ANNALES
POLONICI MATHEMATICI
116.2 (2016)

Asymptotic behaviour of Besov norms via wavelet type basic expansions

Anna Kamont (Gdańsk)

Abstract. J. Bourgain, H. Brezis and P. Mironescu [in: J. L. Menaldi et al. (eds.), Optimal Control and Partial Differential Equations, IOS Press, Amsterdam, 2001, 439–455] proved the following asymptotic formula: if $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, $1 \le p < \infty$ and $f \in W^{1,p}(\Omega)$, then

$$\lim_{s\nearrow 1} (1-s) \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\|x-y\|^{d+sp}}\, dx\, dy = K \int_\Omega |\nabla f(x)|^p\, dx,$$

where K is a constant depending only on p and d.

The double integral on the left-hand side of the above formula is an equivalent seminorm in the Besov space $B_p^{s,p}(\Omega)$. The purpose of this paper is to obtain analogous asymptotic formulae for some other equivalent seminorms, defined using coefficients of the expansion of f with respect to a wavelet or wavelet type basis. We cover both the case of the usual (isotropic) Besov and Sobolev spaces, and the Besov and Sobolev spaces with dominating mixed smoothness. We also treat Besov type spaces defined in terms of a Ditzian–Totik modulus of smoothness, but for a restricted range of parameters only.

1. Introduction. The starting point for this paper is the following result by J. Bourgain, H. Brezis and P. Mironescu [5]: if $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, $1 \le p < \infty$ and $f \in W^{1,p}(\Omega)$, then

$$\lim_{s\nearrow 1} (1-s) \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\|x-y\|^{d+sp}}\, dx\, dy = K \int_\Omega |\nabla f(x)|^p\, dx, \tag{1.1}$$

where K is a constant depending only on p and d, and $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^d$. This result has attracted a lot of interest. V. Maz'ya and T. Shaposhnikova [25] obtained a version of the above result when

2010 *Mathematics Subject Classification*: Primary 46E35.
Key words and phrases: Besov spaces, Sobolev spaces, embeddings, wavelet and wavelet type bases.
Received 7 November 2014; revised 23 November 2015.
Published online 2 March 2016.

DOI: 10.4064/ap3540-11-2015

$s \searrow 0$:

$$\lim_{s\searrow 0} s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{\|x-y\|^{d+sp}}\,dx\,dy = K \int_{\mathbb{R}^d} |f(x)|^p\,dx, \tag{1.2}$$

where K is another constant depending on p and d.

In the terminology of [4] or [33], the double integral on the left-hand sides of these equalities is an equivalent seminorm in the Besov space $B_p^{s,p}(\Omega)$ or $B_p^{s,p}(\mathbb{R}^d)$. Therefore, the above results have been extended by several authors to give the asymptotic behaviour of other natural seminorms in $B_p^{s,q}(\mathbb{R}^d)$ (see G. E. Karadzhov, M. Milman and J. Xiao [21] or H. Triebel [34]). The seminorms considered in those papers are defined in terms of moduli of smoothness or progressive differences. An example of these results is the following (see [21]): if

$$\|f\|_{p,q,s,k} = \Big(\sum_{|\underline{\alpha}|=k} \int_{\mathbb{R}^d} \frac{\|\Delta_{\underline{h}}^{\underline{\alpha}} f\|_p^q}{\|\underline{h}\|^{d+skq}}\,d\underline{h}\Big)^{1/q}, \tag{1.3}$$

then

$$\lim_{s\nearrow 1}(1-s)\|f\|_{p,q,s,k}^q = \sum_{|\underline{\alpha}|=k} c_{\underline{\alpha}}\|D^{\underline{\alpha}} f\|_p^q, \qquad \lim_{s\searrow 0} s\|f\|_{p,q,s,k}^q = C\|f\|_p^q, \tag{1.4}$$

where $c_{\underline{\alpha}}, C$ are some constants depending on $\underline{\alpha}$, k, d, q; another seminorm considered in [21] is $(\int_0^\infty (\omega_{k,p}(f,t)/t^{sk})^q\,dt/t)^{1/q}$, while in [34] the asymptotic behaviour of the norm $\|f\|_p + (\int_0^1 (\omega_{k,p}(f,t)/t^{sk})^q\,dt/t)^{1/q}$ is discussed.

The Besov spaces $B_p^{s,q}$ with $0 < s < m$ can be identified with real interpolation spaces between L^p and the Sobolev space $W^{m,p}$, with parameters s/m and q. Therefore, these results should also be seen in the context of the paper of M. Milman [29], where a variant of the above results for real interpolation spaces for normal interpolation pairs is obtained. That is, if (X_0, X_1) is an interpolation pair, and $K(f,t)$ is the K-functional for the pair (X_0, X_1), and

$$\lim_{t\to 0} \frac{K(f_0,t)}{t} = \|f_0\|_{X_0}, \qquad \lim_{t\to\infty} K(f_1,t) = \|f_1\|_{X_1}, \tag{1.5}$$

and for $0 < s < 1$, $1 \le q \le \infty$,

$$\|f\|_{(X_0,X_1),s,q} = s^{1/q}(1-s)^{1/q} q^{1/q}\Big(\int_0^\infty \Big(\frac{K(f,t)}{t^s}\Big)^q \frac{dt}{t}\Big)^{1/q},$$

then

$$\lim_{s\nearrow 1} \|f\|_{(X_0,X_1),s,q} = \|f\|_{X_1} \quad \text{and} \quad \lim_{s\searrow 0} \|f\|_{(X_0,X_1),s,q} = \|f\|_{X_0}. \tag{1.6}$$

Let us note a recent paper by R. Arcangéli and J. J. Torrens [2], which can be seen as an extension of the original formulation of (1.1) or (1.2) to higher order of smoothness. Some other related results, including the investigation of the best constants in various embedding theorems, can be found e.g. in J. Bourgain, H. Brezis and P. Mironescu [6], V. Maz'ya and T. Shaposhnikova [26], V. I. Kolyada and A. Lerner [22], and M. Milman and J. Xiao [30].

Another important tool used in the study of Besov and Sobolev spaces is wavelet or wavelet type bases. In this paper, we present a version of the above results using such bases. It is well known that the Sobolev spaces $W^{m,p}$ with $1 < p < \infty$ have equivalent norms, defined in terms of multipliers on wavelet bases. In the case of Besov spaces, there are equivalent norms, which are some weighted norms of the type $\ell^q(\ell^p)$ applied to the sequence of coefficients of the expansion of a function with respect to a wavelet or wavelet type basis. However, for such norms, we cannot expect asymptotic results of the type described above (an obvious counterexample is presented in Section 3.2).

Nevertheless, we shall see that Besov spaces also have equivalent norms, defined with the use of some multipliers on wavelet bases; more precisely, when $1 < p < \infty$, we shall see that the modulus of smoothnes $\omega_{m,p}(f, t)$ is equivalent to the norm of some multiplier on wavelet bases. The idea of such an approach can be traced back to Z. Ciesielski [8]. An important property which we use is unconditionality of wavelet bases in L^p and $W^{m,p}$, so the cases $p = 1$ and $p = \infty$ are excluded from our analysis. For the norms in Besov spaces defined in this way, we get counterparts of (1.1), (1.2) or (1.4). We also get a variant of these results for spaces with dominating mixed smoothness, and for Besov type spaces corresponding to moduli of smoothness introduced by Z. Ditzian and V. Totik [13], with the step of the difference depending on the point, but in the latter case for a restricted range of parameters only.

Let us mention that such an analysis can also be applied in other settings, for example in the case of Besov and Sobolev spaces on smooth manifolds, with the use of wavelet type bases constructed by Z. Ciesielski and T. Figiel [11]. On the other hand, one can consider function spaces on fractal sets and piecewise linear bases in those spaces, constructed by A. Jonsson and A. Kamont [19].

The main fact we use is unconditionality of wavelet bases in L^p, $1 < p < \infty$. Therefore, we have decided to present the technical part of the results in an abstract formulation, starting with a Banach space with an unconditional basis. This is done in Section 2. In that setting, we consider two scales of spaces, which correspond to Sobolev and Besov spaces. First we get some estimates between the norms from these two scales. The counterparts of (1.1), (1.2) or (1.4) follow directly from these estimates. We consider two versions: a one-parameter version, which corresponds to Besov and Sobolev spaces as above, and a multiparameter version, which corresponds to Besov and Sobolev spaces with dominating mixed smoothness. This is done in Sections 2.1 and 2.2, respectively.

Then, in Section 3 we use wavelet or wavelet type bases to translate the results of Section 2 to Besov and Sobolev spaces. More specifically, in Section 3.2 we treat the case of Besov and Sobolev spaces on $\mathbb{R}^d$, and wavelet bases. In Section 3.3, we treat spaces and wavelet type bases on the cube $[0,1]^d$: Section 3.3.2 deals with the isotropic case, while Section 3.3.3 deals with spaces with dominating mixed smoothness. Finally, Section 3.4 treats the spaces corresponding to moduli of smoothness defined by Z. Ditzian and V. Totik.

Some notation. The following notation is used. By $\mathbb{N}$ we denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For fixed $d \in \mathbb{N}$, we set $\mathcal{D} = \{1, \ldots, d\}$. Vectors in $\mathbb{R}^d$ or $\mathbb{N}_0^d$ are denoted by $\underline{t}, \underline{h}, \underline{\alpha}, \underline{j}, \underline{n}$, etc.; in particular, $\underline{1} = (1, \ldots, 1), \underline{0} = (0, \ldots, 0) \in \mathbb{N}_0^d$. If $\underline{k} = (k_1, \ldots, k_d)$ and $A \subset \mathcal{D}$, then we write $\underline{k}_A = (k_{1,A}, \ldots, k_{d,A})$ with $k_{i,A} = k_i$ if $i \in A$ and $k_{i,A} = 0$ if $i \notin A$. For $A \subset \mathcal{D}$ we denote $A^c = \mathcal{D} \setminus A$. Occasionally, to simplify the notation, we use only the "active" parameters of $\underline{k}_A$ or $\underline{t}_A$, i.e. with $i \in A$; thus we identify $\underline{k}_A \in \mathbb{Z}^d$, $\underline{t}_A \in (0, \infty)^d$, etc. with elements of $\mathbb{Z}^{|A|}$, $(0, \infty)^{|A|}$, etc.

For a vector $\underline{l} = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$ denote $|\underline{l}| = l_1 + \cdots + l_d$ and $|\underline{l}|_\infty = \max(l_1, \ldots, l_d)$. We use the following vector notation: for $\underline{n} = (n_1, \ldots, n_d)$ and $\underline{j} = (j_1, \ldots, j_d)$, we denote $\underline{n} \cdot \underline{j} = n_1 j_1 + \cdots + n_d j_d$, $\underline{n}^{\underline{j}} = n_1^{j_1} \cdot \ldots \cdot n_d^{j_d}$, $\underline{n} < \underline{j}$ means that $n_i < j_i$ for all $i = 1, \ldots, d$; analogously, $\underline{n} \le \underline{j}$ means that $n_i \le j_i$ for all $i = 1, \ldots, d$.

We consider various function spaces: L^p spaces, Sobolev spaces $W^{m,p}$, Besov spaces $B_p^{\alpha,q}$ etc., both over $\mathbb{R}^d$ and $[0,1]^d$. If the domain is not explicitly indicated, we have in mind both versions simultaneously.

As usual, for an exponent $1 \le q \le \infty$, we denote by q' the conjugate exponent, $1/q + 1/q' = 1$.

The notation $a(x) \sim b(x)$ means that there are constants $0 < c_1, c_2 < \infty$, independent of the parameter x, such that $c_1 a(x) \le b(x) \le c_2 a(x)$. We also denote $a \wedge b = \min(a, b)$.

