"] & E\big(X\otimes (Y\otimes Z)\big) \\
E(X\otimes Y)\overset{E}{\otimes}E(Z) \arrow[u, "E^{\otimes}_{X\otimes Y,Z}"] & E(X)\overset{E}{\otimes}E(Y\otimes Z) \arrow[u, "E^{\otimes}_{X,Y\otimes Z}"'] \\
(E(X)\overset{E}{\otimes}E(Y))\overset{E}{\otimes}E(Z) \arrow[u, "E^{\otimes}_{X,Y}\overset{E}{\otimes}1_{E(X)}"] \arrow[r, "\alpha^E_{E(X),E(Y),E(Z)}"] & E(X)\overset{E}{\otimes}(E(Y)\overset{E}{\otimes}E(Z)) \arrow[u, "1_{E(X)}\overset{E}{\otimes}E^{\otimes}_{Y,Z}"']
\end{tikzcd} \tag{3.141}$$

is translated here as follows:

$$\begin{tikzcd}
E\big((X\otimes Y)\otimes Z\big) \arrow[rr, "E(\alpha_{X,Y,Z})"] \arrow[d, "E(e_{X\otimes Y}\otimes e_Z)"'] & & E\big(X\otimes (Y\otimes Z)\big) \arrow[d, "E(e_X\otimes e_{Y\otimes Z})"] \\
E(E(X\otimes Y)\otimes E(Z)) \arrow[d, "E(E(e_X\otimes e_Y)\otimes 1_{E(Z)})"'] & & E(E(X)\otimes E(Y\otimes Z)) \arrow[d, "E(1_{E(X)}\otimes E(e_Y\otimes e_Z))"] \\
E\big(E(E(X)\otimes E(Y))\otimes E(Z)\big) \arrow[rr, "\alpha^E_{E(X),E(Y),E(Z)}"] & & E\big(E(X)\otimes E(E(Y)\otimes E(Z))\big)
\end{tikzcd}$$

To see that it is commutative, let us represent it as the perimeter of the following diagram:

$$\begin{tikzcd}
& E((X\otimes Y)\otimes Z) \arrow[r, "E(\alpha_{X,Y,Z})"] \arrow[dl, bend right, "E(e_{X\otimes Y}\otimes e_Z)"'] \arrow[d, "E((e_X\otimes e_Y)\otimes e_Z)"] & E(X\otimes (Y\otimes Z)) \arrow[d, "E(e_X\otimes (e_Y\otimes e_Z))"] \arrow[dr, bend left, "E(e_X\otimes e_{Y\otimes Z})"] & \\
E(E(X\otimes Y)\otimes E(Z)) \arrow[dr, bend right, "E(E(e_X\otimes e_Y)\otimes 1_{E(Z)})"'] & E((E(X)\otimes E(Y))\otimes E(Z)) \arrow[r, "E(\alpha_{E(X),E(Y),E(Z)})"] \arrow[d, "E(e_{E(X)\otimes E(Y)}\otimes 1_{E(Z)})"] & E(E(X)\otimes (E(Y)\otimes E(Z))) \arrow[d, "E(1_{E(X)}\otimes e_{E(Y)\otimes E(Z)})"] & E(E(X)\otimes E(Y\otimes Z)) \arrow[dl, bend left, "E(1_{E(X)}\otimes E(e_Y\otimes e_Z))"] \\
& E(E(E(X)\otimes E(Y))\otimes E(Z)) \arrow[r, "\alpha^E_{E(X),E(Y),E(Z)}"'] & E(E(X)\otimes E(E(Y)\otimes E(Z))) &
\end{tikzcd} \tag{3.142}$$

Here the left inner triangle can be represented in the form

$$\begin{tikzcd}
E((X\otimes Y)\otimes Z) \arrow[r, "E((e_X\otimes e_Y)\otimes e_Z)"] \arrow[d, "E(e_{X\otimes Y}\otimes e_Z)"'] & E((E(X)\otimes E(Y))\otimes E(Z)) \arrow[d, "E(e_{E(X)\otimes E(Y)}\otimes 1_{E(Z)})"] \\
E(E(X\otimes Y)\otimes E(Z)) \arrow[r, "E(E(e_X\otimes e_Y)\otimes 1_{E(Z)})"] & E(E(E(X)\otimes E(Y))\otimes E(Z))
\end{tikzcd}$$

This is the result of applying E to the diagram

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "(e_X\otimes e_Y)\otimes e_Z"] \arrow[d, "e_{X\otimes Y}\otimes e_Z"'] & (E(X)\otimes E(Y))\otimes E(Z) \arrow[d, "e_{E(X)\otimes E(Y)}\otimes 1_{E(Z)}"] \\
E(X\otimes Y)\otimes E(Z) \arrow[r, "E(e_X\otimes e_Y)\otimes 1_{E(Z)}"] & E(E(X)\otimes E(Y))\otimes E(Z)
\end{tikzcd}$$

which in turn is the product of the two diagrams

$$\begin{tikzcd} X\otimes Y \arrow[r, "(e_X\otimes e_Y)"] \arrow[d, "e_{X\otimes Y}"'] & E(X)\otimes E(Y) \arrow[d, "e_{E(X)\otimes E(Y)}"] \\ E(X\otimes Y) \arrow[r, "E(e_X\otimes e_Y)"] & E(E(X)\otimes E(Y)) \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "e_Z"] \arrow[d, "e_Z"'] & E(Z) \arrow[d, "1_{E(Z)}"] \\ E(Z) \arrow[r, "1_{E(Z)}"] & E(Z) \end{tikzcd}$$

The left one is trivial, and the right one is diagram (3.130) transposed.

Further, the upper inner quadrangle in (3.142):

$$\begin{tikzcd} E((X\otimes Y)\otimes Z) \arrow[r, "E(\alpha_{X,Y,Z})"] \arrow[d, "E((e_X\otimes e_Y)\otimes e_Z)"'] & E(X\otimes(Y\otimes Z)) \arrow[d, "E(e_X\otimes(e_Y\otimes e_Z))"] \\ E\big((E(X)\otimes E(Y))\otimes E(Z)\big) \arrow[r, "E(\alpha_{E(X),E(Y),E(Z)})"] & E\big(E(X)\otimes(E(Y)\otimes E(Z))\big) \end{tikzcd}$$

is the result of applying E to the diagram

$$\begin{tikzcd} (X\otimes Y)\otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "(e_X\otimes e_Y)\otimes e_Z"'] & X\otimes(Y\otimes Z) \arrow[d, "e_X\otimes(e_Y\otimes e_Z)"] \\ (E(X)\otimes E(Y))\otimes E(Z) \arrow[r, "\alpha_{E(X),E(Y),E(Z)}"] & E(X)\otimes(E(Y)\otimes E(Z)) \end{tikzcd}$$

and the latter is a special case of (3.136).

Then, the lower inner quadrangle in (3.142):

$$\begin{tikzcd} E((E(X)\otimes E(Y))\otimes E(Z)) \arrow[r, "E(\alpha_{E(X),E(Y),E(Z)})"] \arrow[d, "E(e_{E(X)\otimes E(Y)}\otimes 1_{E(Z)})"'] & E(E(X)\otimes(E(Y)\otimes E(Z))) \arrow[d, "E(1_{E(X)}\otimes e_{E(Y)\otimes E(Z)})"] \\ E\big(E(E(X)\otimes E(Y))\otimes E(Z)\big) \arrow[r, "\alpha^E_{E(X),E(Y),E(Z)}"] & E\big(E(X)\otimes E(E(Y)\otimes E(Z))\big) \end{tikzcd}$$

is a special case of (3.135).

Finally, it is useful to represent the right inner quadrangle in (3.142) in the form

$$\begin{tikzcd} E(X\otimes(Y\otimes Z)) \arrow[r, "E(e_X\otimes e_{Y\otimes Z})"] \arrow[d, "E(e_X\otimes(e_Y\otimes e_Z))"'] & E(E(X)\otimes E(Y\otimes Z)) \arrow[d, "E(1_{E(X)}\otimes E(e_Y\otimes e_Z))"] \\ E\big(E(X)\otimes(E(Y)\otimes E(Z))\big) \arrow[r, "E(1_{E(X)}\otimes e_{E(Y)\otimes E(Z)})"] & E\big(E(X)\otimes E(E(Y)\otimes E(Z))\big) \end{tikzcd}$$

This is the result of applying E to the diagram

$$\begin{tikzcd} X\otimes(Y\otimes Z) \arrow[r, "e_X\otimes e_{Y\otimes Z}"] \arrow[d, "e_X\otimes(e_Y\otimes e_Z)"'] & E(X)\otimes E(Y\otimes Z) \arrow[d, "1_{E(X)}\otimes E(e_Y\otimes e_Z)"] \\ E(X)\otimes(E(Y)\otimes E(Z)) \arrow[r, "1_{E(X)}\otimes e_{E(Y)\otimes E(Z)}"] & E(X)\otimes E(E(Y)\otimes E(Z)) \end{tikzcd}$$

which in turn is the product of the two diagrams

$$
\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "e_X"'] & E(X) \arrow[d, "1_{E(X)}"] \\
E(X) \arrow[r, "1_{E(X)}"] & E(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \otimes Z \arrow[r, "e_{Y\otimes Z}"] \arrow[d, "e_Y \otimes e_Z"'] & E(Y \otimes Z) \arrow[d, "E(e_Y \otimes e_Z)"] \\
E(Y) \otimes E(Z) \arrow[r, "e_{E(Y)\otimes E(Z)}"] & E(E(Y) \otimes E(Z))
\end{tikzcd}
$$

The left one here is trivial, and the right one is (3.130) changed a little.

It remains to verify the commutativity of the diagrams for the left and for the right identities:

$$
\begin{tikzcd}
E(I \otimes X) \arrow[r, "E(\lambda_X)"] & E(X) \\
I \overset{E}{\otimes} E(X) \arrow[u, "E^{\otimes}_{I,X}"] & I' \overset{E}{\otimes} E(X) \arrow[l, "E^I \overset{E}{\otimes} 1_{E(X)}"'] \arrow[u, "\lambda'_{E(X)}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E(X \otimes I) \arrow[r, "E(\rho_X)"] & E(X) \\
E(X) \overset{E}{\otimes} I \arrow[u, "E^{\otimes}_{X,I}"] & E(X) \overset{E}{\otimes} I' \arrow[l, "1_{E(X)} \overset{E}{\otimes} E^I"'] \arrow[u, "\rho'_{E(X)}"']
\end{tikzcd}
$$

In our situation they have the form

$$
\begin{tikzcd}
E(I \otimes X) \arrow[r, "E(\lambda_X)"] \arrow[d, "E(e_I \otimes e_X)"'] & E(X) \\
E(I \otimes E(X)) \arrow[r, "E(1_{E(X)} \otimes 1_{E(X)})"] \arrow[d, dash, dashed] & E(I \otimes E(X)) \arrow[u, "E(\lambda_{E(X)})"'] \arrow[d, dash, dashed] \\
E(X \otimes I) \arrow[r, "E(\rho_X)"] \arrow[d, "E(e_X \otimes I)"'] & E(X) \\
E(E(X) \otimes I) \arrow[r, "E(1_{E(X)} \otimes 1_{E(X)})"] & E(E(X) \otimes I) \arrow[u, "E(\rho_{E(X)})"']
\end{tikzcd}
$$

and this is the result of applying E to (3.140) with $X' = E(X)$ and $\varphi = e_X$. ■

Corollary 3.65. *Suppose* $\mathsf{Env}^{\Omega}_{\Phi}$ *is a regular envelope coherent with the tensor product in* $\mathtt{K}$. *The operation* $\mathsf{Env}^{\Omega}_{\Phi}$ *turns each algebra* (*respectively, coalgebra, bialgebra, Hopf algebra*) A *in* $\mathtt{K}$ *into an algebra* (*respectively, coalgebra, bialgebra, Hopf algebra*) $\mathsf{Env}^{\Omega}_{\Phi} A$ *in* $\mathtt{L}$.

Proof. For the case of algebras and general monoidal functors this fact is pointed out in [37]. ■

4. The category $\mathtt{Ste}$ of stereotype spaces

In this section we discuss applications of the above results to the theory of stereotype spaces. To make the exposition more self-contained we give a brief summary of the simplest facts of the theory (for details see [2] and [3]).

4.1. Pseudocomplete and pseudosaturated spaces

4.1.1. Totally bounded and capacious sets. A set S in a locally convex space X is said to be *totally bounded* (or *precompact*) [38] if for each zero neighborhood U in X there is a finite set A such that the shifts of U by elements of A cover S, i.e. $S \subseteq U + A$. This is equivalent to S being totally bounded in the sense of the uniform structure [13] induced from X (i.e. A *can be chosen as a subset in* S).

A set $D \subseteq X$ is said to be *capacious* if for any totally bounded set $S \subseteq X$ there is a finite set $A \subseteq X$ such that the shifts of D by elements of A cover S. (If D is convex, then A can be chosen to be a subset in S.)

Let X be a locally convex space over the field $\mathbb{C}$ of complex numbers. Denote by $X^\star$ the set of continuous linear functionals $f : X \to \mathbb{C}$ endowed with the *topology of uniform convergence on totally bounded sets* in X. We call $X^\star$ the *dual space* of X.

If $B \subseteq X$ and $F \subseteq X^\star$ are arbitrary sets, then we denote by B° and ${}^\circ F$ their (direct and inverse) polars (in $X^\star$ and in X):

$$B^\circ = \Big\{ f \in X^\star : |f|_B := \sup_{x \in B} |f(x)| \le 1 \Big\}, \qquad {}^\circ F = \Big\{ x \in X : |x|_F := \sup_{f \in F} |f(x)| \le 1 \Big\}.$$

Similarly, the *annihilators* of B and F are the sets

$$B^\perp = \{ f \in X^\star : \forall x \in B \ \ f(x) = 0 \}, \qquad {}^\perp F = \{ x \in X : \forall f \in F \ \ f(x) = 0 \}.$$

LEMMA 4.1. *For each locally convex space X:*

(a) *if $B \subseteq X$ is totally bounded, then $B^\circ \subseteq X^\star$ is capacious;*
(b) *if $B \subseteq X$ is capacious, then $B^\circ \subseteq X^\star$ is totally bounded;*
(c) *if $F \subseteq X^\star$ is totally bounded, then ${}^\circ F \subseteq X$ is capacious;*
(d) *if $F \subseteq X^\star$ is capacious, then ${}^\circ F \subseteq X$ is totally bounded.*

LEMMA 4.2. *For each LCS X, every set $A \subseteq X$ and every subspace $E \subseteq X$,*

$$A^\circ \cap E^\perp = (A + E)^\circ. \tag{4.1}$$

Proof. When $A = \emptyset$ or $E = 0$ there is nothing to prove, so we assume that $A \neq \emptyset$ and $E \neq 0$. Then

$$f \in A^\circ \cap E^\perp \ \Rightarrow\ \sup_{a\in A} |f(a)| \le 1 \ \ \&\ \ \forall x \in E \ \ f(x) = 0$$
$$\Rightarrow\ \sup_{a\in A, x\in E} |f(a+x)| = \sup_{a\in A, x\in E} |f(a) + \underbrace{f(x)}_{\substack{\parallel\\ 0}}| \le 1 \ \Rightarrow\ f \in (A+E)^\circ$$

and

$$f \in (A+E)^\circ \ \Rightarrow\ \sup_{a\in A, x\in E} |f(a+x)| \le 1$$
$$\Rightarrow\ \sup_{a\in A} |f(a)| = \sup_{a\in A, x=0} |f(a+x)| \le 1 \ \&\ \exists a \in A \ \ \forall x \in E \ \ |f(a+x)| \le 1$$
$$\Rightarrow\ \sup_{a\in A} |f(a)| \le 1 \ \&\ \forall x \in E \ \ f(x) = 0 \ \Rightarrow\ f \in A^\circ \cap E^\perp. \ \blacksquare$$

4.1.2. Pseudocomplete and pseudosaturated spaces

- A locally convex space X is said to be *pseudocomplete* if every totally bounded Cauchy net in X converges. This is equivalent to every closed totally bounded set in X being compact.

This notion is connected with the usual completeness and quasicompleteness ([1]) by the implications

$$X \text{ is complete } \Rightarrow\ X \text{ is quasicomplete } \Rightarrow\ X \text{ is pseudocomplete.}$$

In the metrizable case these properties are equivalent.

- A locally convex space X is said to be *pseudosaturated* if each closed convex balanced capacious set D in X is a neighborhood of zero.

Example 4.3. Every barreled space is pseudosaturated.

Example 4.4. Every metrizable (not necessarily complete) space is pseudosaturated.

Theorem 4.5 (Criterion for being pseudosaturated). *For a locally convex space X the following conditions are equivalent:*

(i) *X is pseudosaturated;*
(ii) *if a set $F \subseteq X'$ of continuous linear functionals is equicontinuous on each totally bounded set $S \subseteq X$, then F is equicontinuous on X;*
(iii) *if Y is a locally convex space and Φ is a set of continuous linear maps $\varphi : X \to Y$, equicontinuous on each totally bounded set $S \subseteq X$, then Φ is equicontinuous on X.*

Theorem 4.6. *For an arbitrary locally convex space X:*

– *if X is pseudocomplete, then $X^\star$ is pseudosaturated;*
– *if X is pseudosaturated, then $X^\star$ is pseudocomplete.*

([1]) A locally convex space X is said to be *quasicomplete* if every bounded Cauchy net in X converges.

Lemma 4.7. *Let $\varphi : X \to Y$ be a morphism of LCS. Then*

$$\begin{aligned} &\forall A \subseteq X \quad \varphi(A)^\circ_{Y^\star} = (\varphi^\star)^{-1}(A^\circ_{X^\star}), \\ &(\varphi^\star)^{-1}(0) = (\overline{\varphi(X)})^\perp, \qquad (\varphi^\star)^{-1}(0)^\perp = \overline{\varphi(X)}, \end{aligned} \tag{4.2}$$

and if X is pseudocomplete, then

$$\begin{aligned} &\forall B \subseteq Y \quad \varphi^{-1}(B)^\circ_{X^\star} = \overline{\varphi^\star(B^\circ_{Y^\star})}, \\ &\varphi^{-1}(0) = (\overline{\varphi^\star(Y^\star)})^\perp, \qquad \varphi^{-1}(0)^\perp = \overline{\varphi^\star(Y^\star)}. \end{aligned} \tag{4.3}$$

4.1.3. The map $i_X : X \to X^{\star\star}$. The *second dual space* $X^{\star\star}$ of a locally convex space X is the space dual to the first dual:

$$X^{\star\star} = (X^\star)^\star$$

(each star $\star$ means that we take the topology of uniform convergence on totally bounded sets). The formula

$$i_X(x)(f) = f(x)$$

defines a natural map $i_X : X \to X^{\star\star}$.

- Let us say that a linear map $\varphi : X \to Y$ of locally convex spaces is *open* (2) if the image $\varphi(U)$ of any zero neighborhood $U \subseteq X$ is a zero neighborhood in $\varphi(X)$ (with the topology inherited from Y):

  $$\forall U \in \mathcal{U}(X)\ \exists V \in \mathcal{U}(Y) \quad \varphi(U) \supseteq \varphi(X) \cap V.$$

  Clearly, it is sufficient to claim that U is open and absolutely convex. By the obvious formula

  $$\varphi(X) \cap V = \varphi(\varphi^{-1}(V)), \qquad V \subseteq Y \tag{4.4}$$

  (valid for any map $\varphi : X \to Y$ of sets), this condition can be rewritten as follows:

  $$\forall U \in \mathcal{U}(X)\ \exists V \in \mathcal{U}(Y) \quad \varphi(U) \supseteq \varphi(\varphi^{-1}(V)).$$

Theorem 4.8. *For each LCS X the map $i_X : X \to X^{\star\star}$ is injective, open and has dense set of values in $X^{\star\star}$.*

Theorem 4.9. *For an arbitrary LCS X the following conditions are equivalent:*

(i) *X is pseudocomplete;*
(ii) *$i_X : X \to X^{\star\star}$ is surjective (and hence bijective).*

Theorem 4.10. *For an arbitrary LCS X the following conditions are equivalent:*

(i) *X is pseudosaturated;*
(ii) *$i_X : X \to X^{\star\star}$ is continuous.*

Theorem 4.11. *For an arbitrary LCS X:*

– *if X is pseudocomplete, then $X^\star$ is pseudosaturated;*
– *if X is pseudosaturated, then $X^\star$ is pseudocomplete.*

(2) We use the notion of *open map* in the sense different from the one used in General topology [13].

4.2. Variations of openness and closure

4.2.1. Open and closed morphisms. In the stereotype theory the condition dual to the openness defined above is:

- A continuous linear map $\varphi : X \to Y$ of locally convex spaces is *closed* if for any totally bounded set $T \subseteq \overline{\varphi(X)} \subseteq Y$ there is a totally bounded set $S \subseteq X$ such that $T \subseteq \varphi(S)$. This means in particular that $\varphi(X)$ is closed in Y.

THEOREM 4.12. *For a continuous linear map $\varphi : X \to Y$ of locally convex spaces:*

(a) *if X pseudosaturated, Y is pseudocomplete and $\varphi : X \to Y$ is open, then $\varphi^\star : Y^\star \to X^\star$ is closed;*
(b) *if Y is pseudocomplete and $\varphi : X \to Y$ is closed, then $\varphi^\star : Y^\star \to X^\star$ is open.*

For the proof we need

LEMMA 4.13. *Let X be a closed subspace in a LCS Y, T an absolutely convex compact set in Y, and $f : X \to \mathbb{C}$ a continuous linear functional such that*

$$\sup_{x\in T\cap X} |f(x)| < 1. \tag{4.5}$$

Then there exists a continuous linear extension $g : Y \to \mathbb{C}$ of f such that

$$\sup_{y\in T} |g(y)| < 1. \tag{4.6}$$

Proof. Take $\varepsilon > 0$ such that

$$\sup_{x\in T\cap X} |f(x)| < 1 - \varepsilon. \tag{4.7}$$

Since f is continuous on X, the set $Z = \{x \in X : |f(x)| \geq 1 - \varepsilon\}$ is closed in X, and in Y as well. On the other hand, by (4.7), Z is disjoint from T. As a corollary, there is an absolutely convex closed zero neighborhood V in Y such that

$$Z \cap (T + V) = \emptyset.$$

This means, in particular, that

$$\sup_{x\in(T+V)\cap X} |f(x)| < 1 - \varepsilon.$$

If we denote by p the Minkowski functional of $T+V$ (which is a closed absolutely convex zero neighborhood in Y), we obtain

$$|f(x)| \leq (1 - \varepsilon) \cdot p(x), \qquad x \in X.$$

By the Hahn–Banach theorem there is a continuous linear extension $g : Y \to \mathbb{C}$ of f such that

$$|g(y)| \leq (1 - \varepsilon) \cdot p(y), \qquad y \in Y.$$

On $T + V$ we have

$$\sup_{y\in T+V} |g(y)| \leq (1 - \varepsilon) \cdot \sup_{y\in T+V} p(y) = 1 - \varepsilon < 1. \blacksquare$$

Proof of Theorem 4.12. (a) Let $\varphi : X \to Y$ be open. Take a totally bounded set $F \in \mathcal{BS}(\overline{\varphi^\star(Y^\star)})$, i.e. $F \in \mathcal{BS}(X^\star)$ and $F \subseteq \overline{\varphi^\star(Y^\star)}$. By Lemma 4.1(d), the polar $U = {}^\circ F$ is a capacious set in X, and since X is pseudosaturated, U is a zero neighborhood in X. Therefore, since φ is open, there exists a zero neighborhood $V \in \mathcal{BU}(Y)$ such that $\varphi(U) \supseteq \varphi(X) \cap V$. By Lemma 4.1(b), the polar $G = V^\circ$ is a totally bounded set in $Y^\star$. Let us show that $F \subseteq \varphi^\star(G)$.

Take $f \in F$; we will show that there exists $g \in G$ such that $f = \varphi^\star(g)$. Since Y is pseudocomplete, we have $F \subseteq \overline{\varphi^\star(Y^\star)} = (4.3) = \varphi^{-1}(0)^\perp$, so $\varphi^{-1}(0) \subseteq f^{-1}(0)$. Therefore $f = h \circ \varphi$, where h is a (uniquely determined) functional on $\varphi(X)$ (and we need to prove that h is continuous). We have

$$1 \geq \sup_{x \in U} |f(x)| = \sup_{x \in U} |h(\varphi(x))| = \sup_{y \in \varphi(U)} |h(y)| \geq \sup_{y \in \varphi(X) \cap V} |h(y)|,$$

i.e. h is bounded by 1 on the intersection of the unit ball V of the seminorm $p(y) = \inf\{\lambda > 0 : y \in \lambda \cdot V\} = \sup_{g \in G} |g(y)|$ with $\varphi(X)$ where h is defined. In other words, h is subordinated to p on $\varphi(X)$. By the Hahn–Banach theorem, h can be extended to some continuous linear functional $g \in Y^\star$, also subordinated to p, and as a corollary, lying in $V^\circ = G$. Since on $\varphi(X)$ the functionals h and g coincide, we have

$$f = h \circ \varphi = g \circ \varphi = \varphi^\star(g), \qquad g \in G.$$

(b) Suppose Y is pseudocomplete and $\varphi : X \to Y$ is closed. Consider a basic open zero neighborhood V in $Y^\star$, i.e. a set of the form

$$V = \Big\{ g \in Y^\star : \sup_{y \in T} |g(y)| < 1 \Big\},$$

where T is a convex balanced compact set in Y (since Y is pseudocomplete, each closed totally bounded set in Y is compact). The map φ is closed, hence there is a totally bounded set $S \in \mathcal{BS}(X)$ such that $\varphi(S) \supseteq T \cap \overline{\varphi(X)}$. Let

$$U = \Big\{ f \in X^\star : \sup_{x \in S} |f(x)| < 1 \Big\}.$$

If $f \in U \cap \varphi^\star(Y^\star)$, then $\sup_{x \in S} |f(x)| < 1$ and $f = \varphi^\star(g) = g \circ \varphi$ for some $g \in Y^\star$. Let $h = g|_{\overline{\varphi(X)}}$. Then

$$\sup_{y \in T \cap \overline{\varphi(X)}} |h(y)| \leq \sup_{y \in \varphi(S)} |h(y)| = \sup_{x \in S} |h(\varphi(x))| = \sup_{x \in S} |f(x)| < 1.$$

By Lemma 4.13, there is an extension $h' \in Y^\star$ of h such that

$$\sup_{y \in T} |h'(y)| < 1.$$

This means that $h' \in V$, and we obtain $f = \varphi^\star(h') \in \varphi^\star(V)$. So $U \cap \varphi^\star(Y^\star) \subseteq \varphi^\star(V)$. ∎

4.2.2. Weakly open and weakly closed morphisms. Here we consider weakenings of the properties defined above.

- Let us say that a continuous linear map $\varphi : X \to Y$ of locally convex spaces is *weakly open* if it satisfies the following equivalent conditions:
  - (i) each functional $f \in X^\star$ with $f|_{\mathsf{Ker}\,\varphi} = 0$ can be extended along φ to a functional $g \in Y^\star$: $f = g \circ \varphi$;

(ii) the image $\varphi(U)$ of any $X^\star$-weak zero neighborhood $U \subseteq X$ is a $Y^\star$-weak neighborhood of zero in $\varphi(X)$ (with the topology induced from Y):

$$\forall U \in \mathcal{U}(X_w)\ \exists V \in \mathcal{U}(Y_w) \quad \varphi(U) \supseteq \varphi(X) \cap V \tag{4.8}$$

(here X_w denotes X with the $X^\star$-weak topology, and similarly Y_w);

(iii) the image $\varphi(U)$ of any $X^\star$-weak zero neighborhood $U \subseteq X$ is a zero neighborhood (not necessarily $Y^\star$-weak) in $\varphi(X)$:

$$\forall U \in \mathcal{U}(X_w)\ \exists V \in \mathcal{U}(Y) \quad \varphi(U) \supseteq \varphi(X) \cap V. \tag{4.9}$$

Proof of equivalence. (i)$\Rightarrow$(ii). Let U be an $X^\star$-weak neighborhood of zero in X. Then so is $\widetilde{U} = U + \varphi^{-1}(0)$, and in addition

$$\varphi(U) = \varphi(\widetilde{U}), \qquad \widetilde{U} + \varphi^{-1}(0) = \widetilde{U}.$$

From the second equality it follows that $\widetilde{U}$ contains the polar ${}^\circ\{f_1, \dots, f_k\}$ of some finite sequence of functionals $f_i \in X^\star$ such that $\varphi^{-1}(0) \subseteq f_i^{-1}(0)$. By (i), each f_i can be extended to some functional $g_i \in Y^\star$:

$$f_i = g_i \circ \varphi.$$

Letting $V = {}^\circ\{g_1, \dots, g_k\}$, we have (4.9):

$$\begin{aligned} y \in \varphi(U) = \varphi(\widetilde{U}) \ &\Leftarrow\ y \in \varphi({}^\circ\{f_1, \dots, f_k\}) \ \Leftrightarrow\ \exists x \in {}^\circ\{f_1, \dots, f_k\}\ \ y = \varphi(x) \\ &\Leftrightarrow\ \exists x \in X\ \ y = \varphi(x)\ \&\ \sup_i |f_i(x)| \le 1 \\ &\Leftrightarrow\ \exists x \in X\ \ y = \varphi(x)\ \&\ \sup_i |g_i(\varphi(x))| \le 1 \\ &\Leftrightarrow\ \exists x \in X\ \ y = \varphi(x)\ \&\ \sup_i\ |g_i(y)| \le 1 \\ &\Leftrightarrow\ y \in \varphi(X)\ \&\ y \in V \ \Leftrightarrow\ y \in \varphi(X) \cap V. \end{aligned}$$

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). Let $f \in X^\star$ be such that $\operatorname{Ker}\varphi \subseteq \operatorname{Ker} f$. Its polar $U = {}^\circ f$ is an $X^\star$-weak neighborhood of zero in X, so $\varphi(U) \supseteq \varphi(X) \cap V$ for some zero neighborhood V in Y. This means that f can be extended to a functional h on $\varphi(X)$, which is bounded on $\varphi(X) \cap V$:

$$f = h \circ \varphi, \qquad \sup_{y \in \varphi(X) \cap V} |h(y)| \le 1.$$

Hence, h is a continuous functional on $\varphi(X)$ (with respect to the topology induced from Y). By the Hahn–Banach theorem it can be extended to a functional $g \in Y^\star$, and we have $f = h \circ \varphi = g \circ \varphi$. ∎

- Let us say that a continuous linear map $\varphi : X \to Y$ is *weakly closed* if $\varphi(X)$ is closed in Y.

Proposition 4.14. *For a continuous linear map $\varphi : X \to Y$ of locally convex spaces:*

– *if φ is open, then it is weakly open;*
– *if φ is closed, then it is weakly closed.*

Proof. The first part follows from condition (iii) in the definition of weak openness on p. 111, and the second part is obvious, as already noticed when we defined closure on p. 110. ■

THEOREM 4.15. *For a continuous linear map $\varphi : X \to Y$ of locally convex spaces:*

(a) *$\varphi : X \to Y$ is weakly open $\Leftrightarrow$ $\varphi^\star : Y^\star \to X^\star$ is weakly closed;*
(b) *if Y is pseudosaturated and $\varphi : X \to Y$ is weakly closed, then $\varphi^\star : Y^\star \to X^\star$ is weakly open.*

Proof. The first assertion is exactly the equivalence of (i) and (ii) in the definition of weak openness. Let us prove the second one. Suppose Y is pseudosaturated and $\varphi : X \to Y$ is weakly closed. By (a), to prove that $\varphi^\star : Y^\star \to X^\star$ is weakly open, it is sufficient to verify that $\varphi^{\star\star} : X^{\star\star} \to Y^{\star\star}$ is closed. Take $h \in \overline{\varphi^{\star\star}(X^{\star\star})}$. Since Y is pseudosaturated, $Y^\star$ is pseudocomplete by Theorem 4.11. Therefore,

$$h \in \overline{\varphi^{\star\star}(X^{\star\star})} \overset{(4.3)}{=} (\varphi^\star)^{-1}(0)^\perp \overset{(4.2)}{=} \left(\overline{\varphi(X)}\right)^{\perp\perp} = i_Y\left({}^\perp\left(\overline{\varphi(X)}^\perp\right)\right)$$

(the last equality means that the map $i_Y : Y \to Y^{\star\star}$, bijective by Theorem 4.9, turns the annihilator of the space $\overline{\varphi(X)}^\perp$, meant as a subspace in Y, into its annihilator, meant as a subspace in $Y^{\star\star}$). This in turn means that there is $y \in \overline{\varphi(X)}$ such that $h = i_Y(y)$. Since φ is weakly closed, there exists $x \in X$ such that $y = \varphi(x)$. If we denote $g = i_X(x)$, then

$$h = i_Y(y) = i_Y(\varphi(x)) = \varphi^{\star\star}(i_x(x)) = \varphi^{\star\star}(g). \ ■$$

4.2.3. Relatively open and relatively closed morphisms. Another weakening of openness and closure of morphisms is the following.

- We say that a continuous linear map $\varphi : X \to Y$ of locally convex spaces is
  - *relatively open* if for each zero neighborhood U in X (without loss of generality we may assume that U is closed and absolutely convex) such that every functional $f \in X^\star$ bounded on U can be extended along φ to some functional $g \in Y^\star$,
  $$\forall f \in X^\star \quad \Big(\sup_{x\in U} |f(x)| < \infty \ \Rightarrow\ \exists g \in Y^\star \ \ f = g \circ \varphi\Big), \tag{4.10}$$
  its image $\varphi(U)$ is a neighborhood of zero in $\varphi(X)$ (with the topology inherited from Y);
  - *relatively closed* if for each absolutely convex compact set $T \subseteq Y$, if $T \subseteq \varphi(X)$, then there is a compact set $S \subseteq X$ such that $T \subseteq \varphi(S)$.

The following is obvious:

PROPOSITION 4.16. *For a morphism $\varphi : X \to Y$ of locally convex spaces:*

– *if φ is open, then it is relatively open;*
– *if φ is closed, then it is relatively closed.*

THEOREM 4.17. *For a continuous linear map $\varphi : X \to Y$ of locally convex spaces:*

(a) *$\varphi : X \to Y$ is relatively open $\Leftrightarrow$ $\varphi^\star : Y^\star \to X^\star$ is relatively closed;*
(b) *if X is pseudocomplete and $\varphi : X \to Y$ is relatively closed, then $\varphi^\star : Y^\star \to X^\star$ is relatively open.*

Proof. (a) Suppose φ is relatively open, and T is a closed absolutely convex totally bounded set in $X^\star$, contained in $\varphi^\star(Y^\star)$:

$$\forall f \in T\ \exists g \in Y^\star \quad f = \varphi^\star(g) = g \circ \varphi. \tag{4.11}$$

For the polar $U = {}^\circ T$ this means condition (4.10) holds, and, since U is a zero neighborhood in X, $\varphi(U)$ is a zero neighborhood in $\varphi(X)$ (with the topology induced from Y). That is, there exists a zero neighborhood V in Y such that

$$\varphi(U) \supseteq V \cap \varphi(X).$$

Clearly, V can be chosen to be closed and absolutely convex in Y. Let $S = V^\circ$; we will show that $T \subseteq \varphi^\star(S)$, i.e.

$$\forall f \in U^\circ\ \exists h \in V^\circ \quad f = \varphi^\star(h) = h \circ \varphi. \tag{4.12}$$

Indeed, take $f \in T = U^\circ$. Then by (4.11) one can choose $g \in Y^\star$ such that $f = g \circ \varphi$. The restriction $g|_{\varphi(X)}$ is bounded by 1 on the zero neighborhood $V \cap \varphi(X)$:

$$\sup_{y \in V \cap \varphi(X)} |g(y)| \le \sup_{y \in \varphi(U)} |g(y)| \le \sup_{x \in U} |g(\varphi(x))| = \sup_{x \in U} |f(x)| \le 1.$$

In other words, $g|_{\varphi(X)}$ on $\varphi(X)$ is subordinated to the seminorm

$$p(y) = \inf\{\lambda > 0 : y \in \lambda \cdot V\}.$$

By the Hahn–Banach theorem, $g|_{\varphi(X)}$ can be extended to some functional h on Y, subordinated to p:

$$|h(y)| \le p(y) \quad (y \in Y), \qquad h|_{\varphi(X)} = g.$$

From the first condition it follows that $\sup_{y \in V} |h(y)| \le \sup_{y \in V} p(y) \le 1$, i.e. $h \in V^\circ = S$. And from the second one, $h(\varphi(x)) = g(\varphi(x)) = f(x)$. Together this means (4.12).