2. The abstract version. Let X be a Banach space with an unconditional basis $\mathcal{X}$. We assume that the basis is 1-*unconditional*, that is, if $\mathcal{X} = \{x_v : v \in V\}$, where V is a countable set of indices, then for each sequence $(a_v)_{v \in V}$ of coefficients with finitely many non-zero terms and a sequence $(\theta_v)_{v \in V}$ of scalars with $|\theta_v| \le 1$, $v \in V$, we have

$$\Big\| \sum_{v \in V} \theta_v a_v x_v \Big\| \le \Big\| \sum_{v \in V} a_v x_v \Big\|.$$

We will consider two ways to enumerate $\mathcal{X}$. The first one is suitable for one-parameter results. In this version, we write $V = \bigcup_{j=0}^{\infty} V_j$ with V_j finite or countable and pairwise disjoint, and

$$\mathcal{X} = \bigcup_{j=0}^{\infty} \mathcal{X}_j \quad \text{with} \quad \mathcal{X}_j = \{x_v : v \in V_j\}.$$

In the multiparameter version, we set $V = \bigcup_{\underline{j} \in \mathbb{N}_0^d} V_{\underline{j}}$, with $V_{\underline{j}}$ finite or countable and pairwise disjoint, and

$$\mathcal{X} = \bigcup_{\underline{j} \in \mathbb{N}_0^d} \mathcal{X}_{\underline{j}} \quad \text{with} \quad \mathcal{X}_{\underline{j}} = \{x_v : v \in V_{\underline{j}}\}.$$

2.1. One-parameter version. For each $x \in X$, there is a unique sequence $a(x) = (a_v(x))_{v \in V}$ of coefficients such that $x = \sum_{j=0}^{\infty} \sum_{v \in V_j} a_v(x) x_v$. To simplify the notation, we write a_v instead of $a_v(x)$, and we set $Q_j(x) = \sum_{v \in V_j} a_v x_v$.

Now, we define two scales of spaces, w^α and $b_m^{\alpha;q}$. The model for w^α is the scale of fractional order Sobolev spaces, obtained by complex interpolation of Sobolev spaces of integer order. The model for $b_m^{\alpha;q}$ is the scale of Besov spaces.

Definition 2.1. Let $\alpha \geq 0$. Define

$$w^\alpha = \Big\{x \in X : \sum_{j=0}^{\infty} 2^{j\alpha} Q_j(x) \text{ converges in } X\Big\},$$

with the norm

$$\|x\|_{w^\alpha} = \Big\|\sum_{j=0}^{\infty} 2^{j\alpha} Q_j(x)\Big\|.$$

Remark 2.1. It can be checked that if X is a Banach space over $\mathbb{C}$ and $0 < \alpha < m$, then w^α is a complex interpolation space between w^0 and w^m, more precisely $w^\alpha = (w^0, w^m)_{[\alpha/m]}$, with equivalence of norms, and with equivalence constants not depending on α.

Next, for $x \in X = w^0$, let us estimate the K-functional for the pair (w^0, w^m), i.e.

$$K\Big(x, \frac{1}{2^{mn}}\Big) = \inf\Big\{\|x - y\|_{w^0} + \frac{1}{2^{mn}}\|y\|_{w^m} : y \in w^m\Big\}.$$

Proposition 2.2. *Let w^0, w^m be given by Definition 2.1. Then for each $x \in w^0$ and $n \in \mathbb{Z}$,*

$$\frac{1}{2}\Big\|\sum_{j=0}^{\infty}(2^{m(j-n)} \wedge 1) Q_j(x)\Big\| \leq K\Big(x, \frac{1}{2^{mn}}\Big) \leq \Big\|\sum_{j=0}^{\infty}(2^{m(j-n)} \wedge 1) Q_j(x)\Big\|. \tag{2.1}$$

Proof. We give the proof of Proposition 2.2 for reference, since the argument used here is repeated in other cases as well.

The upper estimate follows by taking $y_n = 0$ for $n < 0$ and $y_n = \sum_{j=0}^{n} Q_j(x)$ for $n \geq 0$. To check the lower estimate, take $y \in w^m$ and consider $z = \sum_{j=0}^{\infty} \sum_{v \in V_j} u_v x_v$, where $u_v = \max(|a_v(x) - a_v(y)|, 2^{m(j-n)}|a_v(y)|)$ for $v \in V_j$. Then by 1-unconditionality of the basis,

$$\|z\| \leq \|x - y\|_{w^0} + \frac{1}{2^{mn}} \|y\|_{w^m}.$$

Considering separately the cases $|a_v(y)| \geq |a_v(x)|/2$ and $|a_v(y)| \leq |a_v(x)|/2$ we get $u_v \geq (2^{m(j-n)} \wedge 1)|a_v(x)|/2$, hence, again by 1-unconditionality,

$$\|z\| \geq \frac{1}{2} \Big\| \sum_{j=0}^{\infty} (2^{m(j-n)} \wedge 1) Q_j(x) \Big\|.$$

This concludes the proof of Proposition 2.4. ■

Because of Proposition 2.2, the space $b_m^{\alpha,q}$ defined below is in fact a real interpolation space $(w^0, w^m)_{\alpha/m,q}$, with an equivalent norm, and with equivalence constants independent of $0 < \alpha < m$:

Definition 2.2. Fix $m > 0$, and let $0 < \alpha < m$ and $1 \leq q \leq \infty$. Define

$$b_m^{\alpha,q} = \Big\{ x \in X : \Big\| \Big(2^{\alpha n} \Big\| \sum_{j=0}^{\infty} (2^{m(j-n)} \wedge 1) Q_j(x) \Big\| \Big)_{n \in \mathbb{Z}} \Big\|_{\ell^q} < \infty \Big\},$$

with the norm

$$\|x\|_{b_m^{\alpha,q}} = \Big\| \Big(2^{\alpha n} \Big\| \sum_{j=0}^{\infty} (2^{m(j-n)} \wedge 1) Q_j(x) \Big\| \Big)_{n \in \mathbb{Z}} \Big\|_{\ell^q}.$$

We are interested in the asymptotic behaviour of $(m - \alpha)^{1/q} \|x\|_{b_m^{\alpha,q}}$ as $s \nearrow m$ and of $\alpha^{1/q} \|x\|_{b_m^{\alpha,q}}$ as $\alpha \searrow 0$. Because of (2.1), we have

$$\limsup_{n \to \infty} 2^{mn} K\Big(x, \frac{1}{2^{mn}}\Big) \sim \|x\|_{w^m}, \quad \liminf_{n \to \infty} 2^{mn} K\Big(x, \frac{1}{2^{mn}}\Big) \sim \|x\|_{w^m},$$
$$\limsup_{n \to \infty} K(x, 2^{mn}) \sim \|x\|_{w^0}, \quad \liminf_{n \to \infty} K(x, 2^{mn}) \sim \|x\|_{w^0}.$$

Therefore, we are in a situation similar to that in M. Milman [29], but with equivalence instead of equality in (1.5). However, in the setting of this section, it is possible to get some inequalities between the norms $\|\cdot\|_{b_m^{\alpha,q}}$ and $\|\cdot\|_{w^{\alpha \pm \epsilon}}$ (Propositions 2.4 and 2.5). Then the asymptotic result (Theorem 2.6) is a consequence of these estimates.

For later convenience, let

$$s(\xi) = \sum_{j=0}^{\infty} 2^{-j\xi} = \frac{2^\xi}{2^\xi - 1} \quad \text{for } \xi > 0.$$

For reference, let us formulate

LEMMA 2.3. *We have*

$$\lim_{\xi\to 0} \xi \cdot s(\xi) = \frac{1}{\ln 2 - 1}.$$

The main result of this section will be a direct consequence of Propositions 2.4 and 2.5 below. Proposition 2.4 contains the upper estimate:

PROPOSITION 2.4. *Let $0 < \alpha < m$ and $\epsilon > 0$ with $\alpha + \epsilon \le m$, and let $1 \le q < \infty$. Then for $x \in w^{\alpha+\epsilon}$ we have*

$$\|x\|^q_{b^{\alpha,q}_m} \le s(q\epsilon)\|x\|^q_{w^{\alpha+\epsilon}} + 2^{-q\alpha} s(q\alpha)\|x\|^q_{w^0}. \tag{2.2}$$

Moreover, for $q = \infty$ and $x \in w^\alpha$ we have

$$\|x\|_{b^{\alpha,\infty}_m} \le \|x\|_{w^\alpha}. \tag{2.3}$$

Proof. Denote

$$F_n(x) = \Big\|\sum_{j=0}^{\infty}(2^{m(j-n)} \wedge 1)Q_j(x)\Big\|.$$

First, consider $n \ge 0$. Then

$$2^{n(\alpha+\epsilon)}F_n(x) = \Big\|\sum_{j=0}^{\infty}(2^{m(j-n)} \wedge 1)2^{(n-j)(\alpha+\epsilon)}2^{j(\alpha+\epsilon)}Q_j(x)\Big\|.$$

Now we have:

$$\begin{aligned} &\text{for } j \le n, \quad (2^{m(j-n)} \wedge 1)2^{(n-j)(\alpha+\epsilon)} = 2^{(n-j)(\alpha+\epsilon-m)} \le 1,\\ &\text{for } j \ge n, \quad (2^{m(j-n)} \wedge 1)2^{(n-j)(\alpha+\epsilon)} = 2^{(n-j)(\alpha+\epsilon)} \le 1. \end{aligned}$$

Therefore, by 1-unconditionality of the basis under consideration,

$$2^{n(\alpha+\epsilon)}F_n(x) \le \Big\|\sum_{j=0}^{\infty} 2^{j(\alpha+\epsilon)}Q_j(x)\Big\| = \|x\|_{w^{\alpha+\epsilon}}.$$

In particular, this proves (2.3).

In case $q < \infty$, the above inequality gives

$$\begin{aligned} \sum_{n=0}^{\infty}(2^{n\alpha}F_n(x))^q &= \sum_{n=0}^{\infty}\big(2^{-n\epsilon}2^{n(\alpha+\epsilon)}F_n(x)\big)^q \\ &\le \|x\|^q_{w^{\alpha+\epsilon}}\sum_{n=0}^{\infty}\frac{1}{2^{n\epsilon q}} = s(q\epsilon)\|x\|^q_{w^{\alpha+\epsilon}}. \end{aligned}$$

For $n < 0$, note that

$$F_n(x) = \Big\|\sum_{j=0}^{\infty}(2^{m(j-n)} \wedge 1)Q_j(x)\Big\| = \Big\|\sum_{j=0}^{\infty} Q_j(x)\Big\| = \|x\|_{w^0}.$$

Therefore

$$\sum_{n<0}(2^{n\alpha}F_n(x))^q = \|x\|_{w^0}^q \sum_{n<0} 2^{nq\alpha} = \frac{s(q\alpha)}{2^{q\alpha}}\|x\|_{w^0}^q.$$

Combining the above results we get (2.2). ■

The lower estimate will be a consequence of the following:

PROPOSITION 2.5. *Let $1 \le q < \infty$, and let $0 < \alpha < m$ and $\eta > 0$ be such that $0 < \alpha - \eta/q' < m$. Then for $x \in b_m^{\alpha,q}$ we have*

$$\frac{s(m-\alpha+\eta/q')^q}{s(\eta)^{q-1}}\|x\|_{w^{\alpha-\eta/q'}}^q + 2^{-q\alpha}s(q\alpha)\|x\|_{w^0}^q \le \|x\|_{b_m^{\alpha,q}}^q. \tag{2.4}$$