Now suppose $\varphi^\star : Y^\star \to X^\star$ is relatively closed and let U be an absolutely convex zero neighborhood in X satisfying (4.10). The polar $T = U^\circ$ is a closed absolutely convex totally bounded set in $X^\star$, and for it the condition (4.10) is equivalent to (4.11). This in turn means $T \subseteq \varphi^\star(Y^\star)$, and since $\varphi^\star$ is relatively closed, there exists an absolutely convex totally bounded set $S \subseteq Y^\star$ such that

$$T \subseteq \varphi^\star(S).$$

Hence

$$\begin{aligned} {}^\circ T = \{x \in X : \forall f \in T \quad |f(x)| \le 1\} &\supseteq \{x \in X : \forall g \in S \quad |g(\varphi(x))| \le 1\} \\ &= \{x \in X : \varphi(x) \in {}^\circ S\} = \varphi^{-1}({}^\circ S). \end{aligned}$$

Letting $V = {}^\circ S$ (a zero neighborhood in Y) we obtain

$$U \supseteq \varphi^{-1}(V) \ \Rightarrow\ \varphi(U) \supseteq \varphi(\varphi^{-1}(V)) \overset{(4.4)}{=} \varphi(X) \cap V.$$

(b) Suppose U is an absolutely convex zero neighborhood in $Y^\star$, satisfying (4.10), i.e.

$$\forall \upsilon \in Y^{\star\star} \quad \Big(\sup_{g \in U} |\upsilon(g)| < \infty \ \Rightarrow\ \exists \xi \in X^{\star\star} \ \ \upsilon = \xi \circ \varphi^\star \Big).$$

In particular, for any $y \in T = {}^\circ U$ there exists $\xi \in X^{\star\star}$ such that

$$i_Y(y) = \xi \circ \varphi^\star.$$

Since X is pseudocomplete, by Theorem 4.9 there exists $x \in X$ such that $i_X(x) = \xi$. Then

$$\forall g \in Y^\star \quad g(y) = i_Y(y)(g) = (i_X(x) \circ \varphi^\star)(g) = i_X(x)(\varphi^\star(g)) = \varphi^\star(g)(x) = g(\varphi(x)),$$

and therefore $y = \varphi(x)$. We have proved that $T \subseteq \varphi(X)$, and since φ is relatively closed, there exists an absolutely convex totally bounded set $S \subseteq X$ such that

$$T \subseteq \varphi(S).$$

We have

$$T^\circ \supseteq (\varphi(S))^\circ \overset{(4.2)}{=} (\varphi^\star)^{-1}(S^\circ).$$

Now if we put $V = S^\circ$ (a zero neighborhood in $X^\star$), then

$$U \supseteq (\varphi^\star)^{-1}(V) \;\Rightarrow\; \varphi^\star(U) \supseteq \varphi^\star\big((\varphi^\star)^{-1}(V)\big) \overset{(4.4)}{=} \varphi^\star(Y^\star) \cap V.$$

This is what we need. ∎

4.2.4. Connections between the three variations of openness and closure. Propositions 4.14 and 4.16 can be strengthened as follows.

THEOREM 4.18. *For a morphism $\varphi : X \to Y$ of locally convex spaces:*

(a) *φ is open $\Leftrightarrow$ φ is weakly open and relatively open;*
(b) *φ is closed $\Leftrightarrow$ φ is weakly closed and relatively closed.*

Proof. In both cases the direction from left to right was already noticed in Propositions 4.14 and 4.16, so we must check the reverse implications.

(a) For each zero neighborhood U in X the set $U+\varphi^{-1}(0)$ is also a zero neighborhood in X. If $f \in X^\star$ is bounded on $U + \varphi^{-1}(0)$, then, $f|_{\varphi^{-1}(0)} = 0$, so by the weak openness of φ, f can be extended to a functional $g \in Y^\star$. This means that the zero neighborhood $U + \varphi^{-1}(0)$ satisfies (4.10). Since φ is relatively open, we have

$$\varphi(U) = \varphi(U + \varphi^{-1}(0)) \supseteq \varphi(X) \cap V$$

for some zero neighborhood V in Y.

(b) First, $\overline{\varphi(X)} = \varphi(X)$, and second, each closed absolutely convex totally bounded set $T \subseteq \varphi(X)$ is the image of some totally bounded set $S \subseteq X$ under φ. Together this means that T can be chosen to be a subset in $\overline{\varphi(X)}$. Therefore φ is closed. ∎

4.2.5. Embeddings and coverings

- A continuous linear map $\varphi : X \to Y$ of locally convex spaces will be called:
  - an *embedding* (respectively, a *weak embedding*, a *relative embedding*) if it is injective and open (respectively, weakly open, relatively open);
  - a *dense embeddding* (respectively, a *dense weak embedding*, a *dense relative embedding*) if in addition $\varphi(X)$ is dense in Y;
  - a *covering* (respectively, a *weak covering*, a *relative covering*) if it is surjective and closed (respectively, weakly closed, relatively closed);
  - an *exact covering* (respectively, an *exact weak covering*, an *exact relative covering*) if in addition it is injective.

REMARK 4.19. If a LCS X is pseudocomplete and $\varphi : X \to Y$ is an exact covering, then for any totally bounded set $S \subseteq X$ the restriction $\varphi|_S : S \to \varphi(S)$ is a homeomorphism of topological spaces.

EXAMPLE 4.20. If a locally convex space X is pseudocomplete, then the (continuous and bijective) map $i_X^{-1} : X^{\star\star} \to X$ is defined, and it is an exact covering.

EXAMPLE 4.21. If a locally convex space X is pseudosaturated, then $i_X : X \to X^{\star\star}$ is a dense embedding.

The following is proved in [2, Theorems 3.2, 3.1].

THEOREM 4.22. *For a continuous linear map $\varphi : X \to Y$ of locally convex spaces:*

– *if X is pseudosaturated and $\varphi : X \to Y$ is a dense embedding, then $\varphi^\star : Y^\star \to X^\star$ is an exact covering;*
– *if X is pseudocomplete and $\varphi : X \to Y$ is an exact covering, then $\varphi^\star : Y^\star \to X^\star$ is a dense embedding.*

4.3. Pseudocompletion and pseudosaturation

4.3.1. Pseudocompletion. As in the case of completeness, each locally convex space X has a pseudocompletion, i.e. the "outward-nearest" pseudocomplete space. Formally this construction is described in the following

THEOREM 4.23. *There exists a map $X \mapsto \triangledown_X$ that assigns to each locally convex space X a continuous linear map $\triangledown_X : X \to X^\triangledown$ into a pseudocomplete locally convex space $X^\triangledown$ in such a way that:*

(i) *X is pseudocomplete if and only if $\triangledown_X : X \to X^\triangledown$ is an isomorphism;*
(ii) *for any continuous linear map $\varphi : X \to Y$ of locally convex spaces there is a unique continuous linear map $\varphi^\triangledown : X^\triangledown \to Y^\triangledown$ such that*

$$\begin{tikzcd} X \arrow[r, "\triangledown_X"] \arrow[d, "\varphi"'] & X^\triangledown \arrow[d, dashed, "\varphi^\triangledown"] \\ Y \arrow[r, "\triangledown_Y"] & Y^\triangledown \end{tikzcd} \tag{4.13}$$

From (i), (ii) it follows that for any continuous linear map $\varphi : X \to Y$ into a pseudocomplete space Y there exists a unique continuous linear map $X^\triangledown \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "\triangledown_X"] \arrow[dr, "\varphi"'] & & X^\triangledown \arrow[dl, dashed] \\ & Y & \end{tikzcd} \tag{4.14}$$

This means that $\triangledown_X : X \to X^\triangledown$ is an extension of X in $\mathsf{Ob}(\mathsf{LCS})$ with respect to the object $\mathbb{C}$. Since $\mathbb{C}$ separates morphisms on the outside in LCS, by Theorem 3.8, $\triangledown_X : X \to X^\triangledown$ is a bimorphism. This in turn implies that the morphism $\triangledown_X : X \to X^\triangledown$ is unique up to an isomorphism in Epi^X.

- The space $X^\triangledown$ is called the *pseudocompletion*, and the map $\triangledown_X : X \to X^\triangledown$ the *pseudocompletion map*, of the locally convex space X. From (ii) it also follows that $\varphi \mapsto \varphi^\triangledown$ is a covariant functor from the category $\mathtt{LCS}$ into itself: $(\psi \circ \varphi)^\triangledown = \psi^\triangledown \circ \varphi^\triangledown$. We call it the *pseudocompletion functor*.

THEOREM 4.24. *For any locally convex space X the pseudocompletion map $\triangledown_X : X \to X^\triangledown$ is a dense embedding.*

Like the usual completion, the pseudocompletion operation $X \mapsto X^\triangledown$ adds new elements to X, but does not change the topology of X.

4.3.2. Pseudosaturation. It is remarkable that there exists a dual construction, which assigns to each locally convex space X an "inward-nearest" pseudosaturated locally convex space $X^\vartriangle$:

THEOREM 4.25. *There exists a map $X \mapsto \vartriangle_X$ that assigns to each locally convex space X a continuous linear map $\vartriangle_X \colon X^\vartriangle \to X$ from a pseudosaturated locally convex space $X^\vartriangle$ in such a way that:*

(i) *X is pseudosaturated if and only if $\vartriangle_X \colon X^\vartriangle \to X$ is an isomorphism;*
(ii) *for any continuous linear map $\varphi : Y \to X$ of locally convex spaces there is a unique continuous linear map $\varphi^\vartriangle : Y^\vartriangle \to X^\vartriangle$ such that*

$$\begin{tikzcd} X & X^\vartriangle \arrow[l, "\vartriangle_X"'] \\ Y \arrow[u, "\varphi"] & Y^\vartriangle \arrow[l, "\vartriangle_Y"'] \arrow[u, dashed, "\varphi^\vartriangle"'] \end{tikzcd} \tag{4.15}$$

From (i), (ii) it follows that for any continuous linear map $\varphi : Y \to X$ from a pseudosaturated locally convex space Y there is a unique continuous linear map $Y \to X^\vartriangle$ such that

$$\begin{tikzcd} X & & X^\vartriangle \arrow[ll, "\vartriangle_X"'] \\ & Y \arrow[ul, "\varphi"] \arrow[ur, dashed] & \end{tikzcd} \tag{4.16}$$

This means that $\vartriangle_X \colon X^\vartriangle \to X$ is an enrichment of X in the class $\mathsf{Ob}(\mathtt{LCS})$ by means of the object $\mathbb{C}$. Since $\mathbb{C}$ separates morphisms on the inside in $\mathtt{LCS}$, by Theorem 3.19, $\vartriangle_X \colon X^\vartriangle \to X$ is a bimorphism. This implies that the morphism $\vartriangle_X \colon X^\vartriangle \to X$ is unique up to an isomorphism in Mono_X.

- The space $X^\vartriangle$ is called the *pseudosaturation*, and the map $\vartriangle_X \colon X^\vartriangle \to X$ the *pseudosaturation map*, of X. From (ii) it follows that $\varphi \mapsto \varphi^\vartriangle$ is a covariant functor from the category $\mathtt{LCS}$ into itself: $(\psi \circ \varphi)^\vartriangle = \psi^\vartriangle \circ \varphi^\vartriangle$. We call it the *pseudosaturation functor*.

THEOREM 4.26. *For any locally convex space X the pseudosaturation map $\vartriangle_X \colon X^\vartriangle \to X$ is an exact covering.*

The pseudosaturation $X^\vartriangle$ can be viewed as a new, stronger topologization of X, which preserves the system of totally bounded sets in X and the topology on each of them.

Each of the operations $X \mapsto X^\triangledown$ and $X \mapsto X^\vartriangle$ preserves the properties of being pseudocomplete and pseudosaturated:

THEOREM 4.27. *For a locally convex space X:*

– *if X is pseudocomplete, then so is its pseudosaturation $X^\vartriangle$;*
– *if X is pseudosaturated, then so is its pseudocompletion $X^\triangledown$.*

The following examples show that pseudocompletion and psudosaturation are independent.

EXAMPLE 4.28. Let X be an infinite-dimensional Banach space, and $Y = X'_\sigma$ its dual space with the X-weak topology. Then Y is pseudocomplete, but not pseudosaturated.

EXAMPLE 4.29. An arbitrary non-complete metrizable locally convex space is pseudosaturated, but not pseudocomplete.

4.3.3. Duality between pseudocompletion and pseudosaturation. The passage to the dual space $X \mapsto X^\star$ interchanges pseudocompleteness and pseudosaturatedness:

THEOREM 4.30. *Let X be a pseudocomplete LCS. Then:*

(a) *there is a unique isomorphism of locally convex spaces*

$$(X^\vartriangle)^\star \rightsquigarrow (X^\star)^\triangledown \tag{4.17}$$

such that

$$\begin{tikzcd} (X^\vartriangle)^\star \arrow[rr, rightsquigarrow] & & (X^\star)^\triangledown \\ & X^\star \arrow[ul, "(\vartriangle_X)^\star"] \arrow[ur, "\triangledown_{X^\star}"'] & \end{tikzcd} \tag{4.18}$$

(b) *for any continuous linear map $\varphi : X \to Y$ of locally convex spaces we have*

$$\begin{tikzcd} (X^\vartriangle)^\star \arrow[r, rightsquigarrow] & (X^\star)^\triangledown \\ (Y^\vartriangle)^\star \arrow[r, rightsquigarrow] \arrow[u, "(\varphi^\vartriangle)^\star"] & (Y^\star)^\triangledown \arrow[u, "(\varphi^\star)^\triangledown"'] \end{tikzcd} \tag{4.19}$$

THEOREM 4.31. *Let X be a pseudosaturated LCS. Then:*

(a) *there is a unique isomorphism of locally convex spaces*

$$(X^\triangledown)^\star \rightsquigarrow (X^\star)^\vartriangle \tag{4.20}$$

such that

$$\begin{tikzcd} (X^\triangledown)^\star \arrow[rr, rightsquigarrow] \arrow[dr, "(\triangledown_X)^\star"'] & & (X^\star)^\vartriangle \arrow[dl, "\vartriangle_{X^\star}"] \\ & X^\star & \end{tikzcd} \tag{4.21}$$

(b) *for any continuous linear map $\varphi : X \to Y$ of locally convex spaces we have*

$$\begin{tikzcd} (X^\triangledown)^\star \arrow[r, rightsquigarrow] & (X^\star)^\vartriangle \\ (Y^\triangledown)^\star \arrow[r, rightsquigarrow] \arrow[u, "(\varphi^\triangledown)^\star"] & (Y^\star)^\vartriangle \arrow[u, "(\varphi^\star)^\vartriangle"'] \end{tikzcd} \tag{4.22}$$

4.4. Stereotype spaces

- A locally convex space X is said to be *stereotype* if its natural map to the second dual space
$$i_X : X \to (X^\star)^\star \quad i_X(x)(f) = f(x), \quad x \in X,\ f \in X^\star,$$
is an isomorphism of locally convex spaces (both $\star$'s means the dual space in the sense of the definition on p. 107).

Clearly, if X is a stereotype space, then so is $X^\star$. Theorems 4.9 and 4.10 imply the following criterion:

THEOREM 4.32. *A locally convex space X is stereotype if and only if it is pseudocomplete and pseudosaturated.*

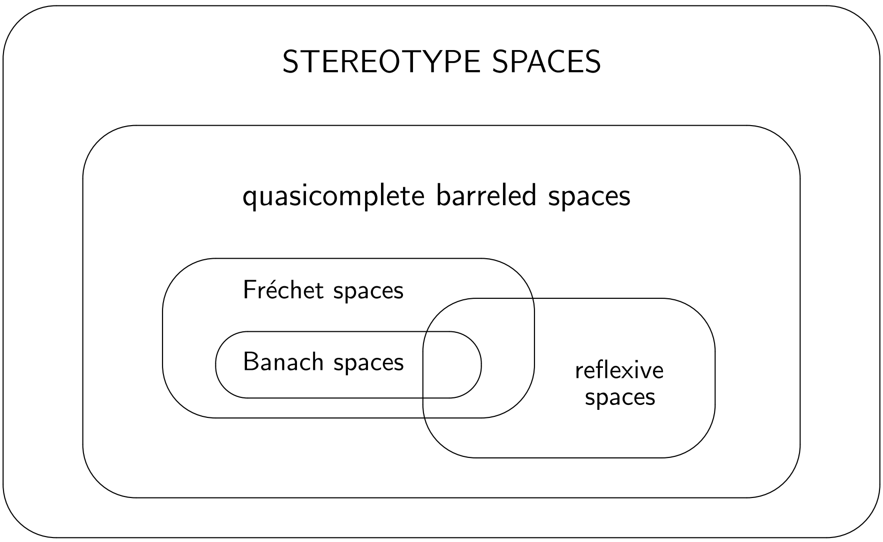
This means in particular that there are non-stereotype locally convex spaces (since there are non-pesudocomplete and non-pseudosaturated spaces: see Examples 4.28 and 4.29). Nevertheless, the class `Ste` of stereotype spaces turns out to be amazingly wide. This is seen from the following series of examples, generalizing each other.

EXAMPLE 4.33. All Banach spaces are stereotype.

EXAMPLE 4.34. All Fréchet spaces are stereotype.

EXAMPLE 4.35. All quasicomplete barreled spaces are stereotype.

As a corollary, the place of stereotype spaces among other frequently used classes of spaces can be illustrated by the following diagram:



(4.23)

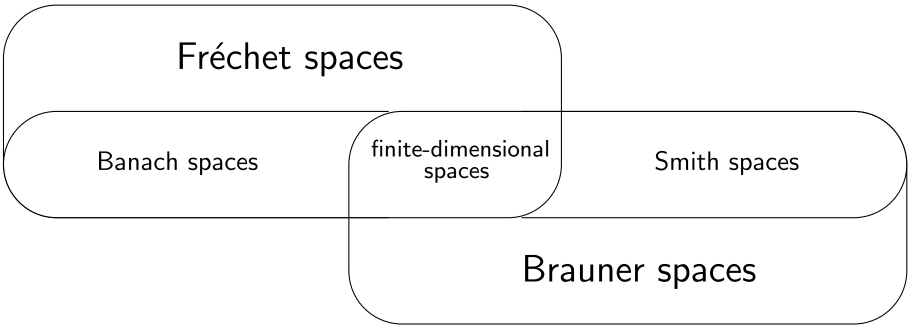
This picture is supplemented by the examples of spaces, dual to the already mentioned, and having quite unwonted ([3]) properties:

([3]) Because of the non-standard notion of dual space.

Example 4.36. A locally convex space X is called a *Smith space* (4) if it is a complete k-space (5) and has a *universal compact set*, i.e. a compact set $K \subset X$ that absorbs any other compact set $T \subset X$: $T \subseteq \lambda K$ for some $\lambda \in \mathbb{C}$. It is known that *a locally convex space X is a Smith space if and only if it is stereotype and its dual space $X^\star$ is a Banach space.*

Example 4.37. A locally convex space X is called a *Brauner space* (6) if it is a complete k-space and has a *countable fundamental system of compact sets*, i.e. a sequence of compact sets $K_n \subseteq X$ such that every compact set $T \subseteq X$ is contained in some K_n. *A locally convex space X is a Brauner space if and only if it is stereotype and its dual space $X^\star$ is a Fréchet space.*

The connections between the spaces of Fréchet, Brauner, Banach, and Smith are illustrated in the following diagram (where the turnover corresponds to the passage to the dual class):



It is clear from the definition that each stereotype space X can be recovered from its dual space $X^\star$. So different properties of X have their dual analogs in $X^\star$. The most obvious facts of that type are listed in the following

Theorem 4.38. *Let X be a stereotype space. Then:*

(a) *X is normable $\Leftrightarrow$ X is a Banach space $\Leftrightarrow$ $X^\star$ is a Smith space;*
(b) *X is metrizable $\Leftrightarrow$ X is a Fréchet space $\Leftrightarrow$ $X^\star$ is a Brauner space;*
(c) *X is barreled $\Leftrightarrow$ $X^\star$ has the Heine–Borel psoperty;*
(d) *X is quasibarreled $\Leftrightarrow$ in $X^\star$ each subset absorbed by any barrel is totally bounded;*
(e) *X is a Mackey space $\Leftrightarrow$ in $X^\star$ each $(X^\star)^\star$-weak compact set is compact;*
(f) *X is a Montel space $\Leftrightarrow$ X is barreled and has the Heine–Borel property $\Leftrightarrow$ $X^\star$ is a Montel space;*
(g) *X has a weak topology $\Leftrightarrow$ in $X^\star$ every compact set is finite-dimensional;*
(h) *X is separable (i.e. has a countable everywhere dense set) $\Leftrightarrow$ in $X^\star$ there is a sequence of closed subspaces L_n of finite codimension with $\bigcap_{n=1}^{\infty} L_n = \{0\}$;*

(4) After M. F. Smith [43].
(5) A topological space X is called a *k-space* or a *Kelley space* if every set $M \subseteq X$ having closed trace $M \cap K$ on each compact set $K \subseteq X$ is closed in X.
(6) After K. Brauner [10].

(i) *X has the (classical) approximation property $\Leftrightarrow$ $X^\star$ has the approximation property;*
(j) *X is complete $\Leftrightarrow$ $X^\star$ cocomplete* (7) *$\Leftrightarrow$ $X^\star$ is saturated* (8)*;*
(k) *X is a Pták space* (9) *$\Leftrightarrow$ in $X^\star$ a subspace L is closed if it leaves a closed trace $L \cap K$ on each compact set $K \subseteq X^\star$;*
(l) *X is hypercomplete* (10) *$\Leftrightarrow$ in $X^\star$ an absolutely convex set B is closed if it leaves a closed trace $B \cap K$ on each compact set $K \subseteq X^\star$.*

PROPOSITION 4.39. *Let E be a closed subspace in a locally convex space X, considered as a locally convex space with the topology induced from X, and let the annihilator $E^\perp$ be also endowed with the topology induced from $X^\star$. Then:*

(a) *there is a natural isomorphism of locally convex spaces*
$$E^\star \cong X^\star / E^\perp, \tag{4.24}$$
and if in addition E is pseudocomplete (for example, if X is pseudocomplete), then (4.24) *generates isomorphisms of stereotype spaces*
$$(E^\vartriangle)^\star \cong (X^\star/E^\perp)^\triangledown, \qquad E^\vartriangle \cong [(X^\star/E^\perp)^\triangledown]^\star; \tag{4.25}$$
(b) *if X is stereotype, then there is a natural isomorphism of locally convex spaces*
$$(E^\perp)^\star \cong X/E \tag{4.26}$$
generating isomorphisms of stereotype spaces
$$((E^\perp)^\vartriangle)^\star \cong (X/E)^\triangledown, \qquad (E^\perp)^\vartriangle \cong [(X/E)^\triangledown]^\star. \tag{4.27}$$

The following example is due to O. G. Smolyanov [44] and it was mentioned in [2] (as Example 3.22). We will use it later as an important technical result:

EXAMPLE 4.40. There is a stereotype space Z with the following properties:

(i) Z and $Z^\star$ are complete and saturated;
(ii) Z has a closed subspace Y such that
  (a) the quotient space Z/Y is metrizable, but not complete;
  (b) the annihilator $Y^\perp$ (with the topology induced from $Z^\star$) is not a pseudosaturated space.

Proof. An example is the space $Z = \mathcal{D}(\mathbb{R})$ of smooth functions with compact support on $\mathbb{R}$. It is complete (as the strong inductive limit of a sequence of complete spaces [35]) and saturated (as the inductive limit of a system of saturated spaces). By Theorem 4.11,

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(7) A locally convex space X is said to be *cocomplete* [2] if each linear functional $f : X \to \mathbb{C}$ continuous on each totally bounded set $S \subseteq X$ is continuous on X.

(8) A locally convex space X is said to be *saturated* [2] if for an absolutely convex set B, being a zero neighborhood in X is equivalent to the following: for any totally bounded set $S \subseteq X$ there is a closed zero neighborhood U in X such that $B \cap S = U$.

(9) A locally convex space X is called a *Pták space* [38] or a *fully complete* space [35] it in the dual space $X^\star$ every subspace $Q \subseteq X^\star$ is X-weakly closed when it leaves an X-weakly closed trace $Q \cap U^\circ$ on the polar U° of each zero neighborhood $U \subseteq X$.

(10) A locally convex space X is said to be *hypercomplete* [35] if in the dual space $X^\star$ an absolutely convex set $Q \subseteq X^\star$ is X-weakly closed when it leaves an X-weakly closed trace $Q \cap U^\circ$ on the polar U° of each zero neighborhood $U \subseteq X$.

$Z^\star = \mathcal{D}^\star(\mathbb{R})$ is also complete and saturated. In [44], O. G. Smolyanov showed that Z contains a closed subspace Y such that Z/Y is metrizable, but not complete. Hence, Z/Y is not pseudocomplete.

Set $X = Z^\star$, $E = Y^\perp$. By Proposition 4.39(a), $Z/Y = X^\star/E^\perp = E^\star$. So if E were pseudosaturated, then Z/Y would be pseudocomplete by Theorem 4.11. ■

EXAMPLE 4.41. There exists a complete locally convex space E (and thus E can be represented as a projective limit of Banach spaces in the category `LCS`) such that $E^\star$ is metrizable, but not complete. As a corollary, E is not pseudosaturated, and there is a discontinuous linear functional $f : E \to \mathbb{C}$ which is continuous with respect to the topology of the pseudosaturation $E^\vartriangle$.

Proof. An example is the space $E = Y^\perp$ from Example 4.40. It is complete, since it is a closed subspace in the complete space $Z^\star = \mathcal{D}^\star(\mathbb{R})$. On the other hand, by Proposition 4.39(a), $E^\star \cong X^\star/E^\perp \cong Z/Y$, and the last space is metrizable, but not complete. That is, $E^\star \neq (E^\star)^\triangledown$, and this can be extended to

$$E^\star \neq (E^\star)^\triangledown \overset{(4.17)}{\cong} (E^\vartriangle)^\star,$$

which means that there exists $f \in (E^\vartriangle)^\star \setminus E^\star$. (It is important here that E is pseudocomplete, while $E^\star$ is not.) ■

4.4.1. Spaces of operators and continuous bilinear maps

- Let X and Y be stereotype spaces. Let us denote:
  - by $Y : X$ the space of continuous linear maps $\varphi : X \to Y$ endowed with the topology of uniform convergence on totally bounded sets in X;
  - by $Y \oslash X$ the pseudosaturation of the space $Y : X$,

$$Y \oslash X = (Y : X)^\vartriangle. \tag{4.28}$$

  The space $Y \oslash X$ is stereotype, and we call it the *inner space of operators* from X into Y. Again, it consists of all continuous linear maps $\varphi : X \to Y$, but its topology is formally stronger than the topology of uniform convergence on totally bounded sets in X ([11]).

THEOREM 4.42. *Let X and Y be locally convex spaces. A set of morphisms $\varPhi \subseteq Y : X$ is totally bounded in $Y : X$ if and only if it satisfies the following two conditions:*

(a) *equicontinuity on totally bounded sets:*

$$\forall S \in \mathcal{S}(X)\ \forall V \in \mathcal{U}(Y)\ \exists U \in \mathcal{U}(X)\ \forall a, b \in S \qquad a - b \in U \ \Rightarrow\ \forall \varphi \in \varPhi\ \ \varphi(a) - \varphi(b) \in V;$$

(b) *uniform total boundedness on totally bounded sets:*

$$\forall S \in \mathcal{S}(X) \quad \varPhi(S) = \{\varphi(x);\ x \in S,\ \varphi \in \varPhi\} \in \mathcal{S}(Y).$$

([11]) Thus, $Y : X$ and $Y \oslash X$ coincide as linear spaces, but may have different topologies. So far, however, it is not clear whether $Y : X$ and $Y \oslash X$ are indeed different, since examples of non-pseudosaturated spaces of the form $Y : X$ (with stereotype X and Y) have not been constructed yet.

Condition (b) *can be replaced by the weakened condition*

(c) *pointwise total boundedness:*

$$\forall x \in X \quad \Phi(x) = \{\varphi(x);\ \varphi \in \Phi\} \in \mathcal{S}(Y).$$

Moreover,

– *if* Y *is a Heine–Borel space, then* (a) $\Rightarrow$ (b) & (c)*;*
– *if* X *is barreled, then* (c) $\Rightarrow$ (a) & (b)*.*

For any continuous linear map $\varphi : X \to Y$ its *dual map* $\varphi^\star : Y^\star \to X^\star$ is defined by

$$\varphi^\star(f) = f \circ \varphi, \qquad f \in Y^\star.$$

Theorem 4.43. *The map* $\varphi \mapsto \varphi^\star$ *is an isomorphism of stereotype spaces:*

$$X^\star \oslash Y^\star \cong Y \oslash X.$$

Example 4.44. If X is a Smith space and Y a Banach space, then $Y \oslash X = Y : X$ is a Banach space.

Example 4.45. If X is a Banach space and Y a Smith space, then $Y \oslash X = Y : X$ is a Smith space.

Example 4.46. If X is a Brauner space and Y a Fréchet space, then $Y \oslash X = Y : X$ is a Fréchet space.

Example 4.47. If X is a Fréchet space and Y a Brauner space, then $Y \oslash X = Y : X$ is a Brauner space.

- Let X, Y, Z be stereotype spaces. Then:
  – we say that a bilinear map $\beta : X \times Y \to Z$ is *continuous* ([12]) if
    (1) for each compact set K in X and for each zero neighborhood W in Z there is a zero neighborhood V in Y such that
    $$\beta(K, V) \subseteq W,$$
    (2) for each compact set L in Y and for each zero neighborhood W in Z there is a zero neighborhood U in X such that
    $$\beta(U, L) \subseteq W;$$
  – we denote by $Z : (X, Y)$ the space of continuous bilinear maps $\beta : X \times Y \to Z$ endowed with the topology of uniform convergence on compact sets in $X \times Y$;
  – we denote by $Z \oslash (X, Y)$ the pseudosaturation of $Z : (X, Y)$,
  $$Z \oslash (X, Y) = (Z : (X, Y))^\vartriangle. \tag{4.29}$$

  The space $Z \oslash (X, Y)$ is stereotype, and we call it the *inner space of bilinear maps* from $X \times Y$ into Z. Like $Z : (X, Y)$, it consists of continuous bilinear maps $\beta : X \times Y \to Z$, but the topologies of $Z : (X, Y)$ and $Z \oslash (X, Y)$ may be different ([13]).

([12]) This type of continuity is sometimes called $(\mathcal{K}(X), \mathcal{K}(Y))$-hypocontinuity (cf. [38]), where $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are systems of compact sets in X and Y respectively.

([13]) Cf. footnote ([11]); the situation with $Z : (X, Y)$ and $Z \oslash (X, Y)$ is the same.

Example 4.48. If X and Y are Smith spaces, and Z a Banach space, then $Z \oslash (X,Y) = Z : (X,Y)$ is a Banach space.

Example 4.49. If X and Y are Banach spaces, and Z a Smith space, then $Z \oslash (X,Y) = Z : (X,Y)$ is a Smith space.

Example 4.50. If X and Y are Brauner spaces, and Z a Fréchet space, then $Z \oslash (X,Y) = Z : (X,Y)$ is a Fréchet space.

Example 4.51. If X and Y are Fréchet spaces, and Z a Brauner space, then $Z \oslash (X,Y) = Z : (X,Y)$ is a Brauner space.

Theorem 4.52. *If X, Y, Z are stereotype spaces, then the formula*

$$\beta(x,y) = \varphi(y)(x) \tag{4.30}$$

defines an isomorphism of stereotype spaces

$$Z \oslash (X,Y) = (Z \oslash X) \oslash Y. \tag{4.31}$$

Remark 4.53. In the special case when $Z = \mathbb{C}$ we have

$$\mathbb{C} \oslash (X,Y) = X^\star \oslash Y, \tag{4.32}$$

$$Y \oslash X = \mathbb{C} \oslash (Y^\star, X). \tag{4.33}$$

Theorem 4.54. *For all stereotype spaces X, Y, Z the composition map*

$$(\beta, \alpha) \in (Z \oslash Y) \times (Y \oslash X) \mapsto \beta \circ \alpha \in Z \oslash X$$

is a continuous bilinear map.

- Let $\alpha : E \to F$ and $\beta : G \to H$ be continuous linear maps of stereotype spaces. Define $\beta \oslash \alpha : G \oslash F \to H \oslash E$ by

$$(\beta \oslash \alpha)(\psi) = \beta \circ \psi \circ \alpha. \tag{4.34}$$

Theorem 4.55. *For all stereotype spaces X, Y, Z the bilinear map*

$$(\beta, \alpha) \in (H \oslash G) \times (F \oslash E) \mapsto \beta \oslash \alpha \in (H \oslash E) \oslash (G \oslash F) \tag{4.35}$$

is continuous.

4.4.2. Tensor products. A *projective* (*stereotype*) *tensor product* $X \circledast Y$ of stereotype spaces X and Y is defined by

$$X \circledast Y = (X^\star \oslash Y)^\star, \tag{4.36}$$

or equivalently, due to (4.32),

$$X \circledast Y = (\mathbb{C} \oslash (X,Y))^\star. \tag{4.37}$$

For $x \in X$ and $y \in Y$ the elementary tensor $x \circledast y \in X \circledast Y$ is defined by

$$(x \circledast y)(\varphi) = \varphi(y)(x) \tag{4.38}$$

(where $\varphi \in X^\star \oslash Y$, and $x \circledast y$ is considered as an element of $(X^\star \oslash Y)^\star$), or equivalently,

$$(x \circledast y)(\beta) = \beta(x,y) \tag{4.39}$$

(where $\beta \in \mathbb{C} \oslash (X,Y)$, and $x \circledast y$ is considered as an element of $\mathbb{C} \oslash (X,Y)^\star$).

PROPOSITION 4.56. *The map* $\iota : (x,y) \in X \times Y \mapsto x \circledast y \in X \circledast Y$ *is a continuous bilinear map.*

PROPOSITION 4.57. *The algebraic tensor product* $X \otimes Y$ *is injectively and densely embedded into the projective tensor product* $X \circledast Y$ *by*

$$x \otimes y \mapsto x \circledast y.$$

THEOREM 4.58 (Universality of projective tensor product). *For any stereotype spaces* X, Y, Z *and for any continuous bilinear form* $\beta : X \times Y \to Z$ *there is a unique continuous linear map* $\widetilde{\beta} : X \circledast Y \to Z$ *such that*

$$\begin{tikzcd} X \times Y \arrow[rr, "\iota"] \arrow[dr, "\beta"'] & & X \circledast Y \arrow[dl, "\widetilde{\beta}"] \\ & Z & \end{tikzcd}$$

where ι *is defined in Proposition* 4.56*. Moreover,* $\beta \mapsto \widetilde{\beta}$ *is an isomorphism of stereotype spaces*

$$Z \oslash (X, Y) = Z \oslash (X \circledast Y). \tag{4.40}$$

An *injective* (*stereotype*) *tensor product* $X \odot Y$ of stereotype spaces X and Y is defined by the formula

$$X \odot Y = Y \oslash X^\star, \tag{4.41}$$

or equivalently, due to (4.33), by

$$X \odot Y = \mathbb{C} \oslash (X^\star, Y^\star). \tag{4.42}$$

For $x \in X$ and $y \in Y$ the elementary operator $x \odot y \in X \odot Y$ is defined by

$$(x \odot y)(f) = f(x)y, \qquad f \in X^\star \tag{4.43}$$

(if $x \odot y$ is considered as an element of $Y \oslash X^\star$), or by

$$(x \odot y)(f, g) = f(x)g(y), \quad f \in X^\star,\ g \in Y^\star \tag{4.44}$$

(if $x \odot y$ is considered as an element of $\mathbb{C} \oslash (X^\star, Y^\star)$).

PROPOSITION 4.59. *The map* $\iota : (x,y) \in X \times Y \mapsto x \odot y \in X \odot Y$ *is a continuous bilinear map.*

PROPOSITION 4.60. *The algebraic tensor product* $X \otimes Y$ *is injectively* (*but not necessarily densely*) *embedded into the injective tensor product* $X \odot Y$ *by*

$$x \otimes y \mapsto x \odot y.$$

EXAMPLE 4.61. If X and Y are Banach spaces, then so are $X \circledast Y$ and $X \odot Y$.