Proof. We keep the notation $F_n(x) = \|\sum_{j=0}^\infty (2^{m(j-n)} \wedge 1)Q_j(x)\|$. By Jensen's inequality,

$$\begin{aligned}\sum_{n=0}^\infty (2^{n\alpha}F_n(x))^q &= s(\eta)\sum_{n=0}^\infty 2^{-n\eta}s(\eta)^{-1}(2^{n(\alpha+\eta/q)}F_n(x))^q \\ &\ge s(\eta)\Big(\sum_{n=0}^\infty s(\eta)^{-1}2^{n(\alpha-\eta/q')}F_n(x)\Big)^q.\end{aligned}$$

By the triangle inequality and 1-unconditionality of the basis,

$$\begin{aligned}\sum_{n=0}^\infty 2^{n(\alpha-\eta/q')}F_n(x) &\ge \Big\|\sum_{n=0}^\infty\sum_{j=0}^\infty 2^{n(\alpha-\eta/q')}(2^{m(j-n)}\wedge 1)Q_j(x)\Big\| \\ &= \Big\|\sum_{j=0}^\infty Q_j(x)\sum_{n=0}^\infty 2^{n(\alpha-\eta/q')}(2^{m(j-n)}\wedge 1)\Big\| \\ &\ge \Big\|\sum_{j=0}^\infty Q_j(x)\sum_{n=j}^\infty 2^{n(\alpha-\eta/q')}2^{m(j-n)}\Big\| \\ &= s(m-\alpha+\eta/q')\Big\|\sum_{j=0}^\infty 2^{j(\alpha-\eta/q')}Q_j(x)\Big\| \\ &= s(m-\alpha+\eta/q')\|x\|_{w^{\alpha-\eta/q'}}.\end{aligned}$$

Putting together these inequalities we get

$$\sum_{n=0}^\infty (2^{n\alpha}F_n(x))^q \ge s(m-\alpha+\eta/q')^q s(\eta)^{1-q}\|x\|_{w^{\alpha-\eta/q'}}^q.$$

As in the proof of Proposition 2.4 we have

$$\sum_{n<0}(2^{n\alpha}F_n(x))^q = \frac{s(q\alpha)}{2^{q\alpha}}\|x\|_{w^0}^q.$$

Combining the above results we get (2.4). ■

The main result of this section is the following:

THEOREM 2.6. *Let $1 \le q < \infty$. Then for $x \in w^m$ we have*

$$\lim_{\alpha \nearrow m} (m-\alpha)^{1/q} \|x\|_{b_m^{\alpha,q}} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q} \|x\|_{w^m}, \tag{2.5}$$

and for $x \in \bigcup_{0<\alpha<m} b_m^{\alpha,q}$,

$$\lim_{\alpha \searrow 0} \alpha^{1/q} \|x\|_{b_m^{\alpha,q}} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q} \|x\|_{w^0}. \tag{2.6}$$

For $q = \infty$ the above formulae take the form

$$\lim_{\alpha \nearrow m} \|x\|_{b_m^{\alpha,\infty}} = \|x\|_{w^m} \quad \text{and} \quad \lim_{\alpha \searrow 0} \|x\|_{b_m^{\alpha,\infty}} = \|x\|_{w^0} \tag{2.7}$$

for $x \in w^m$ or $x \in \bigcup_{0<\alpha<m} b_m^{\alpha,\infty}$, respectively.

Comment. Before we proceed with the proof of Theorem 2.6, let us comment on its assumptions in case $\alpha \searrow 0$. Proposition 2.4 implies that if $x \in w^\alpha$ for some $0 < \alpha \le m$, then for each $1 \le q \le \infty$ there is $0 < \beta < \alpha$ such that $x \in b_m^{\beta',q}$ for all $0 < \beta' < \beta$. Conversely, applying Proposition 2.5 (directly in case $1 \le q < \infty$ or in combination with the straightforward embedding $b_m^{\alpha,\infty} \subset b_m^{\beta,q}$ for $0 < \beta < \alpha < m$ and $1 \le q < \infty$), we find that if there are α, q such that $x \in b_m^{\alpha,q}$, then there is $0 < \beta < \alpha$ such that $x \in w^{\beta'}$ for all $0 < \beta' < \beta$. Therefore, without loss of generality we can formulate the assumption in case $\alpha \searrow 0$ as $x \in \bigcup_{0<\alpha\le m} w^\alpha$.

Proof of Theorem 2.6. Clearly, for $x \in w^m$ we have $\lim_{\beta \nearrow m} \|x\|_{w^\beta} = \|x\|_{w^m}$, and if $x \in w^\alpha$ for some $\alpha > 0$ then $\lim_{\beta \searrow 0} \|x\|_{w^\beta} = \|x\|_{w^0}$.

Consider first the case of $1 \le q < \infty$.

Applying Proposition 2.4 with $\epsilon = m - \alpha$ we find

$$(m-\alpha)\|x\|^q_{b_m^{\alpha,q}} \le (m-\alpha)s(q(m-\alpha))\|x\|^q_{w^m} + (m-\alpha)2^{-q\alpha}s(q\alpha)\|x\|^q_{w^0}. \tag{2.8}$$

By Lemma 2.3 we have

$$\lim_{\alpha \nearrow m} (m-\alpha)s(q(m-\alpha)) = \frac{1}{q(\ln 2 - 1)}. \tag{2.9}$$

Moreover,

$$\lim_{\alpha \nearrow m} (m-\alpha)2^{-q\alpha}s(q\alpha) = 0. \tag{2.10}$$

Thus, letting $\alpha \nearrow m$ in (2.8), we get

$$\limsup_{\alpha \nearrow m} (m-\alpha)\|x\|^q_{b_m^{\alpha,q}} \le \frac{1}{q(\ln 2 - 1)} \|x\|^q_{w^m}.$$

To obtain the lower estimate, we apply Proposition 2.5 with $\eta = q(m-\alpha)$ to find

$$(2.11)\qquad (m-\alpha)s(q(m-\alpha))\|x\|^q_{w^{\alpha-(m-\alpha)q/q'}} + (m-\alpha)2^{-q\alpha}s(q\alpha)\|x\|^q_{w^0} \le (m-\alpha)\|x\|^q_{b^{\alpha,q}_m}.$$

Since $\lim_{\alpha\nearrow m}\|x\|_{w^{\alpha-(m-\alpha)q/q'}} = \|x\|_{w^m}$, by (2.9) and (2.10) we get

$$\frac{1}{q(\ln 2-1)}\|x\|^q_{w^m} \le \liminf_{\alpha\nearrow m}(m-\alpha)\|x\|^q_{b^{\alpha,q}_m}.$$

This completes the proof of (2.5).

To prove (2.6), first apply Proposition 2.4 with $\epsilon = \sqrt{\alpha}$ to get

$$\alpha\|x\|^q_{b^{\alpha,q}_m} \le \alpha s(q\sqrt{\alpha})\|x\|^q_{w^{\alpha+\sqrt{\alpha}}} + \alpha 2^{-q\alpha}s(q\alpha)\|x\|^q_{w^0}.$$

Since

$$\lim_{\alpha\searrow 0}\alpha s(q\sqrt{\alpha}) = 0 \quad\text{and}\quad \lim_{\alpha\searrow 0}\alpha 2^{-q\alpha}s(q\alpha) = \frac{1}{q(\ln 2-1)},$$

it follows that

$$\limsup_{\alpha\searrow 0}\alpha\|x\|^q_{b^{\alpha,q}_m} \le \frac{1}{q(\ln 2-1)}\|x\|^q_{w^0}.$$

On the other hand, by Proposition 2.5,

$$\alpha 2^{-q\alpha}s(q\alpha)\|x\|^q_{w^0} \le \alpha\|x\|^q_{b^{\alpha,q}_m},$$

which implies

$$\frac{1}{q(\ln 2-1)}\|x\|^q_{w^0} \le \liminf_{\alpha\searrow 0}\alpha\|x\|^q_{b^{\alpha,q}_m}.$$

Altogether we get (2.6).

In case $q=\infty$, by (2.3) we have

$$\limsup_{\alpha\nearrow m}\|x\|_{b^{\alpha,\infty}_m} \le \|x\|_{w^m} \quad\text{and}\quad \limsup_{\alpha\searrow 0}\|x\|_{b^{\alpha,\infty}_m} \le \|x\|_{w^0}.$$

To get the lower estimate, recall that $F_n(x) = \|\sum_{j=0}^\infty (2^{m(j-n)}\wedge 1)Q_j(x)\|$. Taking $n=0$, we see that

$$\sup_{n\ge 0} 2^{n\alpha}F_n(x) \ge F_0(x) = \|x\|_{w^0}.$$

This implies

$$\liminf_{\alpha\searrow 0}\|x\|_{b^{\alpha,\infty}_m} \ge \|x\|_{w^0}.$$

To consider the case $\alpha\nearrow m$, note that by 1-unconditionality of the basis under consideration, for each fixed $n\ge 0$ we have

$$\|x\|_{b^{\alpha,\infty}_m} \ge 2^{n\alpha}F_n(x) \ge 2^{n(\alpha-m)}\Big\|\sum_{j=0}^n 2^{mj}Q_j(x)\Big\|.$$

Letting $\alpha \nearrow m$ we find

$$\liminf_{\alpha \nearrow m} \|x\|_{b_m^{\alpha,\infty}} \geq \Big\|\sum_{j=0}^{n} 2^{mj} Q_j(x)\Big\|.$$

Letting $n \to \infty$ yields

$$\liminf_{\alpha \nearrow m} \|x\|_{b_m^{\alpha,\infty}} \geq \|x\|_{w^m}. \ \blacksquare$$

Comment (continued). While considering the case $\alpha \nearrow m$, we assume in Theorem 2.6 that $x \in w^m$. Note that in case $1 < p < \infty$, the result of [5] is stronger: it says that if the left-hand side of (1.1) is finite, then $f \in W^{1,p}(\Omega)$. Therefore it is natural to ask if the conditions

$$\sup_{0<\alpha<m} (m-\alpha)^{1/q} \|x\|_{b_m^{\alpha,q}} < \infty \quad \text{for } 1 \leq q < \infty \tag{2.12}$$

or

$$\sup_{0<\alpha<m} \|x\|_{b_m^{\alpha,\infty}} < \infty \tag{2.13}$$

guarantee that $x \in w^m$. In fact, Proposition 2.5 implies that for $1 \leq q < \infty$ there is a constant $C = C(m,q)$ such that

$$\sup_{0\leq\alpha<m} \|x\|_{w^\alpha} \leq C \sup_{0<\alpha<m} (m-\alpha)^{1/q} \|x\|_{b_m^{\alpha,q}}.$$

For $q = \infty$, the inequality $2^{n\alpha} F_n(x) \leq \|x\|_{b_m^{\alpha,\infty}}$ implies

$$\sup_{n\geq 0} \Big\|\sum_{j=0}^{\infty} (2^{mj} \wedge 2^{mn}) Q_j(x)\Big\| \leq \sup_{0<\alpha<m} \|x\|_{b_m^{\alpha,\infty}}.$$

Therefore, conditions (2.12) or (2.13) imply that $x \in w^m$ if the space X has the following version of the Fatou property (cf. e.g. [24, p. 30]): for each sequence $\{y_n : n \in \mathbb{N}\} \subset X$ with $y_n = \sum_{v\in V} a_v(y_n) x_v$ such that $\sup_{n\in\mathbb{N}} \|y_n\| < \infty$ and for each $v \in V$ we have $|a_v(y_n)| \leq |a_v(y_{n+1})|$ and the (finite) limit $\lim_{n\to\infty} a_v(y_n) = a_v$ exists, the series $\sum_{v\in V} a_v x_v$ converges in X. Observe that the spaces considered in Sections 3.2, 3.3.2 and 3.4 have this property. (But e.g. $X = c_0$ does not.)