EXAMPLE 4.62. If X and Y are Smith spaces, then so are $X \circledast Y$ and $X \odot Y$.

EXAMPLE 4.63. If X and Y are Fréchet spaces, then so are $X \circledast Y$ and $X \odot Y$.

EXAMPLE 4.64. If X and Y are Brauner spaces, then so are $X \circledast Y$ and $X \odot Y$.

4.4.3. The category of stereotype spaces. The class Ste of stereotype spaces forms a category with continuous linear maps as morphisms.

Properties of the category Ste.

1° Ste *is pre-abelian.*

2° Ste *is complete: each covariant (and each contravariant) system has injective and projective limits. In the case of direct coproducts and direct products these constructions coincide with the standard ones in the category* LCS *of locally convex spaces, while in the general case the difference is that the injective limits in* LCS *must be pseudocomplete, while the projective limits must be pseudosaturated:*

$$\mathtt{Ste}\text{-}\bigoplus_{i\in I} X_i = \mathtt{LCS}\text{-}\bigoplus_{i\in I} X_i, \qquad \mathtt{Ste}\text{-}\prod_{i\in I} X_i = \mathtt{LCS}\text{-}\prod_{i\in I} X_i, \tag{4.45}$$

$$\mathtt{Ste}\text{-}\varinjlim_{i\to\infty} X_i = \Big(\mathtt{LCS}\text{-}\varinjlim_{i\to\infty} X_i\Big)^{\triangledown}, \qquad \mathtt{Ste}\text{-}\varprojlim_{i\to\infty} X_i = \Big(\mathtt{LCS}\text{-}\varprojlim_{i\to\infty} X_i\Big)^{\vartriangle}. \tag{4.46}$$

3° *The tensor products* $\circledast$ *and* $\odot$ *and the fraction* $\oslash$ *are related through the following isomorphisms of functors:*

$$(X\circledast Y)^\star \cong Y^\star \odot X^\star, \qquad (X\odot Y)^\star \cong Y^\star \circledast X^\star \tag{4.47}$$

$$Z\oslash (X\circledast Y) \cong (Z\oslash X)\oslash Y, \qquad (X\odot Y)\oslash Z \cong X\odot (Y\oslash Z). \tag{4.48}$$

4° Ste *is a symmetric monoidal category with respect to each of* $\circledast$ *and* $\odot$*:*

$$\mathbb{C}\circledast X \cong X \cong X\circledast \mathbb{C}, \qquad \mathbb{C}\odot X \cong X \cong X\odot \mathbb{C}, \tag{4.49}$$

$$X\circledast Y \cong Y\circledast X, \qquad X\odot Y \cong Y\odot X \tag{4.50}$$

$$(X\circledast Y)\circledast Z \cong X\circledast (Y\circledast Z), \qquad (X\odot Y)\odot Z \cong X\odot (Y\odot Z). \tag{4.51}$$

5° *The projective tensor product in* Ste *commutes with injective limits, and the injective product commutes with projective limits:*

$$\Big(\bigoplus_{i\in I} X_i\Big)\circledast\Big(\bigoplus_{j\in J} Y_j\Big) \cong \bigoplus_{i\in I, j\in J} (X_i\circledast Y_j), \tag{4.52}$$

$$\Big(\prod_{i\in I} X_i\Big)\odot\Big(\prod_{j\in J} Y_j\Big) \cong \prod_{i\in I, j\in J} (X_i\odot Y_j), \tag{4.53}$$

$$\Big(\varinjlim_{i\to\infty} X_i\Big)\circledast\Big(\varinjlim_{j\to\infty} Y_j\Big) \cong \varinjlim_{i,j\to\infty} (X_i\circledast Y_j), \tag{4.54}$$

$$\Big(\varprojlim_{i\to\infty} X_i\Big)\odot\Big(\varprojlim_{j\to\infty} Y_j\Big) \cong \varprojlim_{i,j\to\infty} (X_i\odot Y_j), \tag{4.55}$$

4.5. Subspaces

- Let Y be a subset in a stereotype space X endowed with the structure of stereotype space in such a way that the set-theoretic inclusion $Y\subseteq X$ is a morphism of stereotype spaces (i.e. a continuous linear map). Then Y is called a *subspace* of X, and the set-theoretic inclusion $\sigma : Y\subseteq X$ its *representing monomorphism.* We then write

$$Y \hookrightarrow X \quad\text{or}\quad X \hookleftarrow Y.$$

In this context, $Y = X$ means that the stereotype spaces Y and X coincide not only as sets but also with their algebraic and topological structure.

- The system of subspaces of a stereotype space X will be denoted by $\mathsf{Sub}(X)$.

Proposition 4.65. *For a morphism $\mu : Z \to X$ in* `Ste` *the following conditions are equivalent:*

(i) *μ is a monomorphism;*
(ii) *there exists a subspace Y in X with representing monomorphism $\sigma : Y \subseteq X$ and an isomorphism $\theta : Z \to Y$ of stereotype spaces such that*

$$\begin{tikzcd} Z \arrow[dr, "\mu"] \arrow[dd, squiggly, "\theta"'] & \\ & X \\ Y \arrow[ur, dashed, "\sigma"'] & \end{tikzcd}$$

Corollary 4.66. *For any stereotype space X the system $\mathsf{Sub}(X)$ is a system of subobjects in X (in the sense of the definition on p. 22).*

Clearly, for a stereotype space P the relation $\subseteq$ is a partial order on $\mathsf{Sub}(P)$.

4.5.1. Immediate subspaces

- Suppose
$$Z \subseteq Y \subseteq X,$$
and $Z \subseteq Y$ is a bimorphism of stereotype spaces, i.e. in addition to the other requirements, Z is dense in Y (with respect to the topology of Y). Then we will say that Y is a *mediator* for Z in X.
- We call a subspace Z of a stereotype space X an *immediate subspace* in X if it has no non-isomorphic mediators, i.e. for any mediator Y in X the inclusion $Z \subseteq Y$ is an isomorphism. In this case we use the notation $Z \overset{\circ}{\subseteq} X$:
$$Z \overset{\circ}{\subseteq} X \iff \forall Y \ \Big((Z \subseteq Y \subseteq X \ \& \ \overline{Z}^{Y} = Y) \ \Rightarrow \ Z = Y \Big).$$

Remark 4.67. In `LCS` the same construction gives a widely used object: immediate subspaces in a locally convex space X are exactly closed subspaces in X with the topology inherited from X. In Examples 4.70 and 4.71 below we will see that in `Ste` the situation more complicated.

Recall that immediate monomorphisms were defined on p. 16.

Proposition 4.68. *For a morphism $\mu : Z \to X$ in* `Ste` *the following conditions are equivalent:*

(i) *μ is an immediate monomorphism;*
(ii) *there exists an immediate subspace Y of X with representing monomorphism $\sigma : Y \subseteq X$ and an isomorphism $\theta : Z \to Y$ such that*

$$\begin{tikzcd} Z \arrow[dr, "\mu"] \arrow[dd, squiggly, "\theta"'] & \\ & X \\ Y \arrow[ur, dashed, "\sigma"'] & \end{tikzcd} \tag{4.56}$$

Here Y and θ are uniquely determined by Z and μ.

Proof. The implication (i)$\Leftarrow$(ii) is obvious, so we only need to prove (i)$\Rightarrow$(ii). Set $Y = \mu(Z)$, and denote by $\theta : Z \to Y$ the corestriction of μ into Y, i.e. θ is the same map as μ but viewed as acting into Y. Since μ is injective, θ is bijective. Let us endow Y with the topology under which θ is an isomorphism of locally convex spaces. Then Y becomes a subspace of X, since for any zero neighborhood U in X, $\mu^{-1}(U)$ is a zero neighborhood in Z, and thus $Y \cap U = \theta(\mu^{-1}(U))$ is a zero neighborhood in Y. ■

PROPOSITION 4.69. ([14]) *For an immediate subspace Y of a stereotype space X with representing monomorphism $\sigma : Y \subseteq X$ the following conditions are equivalent:*

(i) *σ is a closed map;*
(ii) *σ is a weakly closed map;*
(iii) *Y as a set is a closed subspace in the locally convex space X, and the topology of Y is a pseudosaturation of the topology inherited from X.*

- If the above conditions (i)–(iii) are fulfilled, we say that the immediate subspace Y of X is *closed.*

Proof of Proposition 4.69. (i)$\Rightarrow$(ii) is a special case of the situation in Proposition 4.14.

(ii)$\Rightarrow$(iii). Let $\sigma : Y \subseteq X$ be a weakly closed map, i.e. Y as a set is closed in X. Denote by E the space Y with the topology inherited from X. Clearly, Y is continuously embedded into E, and since Y is pseudosaturated, this inclusion preserves its continuity after passage from E to its pseudosaturation $E^\vartriangle$ (we use here the reasoning presented in [2, diagram (1.26)]). Thus, we obtain a sequence of subspaces

$$Y \hookrightarrow E^\vartriangle \hookrightarrow X,$$

and since Y and $E^\vartriangle$ coincide as sets, the first of these monomorphisms is a bimorphism. Hence, $E^\vartriangle$ is a mediator for Y, and we obtain $Y = E^\vartriangle$.

(iii)$\Rightarrow$(i) follows from the fact that pseudosaturation does not change the system of totally bounded subsets. ■

EXAMPLE 4.70. There exists a stereotype space P with a closed immediate subspace Q whose topology is not inherited from P, and moreover some continuous functionals $g \in Q^\star$ cannot be continuously extended on P (in the formal language this means that the representing monomorphism $Q \overset{\circ}{\hookrightarrow} P$ is closed, but not weakly open).

Proof. Consider the space E from Example 4.41. It is complete, so it can be represented as a complete subspace in some stereotype space P with the topology inherited from P (for example, one can take as P the direct product of all Banach quotient spaces E/F). The space $Q = E^\vartriangle$ has the required properties. Indeed, it is closed in P, since E is closed in P. On the other hand, the functional $f : Q \to \mathbb{C}$ described in Example 4.41 is continuous on $Q = E^\vartriangle$, but it cannot be continuously extended on P, since otherwise it would be continuous on E. ■

EXAMPLE 4.71. There exists a stereotype space X with an immediate subspace Z which is not closed as a subset in X. Hence the inclusion $Z \subseteq X$ is not a weakly closed morphism

([14]) [2, Theorem 4.14], which is equivalent to Proposition 4.69 here, and the more general [2, Theorem 11.7], contain an inaccuracy: the requirement of closure of σ is omitted there.

in the sense of definition on p. 112 (in particular, the inclusion $Z \subseteq X$ is not isomorphic in Mono_X to the kernel of any other morphism $\varphi : X \to A$ in $\mathtt{Ste}$).

Proof. Let E and f be as in Example 4.41. Endow $F = \{x \in E^\vartriangle : f(x) = 0\}$ with the topology inherited from $E^\vartriangle$ (as a locally convex space, F is a closed subspace in $E^\vartriangle$). By [2, Proposition 3.19], $E^\vartriangle$ is complete, hence so is F, and again by [2, Proposition 3.19], the pseudosaturation $Z = F^\vartriangle$ is complete. In addition, Z is pseudosaturated, and thus stereotype. Note that since E is complete, it can be represented (as a locally convex space) as a closed subspace in a direct product X of some Banach spaces (in such a way that the topology of E is inherited from X). We will show that Z is an immediate subspace, but not a closed set, in X.

First let us show that Z is not closed in X. As a set, Z coincides with F, which is dense in E (in the topology of E, which is inherited from X). Hence,

$$\overline{Z}^X = \overline{F}^X = E \neq F = Z$$

(here $\overline{}^X$ means closure in X). Now let us show that Z is an immediate subspace in X. Let Y be a mediator of Z in X. Since Z is dense in Y, we obtain

$$\begin{gathered} \overline{Z}^Y = Y \\ \Downarrow \\ \overline{Y}^X = \overline{\overline{Z}^Y}^X = \overline{Z}^X = E \\ \Downarrow \\ Y \subseteq E. \end{gathered}$$

The latter is an inclusion of sets. Note that since Y is a subspace in X, the topology of Y majorizes the topology inherited from X, or, what is the same, the topology inherited from E. Hence the inclusion $Y \subseteq E$ is continuous, and therefore Y is a subspace in E. This implies that the pseudosaturation of Y is a subspace in the pseudosaturation of E, and since Y is pseudosaturated, we obtain a continuous inclusion

$$Y = Y^\triangledown \subseteq E^\triangledown.$$

Thus, Y is a subspace in $E^\triangledown$.

Let us now forget about X and consider the following chain of subspaces:

$$Z \subseteq Y \subseteq E^\triangledown.$$

Since Z is a dense subspace in Y, we obtain a new logical chain:

$$\begin{gathered} \overline{Z}^Y = Y \\ \Downarrow \\ \overline{Y}^{E^\triangledown} = \overline{\overline{Z}^Y}^{E^\triangledown} = \overline{Z}^{E^\triangledown} = F \\ \Downarrow \\ Y \subseteq F. \end{gathered}$$

Again the latter is an inclusion of sets. Note that since Y is a subspace in $E^\triangledown$, the topology of Y majorizes the topology inherited from $E^\triangledown$, or, what is the same, the topology inherited from F. Thus the inclusion $Y \subseteq F$ is continuous, so Y is a subspace in F. This implies that the pseudosaturation of Y is a subspace in the pseudosaturation of F, and since Y is pseudosaturated, we obtain a continuous inclusion

$$Y = Y^{\triangledown} \subseteq F^{\triangledown} = Z.$$

Thus, Y is a subspace in Z. On the other hand, from the very beginning Z was a subspace in Y. Hence, $Z = Y$. ■

4.5.2. Envelope of a set M of elements in a space X. Theorem 4.107 below justifies the following definition.

- The *envelope* of a set $M \subseteq X$ in a stereotype space X is a subspace in X, denoted by $\mathsf{Env}^X M$ or $\mathsf{Env}\, M$, and defined as the projective limit in $\mathtt{Ste}$

$$\mathsf{Env}^X M = \mathsf{Env}\, M = \mathtt{Ste}\text{-}\varprojlim E_i \tag{4.57}$$

  of a contravariant system $\{E_i;\ i \in \mathtt{Ord}\}$ of subspaces in X, indexed by ordinals and defined by the following inductive rules:

  0) $E_0 = (\overline{\mathsf{span}\, M}^X)^{\vartriangle}$.
  1) Suppose that for some $j \in \mathtt{Ord}$ all the spaces $\{E_i;\ i < j\}$ are already defined; then E_j is defined as follows:

     – if $j = i + 1$ for some i, then

$$E_j = E_{i+1} = (\overline{\mathsf{span}\, M}^{E_i})^{\vartriangle};$$

     – if j is a limit ordinal, then

$$E_j = \lim_{j \leftarrow i} E_i$$

     in $\mathtt{Ste}$; this means that, as a set,

$$E_j = \bigcap_{i<j} E_i,$$

     and the topology in E_j is the weakest stereotype locally convex topology under which all the inclusions $E_j \subseteq E_i$ are continuous.

  Since the transfinite sequence $\{E_i;\ i \in \mathtt{Ord}\}$ cannot be an injective map from $\mathtt{Ord}$ to $\mathsf{Sub}(X)$, it stabilizes, i.e. for some $k \in \mathtt{Ord}$,

$$\forall l \geq k \quad E_l = E_k. \tag{4.58}$$

  This implies that the contravariant system $\{E_i;\ i \in \mathtt{Ord}\}$ indeed has a projective limit, and this is exactly the subspace E_k in X.

EXAMPLE 4.72. If $\overline{\mathsf{span}\, M}^X = X$, then $\mathsf{Env}\, M = X$.

Proof. The equality $\overline{\mathsf{span}\, M}^X = X$ implies $E_0 = (\overline{\mathsf{span}\, M}^X)^{\vartriangle} = X$, and consequently

$$X = E_0 = E_1 = \cdots .$$

Hence, $\mathsf{Env}\, M = X$. ■

EXAMPLE 4.73. If $\overline{\mathsf{span}\, M}^X = M$, then $\mathsf{Env}\, M = M^{\vartriangle}$.

Proof. From $\overline{\mathsf{span}\, M}^X = M$ we have $E_0 = (\overline{\mathsf{span}\, M}^X)^\vartriangle = M^\vartriangle$, then $E_1 = (\overline{\mathsf{span}\, M}^{E_0})^\vartriangle = M^\vartriangle = E_0$, and all the other spaces E_i coincide with E_0. Thus, $\mathsf{Env}\, M = E_0 = M^\vartriangle$. ∎

THEOREM 4.74. *The envelope* $\mathsf{Env}^X M$ *of each set* $M \subseteq X$ *is an immediate subspace in* X*, containing* M *as a total subset:*

$$M \subseteq \mathsf{Env}^X M \mathrel{\overset{\circ}{\hookrightarrow}} X, \qquad \overline{\mathsf{span}\, M}^{\mathsf{Env}^X M} = \mathsf{Env}^X M. \tag{4.59}$$

Proof. First let us verify that M is total in $\mathsf{Env}^X M$. Suppose (4.58) holds. Then $\mathsf{Env}\, M = E_k$, and if M were not total in E_k, then we would have a contradiction with (4.58):

$$E_{k+1} = \overline{\mathsf{span}\, M}^{E_k} \neq E_k.$$

Next let us show that $\mathsf{Env}\, M$ is an immediate subspace in X. Suppose Y is a subspace in X such that

$$\mathsf{Env}\, M \subseteq Y \subseteq X,$$

and $\mathsf{Env}\, M$ is dense in Y. Since, as already established, $\mathsf{span} M$ is dense in $\mathsf{Env}\, M$, we have

$$Y = \overline{\mathsf{span}\, M}^Y. \tag{4.60}$$

Now by induction we see that Y is continuously embedded into each E_i:

0) For $i = 0$ we have

$$\begin{aligned} Y \hookrightarrow X \;\Rightarrow\; & Y \overset{(4.60)}{=} \overline{\mathsf{span}\, M}^Y \hookrightarrow \overline{\mathsf{span}\, M}^X \\ \Rightarrow\; & Y = Y^\vartriangle \hookrightarrow (\overline{\mathsf{span}\, M}^X)^\vartriangle = E_0. \end{aligned}$$

1) Suppose that we have proved $Y \hookrightarrow E_i$ for all i less that some j. Then:

– if $j = i + 1$ for some i, then

$$\begin{aligned} Y \hookrightarrow E_i \;\Rightarrow\; & Y \overset{(4.60)}{=} \overline{\mathsf{span}\, M}^Y \hookrightarrow \overline{\mathsf{span}\, M}^{E_i} \\ \Rightarrow\; & Y = Y^\vartriangle \subseteq (\overline{\mathsf{span}\, M}^{E_i})^\vartriangle = E_{i+1} = E_j; \end{aligned}$$

– if j is a limit ordinal, then from the continuous inclusions $Y \hookrightarrow E_i$ for $i < j$ we obtain a continuous inclusion of locally convex spaces

$$Y \hookrightarrow \mathsf{LCS}\text{-}\lim_{j \leftarrow i} E_i,$$

and this implies a continuous inclusion of stereotype spaces

$$Y = Y^\vartriangle \hookrightarrow \Big(\mathsf{LCS}\text{-}\lim_{j \leftarrow i} E_i\Big)^\vartriangle = \mathtt{Ste}\text{-}\lim_{j \leftarrow i} E_i = E_j.$$

Since Y is continuously embedded into each E_i, we obtain a continuous inclusion $Y \hookrightarrow \mathsf{Env}\, M$. Together with the initial inclusion $\mathsf{Env}\, M \hookrightarrow Y$ this means that $\mathsf{Env}\, M = Y$ (with topologies). ∎

The following theorem shows that in an immediate subspace the topology is automatically defined by the set of its elements:

THEOREM 4.75. *Every subspace* Y *in a stereotype space* X *is a subspace in its envelope* $\mathsf{Env}^X Y$*:*

$$Y \hookrightarrow X \;\Rightarrow\; Y \hookrightarrow \mathsf{Env}^X Y, \tag{4.61}$$

and Y is an immediate subspace in X iff it coincides (topologically) with its envelope in X:

$$Y \mathrel{\mathring{\subsetneq}} X \iff Y = \mathsf{Env}^X Y. \tag{4.62}$$

Proof. The continuity of $Y \subsetneq \mathsf{Env}^X Y$ is proved by induction:

0) We have a continuous inclusion of locally convex spaces

$$Y \subsetneq \overline{\mathsf{span}\, Y}^X = \overline{Y}^X,$$

which implies a continuous inclusion of stereotype spaces

$$Y = Y^\vartriangle \subsetneq (\overline{Y}^X)^\vartriangle = E_0.$$

1) Suppose that the continuous inclusion $Y \subseteq E_i$ is proved for all i less than some j. Then:

– if $j = i+1$ for some i, then we obtain a continuous inclusion of locally convex spaces

$$Y \subseteq \overline{\mathsf{span}\, Y}^{E_i} = \overline{Y}^{E_i},$$

which implies a continuous inclusion of stereotype spaces

$$Y = Y^\vartriangle \subseteq (\overline{Y}^{E_i})^\vartriangle = E_{i+1} = E_j;$$

– if j is a limit ordinal, then from the continuous inclusions $Y \subseteq E_i$ for all $i < j$ we obtain a continuous inclusion of locally convex spaces

$$Y \subseteq \mathsf{LCS}\text{-}\lim_{j \leftarrow i} E_i,$$

which implies a continuous inclusion of stereotype spaces

$$Y = Y^\vartriangle \subseteq \Big(\mathsf{LCS}\text{-}\lim_{j \leftarrow i} E_i\Big)^\vartriangle = \mathtt{Ste}\text{-}\lim_{j \leftarrow i} E_i = E_j.$$

Let us now consider the special case when Y is an immediate subspace in X. Then by Theorem 4.74, Y is dense in $\mathsf{Env}\, Y$, hence in the chain of inclusions

$$Y \subseteq \mathsf{Env}\, Y \subseteq X$$

the second space is a mediator. Therefore, it coincides with the first one: $Y = \mathsf{Env}\, Y$. ■

COROLLARY 4.76. *The representing monomorphism $\sigma : Y \mathrel{\mathring{\subsetneq}} X$ of an immediate subspace Y in a stereotype space X is always relatively closed.*

Proof. By Theorem 4.75,

$$Y = \mathsf{Env}^X Y = \varprojlim_{i \in \mathtt{Ord}} E_i = \bigcap_{i \in \mathtt{Ord}} E_i.$$

Let T be an absolutely convex compact set in X, lying in Y as a set. Then T lies in $E_0 = (\overline{Y}^X)^\vartriangle$, and since in passing from the topology of X to the topology of E_0 the system of compact sets (as well as the topology on each compact set) is inherited from X (this is one of the fundamental properties of the pseudosaturation $\vartriangle$, [2, Theorem 1.17]), we see that T is a compact set in E_0. With the same technique we show that T is compact in E_1, and more generally, in passing from each ordinal i to its successor $i+1$. When we need to pass to a limit ordinal j, we come to the situation where T is a compact set in each E_i with $i<j$. As a corollary, T is compact in $\varprojlim_{i<j} E_i = \bigcap_{i<j} E_i$. When we come to an ordinal large enough, we conclude that T is compact in Y. ■

THEOREM 4.77. *If $\varphi : Y \to X$ is a morphism of stereotype spaces mapping a set $N \subseteq Y$ into a set $M \subseteq X$,*

$$\varphi(N) \subseteq M,$$

then φ continuously maps $\mathsf{Env}^Y N$ into $\mathsf{Env}^X M$:

$$\begin{tikzcd} Y \arrow[r, "\varphi"] & X \\ \mathsf{Env}^Y N \arrow[u] \arrow[r, dashed] & \mathsf{Env}^X M \arrow[u] \end{tikzcd}$$

In the special cases:

$$\begin{Bmatrix} Y \hookrightarrow X \\ \cup\!\!| \quad \cup\!\!| \\ N \subseteq M \end{Bmatrix} \Rightarrow \mathsf{Env}^Y N \hookrightarrow \mathsf{Env}^X M, \tag{4.63}$$

$$\begin{Bmatrix} Y \overset{\circ}{\hookrightarrow} X \\ \cup\!\!| \quad \cup\!\!| \\ N \subseteq M \end{Bmatrix} \Rightarrow \mathsf{Env}^Y N \overset{\circ}{\hookrightarrow} \mathsf{Env}^X M, \tag{4.64}$$

$$\begin{Bmatrix} Y \overset{\circ}{\hookrightarrow} X \\ \cup\!\!| \quad \cup\!\!| \\ N = M \end{Bmatrix} \Rightarrow \mathsf{Env}^Y M = \mathsf{Env}^X M. \tag{4.65}$$

Proof. Let φ be as in the statement. If we denote by $\{F_i;\ i \in \mathtt{Ord}\}$ and $\{E_i;\ i \in \mathtt{Ord}\}$ the sequences of subspaces in Y and X which define $\mathsf{Env}\, N$ and $\mathsf{Env}\, M$ respectively,

$$\mathsf{Env}\, N = \varprojlim F_i, \qquad \mathsf{Env}\, M = \varprojlim E_i,$$

then we can prove by induction that φ continuously maps each F_i into E_i, and hence $\mathsf{Env}\, N$ into $\mathsf{Env}\, M$. Let us now consider the special cases.

If $N \subseteq M$ and $Y \hookrightarrow X$, then we consider the sequences $\{F_i;\ i \in \mathtt{Ord}\}$ and $\{E_i;\ i \in \mathtt{Ord}\}$ of subspaces in X which define $\mathsf{Env}^X N$ and $\mathsf{Env}^X M$. By induction, we obtain an inclusion of subspaces $F_i \hookrightarrow E_i$ for each i, and this gives the inclusion $\mathsf{Env}^X N \hookrightarrow \mathsf{Env}^X M$.

Suppose that $N \subseteq M$ and $Y \overset{\circ}{\hookrightarrow} X$. Then, by (4.63), $\mathsf{Env}^Y N \hookrightarrow \mathsf{Env}^X M$. Let us show that in this inclusion, $\mathsf{Env}^Y N$ is an immediate subspace in $\mathsf{Env}^X M$. Let Z be a mediator for $\mathsf{Env}^Y N$ in $\mathsf{Env}^X M$:

$$\mathsf{Env}^Y N \hookrightarrow Z \hookrightarrow \mathsf{Env}^X M, \qquad \overline{\mathsf{Env}^Y N}^Z = Z.$$

Consider $\mathsf{Env}^X(Y \cup Z)$. We can include it in a diagram (where all the arrows are set-theoretic inclusions, and are continuous maps):

$$\begin{tikzcd} Y \arrow[r] & \mathsf{Env}^X(Y \cup Z) \arrow[r] & X \\ \mathsf{Env}^Y N \arrow[u] \arrow[r] & Z \arrow[u] \arrow[r] & \mathsf{Env}^X M \arrow[u] \end{tikzcd}$$

By Theorem 4.74, N is total in $\mathsf{Env}^Y N$, which in turn is total in Z (since Z is a mediator). Hence, N is total in Z. On the other hand, $N \subseteq Y$, hence Y is dense in Z (in the topology of Z, and thus in the topology of X as well). From this we deduce that Y is dense in the subset $Y \cup Z$ of the space X, and again by Theorem 4.74, Y is dense in $\mathsf{Env}^X(Y \cup Z)$.

This means that $\mathsf{Env}^X(Y \cup Z)$ is a mediator for Y in X:

$$Y \hookrightarrow \mathsf{Env}^X(Y \cup Z) \hookrightarrow X, \qquad \overline{Y}^{\mathsf{Env}^X(Y\cup Z)} = \mathsf{Env}^X(Y \cup Z).$$

The condition $Y \overset{\circ}{\hookrightarrow} X$ implies the equality of stereotype spaces $Y = \mathsf{Env}^X(Y \cup Z)$. This yields $Z \hookrightarrow Y$, i.e. Z is a mediator for $\mathsf{Env}^Y N$ in Y:

$$\mathsf{Env}^Y N \hookrightarrow Z \hookrightarrow Y, \qquad \overline{\mathsf{Env}^Y N}^Z = Z.$$

By Theorem 4.74, $\mathsf{Env}^Y N$ is an immediate subspace in Y, so $\mathsf{Env}^Y N = Z$.

Suppose finally that $N = M \subseteq Y \overset{\circ}{\hookrightarrow} X$. Then by (4.64),

$$\mathsf{Env}^Y M \overset{\circ}{\hookrightarrow} \mathsf{Env}^X M.$$

On the other hand, by (4.63), $M \subseteq Y \overset{\circ}{\hookrightarrow} X$ implies

$$\mathsf{Env}^X M \overset{\circ}{\hookrightarrow} \mathsf{Env}^X Y \overset{(4.62)}{=} Y.$$

Together this gives

$$\mathsf{Env}^Y M \overset{\circ}{\hookrightarrow} \mathsf{Env}^X M \overset{\circ}{\hookrightarrow} Y.$$

By Theorem 4.74, M is total in $\mathsf{Env}^X M$, hence $\mathsf{Env}^Y M$ is total in $\mathsf{Env}^X M$. Thus, $\mathsf{Env}^X M$ is a mediator in this chain, and we obtain $\mathsf{Env}^Y M = \mathsf{Env}^X M$. ■

THEOREM 4.78. *The envelope* $\mathsf{Env}^X M$ *of any set* $M \subseteq X$ *is a minimal subspace among all the immediate subspaces in* X *which contain* M*, and in each of those immediate subspaces* $Y \overset{\circ}{\hookrightarrow} X$ *the space* $\mathsf{Env}^X M$ *is an immediate subspace:*

$$\forall Y \quad \big(M \subseteq Y \overset{\circ}{\hookrightarrow} X \ \Rightarrow\ \mathsf{Env}^X M \overset{\circ}{\hookrightarrow} Y\big). \tag{4.66}$$

Proof. We have

$$\mathsf{Env}^X M \overset{(4.65)}{=} \mathsf{Env}^Y M \overset{(4.59)}{\overset{\circ}{\hookrightarrow}} Y. \ ■$$

PROPOSITION 4.79. *If* $Y \overset{\circ}{\hookrightarrow} X$ *and* $Z \hookrightarrow X$*, then* $Z \subseteq Y$ *implies* $Z \hookrightarrow Y$*. In the special case when* $Y \overset{\circ}{\hookrightarrow} X$ *and* $Z \overset{\circ}{\hookrightarrow} X$*, the condition* $Z \subseteq Y$ *implies* $Z \overset{\circ}{\hookrightarrow} Y$*.*

Proof. If $Y \overset{\circ}{\hookrightarrow} X$, $Z \hookrightarrow X$, $Z \subseteq Y$, then

$$Z \overset{(4.61)}{\hookrightarrow} \mathsf{Env}^X Z \overset{(4.65)}{=} \mathsf{Env}^Y Z \overset{(4.59)}{\overset{\circ}{\hookrightarrow}} Y.$$

If $Y \overset{\circ}{\hookrightarrow} X$, $Z \overset{\circ}{\hookrightarrow} X$, $Z \subseteq Y$, then

$$Z \overset{(4.62)}{=} \mathsf{Env}^X Z \overset{(4.65)}{=} \mathsf{Env}^Y Z \overset{(4.59)}{\overset{\circ}{\hookrightarrow}} Y. \ ■$$

4.6. Quotient spaces

• Let X be a stereotype space, and

1) in X as a locally convex space take a closed subspace E,
2) on the quotient space X/E consider an arbitrary locally convex topology τ which is majorized by the natural quotient topology of X/E,

3) in the completion $(X/E)^{\blacktriangledown}$ of the locally convex space X/E with the topology τ take a subspace Y which contains X/E and is a stereotype space with respect to the topology inherited from $(X/E)^{\blacktriangledown}$.

Then we call the stereotype space Y a *quotient space of the stereotype space* X, and the composition $\upsilon = \sigma \circ \pi$ of the quotient map $\pi : X \to X/E$ and the natural inclusion $\sigma : X/E \to Y$ is called the *representing epimorphism* of the quotient space Y. We then write

$$Y \leftarrow\!\!\smallsetminus X \quad \text{or} \quad X /\!\!\!\rightarrow Y.$$

The class of all quotient spaces of X will be denoted by $\mathsf{Quot}(X)$. It is clear that $\mathsf{Quot}(X)$ is a set.

The following is evident:

PROPOSITION 4.80. *For a morphism $\varepsilon : Z \leftarrow X$ in* $\mathtt{Ste}$ *the following conditions are equivalent:*

(i) *ε is an epimorphism;*
(ii) *there is a quotient space Y of X with representing epimorphism $\upsilon : Y \leftarrow\!\!\smallsetminus X$, and an isomorphism $\theta : Z \leftarrow Y$ such that*

$$\begin{tikzcd} Z & \\ & X \arrow[ul, "\varepsilon"'] \arrow[dl, dashed, "\upsilon"] \\ Y \arrow[uu, squiggly, "\theta"] & \end{tikzcd} \tag{4.67}$$

COROLLARY 4.81. *For a stereotype space X the system* $\mathsf{Quot}(X)$ *is a system of quotient objects for X.*

The formalization of the idea of quotient object we have presented here has a qualitative shortcoming in comparison with the notion of subspace which we considered above: the problem is that the relation $\leftarrow\!\!\smallsetminus$ does not establish a partial order in $\mathsf{Quot}(P)$ for a stereotype space P. In fact, neither reflexivity, antisymmetry or transitivity holds for $\leftarrow\!\!\smallsetminus$. In particular, the first two axioms do not hold since $Y \leftarrow\!\!\smallsetminus X$ and $Y = X$ is impossible. To explain this, let us agree for simplicity that we do not take into account the necessity to pass to a subspace in the completion which was stated in step 3 of our definition—then $Y \leftarrow\!\!\smallsetminus X$ (and $Y \neq \emptyset$) implies by the axiom of regularity [19, Appendix, Axiom VII] that there exists an element $y \in Y$ such that $y \cap Y = \emptyset$. But if in addition $Y = X$, then y, being a coset of X, i.e. a non-empty subset in X, has non-empty intersection $y \cap Y = y \cap X = y \neq \emptyset$ with $X = Y$. As to the transitivity, when $Z \leftarrow\!\!\smallsetminus Y$ and $Y \leftarrow\!\!\smallsetminus X$, the elements of Z are non-empty sets of elements of Y, and each such element is a non-empty set of elements of X. From the point of view of set theory this is not the same as if elements of Z were sets of elements of X, so in this situation the relation $Z \leftarrow\!\!\smallsetminus X$ is also impossible. This forces us to introduce a new binary relation.

- Suppose $Y \leftarrow\!\!\smallsetminus X$ and $Z \leftarrow\!\!\smallsetminus X$. We will say that the quotient space Y *subordinates* the quotient space Z, and we write $Z \leq Y$, if there exists a morphism $\varkappa : Y \to Z$ such that

$$\begin{tikzcd} Y \arrow[dd, "\varkappa"'] & \\ & X \arrow[ul, "\upsilon_Y"'] \arrow[dl, "\upsilon_Z"] \\ Z & \end{tikzcd} \tag{4.68}$$

(here υ_Y and υ_Z are representing epimorphisms for Y and Z). The morphism $\varkappa$, if it exists, is unique and is an epimorphism.

For any stereotype space P the relation $\le$ is a partial order on $\mathsf{Quot}(P)$.

4.6.1. Immediate quotient spaces

- Let Y and Z be two quotient spaces of X such that $Z \le Y$ and the epimorphism $\varkappa : Z \leftarrow Y$ in diagram (4.68) is a monomorphism (and hence a bimorphism) of stereotype spaces. Then we will say that Y is a *mediator* for Z. One can notice that in this case Y is a subset in Z, so we will write $Z \supseteq Y$.
- We call a quotient space Z of a stereotype space X an *immediate quotient space* in X if it has no non-isomorphic mediators, i.e. for any mediator Y in X the corresponding epimorphism $Z \leftharpoondown Y$ is an isomorphism. We write in this case $Z \underset{\circ}{\leftharpoondown} X$:

$$Z \underset{\circ}{\leftharpoondown} X \;\Leftrightarrow\; \forall Y \; \big((Z \le Y \;\&\; Y \leftharpoondown X \;\&\; Z \supseteq Y) \;\Rightarrow\; Z = Y\big).$$

- Let us say that an immediate quotient space $Y \underset{\circ}{\leftharpoondown} X$ *strongly subordinates* an immediate quotient space $Z \underset{\circ}{\leftharpoondown} X$, and write $Z \le_{\circ} Y$, if there exists a strong epimorphism $\varkappa : Y \to Z$ such that diagram (4.68) is commutative.

Remark 4.82. In the category $\mathtt{LCS}$ of locally convex spaces, immediate quotient spaces of a locally convex space X are exactly quotient spaces of X by closed subspaces with the usual quotient topologies. As in the case of subspaces, in $\mathtt{Ste}$ the situation becomes more complicated (see Examples 4.85 and 4.86 below).

Recall that the notion of immediate epimorphism was defined on p. 17. The following statement is dual to Proposition 4.68, and can be proved by the dual reasoning:

Proposition 4.83. *For a morphism $\varepsilon : Z \leftarrow X$ in $\mathtt{Ste}$ the following conditions are equivalent:*

(i) *ε is an immediate epimorphism;*
(ii) *there exists an immediate quotient space Y of X with representing morphism $\upsilon : Y \leftharpoondown X$ and an isomorphism $\theta : Z \leftarrow Y$ such that*

$$\begin{tikzcd} Z & \\ & X \arrow[ul, "\varepsilon"'] \arrow[dl, dashed, "\upsilon"] \\ Y \arrow[uu, squiggly, "\theta"] & \end{tikzcd} \tag{4.69}$$

Here Y and θ are uniquely determined by Z and ε.

Proposition 4.84 [15]. *For an immediate quotient space Y of a stereotype space X with representing epimorphism $\upsilon : Y \leftarrow X$ the following conditions are equivalent:*

(i) *υ is an open map;*
(ii) *υ is a weakly open map;*
(iii) *Y is the pseudocompletion $(X/E)^{\triangledown}$ of the quotient space X/E of the locally convex space X (with the usual quotient topology) by some closed locally convex subspace E.*

- If the above conditions (i)–(iii) are fulfilled, we say that the immediate quotient space Y of X is *open.*

Proof of Proposition 4.84. (i)⇒(ii) is a special case of the situation described in Proposition 4.14.

(ii)⇒(iii). Suppose $\upsilon : Y \leftarrow X$ is a weakly open map. Denote by E its kernel. By definition of stereotype quotient space, Y is a pseudocomplete locally convex subspace in the completion $(X/E)^{\blacktriangledown}$ of the locally convex space X/E under some topology τ which is majorized by the quotient topology X/E, and X/E lies in Y as a set. Thus, we can represent υ as a diagram

$$\begin{tikzcd} X/E \arrow[d, dashed, "\sigma"'] & X \arrow[l, "\pi"'] \arrow[dl, "\upsilon"] \\ Y & \end{tikzcd}$$

where $\pi : X \to X/E$ is the usual quotient map of locally convex spaces, and $\sigma : X/E \to Y$ is a natural bimorphism. Since Y is pseudocomplete, σ can be extended to some morphism $\sigma^{\triangledown}$ on the pseudocompletion $(X/E)^{\triangledown}$ (use the reasoning in [2, diagram (1.13)]):

$$\begin{tikzcd} (X/E)^{\triangledown} \arrow[dr, dashed, "\sigma^{\triangledown}"'] & X/E \arrow[l, "\triangledown_{X/E}"'] \arrow[d, dashed, "\sigma"'] & X \arrow[l, "\pi"'] \arrow[dl, "\upsilon"] \\ & Y & \end{tikzcd}$$

Note that $\sigma^{\triangledown}$ is not only an epimorphism (this follows from property 3° of epimorphisms on p. 15, since $\upsilon = \sigma^{\triangledown} \circ \triangledown_{X/E} \circ \pi$ is an epimorphism), but also a monomorphism. This is proved as follows. The fact that υ is weakly open implies that so is σ. This means that every continuous linear functional on X/E can be extended along the map σ to a continuous linear functional on Y. In other words, the dual map $\sigma' : Y' \to X'$ is a surjection. This implies that the pseudosaturation $\sigma^{\triangledown}$ must be an injection [16].

As a result, we have a chain of epimorphisms

$$Y \xleftarrow{\sigma^{\triangledown}} (X/E)^{\triangledown} \xleftarrow{\triangledown_{X/E} \circ \pi} X,$$

where $\sigma^{\triangledown}$ is a bimorphism. Thus, $(X/E)^{\triangledown}$ is a mediator for Y, and $Y = (X/E)^{\triangledown}$.

---

[15] In [2, Theorem 4.16], which is equivalent to Proposition 4.84 here, as well as in the more general [2, Theorem 11.9], there is an inaccuracy: the requirement of openness of υ is omitted.

[16] We use here the following obvious property of pseudocompletion: *if $\varphi : X \to Y$ is a monomorphism of locally convex spaces such that the dual map $\varphi' : X' \leftarrow Y'$ is a surjection, then its pseudocompletion $\varphi^{\triangledown} : X^{\triangledown} \to Y^{\triangledown}$ is also a monomorphism of locally convex spaces.*

(iii)⇒(i). This follows from the fact that pseudocompletion does not change the topology. ∎

The following is dual to Example 4.70:

EXAMPLE 4.85. There exists a stereotype space P with an immediate quotient space of the form $Y = (P/E)^{\triangledown}$ which cannot be represented as $Y = P/F$ for a subspace $F \subseteq P$ (in formal language this means that the representing epimorphism $Y \leftarrow\!\circ\, P$ is open, but not closed).

Proof. The space Z from [2, 3.22] is such a space. It contains a closed subspace E such that the locally convex quotient space Z/E is metrizable, but not complete. As a corollary, in the stereotype sense the space $(Z/E)^{\triangledown}$ is an immediate quotient space, but it cannot be represented as Z/F, since F is uniquely determined as the kernel of the map $Z \to Y$, and hence must coincide with E. ∎

From Example 4.71 we have

EXAMPLE 4.86. There exists a stereotype space P with an immediate quotient space Y such that the representing epimorphism $Y \leftarrow\!\circ\, P$ is not weakly open (in the sense of the definition on p. 111). As a corollary, Y is not representable in the form $Y = (P/E)^{\triangledown}$ for a subspace $E \subseteq P$ (and hence is not isomorphic in Epi^P to the cokernel of any morphism $\varphi : A \to P$ in $\mathtt{Ste}$).

4.6.2. Refinement $\mathsf{Ref}^X F$ of a set F of functionals on a space X. Theorem 4.108 below justifies the following definition.

- Let F be a set of continuous linear functionals on a stereotype space X. The *refinement* of F on X is a quotient space of X, denoted by $\mathsf{Ref}^X F$ or by $\mathsf{Ref}\, F$, and defined as the injective limit

$$\mathsf{Ref}^X F = \mathsf{Ref}\, F = \mathtt{Ste}\text{-}\varinjlim E_i \tag{4.70}$$

  in the category $\mathtt{Ste}$ of the covariant system $\{E_i;\ i \in \mathtt{Ord}\}$ of quotient spaces of X indexed by ordinal numbers and defined by the following inductive rules:

  0) The space E_0 is the pseudocompletion of the quotient space $X/\mathsf{Ker}\, F$ (with the usual quotient topology) where $\mathsf{Ker}\, F = \bigcap_{f\in F} \mathsf{Ker}\, f$:

  $$E_0 = (X/\mathsf{Ker}\, F)^{\triangledown};$$

  the set F_0 of continuous linear functionals on E_0 is defined as the set of extensions to E_0 of the functionals from F (every $f \in F$ vanishes on $\mathsf{Ker}\, F$, so it can be uniquely extended to a continuous linear functional on $X/\mathsf{Ker}\, F$, and then to its pseudocompletion $E_0 = (X/\mathsf{Ker}\, F)^{\triangledown}$).

  1) If for some $j \in \mathtt{Ord}$ all the spaces $\{E_i;\ i < j\}$ are already defined, then E_j is defined as follows:

  – if $j = i + 1$ for some i, then $E_j = E_{i+1}$ is defined as the pseudocompletion of $E_i/\mathsf{Ker}\, F$ (with the usual quotient topology):

  $$E_j = E_{i+1} = (E_i/\mathsf{Ker}\, F)^{\triangledown};$$

the set F_{i+1} of continuous linear functionals on E_{i+1} is defined as the set of extensions of functionals from F_i;

– if j is a limit ordinal, then E_j is defined as the injective limit in $\mathtt{Ste}$ of the net $\{E_i;\ i \to j\}$:

$$E_j = \mathtt{Ste}\text{-}\lim_{i\to j} E_i = \Big(\mathsf{LCS}\text{-}\lim_{i\to j} E_i\Big)^{\triangledown};$$

the set F_j of continuous linear functionals on E_j is defined as the system of functionals which when restricted to E_i coincide with F_i.

Since the transfinite sequence $\{E_i;\ i \in \mathtt{Ord}\}$ cannot be an injective map from $\mathtt{Ord}$ into the set $\mathsf{Quot}(X)$ of quotient spaces of X, it stabilizes, i.e. from some i on, all the spaces E_i coincide (together with their topologies). As a corollary, the formula (4.70) uniquely determines some quotient space $\mathsf{Ref}\, F$ of X.

Example 4.87. If $\mathsf{Ker}\, F = \bigcap_{f\in F} \mathsf{Ker}\, f = 0$, then $\mathsf{Ref}^X F = X$.

Proof. From $\mathsf{Ker}\, F = \{x \in X : \forall f \in F \ \ f(x) = 0\} = 0$ we have $E_0 = (X/\mathsf{Ker}\, F)^{\triangledown} = X$. As a corollary, all the other spaces E_i coincide with X:

$$X = E_0 = E_1 = \cdots .$$

Thus, $\mathsf{Ref}\, F = X$. ∎

Example 4.88. If $\overline{\mathsf{span}\, F}^{X^\star} = F$, then $\mathsf{Ref}^X F = (X/\mathsf{Ker}\, F)^{\triangledown}$.

Proof. In this case

$$E_0 = (X/\mathsf{Ker}\, F)^{\triangledown} = (X/F^{\perp})^{\triangledown} \overset{[2,\ (4.3)]}{=} (F^{\vartriangle})^{\star},$$

hence $\mathsf{Ker}\, F_0 = \{y \in (F^{\vartriangle})^{\star} :\ \forall f \in F \quad f(y) = 0\} = 0$, and $E_1 = E_0/0 = E_0$. And further all the spaces E_i coincide with E_0. ∎

The following two theorems are dual to Theorems 4.74 and 4.75, and therefore we omit the proofs.

Theorem 4.89. *The refinement* $\mathsf{Ref}^X F$ *of any set* $F \subseteq X^\star$ *on a stereotype space* X *is an immediate quotient space of* X *to which the functionals from* F *can be continuously extended:*

$$\mathsf{Ref}^X F \overset{\upsilon}{\leftarrow\!\!\!\!\circ}\, X, \qquad \forall f \in F\ \exists g \in (\mathsf{Ref}^X F)^\star \quad f = g \circ \upsilon. \tag{4.71}$$

Theorem 4.90. *Every quotient space* Y *of a stereotype space* X *is subordinated to the refinement* $\mathsf{Ref}^X(Y^\star \circ \upsilon)$ *of the system of functionals* $Y^\star \circ \upsilon = \{g \circ \upsilon;\ g \in Y^\star\}$ *on* X*, where* $\upsilon : Y \leftarrow X$ *is the representing epimorphism of* Y*:*

$$\upsilon : Y \leftarrow X \ \Rightarrow\ Y \le \mathsf{Ref}^X(Y^\star \circ \upsilon), \tag{4.72}$$

and Y *is an immediate quotient subspace of* X*, iff* Y *coincides* (*as a locally convex space*) *with this refinement:*

$$\upsilon : Y \leftarrow\!\!\!\!\circ\, X \ \Leftrightarrow\ Y = \mathsf{Ref}^X(Y^\star \circ \upsilon). \tag{4.73}$$

Corollary 4.91. *The representing epimorphism* $\upsilon : Y \leftarrow\!\!\!\!\circ\, X$ *of any continuous quotient space* Y *of a stereotype space* X *is always relatively open.*

The following theorem is dual to Theorem 4.77.

THEOREM 4.92. *If $\varphi : Y \leftarrow X$ is a morphism of stereotype spaces that maps a set of functionals $G \subseteq Y^\star$ into a set of functionals $F \subseteq X^\star$, $G \circ \varphi \subseteq F$, then there exists a unique morphism $\varepsilon : \mathsf{Ref}^Y G \leftarrow \mathsf{Ref}^X F$ such that*

$$\begin{tikzcd} Y \arrow[d] & X \arrow[l, "\varphi"'] \arrow[d] \\ \mathsf{Ref}^Y G & \mathsf{Ref}^X F \arrow[l, dashed, "\varepsilon"'] \end{tikzcd}$$

In the special cases:

$$\left\{\begin{matrix} \varphi : Y \twoheadleftarrow X \\ G \circ \varphi \subseteq F \end{matrix}\right\} \Rightarrow \varepsilon \text{ is an epimorphism,} \tag{4.74}$$

$$\left\{\begin{matrix} \varphi : Y \twoheadleftarrow_{\circ} X \\ G \circ \varphi \subseteq F \end{matrix}\right\} \Rightarrow \varepsilon \text{ is an immediate epimorphism,} \tag{4.75}$$

$$\left\{\begin{matrix} \varphi : Y \twoheadleftarrow_{\circ} X \\ G \circ \varphi = F \end{matrix}\right\} \Rightarrow \varepsilon \text{ is an isomorphism.} \tag{4.76}$$

THEOREM 4.93. *The refinement $\mathsf{Ref}^X F$ of a set $F \subseteq X^\star$ of functionals on a stereotype space X is a minimal quotient space among the immediate quotient spaces of X to which the functionals F can be extended. Moreover, every such quotient space Y strongly subordinates $\mathsf{Ref}^X F$:*

$$\forall Y \quad (F \subseteq Y^\star \ \&\ Y \twoheadleftarrow_{\circ} X \Rightarrow \mathsf{Ref}^X F \leq_{\circ} Y). \tag{4.77}$$

PROPOSITION 4.94. *If $\alpha : Y \twoheadleftarrow_{\circ} X$ and $\beta : Z \twoheadleftarrow X$, then the condition $Z^\star \circ \alpha \subseteq Y^\star \circ \beta$ implies $Z \leq Y$. In the special case when $Y \overset{\circ}{\hookrightarrow} X$ and $Z \overset{\circ}{\hookrightarrow} X$, the condition $Z^\star \circ \alpha \subseteq Y^\star \circ \beta$ implies $Z \leq_{\circ} Y$.*

4.7. Decompositions, factorizations, envelope and refinement in Ste

4.7.1. Pre-abelian property and basic decomposition in Ste. Since any two parallel morphisms

$$\begin{tikzcd} X \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & Y \end{tikzcd}$$

in Ste can be added and subtracted from each other, it is clear that Ste is an additive category. In [2] it was noticed that this category is pre-abelian:

THEOREM 4.95. *In Ste, for each morphism $\varphi : X \to Y$ the formulas*

$$\begin{aligned} \mathsf{Ker}\,\varphi &= (\varphi^{-1}(0))^\vartriangle, & \mathsf{Coker}\varphi &= (Y/\overline{\varphi(X)})^\triangledown, \\ \mathsf{Coim}\,\varphi &= (X/\varphi^{-1}(0))^\triangledown, & \mathsf{Im}\,\varphi &= (\overline{\varphi(X)})^\vartriangle \end{aligned} \tag{4.78}$$

define respectively the kernel, cokernel, coimage and image. The operation $\varphi \mapsto \varphi^\star$ of taking the dual map establishes the following connections between these objects:

$$(\ker\varphi)^\star = \operatorname{coker}\varphi^\star, \quad (\operatorname{coker}\varphi)^\star = \ker\varphi^\star, \quad (\operatorname{im}\varphi)^\star = \operatorname{coim}\varphi^\star, \quad (\operatorname{coim}\varphi)^\star = \operatorname{im}\varphi^\star, \tag{4.79}$$

$$(\mathsf{Ker}\,\varphi)^{\perp\vartriangle} = \mathsf{Im}\,\varphi^\star, \quad (\mathsf{Im}\,\varphi)^{\perp\vartriangle} = \mathsf{Ker}\,\varphi^\star, \quad \mathsf{Ker}\,\varphi = (\mathsf{Im}\,\varphi^\star)^{\perp\vartriangle}, \quad \mathsf{Im}\,\varphi = (\mathsf{Ker}\,\varphi^\star)^{\perp\vartriangle}. \tag{4.80}$$

The pre-abelian property of Ste implies

THEOREM 4.96. *Each morphism $\varphi : X \to Y$ in* Ste *has basic decomposition* (2.30). *The operation $\varphi \mapsto \varphi^\star$ of taking the dual map establishes the following identities:*

$$(\operatorname{im}\varphi)^\star = \operatorname{coim}\varphi^\star, \qquad (\operatorname{coim}\varphi)^\star = \operatorname{im}\varphi^\star, \tag{4.81}$$

$$(\operatorname{Im}\varphi)^\star = \operatorname{Coim}\varphi^\star, \qquad (\operatorname{Coim}\varphi)^\star = \operatorname{Im}\varphi^\star. \tag{4.82}$$

Formulas (4.78) imply

THEOREM 4.97. *For any morphism $\varphi : X \to Y$ of stereotype spaces:*

– $\operatorname{Ker}\varphi$ *and* $\operatorname{Im}\varphi$ *are closed immediate subspaces* (*in X and Y respectively*)*;*
– $\operatorname{Coim}\varphi$ *and* $\operatorname{Coker}\varphi$ *are open immediate quotient spaces* (*of X and Y respectively*).

EXAMPLE 4.98. There exists a morphism φ of stereotype spaces such that the reduced morphism $\operatorname{red}\varphi$ is not a bimorphism.

Proof. Let E be a space from Example 4.41, i.e. a complete locally convex space with a discontinuous linear functional $f : E \to \mathbb{C}$ which is continuous in the topology of the pseudosaturation $E^\vartriangle$. Then $F = \operatorname{Ker} f$ is a closed subspace in $E^\vartriangle$, different from $E^\vartriangle$, but in E the subspace F is dense. Since E is complete, we can embed it as a closed subspace into a direct product of Banach spaces, say Y. Let $\varphi : F^\vartriangle \to Y$ be the composition of the injections

$$F^\vartriangle \subset F \subset E^\vartriangle \subset E \subset Y.$$

Since F is a closed subspace in the pseudocomplete space $E^\vartriangle$, it is pseudocomplete itself. Hence, its pseudosaturation $F^\vartriangle$ is a stereotype space. On the other hand, Y is a direct product of Banach spaces, therefore it is stereotype as well. Finally, since φ is an injection, its kernel is zero, hence $\operatorname{Coim}\varphi = F^\vartriangle$. On the other hand, the image of φ is the pseudosaturation of $\varphi(F^\vartriangle) = F$ in Y, i.e. the pseudosaturation of E:

$$\operatorname{Im}\varphi = \big(\overline{\varphi(F^\vartriangle)}^{\,Y}\big)^\vartriangle = E^\vartriangle.$$

Thus, the reduced morphism $\operatorname{red}\varphi$ is just the inclusion $F^\vartriangle \subset E^\vartriangle$, and it cannot be a bimorphism, since $F^\vartriangle$ is closed in $E^\vartriangle$, but not equal to $E^\vartriangle$. Diagram (2.30) for φ takes the form

$$\begin{tikzcd} F^\vartriangle \arrow[r, "\varphi"] \arrow[d, "\operatorname{coim}\varphi"'] & Y \\ F^\vartriangle \arrow[r, "\operatorname{red}\varphi"] & E^\vartriangle \arrow[u, "\operatorname{im}\varphi"'] \end{tikzcd}$$

■

COROLLARY 4.99. *The category* Ste *is not quasi-abelian in the sense of J.-P. Schneiders* [39].

Proof. Example 4.98 contradicts [39, Corollary 1.1.5]. ■

4.7.2. Nodal decomposition in Ste. In [2, Theorem 4.21] it was noticed that Ste is complete. On the other hand, from Corollaries 4.66 and 4.81 it follows that Ste is well-powered and co-well-powered. Together with the existence of basic decomposition, this means by Theorem 2.42 that Ste is a category with nodal decomposition:

THEOREM 4.100. *In* $\mathtt{Ste}$ *each morphism* $\varphi : X \to Y$ *has nodal decomposition* (2.24). *The operation* $\varphi \mapsto \varphi^\star$ *of taking the dual map establishes the following identities:*

$$(\mathsf{im}_\infty \varphi)^\star = \mathsf{coim}_\infty \varphi^\star, \qquad (\mathsf{coim}_\infty \varphi)^\star = \mathsf{im}_\infty \varphi^\star, \tag{4.83}$$

$$(\mathsf{Im}_\infty \varphi)^\star = \mathsf{Coim}_\infty \varphi^\star, \qquad (\mathsf{Coim}_\infty \varphi)^\star = \mathsf{Im}_\infty \varphi^\star. \tag{4.84}$$

As we noticed above, the basic and the nodal decompositions are connected with each other through diagram (2.31):

$$\begin{tikzcd}
X \arrow[rrr, "\varphi"] \arrow[dr, "\mathsf{coim}_\infty \varphi"] \arrow[dd, "\mathsf{coim}\,\varphi"'] & & & Y \\
& \mathsf{Coim}_\infty \varphi \arrow[r, "\mathsf{red}_\infty \varphi"] & \mathsf{Im}_\infty \varphi \arrow[ur, "\mathsf{im}_\infty \varphi"'] \arrow[dr, dashed, "\tau"'] & \\
\mathsf{Coim}\,\varphi \arrow[ur, dashed, "\sigma"'] \arrow[rrr, "\mathsf{red}\,\varphi"] & & & \mathsf{Im}\,\varphi \arrow[uu, "\mathsf{im}\,\varphi"']
\end{tikzcd}$$

where the morphisms σ and τ are uniquely determined (by φ).

EXAMPLE 4.101. For the morphism described in Example 4.98 diagram (2.31) has the form

$$\begin{tikzcd}
X^\vartriangle \arrow[rrr, "\varphi"] \arrow[dr, "\mathsf{coim}_\infty \varphi"] \arrow[dd, "\mathsf{coim}\,\varphi"'] & & & Y \\
& X^\vartriangle \arrow[r, "\mathsf{red}_\infty \varphi"] & X^\vartriangle \arrow[ur, "\mathsf{im}_\infty \varphi"'] \arrow[dr, dashed, "\tau"'] & \\
X^\vartriangle \arrow[ur, dashed, "1_{X^\vartriangle}"'] \arrow[rrr, "\mathsf{red}\,\varphi"] & & & E^\vartriangle \arrow[uu, "\mathsf{im}\,\varphi"']
\end{tikzcd}$$

This shows that τ is not necessarily an isomorphism. If we consider the dual map $\varphi^\star$, we can conclude that σ is not necessarily an isomorphism either.

THEOREM 4.102. *For any morphism* $\varphi : X \to Y$ *of stereotype spaces:*

– *the nodal image* $\mathsf{Im}_\infty \varphi$ *coincides with the envelope in* Y *of* $\varphi(X)$*:*

$$\mathsf{Im}_\infty \varphi = \mathsf{Env}^Y \varphi(X); \tag{4.85}$$

– *the nodal coimage* $\mathsf{Coim}_\infty \varphi$ *coincides with the refinement of* $\varphi^\star(Y^\star)$ *on* X*:*

$$\mathsf{Coim}_\infty \varphi = \mathsf{Ref}^X \varphi^\star(Y^\star). \tag{4.86}$$

Proof. By Remark 2.43, $\mathsf{Im}_\infty \varphi$ is the projective limit of the sequence of the "usual" images $\mathsf{Im}\,\varphi^i$ of the transfinite system of morphisms defined by $\varphi^{i+1} = \mathsf{red}\,\varphi^i$. And each $\mathsf{Im}\,\varphi^i$ precisely coincides with the space E_i from the definition of $\mathsf{Env}^Y M$ for $M = \varphi(X)$.

Similarly, $\mathsf{Coim}_\infty \varphi$ is the injective limit of the transfinite system of the "usual" coimages $\mathsf{Coim}\,\varphi^i$, and each such space coincides with the space E_i from the definition of $\mathsf{Ref}^X F$ for $F = \varphi^\star(Y^\star)$. ∎

4.7.3. Factorizations in Ste. Recall that by definition on p. 38, a *factorization* of a morphism $X \xrightarrow{\varphi} Y$ is its representation as a composition $\varphi = \mu \circ \varepsilon$ of an epimorphism ε and a monomorphism μ. Theorem 2.42 implies

THEOREM 4.103. *In the category* $\mathtt{Ste}$*:*

(i) *each morphism* φ *has a factorization;*

(ii) *among all factorizations of φ there is a minimal one $(\varepsilon_{\min}, \mu_{\min})$ and a maximal one $(\varepsilon_{\max}, \mu_{\max})$, i.e. for each factorization (ε, μ),*

$$(\varepsilon_{\min}, \mu_{\min}) \le (\varepsilon, \mu) \le (\varepsilon_{\max}, \mu_{\max}).$$

4.7.4. Characterization of strong morphisms in Ste

THEOREM 4.104. *In* Ste*, for a morphism $\mu : Z \to X$ the following conditions are equivalent:*

(i) *μ is an immediate monomorphism;*
(i)′ *in diagram (4.56) the space Y is an immediate subspace in X;*
(ii) *μ is a strong monomorphism;*
(ii)′ *in diagram (4.56) the morphism σ is a strong monomorphism;*
(iii) *$\mu \cong \mathsf{im}_\infty \mu$;*
(iv) *$\mathsf{coim}_\infty \mu$ and $\mathsf{red}_\infty \mu$ are isomorphisms.*

Proof. The equivalences (i)⇔(ii)⇔(iii)⇔(iv) follow from Theorem 2.47, since Ste is a category with nodal decomposition. Proposition 4.68 implies the equivalences (i)⇔(i)′ and (ii)⇔(ii)′. ∎

The dual proposition is proved by analogy:

THEOREM 4.105. *In* Ste*, for a morphism $\varepsilon : Z \to X$ the following conditions are equivalent:*

(i) *ε is an immediate epimorphism;*
(i)′ *in diagram (4.67) the space Y is an immediate quotient space for X;*
(ii) *ε is a strong epimorphism;*
(ii)′ *in diagram (4.67) the morphism π is a strong epimorphism;*
(iii) *$\varepsilon \cong \mathsf{coim}_\infty \varepsilon$;*
(iv) *$\mathsf{im}_\infty \mu$ and $\mathsf{red}_\infty \mu$ are isomorphisms.*

4.7.5. Envelope and refinement in Ste. Since the category Ste is complete, well-powered, co-well-powered and has nodal decomposition, this implies the existence of some envelopes and refinements in Ste.

THEOREM 4.106. *In* Ste*:*

(a) *Each space X has envelopes in the classes* Epi *of all epimorphisms and* SEpi *of all strong epimorphisms with respect to an arbitrary class Φ of morphisms among which there is at least one going from X. In addition:*

  (i) *if Φ separates morphisms on the outside in* Ste*, then*

$$\mathsf{env}_\Phi^{\mathsf{Epi}} X = \mathsf{env}_\Phi^{\mathsf{Bim}} X;$$

  (ii) *if Φ separates morphisms on the outside and is an ideal in* Ste*, then for any class $\Omega \supseteq$* Bim*,*

$$\mathsf{env}_\Phi^{\mathsf{Epi}} X = \mathsf{env}_\Phi^{\mathsf{Bim}} X = \mathsf{env}_\Phi^{\Omega} X = \mathsf{env}_\Phi X.$$

(b) *each space X has refinements in the classes* Mono *of all monomorphisms and* SMono *of all strong monomorphisms by means of an arbitrary class $\varPhi$ of morphisms among which there is at least one coming to X. In addition:*

(i) *if $\varPhi$ separates morphisms on the inside in* $\mathtt{Ste}$*, then*

$$\mathsf{ref}_{\varPhi}^{\mathsf{Mono}} X = \mathsf{ref}_{\varPhi}^{\mathsf{Bim}} X.$$

(ii) *if $\varPhi$ separates morphisms on the inside and is a left ideal in* $\mathtt{Ste}$*, then for any class $\varOmega \supseteq \mathsf{Bim}$,*

$$\mathsf{ref}_{\varPhi}^{\mathsf{Mono}} X = \mathsf{ref}_{\varPhi}^{\mathsf{Bim}} X = \mathsf{ref}_{\varPhi}^{\varOmega} X = \mathsf{ref}_{\varPhi} X.$$

Proof. By duality it is sufficient to prove (a). Let X be a stereotype space, and $\varPhi$ a class of morphisms which contains at least one going from X. Then $\mathsf{env}_{\varPhi}^{\mathsf{Epi}} X$ and $\mathsf{env}_{\varPhi}^{\mathsf{SEpi}} X$ exist by 5° on p. 64. Suppose now that $\varPhi$ separates morphisms on the outside in $\mathtt{Ste}$. Then by Theorem 3.6 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Epi}} X$ automatically implies the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Bim}} X$ and their equality. Finally, suppose that $\varPhi$ separates morphisms on the outside in $\mathtt{Ste}$ and in addition is a right ideal. Then by Theorem 3.7 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Bim}} X$ (which is already proved) implies that for any class $\varOmega \supseteq \mathsf{Bim}$ the envelope $\mathsf{env}_{\varPhi}^{\varOmega} X$ also exists, and these envelopes coincide. ∎

THEOREM 4.107. *The envelope $\mathsf{Env}^X M$ of a set M in a stereotype space X coincides with the envelope of the space* ([17]) *$\mathbb{C}_M$ in the class* Epi *of all epimorphisms of the category* $\mathtt{Ste}$ *with respect to the morphism $\varphi : \mathbb{C}_M \to X$, $\varphi(\alpha) = \sum_{x\in M} \alpha_x \cdot x$:*

$$\mathsf{Env}^X M = \mathsf{Env}_{\varphi}^{\mathsf{Epi}} \mathbb{C}_M.$$

Proof. This follows from 1° on p. 63 and from Theorem 4.102:

$$\mathsf{Env}_{\varphi}^{\mathsf{Epi}} \mathbb{C}_M \overset{(3.56)}{=} \mathsf{Im}_\infty \varphi \overset{(4.85)}{=} \mathsf{Env}^X \varphi(\mathbb{C}_M) = \mathsf{Env}^X \mathsf{span} M = \mathsf{Env}^X M. \ ∎$$

THEOREM 4.108. *The refinement $\mathsf{Ref}^X F$ of a set F of functionals on a stereotype space X coincides with the refinement of the space* ([18]) *$\mathbb{C}^F$ in the class* Mono *of all monomorphisms in* $\mathtt{Ste}$ *by means of the morphism $\varphi : X \to \mathbb{C}^F$, $\varphi(x)^f = f(x)$, $f \in F$:*

$$\mathsf{Ref}^X F = \mathsf{Ref}_{\varphi}^{\mathsf{Mono}} \mathbb{C}^F.$$

Proof. This follows from 1° on p. 64 and from Theorem 4.102:

$$\mathsf{Ref}_{\varphi}^{\mathsf{Mono}} \mathbb{C}^F \overset{(3.58)}{=} \mathsf{Coim}_\infty \varphi \overset{(4.86)}{=} \mathsf{Ref}^X \varphi^\star(Y^\star) = \mathsf{Ref}^X \mathsf{span}\, F = \mathsf{Ref}^X F. \ ∎$$

4.8. On homology in $\mathtt{Ste}$. As is known, in homology theory, in opposition to the well-established methods of abelian categories, there have always been attempts to find alternative approaches, where it is considered desirable to get rid of the abelian property and even of the additivity with the aim to cover the widest spectrum of situations (one can see this from [34], [30], [50], [17], [15], [14], [39], [23], [36], [8], [12], [21], [41], [20]). We hope that the following effect will be interesting in this connection: in the (non-abelian, but pre-abelian) category $\mathtt{Ste}$ of stereotype spaces the standard definition of homology

([17]) We use here the notation of [3, p. 478].
([18]) The notation of [3, p. 477] is used here.

breaks up into two non-equivalent notions. Let us start with the following definition (taken from [20]):

- Suppose in a pre-abelian category `K` we have a pair of morphisms $X \overset{\varphi}{\to} Y \overset{\psi}{\to} Z$ which form a complex:

  $$\psi \circ \varphi = 0.$$

  By the definitions of kernel and cokernel, this equality defines two natural morphisms $X \xrightarrow{\varphi^{\mathsf{Ker}\,\psi}} \mathsf{Ker}\,\psi$ and $\mathsf{Coker}\varphi \xrightarrow{\psi_{\mathsf{Coker}\varphi}} Z$ such that the following diagram is commutative:

  $$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[d, dashed, "\varphi^{\mathsf{Ker}\,\psi}"'] & & Y \arrow[rr, "\psi"] \arrow[drr, "\mathsf{coker}\,\varphi"'] & & Z \\
\mathsf{Ker}\,\psi \arrow[urr, "\mathsf{ker}\psi"'] & & & & \mathsf{Coker}\varphi \arrow[u, dashed, "\psi_{\mathsf{Coker}\varphi}"']
\end{tikzcd}$$

  The cokernel of $\varphi^{\mathsf{Ker}\,\psi}$ is called the *left homology* of the pair (φ, ψ) and is denoted by

  $$\mathsf{H}_-(\psi : \varphi) = \mathsf{Coker}(\varphi^{\mathsf{Ker}\,\psi}). \tag{4.87}$$

  The kernel of $\psi_{\mathsf{Coker}\varphi}$ is called the *right homology* of the pair (φ, ψ) and is denoted by

  $$\mathsf{H}_+(\psi : \varphi) = \mathsf{Ker}(\psi_{\mathsf{Coker}\varphi}). \tag{4.88}$$

The following observation belongs to folklore:

Proposition 4.109. *In a pre-abelian category* `K` *for any pair of morphisms* $X \overset{\varphi}{\to} Y \overset{\psi}{\to} Z$ *forming a complex,* $\psi \circ \varphi = 0$*, there exists a unique morphism* $\mathsf{h}(\psi : \varphi) : \mathsf{H}_-(\psi : \varphi) \to \mathsf{H}_+(\psi : \varphi)$ *such that*

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[d, dashed, "\varphi^{\mathsf{Ker}\,\psi}"'] & & Y \arrow[rr, "\psi"] \arrow[drr, "\mathsf{coker}\,\varphi"'] & & Z \\
\mathsf{Ker}\,\psi \arrow[urr, "\mathsf{ker}\psi"'] \arrow[d, "\mathsf{coker}(\varphi^{\mathsf{Ker}\,\psi})"'] & & & & \mathsf{Coker}\varphi \arrow[u, dashed, "\psi_{\mathsf{Coker}\varphi}"'] \\
\mathsf{Coker}(\varphi^{\mathsf{Ker}\,\psi}) \arrow[rrrr, dashed, "\mathsf{h}(\psi:\varphi)"] \arrow[d, equal] & & & & \mathsf{Ker}(\psi_{\mathsf{Coker}\varphi}) \arrow[u, "\mathsf{ker}(\psi_{\mathsf{Coker}\varphi})"'] \arrow[d, equal] \\
\mathsf{H}_-(\psi : \varphi) & & & & \mathsf{H}_+(\psi : \varphi)
\end{tikzcd} \tag{4.89}$$

In each autodual category (for instance, in `Ste`) a purely categorical duality reasoning gives the following identities:

$$\mathsf{H}_+(\psi : \varphi)^\star \cong \mathsf{H}_-(\varphi^\star : \psi^\star), \qquad \mathsf{H}_-(\psi : \varphi)^\star \cong \mathsf{H}_+(\varphi^\star : \psi^\star). \tag{4.90}$$

Example 4.110. In `Ste`, the morphism

$$\mathsf{H}_-(\psi : \varphi) \xrightarrow{\mathsf{h}(\psi:\varphi)} \mathsf{H}_+(\psi : \varphi)$$

is not always an epimorphism.

Proof. Let E be the space from Example 4.41, i.e. a complete locally convex space with a discontinuous linear functional $f : E \to \mathbb{C}$ which is continuous in the topology of the pseudosaturation $E^\vartriangle$. Then $F = \mathsf{Ker}\, f$ is a dense subspace of E, but in the space $E^\vartriangle$ it is a closed subspace, different from $E^\vartriangle$ (since $f \neq 0$). As a corollary, the natural inclusion $\sigma : F \to E$ is dense (i.e. has a dense image in E), but its pseudosaturation $\sigma^\vartriangle : F^\vartriangle \to E^\vartriangle$ does not have this property.