2.2. Multiparameter version. In this section, all parameters are d-dimensional, i.e. $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\underline{m} = (m_1, \ldots, m_d)$, $\underline{j} = (j_1, \ldots, j_d)$, etc.

Now, each $x \in X$ has a unique representation $x = \sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{v\in V_{\underline{j}}} a_v x_v$. To simplify the notation, set $Q_{\underline{j}}(x) = \sum_{v\in V_{\underline{j}}} a_v x_v$.

Now, we define two scales of spaces, $w^{\underline{\alpha}}$ and $b_{\underline{m}}^{\underline{\alpha},q}$. Their respective models are the scales of Sobolev and Besov spaces with dominating mixed smoothness.

Definition 2.3. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \geq 0$. Define

$$w^{\underline{\alpha}} = \Big\{x \in X : \sum_{\underline{j}\in\mathbb{N}_0^d} 2^{\underline{j}\cdot\underline{\alpha}} Q_{\underline{j}}(x) \text{ converges in } X\Big\},$$

with the norm

$$\|x\|_{w^{\underline{\alpha}}} = \Big\|\sum_{\underline{j}\in\mathbb{N}_0^d} 2^{\underline{j}\cdot\underline{\alpha}} Q_{\underline{j}}(x)\Big\|.$$

Definition 2.4. Fix $\underline{m} = (m_1, \ldots, m_d)$ with $m_i > 0$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < m_i$ and $1 \leq q \leq \infty$. Define

$$b_{\underline{m}}^{\underline{\alpha},q} = \Big\{x \in X : \Big\|\Big(2^{\underline{n}\cdot\underline{\alpha}}\Big\|\sum_{\underline{j}\in\mathbb{N}_0^d}\prod_{i=1}^{d}(2^{m_i(j_i-n_i)} \wedge 1)Q_{\underline{j}}(x)\Big\|\Big)_{\underline{n}\in\mathbb{Z}^d}\Big\|_{\ell^q} < \infty\Big\},$$

with the norm

$$\|x\|_{b_{\underline{m}}^{\underline{\alpha},q}} = \Big\|\Big(2^{\underline{n}\cdot\underline{\alpha}}\Big\|\sum_{\underline{j}\in\mathbb{N}_0^d}\prod_{i=1}^{d}(2^{m_i(j_i-n_i)} \wedge 1)Q_{\underline{j}}(x)\Big\|\Big)_{\underline{n}\in\mathbb{Z}^d}\Big\|_{\ell^q}.$$

The link between the two scales $w^{\underline{\alpha}}$ and $b_{\underline{m}}^{\underline{\alpha},q}$ is the method of real interpolation for 2^d-tuples of spaces by D. L. Fernandez [14], which we recall now. For $A \subset \mathcal{D} = \{1, \ldots, d\}$, let Y_A be a Banach space, and assume that there is a space Y such that $Y_A \subset Y$ for each $A \subset \mathcal{D}$. Then for $y \in \sum_{A\subset\mathcal{D}} Y_A$ and $\underline{t} = (t_1, \ldots, t_d)$ with $t_i > 0$, let

$$K(y, \underline{t}, \{Y_A\}_{A\subset\mathcal{D}}) = \inf\Big\{\sum_{A\subset\mathcal{D}} \underline{t}^{\underline{1}_A}\|y_A\|_{Y_A} : \\ y = \sum_{A\subset\mathcal{D}} y_A \text{ with } y_A \in Y_A \text{ for } A \subset \mathcal{D}\Big\}.$$

The following proposition, which is a multiparameter counterpart of Proposition 2.2, shows that the spaces $b_{\underline{m}}^{\underline{\alpha},q}$ defined in Definition 2.4 are special cases of interpolation spaces considered by D. L. Fernandez [14].

Proposition 2.7. *Fix $\underline{m} = (m_1, \ldots, m_d)$, and consider the family of spaces $\{w^{\underline{m}_A}\}_{A\subset\mathcal{D}}$. Then for each $x \in X$ with $x = \sum_{\underline{j}\in\mathbb{N}_0^d} Q_{\underline{j}}(x)$ and $\underline{t}_{\underline{n},\underline{m}} = (1/2^{n_1m_1}, \ldots, 1/2^{n_dm_d})$, where $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, we have*

$$K(x, \underline{t}_{\underline{n},\underline{m}}, \{w^{\underline{m}_A}\}_{A\subset\mathcal{D}}) \sim \Big\|\sum_{\underline{j}\in\mathbb{N}_0^d}\prod_{i=1}^{d}(2^{m_i(j_i-n_i)} \wedge 1)Q_{\underline{j}}(x)\Big\|.$$

The equivalence constants can be taken to be 1 and 2^{-d}.

Proof. The proof is analogous to that of Proposition 2.2, but we give the main argument for later reference. For the upper estimate, it is enough to take

$$y_A = \sum_{\underline{j}:\, j_i \le n_i \text{ for } i\in A,\, j_i > n_i \text{ for } i\notin A} Q_{\underline{j}}(x).$$

To get the lower estimate, take $y_A \in w^{\underline{m}_A}$ such that $x = \sum_{A\subset\mathcal{D}} y_A$. Then $x = \sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{v\in V_{\underline{j}}} a_v x_v$ and $y_A = \sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{v\in V_{\underline{j}}} b_{v,A} x_v$, with the series convergent in $X = w^{\underline{0}}$. Then take

$$u_v = \max_{A\subset\mathcal{D}} 2^{\underline{m}_A\cdot(\underline{j}_A - \underline{n}_A)} |b_{v,A}| \quad \text{ for } v \in V_{\underline{j}},$$

and consider $z = \sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{v\in V_{\underline{j}}} u_v x_v$. Then

$$\|z\| \le \sum_{A\subset\mathcal{D}} \underline{t}^{\underline{1}_A} \|y_A\|_{w^{\underline{m}_A}}.$$

On the other hand, we have $a_v = \sum_{A\subset\mathcal{D}} b_{v,A}$. Therefore for each v there is $A\subset\mathcal{D}$ such that $|b_{v,A}| \ge 2^{-d}|a_v|$, and consequently for $v \in V_{\underline{j}}$ we have

$$|u_v| \ge 2^{-d}|a_v| \min_{A\subset\mathcal{D}} 2^{\underline{m}_A\cdot(\underline{j}_A - \underline{n}_A)} = 2^{-d}|a_v| \prod_{i=1}^{d} (2^{m_i(j_i-n_i)} \wedge 1),$$

which implies

$$\|z\| \ge 2^{-d} \Big\| \sum_{\underline{j}\in\mathbb{N}_0^d} \prod_{i=1}^{d} (2^{m_i(j_i-n_i)} \wedge 1) Q_{\underline{j}}(x) \Big\|.$$

This implies Proposition 2.7 with equivalence constants 1 and 2^{-d}. ∎

Now we formulate the multiparameter versions of Propositions 2.4 and 2.5:

Proposition 2.8. *Let $\underline{m} = (m_1, \ldots, m_d)$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < m_i$ and $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$ with $\epsilon_i > 0$ be such that $\alpha_i + \epsilon_i \le m_i$, and $1 \le q < \infty$. Then for $x \in w^{\underline{\alpha}+\underline{\epsilon}}$,*

$$\|x\|^q_{b^{\underline{\alpha},q}_{\underline{m}}} \le \sum_{E\subset\mathcal{D}} \prod_{i\in E} s(q\epsilon_i) \prod_{i\in E^c} 2^{-q\alpha_i} s(q\alpha_i) \|x\|^q_{w^{\underline{\alpha}_E+\underline{\epsilon}_E}}. \tag{2.14}$$

In case $q = \infty$ and $x \in w^{\underline{\alpha}}$,

$$\|x\|_{b^{\underline{\alpha},\infty}_{\underline{m}}} \le \|x\|_{w^{\underline{\alpha}}}.$$

Proposition 2.9. *Let $1 \le q < \infty$, $\underline{m} = (m_1, \ldots, m_d)$, and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < m_i$ and $\underline{\eta} = (\eta_i, \ldots, \eta_d)$ with $\eta_i > 0$ be such that $0 < \alpha_i - \eta_i/q' < m_i$. Then for $x \in b^{\underline{\alpha},q}_{\underline{m}}$,*

$$\sum_{E\subset\mathcal{D}} \prod_{i\in E} \frac{s(m_i - \alpha_i + \eta_i/q')^q}{s(\eta_i)^{q-1}} \prod_{i\in E^c} 2^{-q\alpha_i} s(q\alpha_i) \cdot \|x\|^q_{w^{\underline{\alpha}_E - \underline{\eta}_E/q'}} \le \|x\|^q_{b^{\underline{\alpha},q}_{\underline{m}}}. \tag{2.15}$$

Proof. The proofs of Propositions 2.8 and 2.9 are analogous to those of Propositions 2.4 and 2.5, and use the following observation: if $E \subset \mathcal{D}$ and $\underline{n} = (n_1, \ldots, n_d)$ with $n_i \geq 0$ for $i \in E$ and $n_i < 0$ for $i \in E^c$, then

$$F_{\underline{n}}(x) = \Big\| \sum_{j \in \mathbb{N}_0^d} \prod_{i=1}^{d} (2^{m_i(j_i - n_i)} \wedge 1) Q_{\underline{j}}(x) \Big\| = \Big\| \sum_{j \in \mathbb{N}_0^d} \prod_{i \in E} (2^{m_i(j_i - n_i)} \wedge 1) Q_{\underline{j}}(x) \Big\|.$$

Thus, if we denote

$$\mathbb{Z}^d(E) = \{\underline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_i \geq 0 \text{ for } i \in E \text{ and } n_i < 0 \text{ for } i \in E^c\},$$

then we have

$$\sum_{\underline{n} \in \mathbb{Z}^d(E)} (2^{\underline{n} \cdot \underline{\alpha}} F_{\underline{n}}(x))^q$$
$$= \prod_{i \in E^c} 2^{-q\alpha_i} s(q\alpha_i) \sum_{i \in E : n_i = 0}^{\infty} \Big(\prod_{i \in E} 2^{n_i \alpha_i} \Big\| \sum_{\underline{j} \in \mathbb{N}_0^d} \prod_{i \in E} (2^{m_i(j_i - n_i)} \wedge 1) Q_{\underline{j}}(x) \Big\| \Big)^q.$$

Then we follow the argument in the proofs of Propositions 2.4 and 2.5. ∎

The multiparameter analogue of Theorem 2.6 is now the following:

THEOREM 2.10. *Let $1 \leq q < \infty$, $\underline{m} = (m_1, \ldots, m_d)$ and $F \subset \mathcal{D}$. Then*

$$\lim_{\underline{\alpha} \to \underline{m}_F} \prod_{i \in F} (m_i - \alpha_i)^{1/q} \prod_{i \in F^c} \alpha_i^{1/q} \cdot \|x\|_{b_{\underline{m}}^{\underline{\alpha}, q}} = \left(\frac{1}{q(\ln 2 - 1)} \right)^{d/q} \|x\|_{w^{\underline{m}_F}}.$$

For $q = \infty$,

$$\lim_{\underline{\alpha} \to \underline{m}_F} \|x\|_{b_{\underline{m}}^{\underline{\alpha}, \infty}} = \|x\|_{w^{\underline{m}_F}}.$$

More precisely, in the above limits $\alpha_i \nearrow m_i$ for $i \in F$ and $\alpha_i \searrow 0$ for $i \in F^c$, and we consider $x \in X = w^{\underline{0}}$ such that there is $\underline{\beta} = (\beta_1, \ldots, \beta_d)$ with $0 < \beta_i < m_i$ such that $x \in w^{\underline{m}_F + \underline{\beta}_{F^c}}$.

Proof. Proposition 2.8 and the assumption $x \in w^{\underline{m}_F + \underline{\beta}_{F^c}}$ guarantee that the norms $\|x\|_{b_{\underline{m}}^{\underline{\alpha}, q}}$ are well-defined when $\underline{\alpha} \to \underline{m}_F$.

The proof is similar to the proof of Theorem 2.6, with the use of Propositions 2.8 and 2.9, so we just give a sketch.

To get the upper estimate in case $1 \leq q < \infty$, apply Proposition 2.8 with $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$ defined as follows: $\epsilon_i = m_i - \alpha_i$ for $i \in F$ and $\epsilon_i = \sqrt{\alpha_i}$ for $i \in F^c$. Then we are led to consider the following products: for each $E \subset \mathcal{D}$,

$$\gamma(\underline{\alpha}, E) = \prod_{i \in F \cap E} (m_i - \alpha_i) s(q(m_i - \alpha_i)) \prod_{i \in F \cap E^c} (m - \alpha_i) 2^{-q\alpha_i} s(q\alpha_i)$$
$$\times \prod_{i \in F^c \cap E} \alpha_i s(q\sqrt{\alpha_i}) \prod_{i \in F^c \cap E^c} \alpha_i 2^{-q\alpha_i} s(q\alpha_i).$$

Then

$$\lim_{\underline{\alpha}\to\underline{m}_F} \gamma(\underline{\alpha}, F) = \left(\frac{1}{q(\ln 2-1)}\right)^d \quad \text{and} \quad \lim_{\underline{\alpha}\to\underline{m}_F} \gamma(\underline{\alpha}, E) = 0 \quad \text{for } E \neq F.$$

With this in hand, the upper estimate follows by Proposition 2.8.

To get the lower estimate in case $1 \le q < \infty$, apply Proposition 2.9 with $\underline{\eta} = (\eta_1, \dots, \eta_d)$ defined as follows: $\eta_i = q(m_i - \alpha_i)$ for $i \in F$ and $\eta_i = \alpha_i$ for $i \in F^c$; since we are interested in the estimate from below, it is enough to consider only the term with $E = F$. Since

$$\prod_{i\in F}(m_i-\alpha_i)s(q(m_i-\alpha_i)) \prod_{i\in F^c} \alpha_i 2^{-q\alpha_i} s(q\alpha_i) \to \left(\frac{1}{q(\ln 2-1)}\right)^d \text{ as } \underline{\alpha}\to\underline{m}_F,$$

the lower estimate follows.

The case $q = \infty$ is treated separately. The upper estimate is an immediate consequence of the corresponding part of Proposition 2.8. The lower estimate is obtained as in the corresponding part of the proof of Theorem 2.6, by considering $F_{\underline{n}}(x)$ with $\underline{n} = (n_1, \dots, n_d)$ such that $n_i \ge 0$ for $i \in F$ and $n_i = 0$ for $i \in F^c$. ■

Comment. Let us discuss the assumption $x \in w^{\underline{m}_F + \underline{\beta}_{F^c}}$ for some $\underline{\beta} = (\beta_1, \dots, \beta_d)$ with $0 < \beta_i < m_i$. Theorem 2.10 is applied in Section 3.3.3 in a setting where the spaces $b^{\underline{\alpha},q}_{\underline{m}}$ have a direct interpretation as spaces of functions with some smoothness for all $\underline{0} < \underline{\alpha} < \underline{m}$. The spaces $w^{\underline{\beta}}$ have such an interpretation when $\underline{\beta} \in \mathbb{N}_0^d$. Therefore we would like to have an alternative form of the assumptions in Theorem 2.10 which would allow us to avoid the use of the counterpart of $w^{\underline{\beta}}$ with non-integer $\underline{\beta}$ (cf. also the Comment following Theorem 2.6).

For $F = \mathcal{D}$, the assumption $x \in w^{\underline{m}_F + \underline{\beta}_{F^c}}$ means just that $x \in w^{\underline{m}}$. For $F = \emptyset$, we can take $x \in b^{\underline{\alpha},q}_{\underline{m}}$ for some $\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$ with $0 < \alpha_i < m_i$. Then by Proposition 2.9 (directly in case $1 \le q < \infty$, or in combination with the embedding $b^{\underline{\alpha},\infty}_{\underline{m}} \subset b^{\underline{\beta},q}_{\underline{m}}$ for some $\underline{\beta} = (\beta_1, \dots, \beta_d)$ with $0 < \beta_i < \alpha_i$ and $1 \le q < \infty$) we find that there is $\underline{\beta} = (\beta_1, \dots, \beta_d)$ such that $x \in w^{\underline{\beta}'}$ for each $\underline{\beta}' = (\beta'_1, \dots, \beta'_d)$ with $0 < \beta'_i < \beta_i$.

Let us formulate a version of the assumptions for Theorem 2.10 in this form for other $F \subset \mathcal{D}$ as well. For this, we need to discuss scales of b- and w-spaces with $k = |F^c|$ parameters, but constructed with $Y = w^{\underline{\beta}_F}$ as the initial space, and with $\mathcal{X}$ split as $\mathcal{X} = \bigcup_{\underline{j}'\in\mathbb{N}_0^k} \tilde{\mathcal{X}}_{\underline{j}'}$ with $\tilde{\mathcal{X}}_{\underline{j}'} = \bigcup_{\underline{j}\in\mathbb{N}_0^d:\, \underline{j}_{F^c}=\underline{j}'} \mathcal{X}_{\underline{j}}$. This leads to spaces $w^{\underline{\alpha}_{F^c}}_{\underline{\beta},F}$ and $b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{\beta},F}$ defined by

$$\|x\|_{w^{\underline{\alpha}_{F^c}}_{\underline{\beta},F}} = \Big\| \sum_{\underline{j}\in\mathbb{N}_0^d} 2^{\underline{j}_F\cdot\underline{\beta}_F} \cdot 2^{\underline{j}_{F^c}\cdot\underline{\alpha}_{F^c}} Q_{\underline{j}}(x) \Big\| < \infty$$

(in particular, the series converges in $X = w^{\underline{0}}$), and

$$\|x\|_{b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{\beta},F}} = \Big\| \Big(2^{\underline{n}_{F^c}\cdot\underline{\alpha}_{F^c}} \Big\| \sum_{\underline{j}\in\mathbb{N}_0^d} 2^{\underline{j}_F\cdot\underline{\beta}_F} \prod_{i\in F^c} (2^{m_i(j_i-n_i)} \wedge 1) Q_{\underline{j}}(x) \Big\| \Big)_{\underline{n}_{F^c}\in\mathbb{N}^k} \Big\|_{\ell^q} < \infty.$$

Clearly, $w^{\underline{\alpha}_{F^c}}_{\underline{\beta},F} = w^{\underline{\beta}_F+\underline{\alpha}_{F^c}}$. Note also that $x \in b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{\beta},F}$ iff $T_{\underline{\beta},F}(x) \in b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{0},F}$, where $T_{\underline{\beta},F}(x) = \sum_{\underline{j}\in\mathbb{N}_0^d} 2^{\underline{j}_F\cdot\underline{\beta}_F} Q_{\underline{j}}(x)$.

In this setting, a natural assumption for Theorem 2.10 is $x \in b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{m},F}$ for some $\underline{0} < \underline{\alpha} < \underline{m}$. Note that Proposition 2.9, applied to the k-parameter spaces $w^{\underline{\alpha}_{F^c}}_{\underline{m},F}$ and $b^{\underline{\alpha}_{F^c},q}_{\underline{m},\underline{m},F}$, implies that $x \in w^{\underline{\beta}_{F^c}}_{\underline{m},F} = w^{\underline{m}_F+\underline{\beta}_{F^c}}$ for some $\underline{0} < \underline{\beta} < \underline{\alpha}$. That is, we recover the assumption as formulated in Theorem 2.10.

3. Application to wavelet and wavelet type bases on $\mathbb{R}^d$ and $[0,1]^d$. Let us see what the results of Section 2 mean for wavelet bases on $\mathbb{R}^d$ or wavelet type bases on $[0,1]^d$. We need two properties of such bases: they are unconditional in L^p and in the Sobolev spaces $W^{m,p}$ for $1 < p < \infty$, and the Sobolev spaces have equivalent norms given in terms of a multiplier on the basis under consideration.

We shall discuss two types of bases. The first type of bases are localized wavelet or wavelet type bases. We shall discuss them using the example of wavelet bases on $\mathbb{R}^d$, but there are also bases of this type on $[0,1]^d$ (see e.g. Z. Ciesielski and T. Figiel [11]), and the analysis in this case is fully analogous. This is done in Section 3.2.

The second type of bases are tensor products of one-dimensional bases. We shall discuss them using the example of tensor products of one-dimensional wavelet type bases on $[0,1]^d$, but a similar analysis is also possible on $\mathbb{R}^d$. More precisely, the univariate bases we have in mind are spline bases with dyadic knots as discussed in Z. Ciesielski [7, 9], and their tensor products as discussed by Z. Ciesielski and J. Domsta [10]. Here we use the fact that the bases under consideration are not only bases in $W^{p,m}[0,1]$, but also the derivatives of the basic functions form a basis in $L^p[0,1]$. Because of this additional property, the results we obtain for the tensor product bases are more precise than in the case of localized bases (cf. Theorems 3.4 and 3.6). For the tensor product bases, we show how to apply the results of Section 2 in the case of Sobolev and Besov spaces (Section 3.3.2), but also in the case of Sobolev and Besov spaces with dominating mixed smoothness (Section 3.3.3). Finally, we apply the results of Section 2 to analyse Besov type spaces corresponding to moduli of smoothness defined by Z. Ditzian and V. Totik [13] (Section 3.4); however, this is possible for a limited range of parameters only.

The results we get in the case of isotropic Besov spaces (Theorems 3.4 and 3.6) are parallel to the results of G. E. Karadzhov, M. Milman and J. Xiao [21] and H. Triebel [34], mentioned in the Introduction.

3.1. Sobolev and Besov spaces on $\mathbb{R}^d$ and $[0,1]^d$. First, we recall the definitions of Sobolev and Besov spaces.