Let us represent E as a closed subspace in a stereotype space Y (with the topology inherited from Y; for example, we can consider the system of Banach quotient spaces of E and say that Y is the direct product of these spaces). Let

$$\varphi : F^\vartriangle \to E^\vartriangle \to Y$$

be the corresponding composition of monomorphisms, and

$$\psi : Y \to (Y/E^\vartriangle)^\triangledown$$

the corresponding epimorphism. Then, first,

$$\begin{gathered}
\mathsf{Ker}\,\psi = E^\vartriangle \\
\Downarrow \\
\mathsf{Im}\,\varphi^{\mathsf{Ker}\,\psi} = \Big(\overline{\varphi(F)}^{E^\vartriangle}\Big)^\vartriangle = \Big(\overline{F}^{E^\vartriangle}\Big)^\vartriangle = F^\vartriangle \\
\Downarrow \\
\mathsf{Coker}(\varphi^{\mathsf{Ker}\,\psi}) = (E^\vartriangle/F^\vartriangle)^\triangledown \cong \mathbb{C}^\triangledown = \mathbb{C}.
\end{gathered}$$

And second,

$$\begin{gathered}
\mathsf{Im}\,\varphi = \Big(\overline{\varphi(F)}^{Y}\Big)^\vartriangle = \Big(\overline{F}^{Y}\Big)^\vartriangle = E^\vartriangle, \\
\Downarrow \\
\mathsf{Coker}\,\varphi = (Y/E^\vartriangle)^\triangledown \\
\Downarrow \\
\psi_{\mathsf{Coker}\,\varphi} = 1_{(Y/E^\vartriangle)^\triangledown} \\
\Downarrow \\
\mathsf{Ker}(\psi_{\mathsf{Coker}\,\varphi}) = 0.
\end{gathered}$$

As a result, diagram (4.89) takes the form

$$\begin{tikzcd}
F^\vartriangle \arrow[rr, "\varphi"] \arrow[d, dashed, "\varphi^{\mathsf{Ker}\,\psi}"'] & & Y \arrow[rr, "\psi"] \arrow[drr, "\mathsf{coker}\,\varphi"'] & & (Y/E^\vartriangle)^\triangledown \\
E^\vartriangle \arrow[urr, "\mathsf{ker}\,\psi"'] \arrow[d, "\mathsf{coker}(\varphi^{\mathsf{Ker}\,\psi})"'] & & & & (Y/E^\vartriangle)^\triangledown \arrow[u, dashed, "\psi_{\mathsf{Coker}\varphi}"'] \\
(E^\vartriangle/F^\vartriangle)^\triangledown \cong \mathbb{C} \arrow[rrrr, dashed, "\mathsf{h}(\psi:\varphi)"] \arrow[d, equal] & & & & 0 \arrow[u, "\mathsf{ker}(\psi_{\mathsf{Coker}\varphi})"'] \arrow[d, equal] \\
\mathsf{H}_-(\psi:\varphi) & & & & \mathsf{H}_+(\psi:\varphi)
\end{tikzcd}$$

and clearly $\mathsf{h}(\psi : \varphi)$ cannot be an isomorphism. ∎

5. The category $\mathtt{Ste}^{\circledast}$ of stereotype algebras

5.1. Stereotype algebras and stereotype modules

5.1.1. Stereotype algebras. A stereotype space A over $\mathbb{C}$ is called a *stereotype algebra* if A is endowed with a structure of unital associative algebra over $\mathbb{C}$, and multiplication is a continuous bilinear form in the sense of the definition on p. 123: for any compact set K in A and for any zero neighborhood U in A there exists a zero neighborhood V in A such that

$$K \cdot V \subseteq U \quad \text{and} \quad V \cdot K \subseteq U.$$

This is equivalent to A being a monoid in the category $\mathtt{Ste}$ of stereotype spaces with respect to the tensor product $\circledast$ (defined in (4.36)). Clearly, each stereotype algebra A is a topological algebra (but not vice versa). The class of all stereotype algebras is denoted by $\mathtt{Ste}^{\circledast}$. It is a category, where the morphisms are the linear, continuous, multiplicative and unit-preserving maps $\varphi : A \to B$.

In contrast to $\mathtt{Ste}$, the category $\mathtt{Ste}^{\circledast}$ of stereotype algebras is not additive. In addition, in $\mathtt{Ste}^{\circledast}$ an asymmetry arises between monomorphisms and epimorphisms, since epimorphisms are not inherited from $\mathtt{Ste}$:

– A morphism $\varphi : A \to B$ of stereotype algebras is a monomorphism iff φ is an injective map (i.e. a monomorphism of stereotype spaces).
– On the other hand, an epimorphism $\varphi : A \to B$ of stereotype algebras not necessarily has dense image in B (i.e., not necessarily is an epimorphism of stereotype spaces). A counterexample is the inclusion of the algebra $\mathcal{P}(\mathbb{C})$ of polynomials on $\mathbb{C}$ into the algebra $\mathcal{P}(\mathbb{C}^{\times})$ of Laurent polynomials on $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ (both endowed with the strongest locally convex topology).

The following lemma will be useful:

LEMMA 5.1. *Let A and B be topological algebras (with separately continuous multipication), and $\varphi : A \to B$ be a continuous linear map such that*

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \qquad x, y \in A_0,$$

for some dense subalgebra A_0 in A. Then

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \qquad x, y \in A.$$

Proof. For any $x, y \in A$ we find nets $x_i, y_j \in A_0$ such that

$$x_i \xrightarrow{i\to\infty} x, \qquad y_j \xrightarrow{j\to\infty} y.$$

Then we have

$$\varphi(x\cdot y)\underset{\infty\leftarrow j}{\longleftarrow}\varphi(x\cdot y_j)\underset{\infty\leftarrow i}{\longleftarrow}\varphi(x_i\cdot y_j)=\varphi(x_i)\cdot\varphi(y_j)\underset{i\to\infty}{\longrightarrow}\varphi(x_i)\cdot\varphi(y)\underset{j\to\infty}{\longrightarrow}\varphi(x)\cdot\varphi(y).\ \blacksquare$$

Let us give some examples of stereotype algebras. First, two abstract examples.

EXAMPLE 5.2 (Fréchet algebras). For a Fréchet space A, being a stereotype algebra is equivalent to the joint continuity of multiplication. Hence, *each unital Fréchet algebra is a stereotype algebra.*

EXAMPLE 5.3 (The operator algebra $\mathcal{L}(X)$). Theorem 4.54 implies that for any stereotype space X the space $\mathcal{L}(X)=X\oslash X$ of continuous linear maps $\varphi:X\to X$ is a stereotype algebra with respect to composition.

Next, we give a series of function algebras.

EXAMPLE 5.4 (The algebra $\mathcal{C}(M)$ of continuous functions on a paracompact locally compact space M). Let us recall that a topological space M is said to be *σ-compact* if it is the union of a countable system of compact sets. For a locally compact spaces M this condition is equivalent to the *Lindelöf property*: every open covering of M has a countable subcovering (cf. [13, 3.8.C(b)]). As a corollary, if M is a *Lindelöf space* (i.e. has the Lindelöf property) and is locally compact, then the space $\mathcal{C}(M)$ of continuous functions $u:M\to\mathbb{C}$ is a Fréchet space with the topology of uniform convergence on compact sets $S\subseteq M$.

Consider a more general class of topological spaces. Let M be a paracompact locally compact topological space. Then it can be decomposed into a direct sum

$$M=\coprod_{i\in I}M_i$$

of Lindelöf locally compact spaces M_i (see [13, Theorem 5.1.27]). Therefore, $\mathcal{C}(M)$ (with the topology of uniform convergence on compact sets) is a stereotype space, as a direct product of Fréchet spaces:

$$\mathcal{C}(M)=\prod_{i\in I}\mathcal{C}(M_i).$$

Clearly, $\mathcal{C}(M)$ is an algebra with respect to pointwise multiplication, which is easily checked to be a continuous bilinear map. Hence, $\mathcal{C}(M)$ is a stereotype algebra.

EXAMPLE 5.5. The algebra $\mathcal{E}(M)$ of smooth functions on a smooth manifold M (with the usual topology of uniform convergence on compact sets with all derivatives) is a stereotype algebra (with the usual pointwise multiplication).

EXAMPLE 5.6. The algebra $\mathcal{O}(M)$ of holomorphic functions on a Stein manifold M (with the topology of uniform convergence on compact sets in M) is a stereotype algebra (with the pointwise multiplication).

EXAMPLE 5.7. The algebra $\mathcal{P}(M)$ of polynomials (i.e. regular functions) on an affine algebraic variety M (with the strongest locally convex topology) is a stereotype algebra (with pointwise multiplication).

Finally, we present a series of group algebras.

Example 5.8 (The algebra $\mathcal{C}^\star(G)$ of measures on a locally compact group G). As is known, each locally compact group G is paracompact [18, 2.8.13], hence the space $\mathcal{C}(G)$ of continuous functions on G (with the topology of uniform convergence on compact sets) can be considered as a special case of Example 5.4, and what is important for us, $\mathcal{C}(G)$ is stereotype. Its dual space $\mathcal{C}^\star(G)$ consists of measures with compact support on G. The *convolution of measures* $\alpha, \beta \in \mathcal{C}^\star(G)$ is defined by the formula

$$\begin{aligned} \alpha * \beta(u) &= (\alpha \otimes \beta)(w)|_{w(s,t)=u(s\cdot t)} \\ &= \int_G \left(\int_G u(s\cdot t)\, d\alpha(s) \right) d\beta(t) = \int_G \left(\int_G u(s\cdot t)\, d\beta(t) \right) d\alpha(s). \end{aligned} \tag{5.1}$$

This operation is associative and has a unit (the delta-functional δ^{1_G} of the unit in G). In addition, it is continuous as a bilinear map, so *the space $\mathcal{C}^\star(G)$ of measures on a locally compact group G is a stereotype algebra with the convolution $(\alpha, \beta) \mapsto \alpha * \beta$ as multiplication (and with δ^{1_G} as unit).*

Example 5.9 (The algebra $\mathcal{E}^\star(G)$ of distributions on a Lie group G). Let G be a real Lie group [49, 48]. Consider the space $\mathcal{E}^\star(G)$ of distributions with compact support on G (i.e. the dual space to $\mathcal{E}(G)$ from Example 5.5). The *convolution of distributions* $\alpha, \beta \in \mathcal{E}^\star(G)$ is defined by formula (5.1). *The space $\mathcal{E}^\star(G)$ of distributions is a stereotype algebra with the convolution $(\alpha, \beta) \mapsto \alpha * \beta$ as multiplication (and with δ^{1_G} as unit).*

Example 5.10 (The algebra $\mathcal{O}^\star(G)$ of analytic functionals on a Stein group G). Let G be a *Stein group*, i.e. a complex Lie group [9] which is a Stein manifold [42]. Consider the space $\mathcal{O}^\star(G)$ of analytic functionals on G (i.e. the dual space to $\mathcal{O}(G)$ from Example 5.6). The *convolution of analytic functionals* $\alpha, \beta \in \mathcal{O}^\star(G)$ is defined by formula (5.1). *The space $\mathcal{O}^\star(G)$ of analytic functionals is a stereotype algebra with the convolution $(\alpha, \beta) \mapsto \alpha * \beta$ as multiplication (and with δ^{1_G} as unit).*

Example 5.11 (The algebra $\mathcal{P}^\star(G)$ of currents on an affine algebraic group G). Recall some facts from the theory of algebraic groups [48]. The general linear group $\mathrm{GL}(n,\mathbb{C})$ is a basic open subset in the vector space $L(n,\mathbb{C})$, therefore it can be represented as a closed (in the Zariski topology) subset in some affine algebraic space $\mathbb{C}^m$. This means that $\mathrm{GL}(n,\mathbb{C})$ is an affine algebraic variety. Its polynomials (regular functions) have the form

$$u(g) = P(g)/D(g)^k \tag{5.2}$$

where $D(g)$ is the determinant of the matrix $g \in L(n,\mathbb{C})$, k belongs to $\mathbb{N}$, and P is a polynomial on $L(n,\mathbb{C})$ [48].

Let now G be an affine algebraic group, i.e. a Zariski closed subgroup in $\mathrm{GL}(n,\mathbb{C})$ [48], or equivalently, the set of common zeroes of a system of functions $u : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$ of the form (5.2), which is closed under the group operation in $\mathrm{GL}(n,\mathbb{C})$. Since G is a closed subset in $\mathrm{GL}(n,\mathbb{C})$, it is an affine variety.

Therefore the space $\mathcal{P}(G)$ of polynomials on G is a special case of the general construction from Example 5.7. In this case $\mathcal{P}(G)$ consists of functions $v : G \to \mathbb{C}$ which can be extended to functions $u : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$ of the form (5.2). The dual space $\mathcal{P}^\star(G)$ consists of linear (and automatically continuous) functionals $f : \mathcal{P}(G) \to \mathbb{C}$, called *currents* on G.

The *convolution of currents* $\alpha, \beta \in \mathcal{P}^\star(G)$ is defined by (5.1). *The space $\mathcal{P}^\star(G)$ of currents on an affine algebraic group G is a stereotype algebra with the convolution $(\alpha, \beta) \mapsto \alpha * \beta$ as multiplication (and δ^{1_G} as unit).*

5.1.2. Stereotype modules. A stereotype space X over $\mathbb{C}$ with a given structure of left (or right) A-module is called a *stereotype A-module* if multiplication by elements of A is a continuous bilinear map in the sense of the definition on p. 123. Theorem 4.58 implies that X is a stereotype (left) module over A if and only if μ can be continuously factored through the projective stereotype tensor product:

$$\begin{tikzcd} A \times X \arrow[rr] \arrow[dr, "\mu"'] & & A \circledast X \arrow[dl] \\ & X & \end{tikzcd}$$

Example 5.12. Each stereotype space X is a stereotype left module over the stereotype algebra $\mathcal{L}(X)$ (from Example 5.3).

Theorem 5.13 (on representation). *Let A be a stereotype algebra. A stereotype space X with the structure of left (respectively right) A-module is a stereotype A-module if and only if multiplication by elements of A defines a continuous homomorphism (respectively, antihomomorphism) of A into $\mathcal{L}(X)$.*

The classes ${}_A\mathtt{Ste}$ and $\mathtt{Ste}_A$ of left and right stereotype modules over a stereotype algebra A form categories with continuous A-linear maps as morphisms.

Properties of the categories ${}_A\mathtt{Ste}$ and $\mathtt{Ste}_A$.

1° ${}_A\mathtt{Ste}$ *and* $\mathtt{Ste}_A$ *are pre-abelian categories.*
2° ${}_A\mathtt{Ste}$ *and* $\mathtt{Ste}_A$ *are complete: each covariant (and each contravariant) system has an injective and a projective limit.*
3° ${}_A\mathtt{Ste}$ *and* $\mathtt{Ste}_A$ *are enriched categories over the monoidal category* $\mathtt{Ste}$.

5.2. Subalgebras, quotient algebras, limits and completeness of $\mathtt{Ste}^{\circledast}$

5.2.1. Subalgebras, products and projective limits

- Suppose B is a subset in a stereotype algebra A endowed with a structure of stereotype algebra in such a way that the set-theoretic inclusion $B \subseteq A$ is a morphism of stereotype algebras (i.e. a linear, multiplicative and unit-preserving continuous map). Then B is called a *subalgebra* of the stereotype algebra A, and the set-theoretic inclusion $\sigma : B \subseteq A$ its *representing monomorphism.*
- We say that a subalgebra B of a stereotype algebra A is *closed* if its representing monomorphism $\sigma : B \to A$ is a closed map in the sense of the definition on p. 110.

The following fact was stated in [2, Theorem 10.13]:

Theorem 5.14. *Let A be a stereotype algebra and B its subalgebra (in the purely algebraic sense), and at the same time a closed subspace of the locally convex space A. Then the pseudosaturation $B^\vartriangle$ is a (stereotype algebra and a) closed subalgebra in A.*

THEOREM 5.15. *Each family $\{A_i; i \in I\}$ of stereotype algebras has a direct product in the category $\mathtt{Ste}^{\circledast}$, and as a stereotype space this product is exactly the direct product of the family $\{A_i;\ i \in I\}$ of stereotype spaces:*

$$\mathtt{Ste}^{\circledast}\text{-}\prod_{i\in I} A_i = \mathtt{Ste}\text{-}\prod_{i\in I} A_i.$$

Proof. We have to verify that the direct product is the usual direct product of locally convex spaces $A = \prod_{i\in I} A_i$ with coordinatewise multiplication:

$$(x \cdot y)_i = x_i \cdot y_i, \qquad i \in I.$$

By [2, Theorem 4.20], this is a stereotype space, so we only need to prove that multiplication is continuous. Let U be a neighborhood of zero and K a compact set in A. We must find a zero neighborhood V in A such that

$$V \cdot K \subseteq U, \qquad K \cdot V \subseteq U.$$

It is sufficient to consider a base neighborhood U, i.e.

$$U = \{x \in A :\ \forall i \in J\ \ x_i \in U_i\}$$

where $J \subseteq I$ is a finite subset in I, and for any $i \in J$ the set U_i is a neighborhood of zero in A_i, and x_i is the projection of $x \in A$ onto A_i. If U has this form, then for any $i \in J$ we can consider the zero neighborhood U_i in A_i, and (since A_i is a stereotype algebra) we can choose a zero neighborhood V_i such that

$$V_i \cdot K_i \subseteq U_i, \qquad K_i \cdot V_i \subseteq U_i$$

(where K_i is the projection of the compact set $K \subseteq A$ onto A_i). Then we let

$$V = \{x \in A : \forall i \in J\ \ x_i \in V_i\}$$

and for each $x \in V$ and $y \in K$ we get

$$(\forall i \in J\ \ (x \cdot y)_i = x_i \cdot y_i \in V_i \cdot K_i \subseteq U_i) \ \Rightarrow\ x \cdot y \in U.$$

This means that $V \cdot K \subseteq U$. Similarly,

$$(\forall i \in J\ \ (y \cdot x)_i = y_i \cdot x_i \in K_i \cdot V_i \subseteq U_i) \ \Rightarrow\ y \cdot x \in U,$$

and this means that $K \cdot V \subseteq U$. ∎

THEOREM 5.16. *Each covariant system $\{A_i;\ \pi_i^j\}$ of stereotype algebras has a projective limit in $\mathtt{Ste}^{\circledast}$, and as a stereotype space this limit is exactly the projective limit of the covariant system $\{A_i;\ \pi_i^j\}$ of stereotype spaces:*

$$\mathtt{Ste}^{\circledast}\text{-}\varprojlim A_i = \mathtt{Ste}\text{-}\varprojlim A_i.$$

Proof. By Theorem 5.15, the direct product $A = \prod_{i\in I} A_i$ with coordinatewise multiplication is a direct product of the family $\{A_i\}$ of algebras in $\mathtt{Ste}^{\circledast}$, and by Theorem 5.14, the subalgebra B in A consisting of all families $\{x_i;\ i \in I\}$ with

$$x_i = \pi_i^j(x_j), \qquad i \leq j \in I,$$

and endowed with the topology of pseudosaturation of the topology inherited from A is a stereotype algebra. The same reasoning as in the case of stereotype spaces proves that B is the projective limit in $\mathtt{Ste}^{\circledast}$. ∎

5.2.2. Quotient algebras, coproducts and injective limits

- Let A be a stereotype algebra, and let

  1) I be a two-sided ideal in A (as an algebra), and at the same time a closed set in A (as a topological space); we will call such ideals *closed ideals* in A,
  2) τ be a locally convex topology on the quotient algebra A/I such that τ is majorized by the usual quotient topology,
  3) B be a subspace in the completion $(A/I)^{\blacktriangledown}$ of the locally convex space A/I with respect to τ such that B contains A/I and is a stereotype algebra with respect to the algebraic operations and the topology inherited from $(A/I)^{\blacktriangledown}$.

  Then we call the stereotype algebra B the *quotient algebra* of the stereotype algebra A, and the composition $\upsilon = \sigma \circ \pi$ of the quotient map $\pi : A \to A/I$ and the natural embedding $\sigma : A/I \to B$ is called the *representing epimorphism* of B.
- A quotient algebra B of a stereotype algebra A is said to be *open* if its representing epimorphism $\upsilon : B \leftarrow A$ is an open map in the sense of the definition on p. 109.

The symmetry between projective and injective constructions which was obvious for stereotype spaces (see [2]) is preserved in some sense for stereotype algebras, but the difference is that the injective constructions in $\mathtt{Ste}^{\circledast}$ become more complicated and the proofs more difficult (however, the situation here is the same as for algebras in a purely algebraic sense). For example, the analog of Theorem 5.14 uses the theory of modules over algebras (see [2, proof of Theorem 10.14]):

THEOREM 5.17. *Let A be a stereotype algebra and I a closed ideal in A. Then the pseudocompletion $(A/I)^{\triangledown}$ is a stereotype algebra* (*and is called an* open quotient algebra *of A by the ideal I*).

REMARK 5.18. In Theorem 5.17 the unitality requirement (i.e. the existence of identity) for the algebra A is inessential.

Suppose $\{A_i;\ i \in I\}$ is a family of stereotype algebras. Let us construct an algebra $\coprod_{i\in I} A_i$ in the following way. First let us say that a sequence $i = \{i_1, \dots, i_n\} \in I$ of indices *alternates* if

$$\forall k = 1, \dots, n-1 \quad i_k \neq i_{k+1}.$$

The set of all alternating sequences in I of (various) finite lengths will be denoted as $I_{\mathbb{N}}^{\mathsf{alt}}$. Let us introduce multiplication on $I_{\mathbb{N}}^{\mathsf{alt}}$ as follows: if $\iota, \varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}$ have lengths m and n respectively, then their product is

$$\iota * \varkappa = \begin{cases} (\iota_1, \dots, \iota_m, \varkappa_1, \dots, \varkappa_n) & \text{for } \iota_m \neq \varkappa_1, \\ (\iota_1, \dots, \iota_m, \varkappa_2, \dots, \varkappa_n) & \text{for } \iota_m = \varkappa_1 \end{cases}$$

(the length of $\iota * \varkappa$ is $m + n$ if $\iota_m \neq \varkappa_1$, and $m + n - 1$ if $\iota_m = \varkappa_1$). For each sequence $\iota \in I_{\mathbb{N}}^{\mathsf{alt}}$ we set

$$A_\iota = A_{\iota_1} \circledast \cdots \circledast A_{\iota_m}.$$

(where $\circledast$ is the projective tensor product from (4.36)). Note that for all $\iota, \varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}$ the spaces $A_\iota \circledast A_\varkappa$ and $A_{\iota * \varkappa}$ are naturally related through the continuous linear map

$\mu_{\iota,\varkappa} : A_\iota \circledast A_\varkappa \to A_{\iota * \varkappa}$,

$$\mu_{\iota,\varkappa} = \begin{cases} 1_{A_\iota \circledast A_\varkappa}, & \iota_m \neq \varkappa_1, \\ 1_{A_{\iota_1}} \circledast \cdots \circledast 1_{A_{\iota_{m-1}}} \circledast \mu_{\iota_m} \circledast 1_{A_{\varkappa_2}} \circledast \cdots \circledast 1_{A_{\varkappa_n}}, & \iota_m = \varkappa_1, \end{cases} \tag{5.3}$$

where $\mu_i : A_i \circledast A_i \to A_i$ is multiplication in A_i.

Consider the stereotype space

$$A_* = \bigoplus_{\iota \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\iota$$

and note that the formula

$$(a_{i_1} \circledast a_{i_2} \circledast \cdots \circledast a_{i_m}) \cdot (b_{j_1} \circledast b_{j_2} \circledast \cdots \circledast b_{j_n}) \\ = \begin{cases} a_{i_1} \circledast a_{i_2} \circledast \cdots \circledast a_{i_m} \circledast b_{j_1} \circledast b_{j_2} \circledast \cdots \circledast b_{j_n}, & i_m \neq j_1, \\ a_{i_1} \circledast a_{i_2} \circledast \cdots \circledast (a_{i_m} \cdot b_{j_1}) \circledast b_{j_2} \circledast \cdots \circledast b_{j_n}, & i_m = j_1, \end{cases}$$

defines a multiplication in A_*, which is a continuous bilinear map. This becomes obvious if we represent this operation as the composition

$$A_* \times A_* \to A_* \circledast A_* = \Big(\bigoplus_{\iota \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\iota \Big) \circledast \Big(\bigoplus_{\varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\varkappa \Big) \\ \to \bigoplus_{\iota,\varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\iota \circledast A_\varkappa \to \bigoplus_{\iota,\varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}} A_{\iota * \varkappa} \to \bigoplus_{\lambda \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\lambda = A_*.$$

Here the first arrow is described in Proposition 4.56, the second arrow is the natural isomorphism (4.52) that connects the direct sum and the projective tensor product, the third arrow is the direct sum $\bigoplus_{\iota,\varkappa \in I_{\mathbb{N}}^{\mathsf{alt}}} \mu_{\iota,\varkappa}$ of the morphisms (5.3), and the final arrow is the result of identification of each summand of the form $A_{\iota * \varkappa}$ (there can be many of those) with the space A_λ in the sum $\bigoplus_{\lambda \in I_{\mathbb{N}}^{\mathsf{alt}}} A_\lambda$ (which is unique).

Obviously, this multiplication in A_* is associative. If we take the quotient algebra of the (non-unital) algebra A_* by the closed ideal M (here we use Remark 5.18) generated by the elements of the form

$$1_{A_i} - 1_{A_j}, \quad i, j \in I,$$

then the quotient algebra $(A_*/M)^\triangledown$ will be a stereotype algebra with identity

$$1_{(A_*/M)^\triangledown} = \pi(1_{A_i})$$

(here the right side is the image under the quotient map $\pi : A_* \to (A_*/M)^\triangledown$).

- Following [29] we call $(A_*/M)^\triangledown$ the *free product* of $\{A_i;\, i \in I\}$ and we denote it by

$$\mathtt{Ste}^{\circledast}\text{-}\coprod_{i \in I} A_i = (A_*/M)^\triangledown.$$

This is justified by the following theorem.

Theorem 5.19. *For each family $\{A_i; i \in I\}$ of stereotype algebras its free product $\mathtt{Ste}^{\circledast}\text{-}\coprod_{i \in I} A_i$ is a coproduct in $\mathtt{Ste}^{\circledast}$.*

Theorem 5.20. *Each covariant system $\{A_i;\, \iota_i^j\}$ of stereotype algebras has an injective limit in $\mathtt{Ste}^{\circledast}$.*

Proof. The limit is the open quotient algebra $\big((\coprod_{i\in I} A_i)/N\big)^{\triangledown}$ of the free product $\coprod_{i\in I} A_i$ by the closed ideal N generated by elements of the form

$$\iota_i(x) - \iota_j(\iota_i^j(x)), \qquad x \in A_i,$$

where $\iota_k : A_k \to \coprod_{i\in I} A_i$ are natural embeddings. ∎

As an illustration of the difference between projective and injective constructions in $\mathtt{Ste}^{\circledast}$, note that injective limits in $\mathtt{Ste}^{\circledast}$ do not necessarily coincide as stereotype spaces with injective limits in $\mathtt{Ste}$. For instance, for coproducts we have

$$\mathtt{Ste}^{\circledast}\text{-}\coprod_{i\in I} A_i \neq \mathtt{Ste}\text{-}\coprod_{i\in I} A_i$$

(although there is a natural map from right to left). This asymmetry, however, diasppears when the index set I is directed:

THEOREM 5.21. *If $\{A_i;\, \iota_i^j\}$ is a covariant system of stereotype algebras over a directed set I, then the natural map*

$$\mathtt{Ste}\text{-}\varinjlim A_i \to \mathtt{Ste}^{\circledast}\text{-}\varinjlim A_i$$

is an isomorphism of stereotype spaces.

Proof. Write $A = \mathtt{Ste}\text{-}\varinjlim A_i$, and let $\rho_i : A_i \to A$ be the corresponding morphisms of stereotype spaces:

$$\begin{tikzcd} & A & \\ A_i \arrow[ur, "\rho_i"] \arrow[rr, "\iota_i^j"] & & A_j \arrow[ul, "\rho_j"'] \end{tikzcd} \tag{5.4}$$

We will show that A has a natural structure of stereotype algebra, and with this structure A is an injective limit of the covariant system $\{A_i;\, \iota_i^j\}$ of stereotype algebras.

STEP 1. Take $i \in I$ and note that for any $j \geq i$ the homomorphism $\iota_i^j : A_i \to A_j$ induces on A_j a structure of left A_i-module by the formula

$$a \underset{i}{\cdot} b = \iota_i^j(a) \underset{A_j}{\cdot} b, \qquad a \in A_i,\ b \in A_j. \tag{5.5}$$

(here $\underset{i}{\cdot}$ means left multiplication by elements of A_i, and $\underset{A_j}{\cdot}$ multiplication in A_j). Moreover, for $i \leq j \leq k$ the maps $\iota_j^k : A_j \to A_k$ are morphisms of left A_i-modules:

$$\begin{aligned} \iota_j^k(a \underset{i}{\cdot} b) &\overset{(5.5)}{=} \iota_j^k(\iota_i^j(a) \underset{A_j}{\cdot} b) = \iota_j^k(\iota_i^j(a)) \underset{A_k}{\cdot} \iota_j^k(b) = \iota_i^k(a) \underset{A_k}{\cdot} \iota_j^k(b) \\ &\overset{(5.5)}{=} a \underset{i}{\cdot} \iota_j^k(b), \qquad a \in A_i,\ b \in A_j. \end{aligned}$$

This means that $\{A_j;\, j \geq i\}$ can be considered as a covariant system of left stereotype A_i-modules. By [2, Theorem 11.17], it has an injective limit, which as a stereotype space coincides with the injective limit of $\{A_i;\, j \geq i\}$. And the latter coincides with the injective limit of the whole system $\{A_i;\, \iota_i^j\}$, since I is directed:

$$_{A_i}\mathtt{Ste}\text{-}\lim_{i\leq j\to\infty} A_j = \mathtt{Ste}\text{-}\lim_{i\leq j\to\infty} A_j = \mathtt{Ste}\text{-}\lim_{j\to\infty} A_j = A.$$

An important conclusion is that for any $i\in I$ the space A has the structure of a stereotype A_i-module, and under this structure the maps in diagram (5.4) become morphisms of A_i-modules; in particular,

$$\rho_j(a\underset{i}{\cdot}b)=a\underset{i}{\cdot}\rho_j(b),\qquad i\le j,\ a\in A_i,\ b\in A_j. \tag{5.6}$$

Step 2. For $i\le j$ the structures of left A_i-module and of left A_j-module on A are coherent via the identity

$$\iota_i^j(a)\underset{j}{\cdot}x=a\underset{i}{\cdot}x,\qquad a\in A_i,\ x\in A. \tag{5.7}$$

To prove this we first consider the special case when $x=\rho_k(b)$, $b\in A_k$, $k\ge j$. In this situation

$$\begin{aligned}\iota_i^j(a)\underset{j}{\cdot}x&=\iota_i^j(a)\underset{j}{\cdot}\rho_k(b)\overset{(5.6)}{=}\rho_k(\iota_i^j(a)\underset{j}{\cdot}b)\overset{(5.5)}{=}\rho_k\big(\iota_j^k(\iota_i^j(a))\underset{A_k}{\cdot}b\big)\\&=\rho_k\big(\iota_i^k(a)\underset{A_k}{\cdot}b\big)\overset{(5.5)}{=}\rho_k(a\underset{i}{\cdot}b)\overset{(5.6)}{=}a\underset{i}{\cdot}\rho_k(b)=a\underset{i}{\cdot}x.\end{aligned}$$

Next recall that the family of the spaces A_k is dense in its injective limit A (use the left formula of [2, (4.15)] and the fact that I is directed). This means that for any $x\in A$ there is a net $x_k\in\rho_k(A_k)$ tending to x in A. Since for any x_k the equality (5.7) is already proved, we obtain a relation which proves (5.7) for x:

$$\iota_i^j(a)\underset{j}{\cdot}x\xleftarrow[\infty\leftarrow k]{A}\iota_i^j(a)\underset{j}{\cdot}x_k=a\underset{i}{\cdot}x_k\xrightarrow[k\to\infty]{A}a\underset{i}{\cdot}x$$

(we can take limits by the continuity of multiplication in a stereotype module).

Step 3. As A is a left A_i-module, by [2, Theorem 11.2] the formula

$$\varphi_i(a)(x)=a\underset{i}{\cdot}x,\qquad a\in A_i,\ x\in A,$$

defines a homomorphism $\varphi_i:A_i\to\mathcal{L}(A)$ of stereotype algebras. Since we have the identity

$$\varphi_i(a\cdot b)=\varphi_i(a)\circ\varphi_i(b),\qquad a,b\in A_i, \tag{5.8}$$

and the equality

$$\varphi_i(1_{A_i})=\mathsf{id}_A, \tag{5.9}$$

formula (5.6) turns into

$$\varphi_i(a)(\rho_i(b))=a\underset{i}{\cdot}\rho_i(b)=\rho_i(a\underset{A_i}{\cdot}b),\qquad a,b\in A_i, \tag{5.10}$$

and formula (5.7) into

$$\varphi_j(\iota_i^j(a))(x)=\varphi_i(a)(x),\qquad a\in A_i,\ x\in A,$$

which is equivalent to

$$\varphi_j\circ\iota_i^j=\varphi_i,\qquad i\le j. \tag{5.11}$$

The latter means that the following diagram in $\mathtt{Ste}^{\circledast}$ is commutative:

$$\begin{tikzcd} & \mathcal{L}(A) & \\ A_i \arrow[ur, "\varphi_i"] \arrow[rr, "\iota_i^j"] & & A_j \arrow[ul, "\varphi_j"'] \end{tikzcd}$$

One can interpret this as an injective cone of the covariant system $\{A_i;\ \iota_i^j\}$ in $\mathtt{Ste}$. Hence there exists a continuous linear map $\varphi : A = \varinjlim A_i \to \mathcal{L}(A)$ such that for any i the following diagram is commutative:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & \mathcal{L}(A) \\ & A_i \arrow[ul, "\rho_i"] \arrow[ur, "\varphi_i"'] & \end{tikzcd} \tag{5.12}$$

Let

$$x \cdot y = \varphi(x)(y), \qquad x, y \in A; \tag{5.13}$$

we will verify that this multiplication turns A into a stereotype algebra.

STEP 4. The bilinear form $(x,y) \mapsto x \cdot y$ is continuous. Indeed, if K is a compact set in A, then $\varphi(K)$ is compact in $\mathcal{L}(A)$. Hence, $\varphi(K)$ is compact in the space of operators $A : A$. By [2, Theorems 5.1 and 2.5], this means that $\varphi(K)$ is equicontinuous on A. Hence for every zero neighborhood W in A there is a zero neighborhood V in A such that

$$K \cdot V = \varphi(K)(V) \subseteq W.$$

On the other hand, for any compact set K and for any zero neighborhood W in A the set $W \oslash K$ is a neighborhood of zero in $\mathcal{L}(A)$, hence from the continuity of φ it follows that there is a zero neighborhood V in A such that

$$\varphi(V) \subseteq W \oslash K,$$

and this is equivalent to the inclusion

$$V \cdot K = \varphi(V)(K) \subseteq V.$$

STEP 5. The formula

$$1_A = \rho_i(1_{A_i}) \tag{5.14}$$

defines some element of A, so if $i \le j$, then

$$\rho_j(1_{A_j}) = \rho_j(\iota_i^j(1_{A_i})) = \rho_i(1_{A_i}).$$

Furthermore, the chain

$$\varphi(1_A) = \varphi(\rho_i(1_{A_i})) = \varphi_i(1_{A_i}) \overset{(5.9)}{=} \mathsf{id}_A \tag{5.15}$$

implies that this element is the identity for the multiplication (5.13): First, for any $y \in A$,

$$1_A \cdot y = \varphi(1_A)(y) = \mathsf{id}_A(y) = y.$$

Second, for any $x \in A$ we can find a net $a_k \in A_k$ such that

$$\rho_k(a_k) \xrightarrow[k\to\infty]{A} x,$$

and by the continuity of multiplication in A, we have

$$\begin{aligned} x \cdot 1_A \xleftarrow[\infty\leftarrow k]{A} \rho_k(a_k) \cdot 1_A = \rho_k(a_k) \cdot \rho_k(1_{A_k}) &\overset{(5.13)}{=} \varphi(\rho_k(a_k))(\rho_k(1_{A_k})) \\ = \varphi_k(a_k)(\rho_k(1_{A_k})) &\overset{(5.10)}{=} \rho_k(a_k \underset{A_k}{\cdot} 1_{A_k}) = \rho_k(a_k) \xrightarrow[k\to\infty]{A} x, \end{aligned}$$

Thus, $x \cdot 1_A = x$.