For a vector $\underline{l} = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$ denote $D^{\underline{l}} = \frac{\partial^{l_1}}{\partial t_1^{l_1}} \cdots \frac{\partial^{l_d}}{\partial t_d^{l_d}}$. For $m \in \mathbb{N}$, the norm in the Sobolev space $W^{m,p}(\mathbb{R}^d)$ or $W^{m,p}[0,1]^d$ is defined as

$$\|f\|_{W^{m,p}} = \|f\|_p + \sum_{|\underline{l}|=m} \|D^{\underline{l}} f\|_p.$$

We also need Sobolev spaces with dominating mixed smoothness. For a vector $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, the norm in the space $W^{\underline{m},p}(\mathbb{R}^d)$ or $W^{\underline{m},p}[0,1]^d$ is defined as

$$\|f\|_{W^{\underline{m},p}} = \sum_{A \subset \mathcal{D}} \|D^{\underline{m}_A} f\|_p.$$

Now, we recall the definition of Besov spaces. For $\underline{h} \in \mathbb{R}^d$ and $m \in \mathbb{N}$, define $\Delta_{\underline{h}}^m f$, the progressive difference of order m, by

$$\Delta_{\underline{h}} f(\cdot) = f(\cdot + \underline{h}) - f(\cdot),$$
$$\Delta_{\underline{h}}^m f(\cdot) = \Delta_{\underline{h}}(\Delta_{\underline{h}}^{m-1} f)(\cdot) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} f(\cdot + j\underline{h}).$$

Then $\omega_{m,p}(f,t)$, the modulus of smoothness of f of order m in the L^p norm, is defined as

$$\omega_{m,p}(f,t) = \sup_{\|\underline{h}\| \le t} \|\Delta_{\underline{h}}^m f\|_p,$$

where in the case of $[0,1]^d$ the integral in the definition is over the set $\{\underline{t} \in [0,1]^d : \underline{t} + j\underline{h} \in [0,1]^d,\ j = 0, \ldots, m\}$. Fix $0 < \alpha < m$ and $1 \le q \le \infty$. Then the norm in the Besov space $B_p^{\alpha,q}(\mathbb{R}^d)$ or $B_p^{\alpha,q}[0,1]^d$ is, for $1 \le q < \infty$,

$$\|f\|_{B_p^{\alpha,q}} = \|f\|_p + \left(\int_0^\infty \left(\frac{\omega_{m,p}(f,t)}{t^\alpha} \right)^q \frac{dt}{t} \right)^{1/q},$$

while for $q = \infty$,

$$\|f\|_{B_p^{\alpha,\infty}} = \|f\|_p + \sup_{0<t<\infty} t^{-\alpha} \omega_{m,p}(f,t).$$

Recall that the norms for different $m > \alpha$ are equivalent (see e.g. [4] in the case of $\mathbb{R}^d$, or [12] for the argument with the use of the Marchaud inequality in the case of $[0,1]^d$; cf. also [18] for the Marchaud inequality in the multivariate case).

For further reference, we recall the well-known equivalence of moduli of smoothness and a modified K-functional (see e.g. R. A. DeVore and

G. G. Lorentz [12] in the univariate case, and H. Johnen and K. Scherer [18] or C. Bennett and R. Sharpley [3] in the multivariate case):

FACT 3.1. *Let* $1 < p < \infty$ *and* $d, m \in \mathbb{N}$. *Then for each* $f \in L^p$,

$$\omega_{m,p}(f,t) \sim \inf\Big\{\|f-g\|_p + t^m \sum_{|\underline{l}|=m} \|D^{\underline{l}} g\|_p : g \in W^{m,p}\Big\}, \tag{3.1}$$

with equivalence constants independent of f *and* $t > 0$.

Consequently, for $0 < t \le 1$,

$$t^m \|f\|_p + \omega_{m,p}(f,t) \sim \inf\{\|f-g\|_p + t^m \|g\|_{W^{m,p}} : g \in W^{m,p}\},$$

with equivalence constants independent of f *and* t.

In particular, Fact 3.1 explains the well-known relation $(L^p, W^{m,p})_{\alpha/m,p} = B_p^{\alpha,q}$, with equivalence of norms (and equivalence constants independent of $0 < \alpha < m$); see e.g. [4, 3, 12, 33]. Let us mention that Fact 3.1 also holds for $p = 1$ and $f \in L^1$, and for $p = \infty$ and continuous functions, but here we work only with the case $1 < p < \infty$.

We also need Besov spaces with dominating mixed smoothness. To recall their definition, denote $\underline{e}_i = (e_{i,1}, \ldots, e_{i,d})$, where $e_{i,j} = 1$ for $j = i$ and $e_{i,j} = 0$ for $j \ne i$ (i.e. $\underline{e}_i$ is the ith coordinate vector in $\mathbb{R}^d$). Given $\underline{m} = (m_1, \ldots, m_d)$ (possibly with some $m_i = 0$) and $\underline{h} = (h_1, \ldots, h_d)$, denote

$$\Delta_{\underline{h}}^{\underline{m}} = \Delta_{h_1 \underline{e}_1}^{m_1} \circ \cdots \circ \Delta_{h_d \underline{e}_d}^{m_d}.$$

For $\underline{t} = (t_1, \ldots, t_d)$ and $A \subset \mathcal{D}$ define

$$\omega_{\underline{m},p,A}(f, \underline{t}_A) = \sup_{\underline{h}:\, |h_i| \le t_i \text{ for } i \in A} \|\Delta_{\underline{h}}^{\underline{m}_A} f\|_p.$$

(Note that the "active" variables of $\underline{t}_A$ are only those t_i for which $i \in A$; in particular, for $A = \emptyset$, we have $\omega_{\underline{m},p,A}(f, \underline{t}_A) = \|f\|_p$.) Then for $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < m_i$ and $1 \le q < \infty$, the norm in $B_p^{\underline{\alpha},q}$, the Besov space with dominating mixed smoothness, is defined as

$$\|f\|_{B_p^{\underline{\alpha},q}} = \sum_{A \subset \mathcal{D}} \bigg(\int_{(0,\infty)^d} \bigg(\frac{\omega_{\underline{m},p,A}(f, \underline{t}_A)}{\underline{t}_A^{\underline{\alpha}_A}} \bigg)^q \prod_{i \notin A} u(t_i)\, \frac{d\underline{t}_A}{\underline{t}_A^{\underline{1}_A}}\, d\underline{t}_{A^c} \bigg)^{1/q}, \tag{3.2}$$

where $u : \mathbb{R} \to (0, \infty)$ is a fixed function with $\int_{\mathbb{R}} u(t)\, dt = 1$ (the term $\prod_{i \notin A} u(t_i)$ is introduced to take care of the "inactive" variables t_i, $i \notin A$). In case $q = \infty$ we define

$$\|f\|_{B_p^{\underline{\alpha},\infty}} = \sup_{\underline{t} \in (0,1)^d,\, A \subset \mathcal{D}} \underline{t}_A^{-\underline{\alpha}_A} \omega_{\underline{m},p,A}(f, \underline{t}_A). \tag{3.3}$$

We will need the following formula (see e.g. A. Kamont [20]):

FACT 3.2. *Let $1 < p < \infty$, $d \in \mathbb{N}$, $\underline{m} \in \mathbb{N}^d$ and $A \subset \mathcal{D}$. Then for $f \in L^p$,*

$$\omega_{\underline{m},p,A}(f, \underline{t}_A) \sim \inf\Big\{\Big\| f - \sum_{\emptyset \neq B \subset A} g_B \Big\|_p + \sum_{\emptyset \neq B \subset A} \underline{t}^{\underline{m}_B} \omega_{\underline{m},p,A\setminus B}(D^{\underline{m}_B} g_B, \underline{t}_{A\setminus B}) : g_B \in W^{\underline{m}_B,p}\Big\}. \tag{3.4}$$

with equivalence constants independent of f and $\underline{t} \in (0,\infty)^d$.

The proof in [20] is done for $[0,1]^d$ and $0 < t_i \le 1/m_i$, but it can be generalized to $t_i \ge 1/m_i$ by arguments analogous to that in [12, Chapter 6, proof of Theorem 2.4]; it carries over to the case of $\mathbb{R}^d$ as well. As above, there are also versions for $p = 1$ and $f \in L^1$, and for $p = \infty$ and continuous functions, but we will work only with $1 < p < \infty$.

3.2. Sobolev and Besov spaces on $\mathbb{R}^d$ and localized wavelet bases. First, let us consider the case of wavelet bases on $\mathbb{R}^d$.

Let ϕ be an orthonormal scaling function on $\mathbb{R}^d$ with the corresponding set of orthonormal wavelets $\{\psi_l : l = 1, \dots, 2^d - 1\}$. For a function f defined on $\mathbb{R}^d$ we use the usual notation

$$f_{j,k}(\cdot) = 2^{dj/2} f(2^j \cdot - k), \qquad j \in \mathbb{Z},\ k \in \mathbb{Z}^d.$$

Then we can consider two types of wavelet systems:

$$\{\phi_{0,k} : k \in \mathbb{Z}^d\} \cup \{\psi_{l,j,k} : j \ge 0,\ k \in \mathbb{Z}^d,\ l = 1, \dots, 2^d - 1\},$$

or

$$\{\psi_{l,j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ l = 1, \dots, 2^d - 1\}.$$

It is well known that—under suitable conditions on the smoothness and decay of ϕ, ψ_l, e.g. in the terminology of Y. Meyer [28], under the assumption of r-regularity of the wavelet system under consideration with $m < r$ (see [28, Chapter 6])—both these systems are unconditional bases in $L^p(\mathbb{R}^d)$ and $W^{m,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and $m < r$. More precisely, for $f \in L^p(\mathbb{R}^d)$ with

$$\begin{aligned} f &= \sum_{k \in \mathbb{Z}^d} (f, \phi_{0,k}) \phi_{0,k} + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{2^d-1} (f, \psi_{l,j,k}) \psi_{l,j,k} \\ &= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{2^d-1} (f, \psi_{l,j,k}) \psi_{l,j,k}, \end{aligned} \tag{3.5}$$

we have

$$\begin{aligned} \|f\|_p &\sim \Big\| \Big(\sum_{k \in \mathbb{Z}^d} |(f, \phi_{0,k})|^2 \chi_{0,k}^2 + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{2^d-1} |(f, \psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p \\ &\sim \Big\| \Big(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{l=1}^{2^d-1} |(f, \psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p, \end{aligned} \tag{3.6}$$

where $\chi(\cdot) = \chi_{[0,1)^d}(\cdot)$ is the characteristic function of $[0,1)^d$. Moreover, if $f \in W^{p,m}(\mathbb{R})$, $1 < p < \infty$, $m < r$, then

(3.7) $$\|f\|_{W^{p,m}} \sim \Big\| \Big(\sum_{k\in\mathbb{Z}^d} |(f,\phi_{0,k})|^2 \chi_{0,k}^2 + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{2^d-1} 2^{2mj} |(f,\psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p \sim \Big\| \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{2^d-1} \max(1, 2^{2jm}) |(f,\psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p.$$

The equivalence constants depend only on p, m and the wavelet system involved. On the other hand, for $f \in B_p^{\alpha,q}(\mathbb{R}^d)$,

(3.8) $$\|f\|_{B_p^{\alpha,q}} \sim \Big(\Big(\sum_{k\in\mathbb{Z}^d} |(f,\phi_{0,k})|^p \Big)^{q/p} + \sum_{j=0}^{\infty} \Big(2^{pj(d/2-d/p+\alpha)} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{2^d-1} |(f,\psi_{l,j,k})|^p \Big)^{q/p} \Big)^{1/q},$$

but the equivalence constants depend on α as well; cf. e.g. Y. Meyer [28], and analogous results for Besov spaces on the interval $[0,1]$, on the cube $[0,1]^d$ or on a manifold can be found e.g. in earlier papers by S. Ropela [31], or by Z. Ciesielski and T. Figiel [11]. Therefore, we cannot expect asymptotic results as in Section 2 when using the above equivalent norm (3.8) in the Besov space.