Step 6. The map ρ_i in (5.12) is a homomorphism of algebras. Indeed, it maps identity into identity just by the definition of 1_A in (5.14). On the other hand, it preserves multiplication since for all $a, b \in A_i$,

$$\rho_i(a \underset{A_i}{\cdot} b) \overset{(5.10)}{=} \varphi_i(a)(\rho_i(b)) \overset{(5.12)}{=} \varphi(\rho_i(a))(\rho_i(b)) \overset{(5.13)}{=} \rho_i(a) \cdot \rho_i(b). \tag{5.16}$$

Step 7. The same holds for the map φ. Preservation of identities was already stated in (5.14). And to prove multiplicativity we first note that

$$\varphi(\rho_i(a) \cdot \rho_j(b)) = \varphi(\rho_i(a)) \circ \varphi(\rho_j(b)), \qquad i, j \in I,\ a \in A_i,\ b \in A_j. \tag{5.17}$$

Indeed, for $k \in I$ such that $k \geq i$ and $k \geq j$ we have

$$\begin{aligned}\varphi(\rho_i(a) \cdot \rho_j(b)) &\overset{(5.4)}{=} \varphi\big(\rho_k(\iota_i^k(a)) \cdot \rho_k(\iota_j^k(b))\big) \overset{(5.16)}{=} \varphi\big(\rho_k(\iota_i^k(a) \cdot \iota_j^k(b))\big) \\ &\overset{(5.12)}{=} \varphi_k\big(\iota_i^k(a) \cdot \iota_j^k(b)\big) \overset{(5.8)}{=} \varphi_k(\iota_i^k(a)) \circ \varphi_k(\iota_j^k(b)) \\ &\overset{(5.12)}{=} \varphi\big(\rho_k(\iota_i^k(a))\big) \circ \varphi\big(\rho_k(\iota_j^k(b))\big) \overset{(5.4)}{=} \varphi(\rho_i(a)) \circ \varphi(\rho_j(b)).\end{aligned}$$

Then we take $x, y \in A$ and find $a_i \in A_i$ and $b_j \in A_j$ such that

$$\rho_i(a_i) \xrightarrow[i\to\infty]{A} x, \qquad \rho_j(b_j) \xrightarrow[j\to\infty]{A} y.$$

We obtain

$$\begin{aligned}\varphi(x \cdot y) \xleftarrow[\infty\leftarrow i]{\mathcal{L}(A)} \varphi(\rho_i(a_i) \cdot y) \xleftarrow[\infty\leftarrow j]{\mathcal{L}(A)} \varphi\big(\rho_i(a_i) \cdot \rho_j(b_j)\big) \\ \overset{(5.17)}{=} \varphi(\rho_i(a_i)) \circ \varphi(\rho_j(b_j)) \xrightarrow[j\to\infty]{\mathcal{L}(A)} \varphi(\rho_i(a_i)) \circ \varphi(y) \xrightarrow[i\to\infty]{\mathcal{L}(A)} \varphi(x) \circ \varphi(y),\end{aligned}$$

hence

$$\varphi(x \cdot y) = \varphi(x) \circ \varphi(y).$$

This formula proves in addition the associativity of multiplication in A,

$$x \cdot (y \cdot z) = \varphi(x)(y \cdot z) = \varphi(x)(\varphi(y)(z)) = \big(\varphi(x) \circ \varphi(y)\big)(z) = \varphi(x \cdot y)(z) = (x \cdot y) \cdot z,$$

completing the proof that A is a stereotype algebra.

Step 8. We only have to verify that the cone $\{A_i;\ \rho_i\}$ of algebras is an injective limit of the covariant system $\{A_i;\ \iota_i^j\}$ of algebras. Let $\{B_i;\ \sigma_i\}$ be another cone of algebras. Since it is also a cone of stereotype spaces, there exists a unique continuous linear map $\sigma : A \to B$ such that

$$\begin{tikzcd} A \arrow[rr, dashed, "\sigma"] & & B \\ & A_i \arrow[ul, "\rho_i"] \arrow[ur, "\sigma_i"'] & \end{tikzcd} \tag{5.18}$$

We must check that σ is a homomorphism of algebras. Preservation of identities follows from the fact that all σ_i preserve identity:

$$\sigma(1_A) = \sigma(\rho_i(1_{A_i})) = \sigma_i(1_{A_i}) = 1_B.$$

To prove multiplicativity we first note that

$$\sigma(\rho_i(a) \cdot \rho_j(b)) = \sigma(\rho_i(a)) \cdot \sigma(\rho_j(b)), \qquad i, j \in I,\ a \in A_i,\ b \in A_j, \tag{5.19}$$

This can be proved by the same reasoning as (5.17) above: Take $k \in I$ such that $k \geq i$ and $k \geq j$; then

$$\begin{aligned}\sigma(\rho_i(a)\cdot\rho_j(b)) &\overset{(5.4)}{=} \sigma\big(\rho_k(\iota_i^k(a))\cdot\rho_k(\iota_j^k(b))\big) \overset{(5.16)}{=} \sigma\big(\rho_k(\iota_i^k(a)\cdot\iota_j^k(b))\big)\\ &\overset{(5.18)}{=} \sigma_k\big(\iota_i^k(a)\cdot\iota_j^k(b)\big) = \sigma_k(\iota_i^k(a))\cdot\sigma_k(\iota_j^k(b))\\ &\overset{(5.18)}{=} \sigma\big(\rho_k(\iota_i^k(a))\big)\cdot\sigma\big(\rho_k(\iota_j^k(b))\big) \overset{(5.4)}{=} \sigma(\rho_i(a))\cdot\sigma(\rho_j(b)).\end{aligned}$$

Next take $x, y \in A$ and choose $a_i \in A_i$ and $b_j \in A_j$ such that

$$\rho_i(a_i) \xrightarrow[i\to\infty]{A} x, \qquad \rho_j(a_j) \xrightarrow[j\to\infty]{A} y.$$

We obtain

$$\begin{aligned}\sigma(x\cdot y) \xleftarrow[\infty\leftarrow i]{B} \sigma(\rho_i(a_i)\cdot y) \xleftarrow[\infty\leftarrow j]{B} \sigma\big(\rho_i(a_i)\cdot\rho_j(b_j)\big)\\ \overset{(5.19)}{=} \sigma(\rho_i(a_i))\cdot\sigma(\rho_j(b_j)) \xrightarrow[j\to\infty]{B} \sigma(\rho_i(a_i))\cdot\sigma(y) \xrightarrow[i\to\infty]{B} \sigma(x)\cdot\sigma(y),\end{aligned}$$

and thus $\sigma(x\cdot y) = \sigma(x)\cdot\sigma(y)$. ∎

5.2.3. Completeness of $\mathtt{Ste}^{\circledast}$. Theorems 5.16 and 5.20 imply

THEOREM 5.22. *The category $\mathtt{Ste}^{\circledast}$ is complete.*

5.3. Nodal decomposition, envelope and refinement in $\mathtt{Ste}^{\circledast}$

5.3.1. Discerning properties of strong epimorphisms in $\mathtt{Ste}^{\circledast}$.

THEOREM 5.23. *For a morphism $\varepsilon : A \to B$ of stereotype algebras the following conditions are equivalent:*

(i) *ε is an immediate epimorphism in $\mathtt{Ste}^{\circledast}$;*
(ii) *ε is a strong epimorphism in $\mathtt{Ste}^{\circledast}$;*
(iii) *ε is an immediate epimorphism in $\mathtt{Ste}$;*
(iv) *ε is a strong epimorphism in $\mathtt{Ste}$.*

Proof. The implications (i)⇐(ii) and (iii)⇔(iv) are already known. So it is sufficient to prove (i)⇒(iii) and (ii)⇐(iv).

(i)⇒(iii). Let $\varepsilon : A \to B$ be an immediate epimorphism in $\mathtt{Ste}^{\circledast}$. Consider its minimal factorization in $\mathtt{Ste}$, i.e. a diagram with continuous linear maps

$$\begin{tikzcd} A \arrow[rr, "\varepsilon"] \arrow[dr, "\mathsf{coim}_\infty \varepsilon"'] & & B \\ & \mathsf{Coim}_\infty \varepsilon \arrow[ur, "\mu"'] & \end{tikzcd}$$

where $\mathsf{Coim}_\infty \varepsilon$ is the nodal coimage in $\mathtt{Ste}$. Our aim is to show that $\mathsf{Coim}_\infty \varepsilon$ has the structure of a stereotype algebra under which $\mathsf{coim}_\infty \varepsilon$ and μ become morphisms in $\mathtt{Ste}^{\circledast}$—this will mean that the epimorphism $\mathsf{coim}_\infty \varepsilon$ is a mediator for ε in $\mathtt{Ste}^{\circledast}$, and since ε is an immediate epimorphism, μ must be an isomorphism in $\mathtt{Ste}^{\circledast}$, and hence in $\mathtt{Ste}$ as well. This allows us to conclude that the epimorphism ε is isomorphic in $\mathtt{Ste}$ to the epimorphism $\mathsf{Coim}_\infty \varepsilon$, which is an immediate epimorphism in $\mathtt{Ste}$, and thus ε is also an immediate epimorphism in $\mathtt{Ste}$.

The existence of the structure of a stereotype algebra on $\mathsf{Coim}_\infty \varepsilon$ follows from Theorems 5.17 and 5.21: on the one hand, any operation of the form $A' \mapsto (A'/I)^\triangledown$ (where I is a closed two-sided ideal in A') turns each stereotype algebra A' into a stereotype algebra, and on the other hand, the injective limit in $\mathtt{Ste}$ of the system of stereotype algebras that one can form from A in this way is a stereotype algebra. Theorem 5.21 also implies that the natural map of A into this injective limit $\mathsf{Coim}_\infty \varepsilon$ is a morphism of stereotype algebras.

It remains to check that μ is a morphism of stereotype algebras as well, i.e. it is multiplicative and preserves units. The second property follows from the same property for ε and $\mathsf{coim}_\infty \varepsilon(1_A)$:

$$\mu(1_C) = \mu(\mathsf{coim}_\infty \varepsilon(1_A)) = \varepsilon(1_A) = 1_B.$$

The multiplicativity of μ on $\mathsf{coim}_\infty \varepsilon(A)$ follows from the multiplicativity of ε and $\mathsf{coim}_\infty \varepsilon(1_A)$: for any $a, b \in A$ we have

$$\begin{aligned}\mu(\mathsf{coim}_\infty \varepsilon(a) \cdot \mathsf{coim}_\infty \varepsilon(b)) &= \mu(\mathsf{coim}_\infty \varepsilon(a \cdot b)) = \varepsilon(a \cdot b) \\ &= \varepsilon(a) \cdot \varepsilon(b) = \mu(\mathsf{coim}_\infty \varepsilon(a)) \cdot \mu(\mathsf{coim}_\infty \varepsilon(b)).\end{aligned}$$

Now recall that $\mathsf{coim}_\infty \varepsilon$ is an epimorphism in $\mathtt{Ste}$, so the algebra $\mathsf{coim}_\infty \varepsilon(A)$ is dense in $\mathsf{Coim}_\infty \varepsilon$. Hence, by Lemma 5.1, μ is multiplicative on $\mathsf{Coim}_\infty \varepsilon$.

(ii)$\Leftarrow$(iv). Suppose $\varepsilon : A \to B$ is a strong epimorphism in $\mathtt{Ste}$. Consider the following diagram in $\mathtt{Ste}^{\circledast}$:

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\mu"] & D \end{tikzcd}$$

where μ is a monomorphism. It can be considered as a diagram in $\mathtt{Ste}$, and since μ is a monomorphism in $\mathtt{Ste}$ (by Example 5.1.1), and ε a strong epimorphism in $\mathtt{Ste}$, there must exist a morphism δ in $\mathtt{Ste}$ (i.e. a continuous linear map) such that

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] \\ C \arrow[r, "\mu"] & D \end{tikzcd}$$

It remains to check that δ is a homomorphism of algebras. First, δ preserves units, since μ does:

$$\mu(1_C) = 1_D = \beta(\varepsilon(1_A)) = \mu\big(\delta(\varepsilon(1_A))\big) = \mu(\delta(1_B)) \ \Rightarrow \ 1_C = \delta(1_B).$$

For the same reason δ is multiplicative on the subalgebra $\varepsilon(A)$: for all $a, b \in A$,

$$\mu\big(\delta(\varepsilon(a \cdot b))\big) = \beta(\varepsilon(a \cdot b)) = \beta(\varepsilon(a)) \cdot \beta(\varepsilon(b)) = \mu(\delta(\varepsilon(a))) \cdot \mu(\delta(\varepsilon(b))) = \mu\big(\delta(\varepsilon(a)) \cdot \delta(\varepsilon(b))\big)$$

$$\Downarrow$$

$$\delta(\varepsilon(a \cdot b)) = \delta(\varepsilon(a)) \cdot \delta(\varepsilon(b)).$$

The multiplicativity of δ on B follows from Lemma 5.1. ∎

THEOREM 5.24. *If a morphism $\varphi : A \to B$ of stereotype algebras is not a monomorphism, then there exists a decomposition $\varphi = \varphi' \circ \varepsilon$ where ε is a strong epimorphism but not an isomorphism.*

Proof. If φ is not a monomorphism, then $I = \mathsf{Ker}\,\varphi$ is a non-zero closed ideal in A. By Theorem 5.17 the quotient space $(A/I)^{\triangledown}$ is a stereotype algebra. The homomorphism φ can be lifted to some homomorphism $\psi : A/I \to B$ of algebras, which by the definition of the usual quotient topology is a continuous map:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & A/I \arrow[d, dashed, "\psi"] \\ & B \end{tikzcd}$$

Since B is pseudocomplete, ψ can be extended to a continuous map $\varphi' : (A/I)^{\triangledown} \to B$:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & A/I \arrow[r, "\triangledown_{A/I}"] \arrow[d, "\psi"] & (A/I)^{\triangledown} \arrow[dl, dashed, "\varphi'"] \\ & B & \end{tikzcd}$$

By Theorem 5.23, $\upsilon = \triangledown_{A/I} \circ \pi : A \to (A/I)^{\triangledown}$ is a strong epimorphism of stereotype algebras, so we only have to verify that φ' is a homomorphism of algebras. It preserves identities since $1_{(A/I)^{\triangledown}} = 1_{A/I}$:

$$\varphi'(1_{(A/I)^{\triangledown}}) = \psi(1_{A/I}) = 1_B.$$

Multiplicativity follows from Lemma 5.1, since ψ is multiplicative. ■

5.3.2. Discerning properties of strong monomorphisms in $\mathtt{Ste}^{\circledast}$

LEMMA 5.25. *Let A be a stereotype algebra and B a subalgebra in A (in the purely algebraic sense). Then the envelope $\mathsf{Env}^A B$ of B in the stereotype space A is a stereotype algebra.*

Proof. This follows from the completeness of the category $\mathtt{Ste}^{\circledast}$ (Theorem 5.22) and from the fact that the pseudosaturation of the closure $\overline{C}^{\vartriangle}$ of any subalgebra C in A is always a stereotype algebra by Theorem 5.14. ■

LEMMA 5.26. *In $\mathtt{Ste}^{\circledast}$ the immediate monomorphisms coincide with the strong monomorphisms.*

Proof. We already noticed (property 2° on p. 18) that each strong monomorphism is an immediate monomorphism, so we have to verify that in $\mathtt{Ste}^{\circledast}$ the converse is also true. Let $\mu : C \to D$ be an immediate monomorphism of stereotype algebras. Consider a diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\mu"] & D \end{tikzcd}$$

where ε is an epimorphism. Consider the subset $\mu(C) \cup \beta(B)$ in D. Let $\mathsf{alg}(\mu(C) \cup \beta(B))$ be the subalgebra (in the purely algebraic sense) in D generated by $\mu(C) \cup \beta(B)$, and $R = \mathsf{Env}^D(\mathsf{alg}(\mu(C) \cup \beta(B)))$ the envelope in the sense of the definition on p. 130. By

Lemma 5.25, R is a stereotype algebra. Let $\sigma : R \to D$ denote its natural inclusion in D. Since $\mu(C) \subseteq R$, and R is an immediate subspace in D, the morphism μ of stereotype spaces can be factored through σ,

$$\mu = \sigma \circ \pi.$$

Here π is multiplicative, since the identities

$$\sigma(\pi(x \cdot y)) = \mu(x \cdot y) = \mu(x) \cdot \mu(y) = \sigma(\pi(x)) \cdot \sigma(\pi(y)) = \sigma(\pi(x) \cdot \pi(y))$$

imply by monomorphy of σ the identity

$$\pi(x \cdot y) = \pi(x) \cdot \pi(y).$$

So we conclude that π is a morphism of stereotype algebras. Similarly, the inclusion $\beta(B) \subseteq R$ implies that β can be factored through σ,

$$\beta = \sigma \circ \rho,$$

and again the monomorphy of σ implies that ρ is a morphism of stereotype algebras.

So we obtain a diagram in Ste$^{\circledast}$:

$$\begin{tikzcd}
A \arrow[rr, "\varepsilon"] \arrow[dd, "\alpha"'] & & B \arrow[dl, dashed, "\rho"'] \arrow[dd, "\beta"] \\
& R \arrow[dr, dashed, "\sigma"] & \\
C \arrow[ur, dashed, "\pi"] \arrow[rr, "\mu"] & & D
\end{tikzcd}$$

Let us show that π is an epimorphism (in Ste$^{\circledast}$). Let $\zeta, \eta : R \rightrightarrows T$ be two parallel morphisms of stereotype algebras. Then the equality

$$\zeta \circ \pi = \eta \circ \pi$$

implies, on the one hand, the identity

$$\zeta|_{\pi(C)} = \eta|_{\pi(C)},$$

and on the other hand, the chain

$$\zeta \circ \rho \circ \varepsilon = \zeta \circ \pi \circ \alpha = \eta \circ \pi \circ \alpha = \eta \circ \rho \circ \underset{\substack{\Cap \\ \mathsf{Epi}}}{\varepsilon} \;\Rightarrow\; \zeta \circ \rho = \eta \circ \rho \;\Rightarrow\; \zeta|_{\rho(B)} = \eta|_{\rho(B)}.$$

Together they give

$$\zeta|_{\pi(C)\cup\rho(B)} = \eta|_{\pi(C)\cup\rho(B)} \;\Rightarrow\; \zeta|_{\mathsf{alg}(\pi(C)\cup\rho(B))} = \eta|_{\mathsf{alg}(\pi(C)\cup\rho(B))}.$$

Recall that formally R is a subset in B, so $\mathsf{alg}(\pi(C) \cup \rho(B))$ formally coincides with $\mathsf{alg}(\mu(C) \cup \beta(B))$. As a corollary, $\mathsf{alg}(\pi(C) \cup \rho(B)) = \mathsf{alg}(\mu(C) \cup \beta(B))$ is dense in R, and we obtain $\zeta = \eta$.

This proves that π is an epimorphism of stereotype algebras. Thus, μ is a composition of an epimorphism π and a monomorphism σ. Since μ is an immediate monomorphism, π, being a mediator, is an isomorphism. Now we can set $\delta = \pi^{-1} \circ \rho$, and obtain the required diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] \\ C \arrow[r, "\mu"] & D \end{tikzcd}$$

■

THEOREM 5.27. *If a morphism $\varphi : A \to B$ of stereotype algebras is not an epimorphism, then there exists a decomposition $\varphi = \lambda \circ \varphi'$ (in $\mathtt{Ste}^{\circledast}$) where λ is a strong monomorphism but not an isomorphism.*

Proof. Denote

$$P = \mathsf{Env}^B \varphi(A).$$

By Lemma 5.25, P is a stereotype algebra, and the set-theoretic inclusion $\iota : P \to B$ is a monomorphism of stereotype algebras (and an immediate monomorphism of stereotype spaces). Let $\varPhi$ be the class of all factorizations of ι in $\mathtt{Ste}^{\circledast}$,

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\mathsf{Epi}\ni\pi"'] & & B \\ & X \arrow[ur, "\mu\in\mathsf{Mono}"'] & \end{tikzcd} \tag{5.20}$$

where the algebra X as a set lies between P and B:

$$P \subseteq X \subseteq B. \tag{5.21}$$

This class is not empty, since it contains the factorization $\iota = \iota \circ 1$, and it is full in the class of all factorizations (i.e. each factorization of ι is isomorphic to some factorization from $\varPhi$). Every factorization from $\varPhi$ is uniquely determined by the set X in B and a topology on X, i.e. by a subspace X in the topological space B. Since all subspaces of a given topological space form a set, we see that $\varPhi$ is a set (not just a class). For simplicity we can view $\varPhi$ as just a set of subalgebras X in B satisfying (5.21) and endowed with a topology that turns X's into stereotype algebras in such a way that the inclusions (5.21) are continuous maps (this will mean that they are morphisms of stereotype algebras). For any $X \in \varPhi$ the set-theoretic inclusions $P \subseteq X$ and $X \subseteq B$ will be denoted by π_X and μ_X. Thus, diagram (5.20) turns into

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\pi_X"'] & & B \\ & X \arrow[ur, "\mu_X"'] & \end{tikzcd} \tag{5.22}$$

Set

$$Y = \bigcup_{X \in \varPhi} X;$$

then $Q = \mathsf{Env}^B \mathsf{alg} Y$ and $\varkappa$ and λ are the inclusions $P \subseteq Q$ and $Q \subseteq B$ respectively:

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\varkappa"'] & & B \\ & Q \arrow[ur, "\lambda"'] & \end{tikzcd}$$

By Lemma 5.25, Q is a stereotype algebra, and this means that $\varkappa$ and λ are (mono-) morphisms of stereotype algebras. For any $X \in \varPhi$ we denote by σ_X the inclusion $X \subseteq Q$. The topology of X majorizes the topology of Q, hence σ_X is a continuous map, and we obtain a diagram in $\mathtt{Ste}^{\circledast}$:

$$\begin{tikzcd}
P \arrow[rr, "\iota"] \arrow[dr, "\pi_X"'] \arrow[ddr, bend right, "\varkappa"'] & & B \\
& X \arrow[ur, "\mu_X"'] \arrow[d, "\sigma_X"'] & \\
& Q \arrow[uur, bend right, "\lambda"'] &
\end{tikzcd} \tag{5.23}$$

Let us now show that $\varkappa$ is (not only a monomorphism, but also) an epimorphism of stereotype algebras. Indeed, for any two morphisms $\zeta, \eta : Q \rightrightarrows T$ we have

$$\zeta \circ \varkappa = \eta \circ \varkappa \ \Rightarrow \ \forall X \in \varPhi \ \ \zeta \circ \sigma_X \circ \pi_X = \eta \circ \sigma_X \circ \underset{\mathsf{Epi}}{\underset{\Cap}{\pi_X}} \ \Rightarrow \ \forall X \in \varPhi \ \ \zeta \circ \sigma_X = \eta \circ \sigma_X$$
$$\Rightarrow \ \forall X \in \varPhi \ \ \zeta|_X = \eta|_X \ \Rightarrow \ \zeta|_Y = \zeta|_{\bigcup_{X \in \varPhi} X} = \eta|_{\bigcup_{X \in \varPhi} X} = \eta|_Y$$
$$\Rightarrow \ \zeta|_{\mathsf{alg}\, Y} = \eta|_{\mathsf{alg}\, Y} \ \Rightarrow \ \zeta = \zeta|_Q = \eta|_Q = \eta$$

(the last implication follows from the fact that $\mathsf{alg}\, Y$ is dense in its envelope).

Let us show that $\lambda : Q \to B$ is an immediate monomorphism (in $\mathtt{Ste}^{\circledast}$). Suppose $\lambda = \lambda' \circ \varepsilon$. Denote by R the range of ε (and the domain of λ'); then we have

$$\begin{tikzcd}
P \arrow[rr, "\iota"] \arrow[dr, "\varkappa"'] \arrow[ddr, bend right, "\varepsilon\circ\varkappa"'] & & B \\
& Q \arrow[ur, "\lambda"'] \arrow[d, "\varepsilon"'] & \\
& R \arrow[uur, bend right, "\lambda'"'] &
\end{tikzcd} \tag{5.24}$$

The morphism $\varepsilon \circ \varkappa$ is an epimorphism (as a composition of two epimorphisms), so $\iota = \lambda' \circ (\varepsilon \circ \varkappa)$ is a factorization of ι. As a corollary, it is isomorphic to some standard factorization $\iota = \mu_X \circ \pi_X$ for some $X \in \varPhi$:

$$\begin{tikzcd}
P \arrow[rr, "\iota"] \arrow[dr, "\varkappa"'] \arrow[ddr, bend right, "\varepsilon\circ\varkappa"'] \arrow[dddr, bend right=60, "\pi_X"'] & & B \\
& Q \arrow[ur, "\lambda"'] \arrow[d, "\varepsilon"'] & \\
& R \arrow[uur, bend right, "\lambda'"'] \arrow[d, dashed] & \\
& X \arrow[uuur, bend right=60, "\mu_X"'] &
\end{tikzcd}$$

(the dashed arrow is some isomorphism of stereotype algebras). So from the very beginning we can think that in (5.24) some $X \in \varPhi$ stands instead of R:

$$\begin{tikzcd}
P \arrow[rr, "\iota"] \arrow[dr, "\varkappa"] \arrow[ddr, bend right, "\pi_X"'] & & B \\
& Q \arrow[ur, "\lambda"'] \arrow[d, "\varepsilon"'] & \\
& X \arrow[uur, bend right, "\mu_X"'] &
\end{tikzcd}$$

Here every arrow is a set-theoretic inclusion, and the topology of the source of the arrow majorizes the topology of its target. In particular, the arrow ε means that Q is a subset of X, and the topology of Q majorizes that of X. But on the other hand, the arrow σ_X in diagram (5.23) means that X is a subset in Q, and the topology of X majorizes that of Q. Together this means that X and Q coincide together with their topologies. In particular, ε is an isomorphism, as desired.

Since λ is an immediate monomorphism, by Lemma 5.26 it is a strong monomorphism.

Note that since $\varphi(A) \subseteq P$, the morphism φ factors through P:

$$\varphi = \iota \circ \theta$$

for some morphism $\theta : A \to P$. We obtain a diagram in $\mathtt{Ste}^{\circledast}$:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\theta"'] & B \\
P \arrow[r, "\varkappa"] \arrow[ur, "\iota"] & Q \arrow[u, "\lambda"']
\end{tikzcd}$$

We now see that λ cannot be an isomorphism, since otherwise φ would be an epimorphism, as a composition of two epimorphisms θ and $\varkappa$, and an isomorphism λ. So if we set $\varphi' = \varkappa \circ \theta$, we obtain a decomposition $\varphi = \lambda \circ \varphi'$ where λ is a strong monomorphism but not an isomorphism. ■

5.3.3. Nodal decomposition in $\mathtt{Ste}^{\circledast}$. We record the following two properties of the category $\mathtt{Ste}^{\circledast}$.

THEOREM 5.28. *The category* $\mathtt{Ste}^{\circledast}$ *is well-powered.*

Proof. A morphism $\mu : A \to B$ in $\mathtt{Ste}^{\circledast}$ is a monomorphism in $\mathtt{Ste}^{\circledast}$ iff it is a monomorphism in $\mathtt{Ste}$, and the latter category is well-powered. ■

THEOREM 5.29. *The category* $\mathtt{Ste}^{\circledast}$ *is co-well-powered in strong epimorphisms.*

Proof. By Theorem 5.23, a morphism $\varepsilon : A \to B$ in $\mathtt{Ste}^{\circledast}$ is a strong epimorphism in $\mathtt{Ste}^{\circledast}$ iff it is a strong epimorphism in $\mathtt{Ste}$, and the latter category is co-well-powered. ■

On the other hand, $\mathtt{Ste}^{\circledast}$ is complete (by Theorem 5.22), and in $\mathtt{Ste}^{\circledast}$ strong epimorphisms discern monomorphisms, and strong monomorphisms discern epimorphisms (Theorems 5.24 and 5.27). Thus, we can apply Theorem 2.36 to get

THEOREM 5.30. *In* $\mathtt{Ste}^{\circledast}$ *each morphism* $\varphi : X \to Y$ *has a nodal decomposition* (2.24).

REMARK 5.31. Theorem 5.23 implies in addition that the nodal coimage $\mathsf{Coim}_\infty \varphi$ in $\mathtt{Ste}^{\circledast}$ coincides with the nodal coimage in $\mathtt{Ste}$, and as a corollary with the refinement (as a quotient space of a stereotype space) of $\varphi^\star(Y^\star)$ on X:

$$\mathsf{Coim}_\infty \varphi = \mathsf{Ref}^X \varphi^\star(Y^\star). \tag{5.25}$$

For the nodal image $\mathsf{Im}_\infty \varphi$ the analogous proposition is not true.

THEOREM 5.32. *For each morphism $\varphi : A \to B$ in $\mathtt{Ste}^{\circledast}$ its nodal decomposition $\varphi = \mathsf{im}_\infty \varphi \circ \mathsf{red}_\infty \varphi \circ \mathsf{coim}_\infty \varphi$ in $\mathtt{Ste}$ is a decomposition (not necessarily nodal) in $\mathtt{Ste}^{\circledast}$.*

Proof. We need to verify that the stereotype spaces $\mathsf{Coim}_\infty \varphi$ and $\mathsf{Im}_\infty \varphi$ have natural structures of stereotype algebras, and the morphisms of stereotype spaces $\mathsf{coim}_\infty \varphi : A \to \mathsf{Coim}_\infty \varphi$, $\mathsf{red}_\infty \varphi : \mathsf{Coim}_\infty \varphi \to \mathsf{Im}_\infty \varphi$, $\mathsf{im}_\infty \varphi : \mathsf{Im}_\infty \varphi \to B$ are morphisms of stereotype algebras (i.e., homomorphisms of algebras). This follows from the construction of $\mathsf{Coim}_\infty \varphi$ and $\mathsf{Im}_\infty \varphi$: since $\varphi : A \to B$ is a morphism of stereotype algebras, so is $\varphi^1 = \mathsf{red}\,\varphi : \mathsf{Coim}\,\varphi \to \mathsf{Im}\,\varphi$ (together with $\mathsf{coim}\,\varphi : A \to \mathsf{Coim}\,\varphi$ and $\mathsf{im}\,\varphi : \mathsf{Im}\,\varphi \to B$). For the same reason, $\varphi^2 = \mathsf{red}\,\varphi^1$ is a morphism of stereotype algebras, and so on. By transfinite induction, $\mathsf{Coim}_\infty \varphi$ is a stereotype algebra (as injective limit of the stereotype algebras $\mathsf{Coim}\,\varphi^i$), $\mathsf{Im}_\infty \varphi$ is a stereotype algebra (as the projective limit of the stereotype algebras $\mathsf{Im}\,\varphi^i$), and the morphisms $\mathsf{coim}_\infty \varphi : A \to \mathsf{Coim}_\infty \varphi$, $\mathsf{red}_\infty \varphi : \mathsf{Coim}_\infty \varphi \to \mathsf{Im}_\infty \varphi$, $\mathsf{im}_\infty \varphi : \mathsf{Im}_\infty \varphi \to B$ are homomorphisms of algebras. ∎

5.3.4. Envelopes and refinements in $\mathtt{Ste}^{\circledast}$. Since it is not clear whether the category $\mathtt{Ste}^{\circledast}$ is co-well-powered in the class Epi, in the analogue of Theorem 4.106 for the case of envelopes in Epi one should claim that the class of test morphisms $\varPhi$ is a set (so that in the proof property 5° on p. 61 could be replaced by 3°):

THEOREM 5.33. *In $\mathtt{Ste}^{\circledast}$:*

(a) *Each algebra A has an envelope in the class Epi of all epimorphisms (respectively, in the class SEpi of all strong epimorphisms) of $\mathtt{Ste}^{\circledast}$ with respect to an arbitrary* set *(respectively,* class*) $\varPhi$ of morphisms going from A; in addition,*

 (i) *if $\varPhi$ separates morphisms on the outside in $\mathtt{Ste}^{\circledast}$, then*

$$\mathsf{env}^{\mathsf{Epi}}_{\varPhi} A = \mathsf{env}^{\mathsf{Bim}}_{\varPhi} A;$$

 (ii) *if $\varPhi$ separates morphisms on the outside and is a right ideal in $\mathtt{Ste}^{\circledast}$, then for any class $\varOmega \supseteq \mathsf{Bim}$,*

$$\mathsf{env}^{\mathsf{Epi}}_{\varPhi} A = \mathsf{env}^{\mathsf{Bim}}_{\varPhi} A = \mathsf{env}^{\varOmega}_{\varPhi} A = \mathsf{env}_{\varPhi} A.$$

(b) *Each algebra A has a refinement in the class Mono of all monomorphisms (respectively, the class SMono of all strong monomorphisms) in $\mathtt{Ste}^{\circledast}$ by means of an arbitrary class $\varPhi$ of morphisms going to A; in addition,*

 (i) *if $\varPhi$ separates morphisms on the inside in $\mathtt{Ste}^{\circledast}$, then*

$$\mathsf{ref}^{\mathsf{Mono}}_{\varPhi} A = \mathsf{ref}^{\mathsf{Bim}}_{\varPhi} A;$$

 (ii) *if $\varPhi$ separates morphisms on the inside and is a left ideal in $\mathtt{Ste}^{\circledast}$, then for any* *class* $\varGamma \supseteq \mathsf{Bim}$,

$$\mathsf{ref}_{\varPhi}^{\mathsf{Mono}} A = \mathsf{ref}_{\varPhi}^{\mathsf{Bim}} A = \mathsf{ref}_{\varPhi}^{\varGamma} A = \mathsf{ref}_{\varPhi} A.$$

Proof. Consider the case of envelopes. If $\varPhi$ is a set, then the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Epi}(\mathtt{Ste}^{\circledast})} A$ follows from 3° on p. 61. If $\varPhi$ separates morphisms on the outside, then by Theorem 3.6 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Epi}} A$ implies the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Epi}\cap\mathsf{Mono}} A = \mathsf{env}_{\varPhi}^{\mathsf{Bim}} A$, and $\mathsf{env}_{\varPhi}^{\mathsf{Epi}} A = \mathsf{env}_{\varPhi}^{\mathsf{Bim}} A$. If $\varPhi$ separates morphisms on the outside and is a right ideal, then by Theorem 3.7 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{Bim}} A$ implies the existence of $\mathsf{env}_{\varPhi}^{\varOmega} A$ for any $\varOmega \supseteq \mathsf{Bim}$, and $\mathsf{env}_{\varPhi}^{\mathsf{Bim}} A = \mathsf{env}_{\varPhi}^{\varOmega} A$. ∎

From Theorems 3.42 and 3.48 (with $\varOmega = \mathsf{Epi}$) we have

THEOREM 5.34. *Let $\varPhi$ be a class of morphisms in $\mathtt{Ste}^{\circledast}$ which goes from $\mathtt{Ste}^{\circledast}$ and is a right ideal. Then the classes of morphisms Epi and $\varPhi$ define in $\mathtt{Ste}^{\circledast}$ a semiregular envelope $\mathsf{Env}_{\varPhi}^{\mathsf{Epi}}$, which for each object A in $\mathtt{Ste}^{\circledast}$ is described by the formula*

$$\mathsf{red}_{\infty} \varprojlim \mathcal{N}^A \circ \mathsf{coim}_{\infty} \varprojlim \mathcal{N}^A = \mathsf{env}_{\varPhi}^{\mathsf{Epi}} A \tag{5.26}$$

where $\mathcal{N}$ is the net of epimorphisms generated by Epi and $\varPhi$, and $\mathsf{red}_{\infty} \varprojlim \mathcal{N}^A$ and $\mathsf{coim}_{\infty} \varprojlim \mathcal{N}^A$ are elements of the nodal decomposition (2.24) *of $\varprojlim \mathcal{N}^A : A \to A_{\mathcal{N}}$ in $\mathtt{Ste}^{\circledast}$. If in addition Epi pushes $\varPhi$, then $\mathsf{Env}_{\varPhi}^{\mathsf{Epi}}$ is regular* (*and thus it can be defined as an idempotent functor*).

5.3.5. Dense epimorphisms

- Let us say that a morphism $\varphi : A \to B$ of stereotype (or, in general, topological) algebras is *dense* if $\varphi(A)$ is dense in B. Clearly, dense morphisms are epimorphisms, so we also call them *dense epimorphisms*. The class of all dense epimorphisms in $\mathtt{Ste}^{\circledast}$ (or in $\mathtt{TopAlg}$) will be denoted by DEpi. It is related to the classes Epi and SEpi by the inclusions

$$\mathsf{SEpi} \subset \mathsf{DEpi} \subset \mathsf{Epi}. \tag{5.27}$$

REMARK 5.35. The inclusions (5.27) are not equalities. An example of a dense epimorphism which is not strong is the set-theoretic inclusion of the algebra $\mathcal{C}^{\infty}(M)$ of smooth functions into the algebra $\mathcal{C}(M)$ of continuous functions on a smooth manifold M (this inclusion is a bimorphism of stereotype algebras, so if it were a strong epimorphism, this would automatically mean that it is an isomorphism, which is not true). An example of a non-dense epimorphism is the standard inclusion of the algebra $\mathcal{P}(\mathbb{C})$ of polynomials on $\mathbb{C}$ into the algebra $\mathcal{P}(\mathbb{C}^{\times})$ of Laurent polynomials on $\mathbb{C}^{\times}$ (we already mentioned this example on p. 147).

THEOREM 5.36. *The class DEpi is monomorphically complementable in $\mathtt{Ste}^{\circledast}$.*

Proof. The monomorphic complement for DEpi is the class $\mathsf{SMono}_{\mathtt{Ste}}$ of strong monomorphisms in $\mathtt{Ste}^{\circledast}$ which are strong monomorphisms in $\mathtt{Ste}$.

$$\mathsf{SMono}_{\mathtt{Ste}} \circledcirc \mathsf{DEpi} = \mathtt{Ste}^{\circledast}. \ \blacksquare \tag{5.28}$$

For dense epimorphisms the first part of Theorem 5.33 can be strengthened as follows:

THEOREM 5.37. *In* $\mathtt{Ste}^{\circledast}$ *every algebra* A *has an envelope in* DEpi *with respect to an arbitrary* class $\varPhi$ *of morphisms going from* A. *If in addition* $\varPhi$ *separates morphisms on the outside in* $\mathtt{Ste}^{\circledast}$, *then the envelope in* DEpi *is also an envelope in the class* DBim *of all dense bimorphisms:*

$$\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A = \mathsf{env}_{\varPhi}^{\mathsf{DBim}} A.$$

Proof. The existence of $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}(\mathtt{Ste}^{\circledast})} A$ follows from 5° on p. 61. If $\varPhi$ separates morphisms on the outside, then by Theorem 3.6 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A$ implies the existence of $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}\cap\mathsf{Mono}} A = \mathsf{env}_{\varPhi}^{\mathsf{DBim}} A$, and $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A = \mathsf{env}_{\varPhi}^{\mathsf{DBim}} A$. If in addition $\varPhi$ is a right ideal, then by Theorem 3.7 the existence of $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A$ implies the existence of $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}\cap\mathsf{Mono}} A = \mathsf{env}_{\varPhi}^{\mathsf{DBim}} A$, and $\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A = \mathsf{env}_{\varPhi}^{\mathsf{DBim}} A$. ∎

From Theorems 3.42 and 3.48 (with $\varOmega = \mathsf{DEpi}$) we deduce:

THEOREM 5.38. *Let* $\varPhi$ *be a class of morphisms in* $\mathtt{Ste}^{\circledast}$ *which goes from* $\mathtt{Ste}^{\circledast}$ *and is a right ideal. Then* DEpi *and* $\varPhi$ *define a semiregular envelope* $\mathsf{Env}_{\varPhi}^{\mathsf{DEpi}}$ *in* $\mathtt{Ste}^{\circledast}$, *which for any object* A *in* $\mathtt{Ste}^{\circledast}$ *is described by the formula*

$$\mathsf{env}_{\varPhi}^{\mathsf{DEpi}} A = \mathsf{red}_\infty \varprojlim \mathcal{N}^A \circ \mathsf{coim}_\infty \varprojlim \mathcal{N}^A, \tag{5.29}$$

where $\mathcal{N}$ *is the net of epimorphisms generated by* DEpi *and* $\varPhi$, *and* $\mathsf{red}_\infty \varprojlim \mathcal{N}^A$ *and* $\mathsf{coim}_\infty \varprojlim \mathcal{N}^A$ *are elements of the nodal decomposition* (2.24) *of the morphism* $\varprojlim \mathcal{N}^A : A \to A_{\mathcal{N}}$ *in the category* $\mathtt{Ste}$ *of stereotype spaces* (*not algebras!*). *If in addition* DEpi *pushes* $\varPhi$, *then the envelope* $\mathsf{Env}_{\varPhi}^{\mathsf{DEpi}}$ *is regular* (*and thus it can be defined as an idempotent functor*).

5.4. Holomorphic envelope. A. Ya. Helemskii introduced in [16] the notion of the Arens–Michael envelope in the category of topological algebras. The properties of this construction used in the duality theory for complex Lie groups [3] have different formal interpretations (while preserving the essential results) than an envelope in the sense of the definition of Chapter 3 in the category of stereotype algebras. For one of them, which seems to be most natural, we use the (working) name *holomorphic envelope*. The choice of the term is meant to emphasize the connection to complex analysis and the analogy with the continuous envelope (which we define below on p. 179) and the smooth envelope from [5].

5.4.1. Net of Banach quotient maps and the stereotype Arens–Michael envelope. Here we define the analogue of the Arens–Michael envelope in the category $\mathtt{Ste}^{\circledast}$ of stereotype algebras. All the definitions and results can be easily transferred to the category $\mathtt{TopAlg}$ of topological algebras.

Recall that an absolutely convex closed neighborhood U of zero in a topological algebra A is said to be *submultiplicative* if $U \cdot U \subseteq U$. The set of all submultiplicative absolutely convex closed neighbourhoods of zero in A is denoted by $\mathcal{SU}(A)$. To any such neighborhood U in A one can assign a two-sided closed ideal $\mathsf{Ker}\, U = \bigcap_{\varepsilon>0} \varepsilon \cdot U$ in A and a quotient algebra $A/\mathsf{Ker}\, U$ endowed with (not the quotient topology as one could expect, but) the topology of a normed space with the unit ball $U + \mathsf{Ker}\, U$. Then the completion

$(A/\mathsf{Ker}\,U)^{\blacktriangledown}$ is a Banach algebra, and we denote it by A/U and call the *quotient algebra* of A *by the zero neighborhood* U. The natural map from A into A/U given by

$$\begin{tikzcd} A \arrow[r, "\tau_U"] \arrow[rr, bend left, "\rho_U"] & A/\mathsf{Ker}\,U \arrow[r, "\blacktriangledown_{A/\mathsf{Ker}\,U}"] & (A/\mathsf{Ker}\,U)^{\blacktriangledown} = A/U \end{tikzcd}$$

(where τ_U is a quotient map, and $\blacktriangledown_{A/\mathsf{Ker}\,U}$ is the completion map) will be called the *Banach quotient map* of A *by the zero neighborhood* U.

Denote by $\mathcal{B}$ the class $\{\rho_U : A \to A/U\}$ of all Banach quotient maps, where A runs over the class of topological algebras, and U the set of all submultiplicative neighborhoods of zero in A.

PROPOSITION 5.39. *The class $\mathcal{B}$ is a net of epimorphisms in $\mathtt{Ste}^{\circledast}$, and the relation $\to$ of pre-order* ([1]) *is equivalent to the embedding of the corresponding neighborhoods of zero up to a positive scalar multiple:*

$$\rho_V \to \rho_U \iff \exists \varepsilon > 0 \;\; \varepsilon \cdot V \subseteq U. \tag{5.30}$$

Proof. Let us first verify (5.30). Suppose U and V are submultiplicative closed absolutely convex neighborhoods of zero in A, and $\varepsilon \cdot V \subseteq U$ for some $\varepsilon > 0$. Then $\mathsf{Ker}\,V \subseteq \mathsf{Ker}\,U$, and the formula

$$x + \mathsf{Ker}\,V \mapsto x + \mathsf{Ker}\,U$$

defines a continuous linear map $A/\mathsf{Ker}\,V \to A/\mathsf{Ker}\,U$ which can be extended by continuity to an operator

$$\pi_V^U : A/V = (A/\mathsf{Ker}\,V)^{\triangledown} \to (A/\mathsf{Ker}\,U)^{\triangledown} = A/U.$$

Obviously, the following diagram is commutative:

$$\begin{tikzcd} & A \arrow[dl, bend right, "\rho_V"'] \arrow[dr, bend left, "\rho_U"] & \\ A/V \arrow[rr, "\pi_V^U"] & & A/U \end{tikzcd} \tag{5.31}$$

In particular, $\rho_V \to \rho_U$. Conversely, if for some morphism $\iota : A/V \to A/U$ we have a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, bend right, "\rho_V"'] \arrow[dr, bend left, "\rho_U"] & \\ A/V \arrow[rr, "\iota"'] & & A/U \end{tikzcd} \tag{5.32}$$

then we can set $\widetilde{U} = \overline{\rho_U(U)}$ and $\widetilde{V} = \overline{\rho_V(V)}$, and these will be balls centered at the zeroes in A/U and A/V respectively, so the continuity of $\iota : A/V \to A/U$ implies that

$$\varepsilon \cdot \widetilde{V} \subseteq \iota^{-1}(\widetilde{U})$$

([1]) The pre-order $\to$ on the class Epi^X of all epimorphisms going from a given object X of a category $\mathtt{K}$ was defined on p. 25.

for some $\varepsilon > 0$. Therefore,

$$\varepsilon \cdot V = (\rho_V)^{-1}(\varepsilon \cdot \widetilde{V}) \subseteq (\rho_V)^{-1}(\iota^{-1}(\widetilde{U})) = (\rho_U)^{-1}(\widetilde{U}) = U.$$

Let us now check axiom (a) of the net of epimorphisms from p. 70. For each topological algebra A the set $\mathcal{B}_A$ of its Banach quotient maps is non-empty, since there always exists at least one submultiplicative zero neighborhood U in A, namely $U = A$ (and the corresponding quotient map is zero, $\rho_U : A \to 0$). Furthermore, if U and V are two submultiplicative closed absolutely convex neighborhoods of zero in A, then clearly $U \cap V$ is also a submultiplicative (and closed absolutely convex) neighborhood of zero in A. That is, the submultiplicative absolutely convex neighborhoods of zero form a system directed to the contraction in A. Together with the rule (5.30) this means that the system $\{\rho_U : A \to A/U\}$ of epimorphisms is directed to the left with respect to the pre-order $\to$.

Next we check axiom (b). For each topological algebra A the system $\mathsf{Bind}(\mathcal{B}_A)$ of connecting morphisms has a projective limit, since the category $\mathtt{Ste}^{\circledast}$ is complete. This limit can be defined as a map $A \mapsto \varprojlim \mathsf{Bind}(\mathcal{B}_A)$, since it is directly constructed as a set in the product of the algebras A/U.

It remains to check axiom (c). Let $\alpha : A \to B$ be a morphism of topological algebras and $\rho_V : B \to B/V$ a Banach quotient map. The set $U = \alpha^{-1}(V)$ is a submultiplicative closed absolutely convex neighborhood of zero in A. The map

$$x + \mathsf{Ker}\, U \mapsto \alpha(x) + \mathsf{Ker}\, V$$

extends by continuity to a map $\alpha_U^V : A/U \to B/V$ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\rho_U"'] & Y \arrow[d, "\rho_V"] \\ A/U \arrow[r, dashed, "\alpha_\sigma^\tau"] & B/V \end{tikzcd}$$

■

- The net $\mathcal{B}$ will be called the *net of Banach quotient maps.*
- For each algebra A diagram (5.31) means that the family of quotient maps $\rho_U : A \to A/U$ is a projective cone of the contravariant system $\mathsf{Bind}(\mathcal{B}_A) = \{\pi_V^U\}$. The projective limit of this cone in the category $\mathtt{Ste}^{\circledast}$ of stereotype algebras is called the *stereotype Arens–Michael envelope* of the algebra A and is denoted by

$$\varprojlim \mathcal{B}_A : A \to A_{\mathcal{B}} \tag{5.33}$$

(this limit exists since $\mathtt{Ste}^{\circledast}$ is projectively complete). The range of this morphism,

$$A_{\mathcal{B}} = \mathsf{Ran} \varprojlim \mathcal{B}_A = \mathtt{Ste}^{\circledast}\text{-} \varprojlim_{U \in \mathcal{SU}(A)} A/U = \Big(\mathtt{TopAlg}\text{-} \varprojlim_{U \in \mathcal{SU}(A)} A/U\Big)^{\vartriangle}, \tag{5.34}$$

will also be called the *stereotype Arens–Michael envelope* of A.

If U and V are submultiplicative neighborhoods of zero such that $\varepsilon \cdot V \subseteq U$ for some $\varepsilon > 0$, then we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[d, dashed, "\varprojlim \mathcal{B}_A"] \arrow[ldd, bend right, "\rho_V"'] \arrow[rdd, bend left, "\rho_U"] & \\
 & A_{\mathcal{B}} \arrow[ld, "\pi_V"'] \arrow[rd, "\pi_U"] & \\
A/V \arrow[rr, "\pi_V^U"] & & A/U
\end{tikzcd}$$

Theorem 3.36 implies

THEOREM 5.40. *The Arens–Michael envelope is an envelope in the class of all morphisms in* $\mathtt{Ste}^{\circledast}$ *with respect to the system* $\mathcal{B}$ *of Banach quotient maps,*

$$A_{\mathcal{B}} = \mathsf{Env}_{\mathcal{B}}^{\mathsf{Mor}(\mathtt{Ste}^{\circledast})} A, \tag{5.35}$$

and to each morphism $\varphi : A \to B$ *in* $\mathtt{Ste}^{\circledast}$ *the formula*

$$\varphi_{\mathcal{B}} = \varprojlim_{\tau \in \mathcal{B}_B} \varprojlim_{\sigma \in \mathcal{B}_A} \varphi_\sigma^\tau \circ \sigma_{\mathcal{B}} \tag{5.36}$$

assigns a morphism $\varphi_{\mathcal{B}} : A_{\mathcal{B}} \to B_{\mathcal{B}}$ *such that*

$$\begin{tikzcd}
A \arrow[r, "\varprojlim \mathcal{B}_A"] \arrow[d, "\varphi"'] & A_{\mathcal{B}} \arrow[d, dashed, "\varphi_{\mathcal{B}}"] \\
B \arrow[r, "\varprojlim \mathcal{B}_B"] & B_{\mathcal{B}}
\end{tikzcd} \tag{5.37}$$

and the map $(A, \varphi) \mapsto (A_{\mathcal{B}}, \varphi_{\mathcal{B}})$ *can be defined as a functor from* $\mathtt{Ste}^{\circledast}$ *into* $\mathtt{Ste}^{\circledast}$.

5.4.2. Holomorphic envelope of a stereotype algebra. Recall that on p. 166 we defined dense epimorphisms $\varphi : A \to B$ of topological algebras.

- By the *holomorphic envelope* of a stereotype algebra A we mean its envelope in the class DEpi of dense epimorphisms of the category $\mathtt{Ste}^{\circledast}$ with respect to the class $\mathtt{BanAlg}$ of Banach algebras. We use the following notation for this construction:

$$A^{\heartsuit} = \mathsf{Env}_{\mathtt{BanAlg}}^{\mathsf{DEpi}} A, \qquad \heartsuit_A = \mathsf{env}_{\mathtt{BanAlg}}^{\mathsf{DEpi}} A. \tag{5.38}$$

  Thus,

$$(\heartsuit_A : A \to A^{\heartsuit}) = (\mathsf{env}_{\mathtt{BanAlg}}^{\mathsf{DEpi}} A : A \to \mathsf{Env}_{\mathtt{BanAlg}}^{\mathsf{DEpi}} A).$$

PROPERTIES OF HOLOMORPHIC ENVELOPES.

1° *Each stereotype algebra* A *has a holomorphic envelope* $A^{\heartsuit}$.

2° *The holomorphic envelope* $A^{\heartsuit}$ *is connected with the stereotype Arens–Michael envelope* $A_{\mathcal{B}}$ *through the formulas*

$$\heartsuit_A = \mathsf{red}_\infty \varprojlim \mathcal{B}_A \circ \mathsf{coim}_\infty \varprojlim \mathcal{B}_A, \qquad A^{\heartsuit} = \mathsf{Dom}\, \mathsf{im}_\infty \varprojlim \mathcal{B}_A \tag{5.39}$$

where $\mathsf{coim}_\infty \varprojlim \mathcal{B}_A$, $\mathsf{red}_\infty \varprojlim \mathcal{B}_A$, $\mathsf{im}_\infty \varprojlim \mathcal{B}_A$ *are elements of the nodal decomposition of the morphism* $\varprojlim \mathcal{B}_A$ *in the category* $\mathtt{Ste}$ *of stereotype spaces (not algebras!).*

3° *For any morphism $\varphi : A \to B$ of stereotype algebras and for each choice of holomorphic envelopes $\heartsuit_A : A \to A^\heartsuit$ and $\heartsuit_B : B \to B^\heartsuit$ there exists a unique morphism $\varphi^\heartsuit : A^\heartsuit \to B^\heartsuit$ such that*

$$\begin{tikzcd} A \arrow[r, "\heartsuit_A"] \arrow[d, "\varphi"'] & A^\heartsuit \arrow[d, dashed, "\varphi^\heartsuit"] \\ B \arrow[r, "\heartsuit_B"] & B^\heartsuit \end{tikzcd} \tag{5.40}$$

4° *The correspondence $(X, \alpha) \mapsto (X^\heartsuit, \alpha^\heartsuit)$ can be defined as a covariant functor from $\mathtt{Ste}^\circledast$ into $\mathtt{Ste}^\circledast$:*

$$(1_A)^\heartsuit = 1_{A^\heartsuit}, \quad (\beta \circ \alpha)^\heartsuit = \beta^\heartsuit \circ \alpha^\heartsuit, \quad (\alpha^\heartsuit)^\heartsuit = \alpha^\heartsuit. \tag{5.41}$$

5° *If an algebra A is dense in its stereotype Arens–Michael envelope $A_{\mathcal{B}}$, i.e.*

$$\varprojlim \mathcal{B}_A \in \mathsf{DEpi}(\mathtt{Ste}^\circledast),$$

then the holomorphic envelope of A coincides with its envelope in the class Epi of all epimorphisms in $\mathtt{Ste}^\circledast$ and with the stereotype Arens–Michael envelope:

$$A^\heartsuit = \mathsf{Env}^{\mathsf{DEpi}}_{\mathtt{BanAlg}} A = \mathsf{Env}^{\mathsf{Epi}}_{\mathtt{BanAlg}} A = A_{\mathcal{B}}. \tag{5.42}$$

6° *The holomorphic envelope is coherent with the projective tensor product $\circledast$ in $\mathtt{Ste}^\circledast$.*

In the proof we shall need the following

LEMMA 5.41. *In $\mathtt{Ste}^\circledast$ the net $\mathcal{B}$ of Banach quotient maps consists of dense epimorphisms and generates on the inside the class $\mathsf{Mor}(\mathtt{Ste}^\circledast, \mathtt{BanAlg})$ of morphisms with values in Banach algebras:*

$$\mathcal{B} \subseteq \mathsf{Mor}(\mathtt{Ste}^\circledast, \mathtt{BanAlg}) \subseteq \mathsf{Mor}(\mathtt{Ste}^\circledast) \circ \mathcal{B}. \tag{5.43}$$

Proof. The class $\mathcal{B}$ consists of dense epimorphisms, since the image $\rho_U(A)$ of any algebra A is always dense in its Banach quotient algebra $A/U = (A/\mathsf{Ker}\, U)^\blacktriangledown$. Let us show that $\mathcal{B}$ generates the class of morphisms with values in Banach algebras. We have to verify the second embedding in the chain (3.13). Let $\varphi : A \to B$ be a morphism into a Banach algebra B. If V is the unit ball in B, then $U = \varphi^{-1}(V)$ is a neighborhood of zero in A, and the condition $V \cdot V \subseteq V$ implies $U \cdot U \subseteq U$:

$$x, y \in U \;\Rightarrow\; \varphi(x), \varphi(y) \in V \;\Rightarrow\; \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \in V \;\Rightarrow\; x \cdot y \in U = \varphi^{-1}(V).$$

Consider the normed algebra $A/\mathsf{Ker}\, U$ and the quotient map $\tau_U : A \to A/\mathsf{Ker}\, U$. From the obvious equality $\mathsf{Ker}\, \varphi = \mathsf{Ker}\, U$ it follows that φ can be decomposed in the category $\mathtt{Alg}$ of algebras as follows:

$$\begin{tikzcd} A \arrow[r, "\tau_U"] \arrow[dr, "\varphi"'] & A/\mathsf{Ker}\, U \arrow[d, dashed, "\chi"] \\ & B \end{tikzcd}$$

On the other hand, the equality $\chi^{-1}(V) = U + \mathsf{Ker}\, \varphi = U + \mathsf{Ker}\, U$ implies the continuity of χ. So χ continuously extends to the completion $(A/\mathsf{Ker}\,U)^{\blacktriangledown} = A/U$:

$$\begin{tikzcd} A \arrow[r, "\tau_U"'] \arrow[rr, bend left, "\rho_U"] \arrow[dr, "\varphi"'] & A/\mathsf{Ker}\,U \arrow[r, "\blacktriangledown_{A/\mathsf{Ker}\,U}"'] \arrow[d, "\chi"] & A/U \arrow[dl, dashed, "\chi^{\blacktriangledown}"] \\ & B & \end{tikzcd}$$

and since $A/\mathsf{Ker}\,U$ is dense in its completion, $\chi^{\blacktriangledown}$ is multiplicative by Lemma 5.1. At the same time, $\chi^{\blacktriangledown}$ obviously preserves the identity. Hence, $\chi^{\blacktriangledown}$ is a morphism in $\mathtt{Ste}^{\circledast}$. ■

Proof of properties 1°–6°. 1° By Lemma 5.41 the net of Banach quotient maps generates on the inside the class $\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})$ of morphisms with values in Banach algebras. On the other hand, by Theorem 5.36 the class DEpi is monomorphically complementable in $\mathtt{Ste}^{\circledast}$. Therefore by Theorem 3.38 each object A in $\mathtt{Ste}^{\circledast}$ has an envelope in DEpi with respect to $\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})$, and by definition this is the holomorphic envelope of A.

2° & 3° Formulas (5.39) follow immediately from (3.94), and diagram (5.40) from diagram (3.95).

4° The category $\mathtt{Ste}^{\circledast}$ is projectively complete and co-well-powered in the quotient objects of the class DEpi, and the class $\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})$ goes from $\mathtt{Ste}^{\circledast}$ (since each algebra A can be mapped at least into the zero Banach algebra) and is a right ideal. Therefore, the holomorphic envelope $\heartsuit$ is semiregular, and by Theorem 3.42 it can be defined as a functor. Moreover, by Remark 3.44, each class, in particular DEpi, pushes $\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})$, hence the holomorphic envelope is regular, and by Theorem 3.48 it can be defined as an idempotent functor.

5° Suppose $\varprojlim \mathcal{B}_A$ is a dense epimorphism. By Lemma 5.41, the net $\mathcal{B}$ generates on the inside the class of morphisms with values in Banach algebras, hence by Theorem 3.5 (with $\Omega = \mathsf{DEpi}$),

$$\heartsuit_A = \mathsf{env}^{\mathsf{DEpi}}_{\mathtt{BanAlg}} A = \mathsf{env}^{\mathsf{DEpi}}_{\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})} A \overset{(3.14)}{=} \mathsf{env}^{\mathsf{DEpi}}_{\mathcal{B}} A.$$

Further, the condition $\varprojlim \mathcal{B}_A \in \mathsf{DEpi}$ implies by Lemma 3.23 that

$$\mathsf{env}^{\mathsf{DEpi}}_{\mathcal{B}} A \overset{(3.42)}{=} \varprojlim \mathcal{B}_A.$$

Again by Lemma 3.23, from $\varprojlim \mathcal{B}_A \in \mathsf{DEpi} \subseteq \mathsf{Epi}$ we have

$$\varprojlim \mathcal{B}_A \overset{(3.42)}{=} \mathsf{env}^{\mathsf{Epi}}_{\mathcal{B}} A.$$

And again by Theorem 3.5 (now with $\Omega = \mathsf{Epi}$),

$$\mathsf{env}^{\mathsf{Epi}}_{\mathcal{B}} A \overset{(3.14)}{=} \mathsf{env}^{\mathsf{Epi}}_{\mathsf{Mor}(\mathtt{Ste}^{\circledast},\mathtt{BanAlg})} A = \mathsf{env}^{\mathsf{Epi}}_{\mathtt{BanAlg}} A.$$

6° We need to verify that the holomorphic envelope satisfies conditions T.1 and T.2 on p. 94. First, let $\rho : A \to A'$ and $\sigma : B \to B'$ be two holomorphic extensions. Then for any Banach algebra C and for any morphism $\varphi : A \circledast B \to C$ there are morphisms $\varphi_A : A \to C$ and $\varphi_B : B \to C$ such that

$$\varphi(a \circledast b) = \varphi_A(a) \cdot \varphi_B(b) = \varphi_B(b) \cdot \varphi_A(a).$$

Since φ_A and φ_B are morphisms into the Banach algebra C, they can be extended along ρ and σ:

$$\varphi_A = \varphi'_A \circ \rho, \qquad \varphi_B = \varphi'_B \circ \sigma.$$

Set

$$\varphi'(x \circledast y) = \varphi'_A(x) \cdot \varphi'_B(y) = \varphi'_B(y) \cdot \varphi'_A(x), \qquad x \in A',\, y \in B'.$$

Then

$$\varphi'((\rho \circledast \sigma)(a \circledast b)) = \varphi'(\rho(a) \circledast \sigma(b)) = \varphi'_A(\rho(a)) \cdot \varphi'_B(\sigma(b)) = \varphi_A(a) \cdot \varphi_B(b) = \varphi(a \circledast b).$$

Second, let $\sigma : \mathbb{C} \to B$ be a holomorphic extension of the algebra $\mathbb{C}$. It must be a dense epimorphism, and since $\mathbb{C}$ is finite-dimensional, σ is an epimorphism. ∎

- We say that a stereotype algebra A is *holomorphic* if it is a complete object with respect to the envelope $\heartsuit$, i.e. its holomorphic envelope is an isomorphism: $\heartsuit_A \in \mathsf{Iso}$.

Property 6° and Theorems 3.63 and 3.64 give

THEOREM 5.42. *The formulas*

$$A \overset{\heartsuit}{\circledast} B = (A \circledast B)^{\heartsuit}, \qquad \varphi \overset{\heartsuit}{\circledast} \psi = (\varphi \circledast \psi)^{\heartsuit} \tag{5.44}$$

define a monoidal structure on the category of holomorphic algebras, and the functor $A \mapsto A^{\heartsuit}$ is monoidal (from $\mathtt{Ste}^{\circledast}$ with $\circledast$ as tensor product into the category of holomorphic algebras with $\overset{\heartsuit}{\circledast}$ as tensor product).

One can describe the tensor product (5.44) in terms of the net $\mathcal{B}$ of Banach quotient maps as follows. Let A and A' be two stereotype algebras. If U, V, U', V' are submultiplicative closed absolutely convex neighborhoods of zero such that

$$V \subseteq U \subseteq A, \qquad V' \subseteq U' \subseteq A',$$

then by multiplying the arising pair of diagrams (5.31) we get

$$\begin{tikzcd} & A \circledast A' \arrow[dl, bend right, "\rho_V \circledast \rho_{V'}"'] \arrow[dr, bend left, "\rho_U \circledast \rho_{U'}"] & \\ A/V \circledast A'/V' \arrow[rr, "\pi^U_V \circledast \pi^{U'}_{V'}"] & & A/U \circledast A'/U' \end{tikzcd} \tag{5.45}$$

This means that the system of morphisms $\rho_U \circledast \rho_{U'} : A \circledast A' \to A/U \circledast A'/U'$, $U \in \mathcal{SU}(A)$, $U' \in \mathcal{SU}(A')$, is a projective cone of the covariant system $\pi^U_V \circledast \pi^{U'}_{V'}$. As a corollary, there exists a unique morphism

$$\vartheta : A \circledast A' \to \varprojlim_{\substack{U \in \mathcal{SU}(A),\\ U' \in \mathcal{SU}(A')}} A/U \circledast A'/U'$$

such that

$$
\begin{tikzcd}
A \circledast A' \arrow[rr, dashed, "\vartheta"] \arrow[dr, bend right, "\rho_V \circledast \rho_{V'}"'] & & \varprojlim_{U \in \mathcal{SU}(A),\ U' \in \mathcal{SU}(A')} A/U \circledast A'/U' \arrow[dl, bend left, "\pi_{V,V'}"] \\
& A/V \circledast A'/V' &
\end{tikzcd}
$$

where $V \in \mathcal{SU}(A)$, $V' \in \mathcal{SU}(A')$, and $\pi_{V,V'}$ is the cone of morphisms from the projective limit into the covariant system.

PROPOSITION 5.43. *For any stereotype algebras A and A' we have*

$$(A \circledast A')^{\heartsuit} = \mathsf{Im}_\infty \vartheta \tag{5.46}$$

where Im_∞ is the element of the nodal decomposition in the category $\mathtt{Ste}$ *of stereotype spaces (not algebras). In particular, if the algebras A and A' are holomorphic, then*

$$A \overset{\heartsuit}{\circledast} A' = \mathsf{Im}_\infty \vartheta. \tag{5.47}$$

Proof. We need to verify that the map $\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta : A \circledast A' \to \mathsf{Ran}\,\mathsf{Im}_\infty \vartheta$ is a holomorphic envelope of the algebra $A \circledast A'$ (where red_∞ and coim_∞ are elements of the nodal decomposition in $\mathtt{Ste}$).

Let us show first that this is a holomorphic extension. Take a morphism $\varphi : A \circledast A' \to B$ into a Banach algebra B. Set

$$\eta(x) = \varphi(x \circledast 1_{A'}), \quad \eta'(y) = \varphi(1_A \circledast a'), \qquad x \in A,\, y \in A'.$$

Then

$$\eta(x) \cdot \eta'(y) = \eta'(y) \cdot \eta(x), \qquad x \in A,\, y \in A'. \tag{5.48}$$

and

$$\varphi(x \circledast y) = \eta(x) \cdot \eta'(y) = \eta'(y) \cdot \eta(x), \qquad x \in A,\, y \in A'.$$

Let U be the unit ball in B. Consider its preimages in A and A',

$$V = \eta^{-1}(U), \qquad V' = (\eta')^{-1}(U),$$

and morphisms ψ and ψ' such that

$$
\begin{tikzcd}
& A \arrow[dl, "\eta"'] \arrow[dr, "\rho_V"] & \\
B & & A/V \arrow[ll, "\psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[dl, "\eta'"'] \arrow[dr, "\rho_{V'}"] & \\
B & & A'/V' \arrow[ll, "\psi'"]
\end{tikzcd}
$$

From (5.48) we have the identity

$$\psi(s) \cdot \psi'(t) = \psi'(t) \cdot \psi(s), \qquad s \in A/V,\, t \in A'/V', \tag{5.49}$$

which means in turn that

$$\varphi_{V,V'} : A/V \circledast A'/V' \to B, \qquad \varphi_{V,V'}(x \circledast y) = \psi(x) \cdot \psi'(y),$$

is a morphism. We have

$$
\begin{aligned}
\varphi(x \circledast y) = \eta(x) \cdot \eta'(y) &= \psi(\rho_V(x)) \cdot \psi'(\rho_{V'}(y)) \\
&= \varphi_{V,V'}(\rho_V(x) \circledast \rho_{V'}(y)) = \varphi_{V,V'}((\rho_V \circledast \rho_{V'})(x \circledast y)),
\end{aligned}
$$

hence

$$\begin{tikzcd}
& A \circledast A' \arrow[dl, "\varphi"'] \arrow[dr, "\rho_V \circledast \rho_{V'}"] & \\
B & & A/V \circledast A'/V' \arrow[ll, "\varphi_{V,V'}"']
\end{tikzcd}$$

This can be inserted into the diagram

$$\begin{tikzcd}
& A \circledast A' \arrow[r, "\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta"] \arrow[dl, "\varphi"'] \arrow[dr, "\rho_V \circledast \rho_{V'}"] & A \overset{\heartsuit}{\circledast} A' \arrow[r, "\mathsf{im}_\infty \vartheta"] & \varprojlim\limits_{\substack{W \in \mathcal{SU}(A),\\ W' \in \mathcal{SU}(A')}} A/W \circledast A'/W' \arrow[dl, "\pi_{V,V'}"] \\
B & & A/V \circledast A'/V' \arrow[ll, "\varphi_{V,V'}"'] &
\end{tikzcd}$$

which we can transform into

$$\begin{tikzcd}
A \circledast A' \arrow[rr, "\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta"] \arrow[dr, "\varphi"'] & & A \overset{\heartsuit}{\circledast} A' \arrow[dl, "\varphi_{V,V'} \circ \pi_{V,V'} \circ \mathsf{im}_\infty \vartheta"] \\
& B &
\end{tikzcd}$$

and this means that φ extends along $\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta$.

Let us show now that $\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta$ is a holomorphic envelope. Suppose $\sigma : A \circledast A' \to C$ is another holomorphic extension. Then for any submultiplicative zero neighborhoods $V \subseteq A$ and $V' \subseteq A'$ the morphism $\rho_V \circledast \rho_{V'} : A \circledast A' \to A/V \circledast A'/V'$ is a morphism into a Banach algebra, hence there exists a unique morphism $\widetilde{\rho_V \circledast \rho_{V'}}$ such that

$$\begin{tikzcd}
A \circledast A' \arrow[rr, "\sigma"] \arrow[dr, "\rho_V \circledast \rho_{V'}"'] & & C \arrow[dl, "\widetilde{\rho_V \circledast \rho_{V'}}"] \\
& A/V \circledast A'/V' &
\end{tikzcd}$$

At the same time, for a system of submultiplicative neighborhoods $W \subseteq V \subseteq A$ and $W' \subseteq V' \subseteq A'$ we get

$$\begin{tikzcd}
& A \circledast A' \arrow[d, "\sigma"] \arrow[dl, bend right, "\rho_W \circledast \rho_{W'}"'] \arrow[dr, bend left, "\rho_V \circledast \rho_{V'}"] & \\
& C \arrow[dl, dashed, "\widetilde{\rho_W \circledast \rho_{W'}}"'] \arrow[dr, dashed, "\widetilde{\rho_V \circledast \rho_{V'}}"] & \\
A/W \circledast A'/W' \arrow[rr, "\pi_V^W \circledast \pi_{V'}^{W'}"] & & A/V \circledast A'/V'
\end{tikzcd}$$

where the perimeter and the two upper triangles are commutative; since σ is an epimorphism, this means that the lower triangle is also commutative.