Indeed, consider the case $d = 1$. Given a sequence of coefficients $(c_j,\ j \geq 0)$, consider two functions on $\mathbb{R}$:

$$f_1 = \sum_{j\geq 0} \sum_{k=0}^{2^j-1} c_j \psi_{j,k} \quad \text{and} \quad f_2 = \sum_{j\geq 0} \sum_{k=j2^j}^{j2^j+2^j-1} c_j \psi_{j,k}.$$

Then for the equivalent coefficient norm in $B_p^{\alpha,q}(\mathbb{R})$ given by the right-hand side of (3.8) we have

$$\|f_1\|_{B_p^{\alpha,q}} \sim \Big(\sum_{j\geq 0} 2^{qj(1/2+\alpha)} |c_j|^q \Big)^{1/q} \sim \|f_2\|_{B_p^{\alpha,q}(\mathbb{R})}.$$

However, by (3.6) and (3.7) we have

$$\|f_1\|_p \sim \Big(\sum_{j\geq 0} 2^j |c_j|^2 \Big)^{1/2} \quad \text{and} \quad \|f_1\|_{W^{p,m}} \sim \Big(\sum_{j\geq 0} 2^{j(1+2m)} |c_j|^2 \Big)^{1/2},$$

while

$$\|f_2\|_p \sim \Big(\sum_{j\geq 0} 2^{pj/2} |c_j|^p \Big)^{1/p} \quad \text{and} \quad \|f_2\|_{W^{p,m}} \sim \Big(\sum_{j\geq 0} 2^{pj(m+1/2)} |c_j|^p \Big)^{1/p}.$$

On the other hand, we can use the coefficients of the wavelet expansion of $f \in L^p(\mathbb{R}^d)$ to define an equivalent norm in $B_p^{\alpha,q}(\mathbb{R}^d)$ by means of Definition 2.2. That is, our space is now $w^0 = L^p(\mathbb{R}^d)$ with the norm defined by

$$\|f\|_{L^p,1} = \Big\| \Big(\sum_{k\in\mathbb{Z}^d} |(f,\phi_{0,k})|^2 \chi_{0,k}^2 + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{l=1}^{2^d-1} |(f,\psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p \tag{3.9}$$

or

$$\|f\|_{L^p,2} = \Big\| \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{2^d-1} |(f,\psi_{l,j,k})|^2 \chi_{j,k}^2 \Big)^{1/2} \Big\|_p. \tag{3.10}$$

The decomposition of w^0 is defined by

$$Q_0 f = \sum_{k\in\mathbb{Z}^d} (f,\phi_{0,k})\phi_{0,k} = \sum_{j<0} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{2^d-1} (f,\psi_{l,j,k})\psi_{l,j,k},$$

$$Q_j f = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{2^d-1} (f,\psi_{l,j-1,k})\psi_{l,j-1,k} \quad \text{for } j \ge 1.$$

Thus, there are two equivalent norms in $w^m = W^{m,p}(\mathbb{R}^d)$:

$$\|f\|_{W^{m,p},i} = \Big\| \sum_{j\ge 0} 2^j Q_j f \Big\|_{L^p,i}, \quad i = 1,2. \tag{3.11}$$

Applying Fact 3.1 in combination with (3.6) and (3.7), we find

Proposition 3.3. *For fixed $1 < p < \infty$, $m \in \mathbb{N}$ and an r-regular wavelet basis with $m < r$, let $f \in L^p(\mathbb{R}^d)$ be given by (3.5). Then for $n \in \mathbb{Z}$,*

$$(1 \wedge 2^{-nm})\|f\|_p + \omega_{m,p}(f,2^{-n}) \sim \Big\| \sum_{j\ge 0} (2^{m(j-n)} \wedge 1) Q_j f \Big\|_{L^p,i}, \quad i = 1,2,$$

with equivalence constants independent of f and n.

Proof. Once Fact 3.1 and equivalences (3.6), (3.7) are at hand, the proof is analogous to that of Proposition 2.2, so we omit the details. ∎

Thus, we can define equivalent norms in Besov spaces $B_p^{\alpha,q}(\mathbb{R}^d)$ using multipliers on wavelet basis:

$$\|f\|_{B_p^{\alpha,q},i} = \Big\| \Big(2^{n\alpha} \Big\| \sum_{j\ge 0} (2^{m(j-n)} \wedge 1) Q_j f \Big\|_{L^p,i} \Big)_{n\in\mathbb{Z}} \Big\|_{\ell^q}, \quad i = 1,2. \tag{3.12}$$

Because of (3.6) and Proposition 3.3 we have, for $1 \le q < \infty$,

$$\|f\|_{B_p^{\alpha,q},i} \sim \max\big(s(q(m-\alpha)), s(q\alpha)\big)^{1/q} \|f\|_p + \Big(\int_0^1 \Big(\frac{\omega_{m,p}(f,t)}{t^\alpha} \Big)^q \frac{dt}{t} \Big)^{1/q}, \tag{3.13}$$

while for $q = \infty$,

(3.14) $$\|f\|_{B_p^{\alpha,q},i} \sim \max\Big(\|f\|_p, \sup_{0<t\le 1} t^{-\alpha}\omega_{m,p}(f,t)\Big),$$

with equivalence constants independent of $0 < \alpha < m$ and $1 \le q \le \infty$.

Formula (3.12) coincides with the norm obtained via Definition 2.2 for $\|\cdot\|_{L^p,i}$ and $\|\cdot\|_{W^{m,p},i}$, $i = 1, 2$. Therefore, we can apply Theorem 2.6. To summarize, we get

THEOREM 3.4. *For fixed $1 < p < \infty$, $m \in \mathbb{N}$ and an r-regular wavelet basis with $m < r$, let $f \in L^p(\mathbb{R}^d)$ be given by* (3.5)*. For $i = 1, 2$, let the norms $\|\cdot\|_{L^p,i}$, $\|\cdot\|_{W^{m,p},i}$ and $\|\cdot\|_{B_p^{\alpha,q},i}$ be given by* (3.9)–(3.12)*. Then*

$$\|f\|_{L^p,i} \sim \|f\|_p, \qquad \|f\|_{W^{m,p},i} \sim \|f\|_{W^{m,p}},$$

while the equivalent form of $\|\cdot\|_{B_p^{\alpha,q},i}$ is given by (3.13) *and* (3.14)*, with equivalence constants independent of $0 < \alpha < m$ and $1 \le q \le \infty$.*

If $f \in B_p^{\epsilon,q}(\mathbb{R}^d)$ for some $\epsilon > 0$ and $1 \le q < \infty$ then

$$\lim_{\alpha\searrow 0} \alpha^{1/q}\|f\|_{B_p^{\alpha,q},i} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q}\|f\|_{L^p,i},$$

and for $f \in W^{m,p}(\mathbb{R}^d)$,

$$\lim_{\alpha\nearrow m} (m-\alpha)^{1/q}\|f\|_{B_p^{\alpha,q},i} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q}\|f\|_{W^{m,p},i}.$$

If $f \in B_p^{\epsilon,\infty}(\mathbb{R}^d)$ for some $\epsilon > 0$ then

$$\lim_{\alpha\searrow 0} \|f\|_{B_p^{\alpha,\infty},i} = \|f\|_{L^p,i},$$

and for $f \in W^{m,p}(\mathbb{R}^d)$,

$$\lim_{\alpha\nearrow m} \|f\|_{B_p^{\alpha,\infty},i} = \|f\|_{W^{m,p},i}.$$

3.3. Tensor products of univariate wavelet type bases in function spaces on $[0,1]^d$. Now, we present a version of the results on the cube $[0,1]^d$. Here in the d-dimensional case, we use bases consisting of tensor products of univariate bases. We exploit the fact that the univariate bases under consideration are not only (unconditional) bases in $L^p[0,1]$ and $W^{m,p}[0,1]$, but they are simultaneous bases, that is, the system consisting of the derivatives of the basic functions is again an (unconditional) basis in $L^p[0,1]$.

3.3.1. *Bases.* We start by recalling the main properties of the spline bases to be used. The properties listed below can be found in Z. Ciesielski [7, 9], Z. Ciesielski and J. Domsta [10], S. Ropela [31, 32], or they are direct consequences of univariate results; in particular, unconditionality of d-variate tensor product systems considered below in $L^p[0,1]^d$ or $W^{m,p}[0,1]^d$,

$1 < p < \infty$, is a consequence of the unconditionality of the univariate systems in $L^p[0,1]$, $1 < p < \infty$, and C. A. McCarthy's result on boundedness of commuting boolean algebras of projections (see C. A. McCarthy [27]).

1. Fix $r \in \mathbb{N}$. The system under consideration, denoted by $\Psi_r = \{\psi_n : n \ge -r+2\}$, is an orthonormal system in $L^2[0,1]$ with dyadic structure, regularity and decay as described below.

2. Dyadic structure: $\Psi_r = \bigcup_{j\ge 0} \Psi_{r,j}$, where $\Psi_{r,0} = \{\psi_n : -r+2 \le j \le 1\}$ and $\Psi_{r,j} = \{\psi_n : 2^{j-1}+1 \le n \le 2^j\}$. The set of indices of $\Psi_{r,j}$ is $U_{0,r} = \{-r+2,\dots,1\}$ for $j=0$, and $U_j = \{2^{j-1}+1,\dots,2^j\}$ for $j \ge 1$.

3. Regularity: $\Psi_r \subset C^{r-2}[0,1]$, ψ_n is a polynomial of degree $n+r-2$ for $n \le 1$, while for $n \in U_j$ with $j \ge 1$, ψ_n is a piecewise polynomial of degree $r-1$ and with dyadic knots $\{l/2^j : l = 0,\dots,2^j\}$; for $n \in U_j$, the derivative $\psi_n^{(r-1)}$ exists and is constant on each dyadic interval $(l/2^j,(l+1)/2^j)$, $l = 0,\dots,2^j-1$.

4. Exponential decay: there are $0 < \theta < 1$ and $C > 0$ such that

$$|\psi_n(t)| \le C 2^{j/2} \theta^{2^{j-1}|t-k/2^{j-1}|}, \qquad n \in U_j,\ n = 2^{j-1}+k,\ 1 \le k \le 2^{j-1}.$$

5. Together with $\Psi_r = \{\psi_n : n \ge -r+2\}$, we consider systems $\Psi_r^{(s)}$ and $\Psi_r^{(-s)}$ defined as follows. For $0 \le s \le r-1$, let

$$\Psi_{r,0}^{(s)} = \left\{\psi_{s;n} = \frac{d^s}{dt^s}\psi_n : n \in U_{0,r-s}\right\},$$

$$\Psi_{r,j}^{(s)} = \left\{\psi_{s;n} = 2^{-sj}\frac{d^s}{dt^s}\psi_r : n \in U_j\right\}.$$

Moreover, denoting $Hf(t) = \int_t^1 f(u)\,du$, we define

$$\Psi_{r,j}^{(-s)} = \{\psi_{-s,n} = 2^{sj} H^s \psi_r : n \in U_j\},$$

the set of indices being $U_{0,r-s}$ in case $j=0$. Then we set $\Psi_r^{(\pm s)} = \bigcup_{j\ge 0} \Psi_{r,j}^{(\pm s)}$. These functions have the same decay as Ψ_r. The systems $\Psi_r^{(s)}$ and $\Psi_r^{(-s)}$ are biorthogonal.

6. Each system $\Psi_r^{(\pm s)}$ is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$. In addition if $|s| \le r-1$ and

$$f = \sum_{n\in U_{0,r-|s|}} a_n \psi_{s;n} + \sum_{j=1}^{\infty} \sum_{n\in U_j} a_n \psi_{s;n} \quad \text{with } a_n = (f_n, \psi_{-s,n}), \tag{3.15}$$

then

$$\|f\|_p \sim \Big\| \Big(\sum_{n\in U_{0,r-|s|}} |a_n|^2 \kappa_n^2 + \sum_{j=1}^{\infty} \sum_{n\in U_j} |a_n|^2 \kappa_n^2 \Big)^{1/2} \Big\|_p, \tag{3.16}$$

where $\kappa_n = 1$ for $n \in U_{0,r-|s|}$, and $\kappa_n = 2^{(j-1)/2}\chi_{[(k-1)/2^{j-1},k/2^{j-1})}$ for $n \in U_j$, $n = 2^{j-1}+k$, $1 \le k \le 2^{j-1}$.