This diagram implies that the system $\widetilde{\rho_V \circledast \rho_{V'}}$ of morphisms forms a projective cone of the contravariant system $\pi_V^W \circledast \pi_{V'}^{W'}$. As a corollary, there exists a unique morphism $\varkappa$

such that

$$\begin{tikzcd}
C \arrow[rr, dashed, "\varkappa"] \arrow[dr, "\widetilde{\rho_V \circledast \rho_{V'}}"'] & & \varprojlim_{W,W'} A/W \circledast A'/W' \arrow[dl, "\pi_{V,V'}"] \\
& A/V \circledast A'/V' &
\end{tikzcd}$$

In the diagrams

$$\begin{tikzcd}
A/V \circledast A'/V' \arrow[rr, "\sigma"] \arrow[dr, "\vartheta"] \arrow[ddr, bend right, "\rho_V \circledast \rho_{V'}"'] & & C \arrow[dl, dashed, "\varkappa"] \arrow[ddl, bend left, "\widetilde{\rho_V \circledast \rho_{V'}}"] \\
& \varprojlim_{W,W'} A/W \circledast A'/W' \arrow[d, "\pi_{V,V'}"] & \\
& A/V \circledast A'/V' &
\end{tikzcd} \tag{5.50}$$

the perimeter and the two lower triangles are commutative. Hence for all V, V',

$$\pi_{V,V'} \circ \vartheta = \rho_V \circledast \rho_{V'}, \qquad \pi_{V,V'} \circ \varkappa \circ \sigma = \rho_V \circledast \rho_{V'},$$

and from the uniqueness of ϑ satisfying these equalities it follows that

$$\vartheta = \varkappa \circ \sigma,$$

i.e. (5.50) is commutative. Now we obtain a commutative diagram

$$\begin{tikzcd}
A \circledast A' \arrow[rr, "\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta"'] \arrow[rrrr, bend left, "\vartheta"] \arrow[drr, bend right, "\sigma"'] & & A \overset{\heartsuit}{\circledast} A' \arrow[rr, "\mathsf{im}_\infty \vartheta"'] & & \varprojlim_{W,W'} A/W \circledast A'/W' \\
& & C \arrow[urr, bend right, "\varkappa"'] & &
\end{tikzcd}$$

Here $\sigma \in \mathsf{DEpi}(\mathtt{Ste}^{\circledast}) = \mathsf{Epi}(\mathtt{Ste})$, and $\mathsf{im}_\infty \vartheta \in \mathsf{SMono}(\mathtt{Ste})$, so there exists a diagonal δ:

$$\begin{tikzcd}
A \circledast A' \arrow[rr, "\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta"'] \arrow[drr, bend right, "\sigma"'] & & A \overset{\heartsuit}{\circledast} A' \arrow[rr, "\mathsf{im}_\infty \vartheta"'] & & \varprojlim_{W,W'} A/W \circledast A'/W' \\
& & C \arrow[u, dashed, "\delta"] \arrow[urr, bend right, "\varkappa"'] & &
\end{tikzcd}$$

Initially δ is built as a morphism in $\mathtt{Ste}$, but since σ is a dense morphism, δ is a homomorphism of algebras, i.e. a morphism in $\mathtt{Ste}^{\circledast}$. We see that every extension σ is subordinated to $\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta$, thus $\mathsf{red}_\infty \vartheta \circ \mathsf{coim}_\infty \vartheta$ is an envelope. ∎

5.4.3. Fourier transform on a commutative Stein group. Let G be a commutative compactly generated Stein group, $\mathcal{O}(G)$ the algebra of holomorphic functions on G, and $\mathcal{O}^\star(G)$ the algebra of analytic functionals from Examples 5.6 and 5.10. Let $G^\bullet$ be the dual group of complex characters on G, i.e. continuous homomorphisms $\chi : G \to \mathbb{C}^\times$ into the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers ($G^\bullet$ is endowed with pointwise multiplication and the topology of uniform convergence on compact sets in G), and let $\mathcal{F}_G : \mathcal{O}^\star(G) \to \mathcal{O}(G^\bullet)$ be the Fourier transform on G, i.e. the homomorphism of algebras acting by the formula

$$\overbrace{\mathcal{F}_G(\alpha)(\chi)}^{\substack{\text{value of the function } \mathcal{F}_G(\alpha)\in\mathcal{O}(G^\bullet)\\ \text{at the point } \chi\in G^\bullet\\ \downarrow}} = \underbrace{\alpha(\chi)}_{\substack{\uparrow\\ \text{action of the functional } \alpha\in\mathcal{O}^\star(G)\\ \text{on the function } \chi\in G^\bullet\subseteq\mathcal{O}(G)}} \qquad (\chi\in G^\bullet,\ \alpha\in\mathcal{O}^\star(G)).$$

THEOREM 5.44. *For a compactly generated commutative Stein group G its Fourier transform $\mathcal{F}_G : \mathcal{O}^\star(G) \to \mathcal{O}(G^\bullet)$ is a holomorphic envelope of $\mathcal{O}^\star(G)$, and coincides with the stereotype Arens–Michael envelope and with the envelope with respect to the class of Banach algebras in the class* Epi *of all epimorphisms* (*in the categories* `TopAlg` *and* $\mathtt{Ste}^{\circledast}$)*:*

$$\mathcal{F}_G = \heartsuit_{\mathcal{O}^\star(G)} = \mathsf{env}^{\mathsf{DEpi}}_{\mathtt{BanAlg}}\,\mathcal{O}^\star(G) = \mathsf{env}^{\mathsf{Epi}}_{\mathtt{BanAlg}}\,\mathcal{O}^\star(G) = \varprojlim \mathcal{B}_{\mathcal{O}^\star(G)}. \tag{5.51}$$

Proof. In [3] it was shown that in `TopAlg` the local limit of the net of Banach quotient maps on the object $\mathcal{O}^\star(G)$ coincides with $\mathcal{O}(G^\bullet)$:

$$\mathcal{O}(G^\bullet) = \varprojlim \mathcal{B}_{\mathcal{O}^\star(G)}. \tag{5.52}$$

Here $\mathcal{O}(G^\bullet)$ is a Fréchet algebra, so it coincides with its pseudosaturation, and this implies that (5.52) holds in the category of stereotype algebras. In addition, the morphism $\mathcal{F}_G : \mathcal{O}^\star(G) \to \mathcal{O}(G^\bullet)$, being a local limit in `TopAlg`, is a dense epimorphism, therefore it is so in $\mathtt{Ste}^{\circledast}$ as well. Thus, by (5.42) we have (5.51). ■

5.5. Continuous envelope

- Let us say that a stereotype algebra A is *involutive* if an involution $x \mapsto \overline{x}$ is defined on A (in the usual sense, see e.g. [16] or [28]), and this operation is continuous as a map from A into A. The involutive stereotype algebras form a category $\mathtt{InvSte}^{\circledast}$ where morphisms are continuous involutive unital homomorphisms $\varphi : A \to B$:

$$\varphi(\lambda\cdot x + \mu\cdot y) = \lambda\cdot\varphi(x) + \mu\cdot\varphi(y), \qquad \varphi(x\cdot y) = \varphi(x)\cdot\varphi(y),$$
$$\varphi(1) = 1, \qquad \varphi(\overline{x}) = \overline{\varphi(x)}.$$

All C^*-algebras are obvious examples ([16], [28]). Another example is the algebra $\mathcal{C}(M)$ of continuous functions on a paracompact locally compact topological space M from Example 5.4.

5.5.1. Net of C^*-quotient-maps and the Kuznetsova envelope

- By a *C^*-seminorm* on an involutive algebra A we mean any seminorm $p : A \to \mathbb{R}_+$ satisfying

$$p(x\cdot\overline{x}) = p(x)^2, \qquad x \in A. \tag{5.53}$$

By the Z. Sebestyén theorem [40], any such seminorm automatically preserves involution and is submultiplicative:

$$p(\overline{x}) = p(x), \qquad p(x \cdot y) \le p(x) \cdot p(y).$$

The identity (5.53) implies in particular

$$p(1) = p(1 \cdot \overline{1}) = p(1)^2,$$

hence

$$p(1) = 1 \quad \text{or} \quad p(1) = 0,$$

and the second of these equalities means that p vanishes, since in this case

$$p(x) = p(x \cdot 1) \le p(x) \cdot p(1) = p(x) \cdot 0 = 0.$$

Further we will be interested in continuous C^*-seminorms on involutive topological algebras.

- Let us define a *C^*-neighborhood* in a topological algebra A to be any closed absolutely convex zero neighborhood U for which the Minkowski functional

$$p(x) = \inf\{\lambda > 0 : \lambda \cdot x \in U\}$$

  is a C^*-seminorm on A. For any such U the quotient algebra A/U (defined on p. 168) is a C^*-algebra; we call it the *C^*-quotient algebra* of A, and the natural map $\rho_U : A \to A/U$ will be called a *C^*-quotient map* of A. The symbol $\mathcal{C}^*$ will denote the class of all C^*-quotient maps $\{\rho_U : A \to A/U\}$, where A runs over the class of involutive topological algebras, and U over the set of all C^*-neighborhoods of zero in A.

The following fact is an analog of Proposition 5.39.

Proposition 5.45. *The class $\mathcal{C}^*$ is a net of epimorphisms in the category $\mathtt{InvSte}^{\circledast}$ of involutive stereotype algebras, and the semiorder $\to$ in $\mathcal{C}^*$ is equivalent to the embedding of the corresponding neighborhoods of zero:*

$$\rho_V \to \rho_U \iff V \subseteq U. \tag{5.54}$$

Proof. By definition, the relation $\rho_V \to \rho_U$ means the existence of an involutive continuous homomorphism $\iota : A/V \to A/U$ of C^*-algebras such that diagram (5.32) is commutative. By the well-known property of C^*-algebras [28, Theorem 2.1.7], the homomorphism ι cannot increase the C^*-norm: $\|\iota(x)\| \le \|x\|$. Applied to C^*-seminorms p_U and p_V which correspond to the neighborhoods U and V, this means $p_U(x) \le p_V(x)$, which in turn is equivalent to $V \subseteq U$. ∎

- The net $\mathcal{C}^*$ will be called *the net of C^*-quotient maps.*
- For each involutive stereotype algebra A the family of C^*-quotient maps $\rho_U : A \to A/U$ is a projective cone of the covariant system $\mathsf{Bind}(\mathcal{C}_A)$. The projective limit of this cone in the category $\mathtt{InvSte}^{\circledast}$ of involutive stereotype algebras will be called the *Kuznetsova envelope* ([2]) of A and denoted by

$$\varprojlim \mathcal{C}_A : A \to A_{\mathcal{C}} \tag{5.55}$$

([2]) Our terminology and notation differ from those used in [24].

(this limit exists, since $\mathtt{InvSte}^{\circledast}$ is projectively complete). The range of this morphism,

$$A_{\mathcal{C}} = \mathsf{Ran}\varprojlim \mathcal{B}_A = \mathtt{InvSte}^{\circledast}\text{-}\varprojlim_{U\in\mathcal{C}^*\mathcal{U}(A)} A/U = \Big(\mathtt{InvTopAlg}\text{-}\varprojlim_{U\in\mathcal{C}^*\mathcal{U}(A)} A/U\Big)^{\vartriangle}, \tag{5.56}$$

is also called the *Kuznetsova envelope* of A.

Theorem 3.36 implies

THEOREM 5.46. *The Kuznetsova envelope is an envelope in the class of all morphisms in* $\mathtt{InvSte}^{\circledast}$ *with respect to the system of all* C^**-quotient maps* $\mathcal{C}^*$*,*

$$A_{\mathcal{C}} = \mathsf{Env}_{\mathcal{C}^*}^{\mathsf{Mor}(\mathtt{InvSte}^{\circledast})} A, \tag{5.57}$$

and to each morphism $\varphi : A \to B$ *in* $\mathtt{InvSte}^{\circledast}$ *the formula*

$$\varphi_{\mathcal{C}} = \varprojlim_{\tau\in\mathcal{C}_B}\ \varprojlim_{\sigma\in\mathcal{B}_A} \varphi_\sigma^\tau \circ \sigma_{\mathcal{C}} \tag{5.58}$$

assigns a morphism $\varphi_{\mathcal{C}} : A_{\mathcal{C}} \to B_{\mathcal{C}}$ *such that*

$$\begin{tikzcd} A \arrow[r, "\varprojlim \mathcal{C}_A"] \arrow[d, "\varphi"'] & A_{\mathcal{C}} \arrow[d, dashed, "\varphi_{\mathcal{C}}"] \\ B \arrow[r, "\varprojlim \mathcal{C}_B"] & B_{\mathcal{C}} \end{tikzcd} \tag{5.59}$$

and the correspondence $(A,\varphi) \mapsto (A_{\mathcal{C}}, \varphi_{\mathcal{C}})$ *can be defined as a functor from* $\mathtt{InvSte}^{\circledast}$ *into* $\mathtt{InvSte}^{\circledast}$*.*

5.5.2. Continuous envelope of an involutive stereotype algebra. By a *dense epimorphism* of involutive stereotype algebras we mean the same object as for general (non-involutive) stereotype algebras, i.e. a morphism $\varphi : A \to B$ such that $\varphi(A)$ is dense in B.

- A *continuous envelope* of an involutive stereotype algebra A is its envelope in the class DEpi of dense epimorphisms in the category $\mathtt{InvSte}^{\circledast}$ with respect to the class C^* of C^*-algebras. We use the following notation for this construction:

$$A^{\diamondsuit} = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A, \qquad \diamondsuit_A = \mathsf{env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A. \tag{5.60}$$

  Thus,

$$(\diamondsuit_A : A \to A^{\diamondsuit}) = (\mathsf{env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A : A \to \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A).$$

The following properties are proved by analogy with the properties of holomorphic envelopes on p. 170.

PROPERTIES OF CONTINUOUS ENVELOPES.

1° *Each involutive stereotype algebra* A *has a continuous envelope* $A^{\diamondsuit}$*.*

2° *The continuous envelope* $A^{\diamondsuit}$ *is connected with the Kuznetsova envelope* $A_{\mathcal{C}}$ *through the formulas*

$$\diamondsuit_A = \mathsf{red}_\infty \varprojlim \mathcal{C}_A \circ \mathsf{coim}_\infty \varprojlim \mathcal{C}_A, \qquad A^{\diamondsuit} = \mathsf{Dom}\,\mathsf{im}_\infty \varprojlim \mathcal{C}_A \tag{5.61}$$

where $\mathsf{coim}_\infty \varprojlim \mathcal{C}_A$, $\mathsf{red}_\infty \varprojlim \mathcal{C}_A$, $\mathsf{im}_\infty \varprojlim \mathcal{C}_A$ *are elements of the nodal decomposition of the morphism* $\varprojlim \mathcal{C}_A$ *in* $\mathtt{Ste}$.

3° *For any morphism* $\varphi : A \to B$ *of involutive stereotype algebras and for each choice of continuous envelopes* $\diamondsuit_A : A \to A^\diamondsuit$ *and* $\diamondsuit_B : B \to B^\diamondsuit$ *there exists a unique morphism* $\varphi^\diamondsuit : A^\diamondsuit \to B^\diamondsuit$ *such that*

$$\begin{tikzcd} A \arrow[r, "\diamondsuit_A"] \arrow[d, "\varphi"'] & A^\diamondsuit \arrow[d, dashed, "\varphi^\diamondsuit"] \\ B \arrow[r, "\diamondsuit_B"] & B^\diamondsuit \end{tikzcd} \tag{5.62}$$

4° *The correspondence* $(X, \alpha) \mapsto (X^\diamondsuit, \alpha^\diamondsuit)$ *can be defined as an idempotent functor from* $\mathtt{InvSte}^{\circledast}$ *into* $\mathtt{InvSte}^{\circledast}$*:*

$$(1_A)^\diamondsuit = 1_{A^\diamondsuit}, \qquad (\beta \circ \alpha)^\diamondsuit = \beta^\diamondsuit \circ \alpha^\diamondsuit, \qquad (\alpha^\diamondsuit)^\diamondsuit = \alpha^\diamondsuit. \tag{5.63}$$

5° *If an algebra* A *is dense in its Kuznetsova envelope* $A_{\mathcal{C}}$*, i.e.*

$$\varprojlim \mathcal{C}_A \in \mathsf{DEpi}(\mathtt{Ste}^{\circledast}),$$

then the continuous envelope of A *coincides with its envelope in the class* Epi *of all epimorphisms in* $\mathtt{InvSte}^{\circledast}$ *and with the Kuznetsova envelope:*

$$A^\diamondsuit = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{Epi}} A = A_{\mathcal{C}}. \tag{5.64}$$

6° *The continuous envelope is coherent with the projective tensor product* $\circledast$ *in* $\mathtt{InvSte}^{\circledast}$.

The following lemma is used in the proof:

LEMMA 5.47. *In* $\mathtt{InvSte}^{\circledast}$ *the net* $\mathcal{C}$ *of* C^**-quotient maps consists of dense epimorphisms and generates on the inside the class* $\mathsf{Mor}(\mathtt{InvSte}^{\circledast}, \mathsf{C}^*)$ *of all morphisms with values in* C^**-algebras:*

$$\mathcal{C} \subseteq \mathsf{Mor}(\mathtt{InvSte}^{\circledast}, \mathsf{C}^*) \subseteq \mathsf{Mor}(\mathtt{InvSte}^{\circledast}) \circ \mathcal{C}. \tag{5.65}$$

Proof. Let $\varphi : A \to B$ be a morphism into a C^*-algebra B, and V the unit ball in B. Set $U = \varphi^{-1}(V)$. It is a zero neighborhood in A, and its Minkowski functional p is a composition of φ and the norm on B:

$$p(x) = \inf\{\lambda > 0 : \lambda \cdot x \in \varphi^{-1}(V)\} = \inf\{\lambda > 0 : \lambda \cdot \varphi(x) \in V\} = \|\varphi(x)\|.$$

This implies that p is a C^*-seminorm on A:

$$p(x \cdot \overline{x}) = \|\varphi(x \cdot \overline{x})\| = \|\varphi(x) \cdot \overline{\varphi(x)}\| = \|\varphi(x)\|^2 = p(x)^2.$$

That is, U is a C^*-neighborhood of zero in A. Now the proof of Lemma 5.41 works. ■

- An involutive stereotype algebra A is said to be *continuous* if it is a complete object with respect to the envelope $\diamondsuit$, i.e. its continuous envelope is an isomorphism: $\diamondsuit_A \in \mathsf{Iso}$.

Property 6° and Theorems 3.63 and 3.64 give

THEOREM 5.48. *The formulas*

$$A \overset{\diamondsuit}{\circledast} B = (A \circledast B)^\diamondsuit, \qquad \varphi \overset{\diamondsuit}{\circledast} \psi = (\varphi \circledast \psi)^\diamondsuit \tag{5.66}$$

define a monoidal structure on the category of continuous algebras, and the functor $A \mapsto A^\diamondsuit$ *is monoidal (from the category of involutive stereotype algebras with tensor product* $\circledast$ *into the category of continuous algebras with tensor product* $\overset{\diamondsuit}{\circledast}$*).*

The continuous envelope can be described in terms of the net $\mathcal{C}$ of C^*-quotient maps by formula (5.46) with obvious modifications.

5.5.3. The Gelfand transform as a continuous envelope of a commutative algebra

- By the *involutive spectrum* $\mathsf{Spec}(A)$ of an involutive topological (respectively, stereotype) algebra A over $\mathbb{C}$ we mean the set of its involutive characters, i.e. homomorphisms $\chi : A \to \mathbb{C}$ (also continuous, involutive and identity preserving). This set is endowed with the topology of uniform convergence on totally bounded sets in A.
- By the *Gelfand transform* of an involutive stereotype algebra A we mean the natural map $\mathcal{G}_A : A \to C(M)$ of A into the algebra $C(M)$ of functions on the involutive spectrum $M = \mathsf{Spec}(A)$, continuous on each compact set $K \subseteq M$:
$$\mathcal{G}_A(x)(t) = t(x), \qquad t \in M = \mathsf{Spec}(A),\ x \in A. \tag{5.67}$$
We endow $C(M)$ with the topology which is the pseudosaturation (3) of the topology of uniform convergence on compact sets in M; this turns $C(M)$ into a stereotype algebra. In the special case when M is a paracompact locally compact space, the topology of uniform convergence on compact sets in M is already a pseudosaturated (and complete) topology on $C(M)$, so $C(M)$ becomes a stereotype algebra already at this step [2, Sec. 8.1] (and the operation of pseudosaturation does not change this topology anymore).
- For each compact set $K \subseteq M$ consider the restriction map
$$\pi_K : C(M) \to C(K), \qquad y \mapsto y|_K,$$
and let $\mathcal{G}_K = \pi_K \circ \mathcal{G}_A$ be the composition
$$\begin{tikzcd} A \arrow[rr, "\mathcal{G}_A"] \arrow[dr, "\mathcal{G}_K"'] & & C(M) \arrow[dl, "\pi_K"] \\ & C(K) & \end{tikzcd} \tag{5.68}$$
If K and L are two compact sets in M, and $K \subseteq L \subseteq M$, then π^L_K denotes the restriction map
$$\pi^L_K : C(L) \to C(K), \qquad y \mapsto y|_K.$$
Obviously, the algebra $C(M)$ with the system of projections $\rho_K : C(M) \to C(K)$, $K \subseteq M$, is a projective limit of the system of binding morphisms $\pi^L_K : C(L) \to C(K)$, $K \subseteq L \subseteq M$ (in the category $\mathtt{InvSte}^{\circledast}$):
$$C(M) = \mathtt{InvSte}^{\circledast}\text{-}\varprojlim_{K \subseteq M} C(K).$$

(3) The operation of pseudosaturation was defined on p. 117.

PROPOSITION 5.49. *For any involutive stereotype algebra A its Gelfand transform $\mathcal{G}_A : A \to C(M)$ is a morphism of stereotype algebras. If $M = \mathsf{Spec}(A)$ is a paracompact locally compact space, the morphism $\mathcal{G}_A : A \to C(M)$ is a dense epimorphism.*

Proof. In the first assertion only the continuity of $\mathcal{G}_A$ is not obvious. Take a basic zero neighborhood U in $C(M)$, i.e. $U = \{f \in C(M) : \sup_{t\in T} |f(t)| \le \varepsilon\}$ for some compact set $T \subseteq M$ and some $\varepsilon > 0$. Its preimage under $\mathcal{G}_A : A \to C(M)$ is $\{x \in A : \sup_{t\in T} |t(x)| \le \varepsilon\}$ $= \varepsilon \cdot {}^{\circ}T$, the homothety of the polar ${}^{\circ}T$. Since A is stereotype, ${}^{\circ}T$ is a neighborhood of zero in it. This proves that $\mathcal{G}_A : A \to C(M)$ is continuous if the space $C(M)$ is endowed with the topology of uniform convergence on compact sets in M. Since A, being stereotype, is pseudosaturated, this means that under the pseudosaturation of the topology in $C(M)$ the map $\mathcal{G}_A : A \to C(M)$ remains continuous (this follows, for example, from [2, Theorem 1.16]).

Suppose further that $M = \mathsf{Spec}(A)$ is a paracompact locally compact space. For each compact set $K \subseteq M$ the image $\mathcal{G}_K(A)$ is an involutive subalgebra in $C(K)$, and it contains the identity (and hence all constant functions) and separates the points of K. So by the Stone–Weierstrass theorem, $\mathcal{G}_K(A)$ is dense in $C(K)$. This is true for each map $\mathcal{G}_K = \pi_K \circ \gamma$, where K is a compact set in M. Since the topology in $C(M)$ is the projective topology with respect to the maps π_K, we conclude that $\mathcal{G}_A(A)$ is dense in $C(M)$. ∎

THEOREM 5.50. *For each commutative involutive stereotype algebra A the system of morphisms $\mathcal{G}_K : A \to C(K)$ consists of dense epimorphisms and is isomorphic in Epi^A to the system $\rho_U : A \to A/U$ of all C^*-quotient maps of A,*

$$\{\mathcal{G}_K : A \to C(K) : K \subseteq \mathsf{Spec}(A)\} \cong \mathcal{C}^*_A. \tag{5.69}$$

Under this isomorphism:

– *the system of restrictions $\pi^L_K : C(L) \to C(K)$, $K \subseteq L \subseteq M$, turns into the system $\mathsf{Bind}(\mathcal{C}^*_A)$ of binding morphisms of the net $\mathcal{C}^*$ on A:*

$$\{\pi^L_K : C(L) \to C(K) : K \subseteq L \subseteq \mathsf{Spec}(A)\} \cong \mathsf{Bind}(\mathcal{C}^*_A); \tag{5.70}$$

– *the Gelfand transform $\mathcal{G}_A : A \to C(M)$ is a local limit of the net $\mathcal{C}^*$ on A (and hence it coincides with the Kuznetsova envelope of A):*

$$\mathcal{G}_A = \varprojlim \mathcal{C}^*_A. \tag{5.71}$$

Proof. On each compact set $K \subseteq M$ the algebra of functions of the form $\mathcal{G}_A(x)$, where $x \in A$, separates the points, contains the constant functions, and is invariant with respect to involution, so it is dense in $C(K)$ by the Stone–Weierstrass theorem. This implies that $C(M)$, which contains $\mathcal{G}_A(A)$, is also dense in $C(K)$, so both $\mathcal{G}_K : A \to C(K)$ and $\pi_K : C(M) \to C(K)$ are dense epimorphisms (in $\mathtt{InvSte}^{\circledast}$).

The range A/U of each C^*-quotient map $\rho_U : A \to A/U$ is a commutative C^*-algebra, hence it is isomorphic to the algebra $C(T_U)$ of continuous functions on its spectrum T_U. Under the dual map $\rho_U^{\star} : \mathsf{Spec}(A) \leftarrow \mathsf{Spec}(A/U)$ this spectrum T_U is homeomorphically transformed into a compact set $K_U = \rho_U^\star(T_U)$ in $M = \mathsf{Spec}(A)$, and we get the diagram

$$\begin{tikzcd} & A \arrow[dl, "\rho_U"'] \arrow[dr, "\mathcal{G}_{K_U}"] & \\ A/U \arrow[rr, "\mathcal{G}_U"] & & C(K_U) \end{tikzcd}$$

where $\mathcal{G}_U$ is the Gelfand transform of A/U composed with the dual map to the homeomorphism $T_U \cong K_U$.

Conversely, for each compact set $K \subseteq M$ the set

$$U_K = \Big\{ a \in A : \sup_{t\in K} |t(a)| \le 1 \Big\}$$

is a C^*-neighborhood of zero in A. The corresponding quotient algebra A/U_K will be commutative, hence isomorphic to $C(T_K)$, which is in addition homeomorphic to K. If we denote by $\mathcal{G}_K$ the composition of the Gelfand transform of A with the dual map to the homeomorphism $T_K \cong K$, we obtain a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\rho_{U_K}"'] \arrow[dr, "\mathcal{G}_K"] & \\ A/U_K \arrow[rr] & & C(K) \end{tikzcd}.$$

Together this proves (5.69), and (5.70) and (5.71) become its obvious corollaries. ■

LEMMA 5.51. *If the spectrum $M = \mathsf{Spec}(A)$ of a stereotype algebra A is a k-space, then for each extension $\sigma : A \to C$ in the class* Mor *of all morphisms* (*in* $\mathtt{InvSte}^{\circledast}$) *with respect to the class of C^*-algebras the dual map of spectra*

$$\sigma^\star : \mathsf{Spec}(C) \to \mathsf{Spec}(A) = M, \qquad \sigma(s) = s \circ \sigma, \quad s \in \mathsf{Spec}(C),$$

is a homeomorphism of topological spaces.

Proof. First, $\sigma^\star$ must be an injection, since if some characters $s \ne s' \in \mathsf{Spec}(C)$ have the same image under $\sigma^\star$, i.e.

$$s \circ \sigma = \sigma^\star(s) = \sigma^\star(s') = s' \circ \sigma,$$

then the character $s \circ \sigma = s' \circ \sigma : A \to \mathbb{C}$ has two different continuations on C:

$$\begin{tikzcd} A \arrow[rr, "\sigma"] \arrow[dr, "s\circ\sigma = s'\circ\sigma"'] & & C \arrow[dl, bend right, "s"'] \arrow[dl, bend left, "s'"] \\ & \mathbb{C} & \end{tikzcd}$$

This is impossible, as σ is in particular an extension with respect to the C^*-algebra $\mathbb{C}$.

On the other hand, $\sigma^\star$ is a covering, i.e. for each compact set K in M there is a compact set T in $\mathsf{Spec}(C)$ such that $\sigma^\star(T) \supseteq K$. Indeed, if K is a compact set in $M = \mathsf{Spec}(A)$, then, since $\sigma : A \to C$ is an extension with respect to the class of C^*-algebras, the natural homomorphism $\mathcal{G}_K : A \to C(K)$ has a continuation to C:

$$\begin{tikzcd} A \arrow[rr, "\sigma"] \arrow[dr, "\mathcal{G}_K"'] & & C \arrow[dl, dashed, "\tau_K"] \\ & C(K) & \end{tikzcd}$$

If we now set $T = \tau_K^\star(K)$, then

$$\sigma^\star(T) = \sigma^\star(\tau_K^\star(K)) = \mathcal{G}_K^\star(K) = K.$$

In addition, as $\sigma^\star$ is a covering, it is surjective. Hence $\sigma^\star : \mathsf{Spec}(C) \to \mathsf{Spec}(A)$ is a continuous bijective covering. Since $\mathsf{Spec}(A)$ is a k-space, the map $\sigma^\star$ is open, and thus a homeomorphism. ■

The following result supplements the results of Yu. N. Kuznetsova's paper [24]:

THEOREM 5.52. *If A is a commutative involutive stereotype algebra with a paracompact locally compact involutive spectrum $M = \mathsf{Spec}(A)$, then its Gelfand transform $\mathcal{G}_A : A \to C(M)$ is its continuous envelope, its Kuznetsova envelope, and its envelope in the classes of all morphisms and all epimorphisms in the category $\mathtt{InvSte}^{\circledast}$ with respect to the class of C^*-algebras:*

$$C(M) = A^\diamondsuit = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{Epi}} A = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{Mor}} A = \varprojlim \mathcal{C}_A.$$

Proof. The equality $C(M) = \varprojlim \mathcal{C}_A$ was already proved in Theorem 5.50. By Proposition 5.49, the morphism $\mathcal{G}_A : A \to C(M)$ is a dense epimorphism, and by (5.42) we have

$$C(M) = A^\diamondsuit = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{DEpi}} A = \mathsf{Env}_{\mathsf{C}^*}^{\mathsf{Epi}} A = \varprojlim \mathcal{C}_A.$$

It remains to prove

$$\mathsf{Env}_{C^*}^{\mathsf{Mor}} A = C(M),$$

where Mor is the class of all morphisms in $\mathtt{InvSte}^{\circledast}$. Let us first show that $\mathcal{G}_A : A \to C(M)$ is an extension of A with respect to the class of C^*-algebras. Let $\varphi : A \to B$ be a morphism into a C^*-algebra B. To construct a dashed arrow φ' for (3.3), that is,

$$\begin{tikzcd} A \arrow[rr, "\mathcal{G}_A"] \arrow[dr, "\varphi"'] & & C(M) \arrow[dl, dashed, "\varphi'"] \\ & B & \end{tikzcd}$$

it is sufficient to assume that B is commutative and $\varphi(A)$ is dense in B (since otherwise we can replace B by the closure $\overline{\varphi(A)}$ in B, and this is a commutative subalgebra in B). By commutativity, B has the form $C(K)$, and from the density of $\varphi(A)$ in B the compact space K is injectively embedded in $M = \mathsf{Spec}(A)$. Thus our diagram can be represented in the form

$$\begin{tikzcd} A \arrow[rr, "\mathcal{G}_A"] \arrow[dr, "\mathcal{G}_K"'] & & C(M) \arrow[dl, dashed, "\varphi'"] \\ & C(K) & \end{tikzcd}$$

where K is a compact set in M, and $\mathcal{G}_K$ is defined in (5.68). It is clear that φ' can now be defined as the restriction map π_K from M to K, which we considered above:

$$\begin{tikzcd} A \arrow[rr, "\mathcal{G}_A"] \arrow[dr, "\mathcal{G}_K"'] & & C(M) \arrow[dl, dashed, "\pi_K"] \\ & C(K) & \end{tikzcd}$$

And this dashed arrow is unique since $\mathcal{G}_A$ is an epimorphism by Proposition 5.49.

Let us now check that $\mathcal{G}_A : A \to C(M)$ is a maximal extension, i.e. if we take another extension $\sigma : A \to C$, then there exists a morphism $\upsilon : C \to C(M)$ such that

$$\begin{tikzcd} & A \arrow[dl, "\sigma"'] \arrow[dr, "\mathcal{G}_A"] & \\ C \arrow[rr, dashed, "\upsilon"] & & C(M) \end{tikzcd} \tag{5.72}$$

By Lemma 5.51 the dual map $\sigma^\star : \mathsf{Spec}(C) \to \mathsf{Spec}(A) = M$ is a homeomorphism. Therefore, the following map is defined:

$$\upsilon : C \to C(M), \qquad \upsilon(y)(t) = \underbrace{(\sigma^*)^{-1}(t)}_{\substack{\rotatebox{90}{$\in$} \\ \mathsf{Spec}(C)}}(y), \quad y \in C,\, t \in M.$$

It is trivially checked that this is a morphism of involutive stereotype algebras. In addition (5.72) will be commutative:

$$\upsilon(\sigma(x))(t) = (\sigma^*)^{-1}(t)(\sigma(x)) = \sigma^*((\sigma^*)^{-1}(t))(x) = t(x) = \mathcal{G}_A(x)(t), \qquad x \in A,\, t \in M,$$

i.e. $\upsilon \circ \sigma = \mathcal{G}_A$.

It remains to verify that the dashed arrow in (5.72) is unique. Suppose that υ' is another dashed arrow with the same properties:

$$\upsilon \circ \sigma = \mathcal{G}_A = \upsilon' \circ \sigma. \tag{5.73}$$

If υ and υ' are different, they do not coincide on some vector $y \in C$:

$$\upsilon(y) \neq \upsilon'(y).$$

Here on both sides there are functions on M, so for some $t \in M$,

$$\upsilon(y)(t) \neq \upsilon'(y)(t).$$

Let

$$s(z) = \upsilon(z)(t), \quad s'(z) = \upsilon'(z)(t), \qquad z \in C.$$

Then two different characters on C give the same character after composition with σ:

$$s(\sigma(x)) = \upsilon(\sigma(x))(t) \overset{(5.73)}{=} \upsilon'(\sigma(x))(t) = s'(\sigma(x)), \qquad x \in A.$$

By Lemma 5.51 this is impossible, so our initial supposition that $\upsilon \neq \upsilon'$ is also not true. ■

5.5.4. Fourier transform on a commutative locally compact group. Let G be a commutative locally compact group, $\mathcal{C}(G)$ the algebra of continuous functions on G, and $\mathcal{C}^\star(G)$ the algebra of measures with compact support on G (see Examples 5.4 and 5.8). Let $G^\bullet$ be the dual group of characters on G, i.e. continuous homomorphisms $\chi : G \to \mathbb{T}$ into the circle $\mathbb{T}$ ($G^\bullet$ is endowed with the pointwise algebraic operations and the topology of uniform convergence on compact sets in G), and $\mathcal{F}_G : \mathcal{C}^\star(G) \to \mathcal{C}(G^\bullet)$ the Fourier transform on G, i.e. the homomorphism of algebras acting by the formula

$$\overbrace{\mathcal{F}_G(\alpha)(\chi)}^{\substack{\text{value of the function } \mathcal{F}_G(\alpha)\in\mathcal{C}(G^\bullet)\\ \text{at the point } \chi\in G^\bullet\\ \downarrow}} = \underbrace{\alpha(\chi)}_{\substack{\uparrow\\ \text{action of the functional } \alpha\in\mathcal{C}^\star(G)\\ \text{on the function } \chi\in G^\bullet\subseteq\mathcal{C}(G)}} \qquad (\chi\in G^\bullet,\ \alpha\in\mathcal{C}^\star(G))$$

The following observation belongs to Yu. N. Kuznetsova [24]:

THEOREM 5.53. *For each commutative locally compact group G its Fourier transform $\mathcal{F}_G:\mathcal{C}^\star(G)\to\mathcal{C}(G^\bullet)$ is a continuous envelope of the algebra $\mathcal{C}^\star(G)$, and it coincides with the Kuznetsova envelope and with the envelopes with respect to the class of C^*-algebras in the classes* Mor *of all morphisms and* Epi *of all epimorphisms (in the categories* $\mathtt{InvTopAlg}$ *and* $\mathtt{InvSte}^{\circledast}$*):*

$$\mathcal{F}_G=\diamondsuit_{\mathcal{C}^\star(G)}=\mathsf{env}^{\mathsf{DEpi}}_{\mathsf{C}^*}\mathcal{C}^\star(G)=\mathsf{env}^{\mathsf{Epi}}_{\mathsf{C}^*}\mathcal{C}^\star(G)=\mathsf{env}^{\mathsf{Mor}}_{\mathsf{C}^*}\mathcal{C}^\star(G)=\varprojlim\mathcal{C}_{\mathcal{C}^\star(G)}.$$

Proof. The spectrum of the algebra $\mathcal{C}^\star(G)$ is homeomorphic to $G^\bullet$, so everything follows from Theorem 5.52. ∎

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