7. Consequently, each system $\Psi_r^{(s)}$, $0 \le s \le r-1$, is an unconditional basis in $W^{l,p}[0,1]$, $0 \le l \le r-1-s$. Moreover, if $f \in W^{l,p}[0,1]$, $s \ge 0$ and $0 \le s+l \le r-1$, then

$$f^{(l)} = \sum_{n\in U_{0,r-(s+l)}} a_n \psi_{s+l;n} + \sum_{j=1}^{\infty}\sum_{n\in U_j} 2^{jl} a_n \psi_{s+l;n} \tag{3.17}$$

and

$$\|f^{(l)}\|_p \sim \Big\|\Big(\sum_{n\in U_{0,r-(s+l)}} |a_n|^2\kappa_n^2 + \sum_{j=1}^{\infty}\sum_{n\in U_j} 2^{2jl}|a_n|^2\kappa_n^2\Big)^{1/2}\Big\|_p. \tag{3.18}$$

Note also that in this case

$$2^{lj} a_n = 2^{lj}(f_n, \psi_{-s,n}) = (f^{(l)}, \psi_{-s-l,n}).$$

8. In the multivariate case, we consider tensor product systems. Fix $\underline{r} = (r_1,\ldots,r_d)$, $\underline{s} = (s_1,\ldots,s_d)$ with $|s_i| \le r_i - 1$ and systems $\Psi_{r_i}^{(s_i)}$. Set

$$\Psi_{\underline{r}}^{(\underline{s})} = \{\psi_{\underline{s};\underline{n}} = \psi_{s_1;n_1}\otimes\cdots\otimes\psi_{s_d;n_d} : \psi_{s_i;n_i} \in \Psi_{r_i}^{(s_i)}\}.$$

The set of indices is split into blocks: for $\underline{j} \in \mathbb{N}_0^d$, write

$$V_{\underline{j};\underline{r},\underline{s}} = U_{j_1}\times\cdots\times U_{j_d},$$

where U_0 means $U_{0,r_i-|s_i|}$. This splitting will be used in the multiparameter setting, while in the one-parameter setting we will use

$$V_{j;\underline{r},\underline{s}} = \bigcup_{|\underline{j}|_\infty = j} V_{\underline{j};\underline{r},\underline{s}}, \quad \text{where} \quad |\underline{j}|_\infty = \max(j_1,\ldots,j_d),\ \underline{j} \in \mathbb{N}_0^d.$$

Then each system $\Psi_{\underline{r}}^{(\underline{s})}$ is an unconditional basis in $L^p[0,1]^d$, $1 < p < \infty$, and for

$$f = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} a_{\underline{n}}\psi_{\underline{s};\underline{n}} \quad \text{with} \quad a_{\underline{n}} = (f, \psi_{-\underline{s};\underline{n}}) \tag{3.19}$$

we have

$$\|f\|_p \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} |a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p, \tag{3.20}$$

where $\kappa_{\underline{n}} = \kappa_{n_1}\otimes\cdots\otimes\kappa_{n_d}$ for $\underline{n} = (n_1,\ldots,n_d)$.

9. Moreover, if $\underline{l} = (l_1,\ldots,l_d)$ and $\underline{s} = (s_1,\ldots,s_d)$ with $s_i \ge 0$ and $0 \le s_i + l_i \le r_i - 1$, then

$$D^{\underline{l}} f = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}+\underline{l}}} 2^{\underline{j}\cdot\underline{l}} a_{\underline{n}}\psi_{\underline{s}+\underline{l};\underline{n}} \tag{3.21}$$

and

$$\|D^{\underline{l}} f\|_p \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}+\underline{l}}} 2^{2\underline{j}\cdot\underline{l}}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p. \tag{3.22}$$

Note that

$$2^{\underline{j}\cdot\underline{l}}a_{\underline{n}} = 2^{\underline{j}\cdot\underline{l}}(f,\psi_{-\underline{s};\underline{n}}) = (D^{\underline{l}}f,\psi_{-\underline{s}-\underline{l};\underline{n}}). \tag{3.23}$$

Thus, in case $s_i \geq 0$, if $m + s_i \leq r_i - 1$ for each $1 \leq i \leq d$, then $\Psi_{\underline{r}}^{(\underline{s})}$ is an unconditional basis in $W^{m,p}[0,1]^d$, $1 < p < \infty$. In the mixed smoothness case, if $\underline{m} = (m_1,\dots,m_d)$ with $m_i + s_i \leq r_i - 1$ for each $1 \leq i \leq d$, then $\Psi_{\underline{r}}^{(\underline{s})}$ is an unconditional basis in $W^{\underline{m},p}[0,1]^d$, $1 < p < \infty$. In particular,

$$\|f\|_{W^{m,p}} \sim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} 2^{2mj}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p, \tag{3.24}$$

while in the mixed smoothness case

$$\|f\|_{W^{\underline{m},p}} \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} 2^{2\underline{m}\cdot\underline{j}}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p. \tag{3.25}$$

10. For later reference, consider the following procedure: for fixed m, $\underline{r}$ and $\underline{s}$ with $s_i \geq 0$, take the expansion of f with respect to $\Psi_{\underline{r}}^{(\underline{s})}$ and remove from it all terms for which the basic function $\psi_{\underline{s};\underline{n}}$ is a polynomial of total degree $< m$. For this, denote by $V_{\underline{0};\underline{r},\underline{s}}^{(m)}$ the set of indices $\underline{n}$ such that $\psi_{\underline{s};\underline{n}}$ is not a polynomial of total degree $< m$, that is,

$$V_{\underline{0};\underline{r},\underline{s}}^{(m)} = \Big\{\underline{n}\in V_{\underline{0};\underline{r},\underline{s}} : \sum_{i=1}^{d}|n_i + r_i - 2 - s_i| \geq m\Big\}.$$

In this notation, set

$$P_{m;\underline{r},\underline{s}}f = \sum_{\underline{n}\in V_{\underline{0};\underline{r},\underline{s}}\setminus V_{\underline{0};\underline{r},\underline{s}}^{(m)}} a_{\underline{n}}\psi_{\underline{s};\underline{n}}. \tag{3.26}$$

Then

$$\|f - P_{m;\underline{r},\underline{s}}f\|_p \sim \Big\|\Big(\sum_{\underline{n}\in V_{\underline{0};\underline{r},\underline{s}}^{(m)}}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{\underline{j}\neq\underline{0}}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p. \tag{3.27}$$

Moreover, for fixed $m\in\mathbb{N}$,

$$\sum_{|\underline{l}|=m}\|D^{\underline{l}}f\|_p \sim \Big\|\Big(\sum_{\underline{n}\in V_{\underline{0};\underline{r},\underline{s}}^{(m)}}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{\underline{j}\neq\underline{0}}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} 2^{2m|\underline{j}|_\infty}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p. \tag{3.28}$$

The multiparameter counterpart is as follows: for fixed $\underline{l} = (l_1,\dots,l_d)$, we remove from the expansion of f all terms which are polynomials of degree

$< l_i$ in direction i for some $i \in \{1, \ldots, d\}$. Note that such terms appear only for $\underline{j} = (j_1, \ldots, j_d)$ with some $j_i = 0$, i.e. $\underline{j} \in \mathbb{N}_0^d \setminus \mathbb{N}^d$. That is, let

$$P_{\underline{l};\underline{r},\underline{s}} f = \sum_{\underline{j} \in \mathbb{N}_0^d \setminus \mathbb{N}^d} \sum_{\underline{n} \in V_{\underline{j};\underline{r},\underline{s}} \setminus V_{\underline{j};\underline{r},\underline{s}+\underline{l}}} a_{\underline{n}} \psi_{\underline{s};\underline{n}}. \tag{3.29}$$

Then

$$\|f - P_{\underline{l};\underline{r},\underline{s}} f\|_p \sim \Big\| \Big(\sum_{\underline{j} \in \mathbb{N}_0^d} \sum_{\underline{n} \in V_{\underline{j};\underline{r},\underline{s}+\underline{l}}} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p \tag{3.30}$$

and

$$\|D^{\underline{l}} f\|_p \sim \Big\| \Big(\sum_{\underline{j} \in \mathbb{N}_0^d} \sum_{\underline{n} \in V_{\underline{j};\underline{r},\underline{s}+\underline{l}}} 2^{2\underline{j}\cdot\underline{l}} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p. \tag{3.31}$$

3.3.2. *The isotropic* (*one-parameter*) *case.* In this section, we fix $m \in \mathbb{N}$, $\underline{r} = (r_1, \ldots, r_d)$ and $\underline{s} = (s_1, \ldots, s_d)$ such that $s_i \geq 0$ and $m + s_i \leq r_i - 1$ for each $i = 1, \ldots, d$. Recall the grouping of the tensor product basis described in Section 3.3.1:

$$V_{j;\underline{r},\underline{s}} = \bigcup_{\underline{j}: |\underline{j}|_\infty = j} V_{\underline{j};\underline{r},\underline{s}} \quad \text{and set} \quad Q_j f = \sum_{\underline{n} \in V_{j;\underline{r},\underline{s}}} a_{\underline{n}} \psi_{\underline{s};\underline{n}},$$

where $f \in L^p[0,1]^d$ is given by (3.19).

We begin with the following:

Proposition 3.5. *Fix* $1 < p < \infty$ *and* $m \in \mathbb{N}$. *Then for* $f \in L^p[0,1]^d$ *with* $f = \sum_{\underline{j} \in \mathbb{N}_0^d} \sum_{\underline{n} \in V_{\underline{j}}} a_{\underline{n}} \psi_{\underline{s};\underline{n}}$ *and* $\mu \in \mathbb{Z}$ *we have*

$$\begin{aligned} &\omega_{m,p}(f, 1/2^\mu) \sim \\ &\Big\| \Big(\sum_{\underline{n} \in V^{(m)}_{0;\underline{r},\underline{s}}} (2^{-m\mu} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 + \sum_{j>0} \sum_{\underline{n} \in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p, \end{aligned} \tag{3.32}$$

and consequently

$$\begin{aligned} &(1 \wedge 2^{-m\mu}) \|f\|_p + \omega_{m,p}(f, 1/2^\mu) \\ &\qquad \sim \Big\| \Big(\sum_{j \geq 0} \sum_{\underline{n} \in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p. \end{aligned} \tag{3.33}$$

The equivalence constants do not depend on f *or* $\mu \in \mathbb{Z}$.

Proof. Note that for $\underline{n} \in V_{0;\underline{r},\underline{s}} \setminus V^{(m)}_{0;\underline{r},\underline{s}}$ we have $\Delta^m_{\underline{h}} \psi_{\underline{s};\underline{n}} = 0$ for each $\underline{h}$, and $D^{\underline{l}} \psi_{\underline{s};\underline{n}} = 0$ for each $\underline{l}$ with $|\underline{l}| = m$. Therefore, it is enough to consider $\underline{n} \in V^{(m)}_{0;\underline{r},\underline{s}} \cup \bigcup_{j>0} V_{j;\underline{r},\underline{s}}$. Then, because of (3.1) and (3.28) combined with Fact 3.1, we are in the situation of Proposition 2.2, and an analogous argument applies. This gives (3.32). The equivalence (3.33) is a consequence of (3.32) and (3.20). ■

The multipliers appearing on the right-hand sides of (3.32) and (3.33) are used to define two versions of $b_m^{\alpha,q}$ spaces. The version using the right-hand side of (3.32) leads to an expression equivalent to $\int_0^\infty(\omega_m(f,t)/t^\alpha)^q\,dt/t)^{1/q}$, while the use of the right-hand side of (3.33) leads to an expression equivalent to $\|\cdot\|_{B_p^{\alpha,q}}$.

Version 1. Define w^0 as the set of sequences $f \sim (a_{\underline{n}}, \underline{n} \in V^{(m)}_{\underline{0};\underline{r},\underline{s}} \cup \bigcup_{j>0} V_{j;\underline{r},\underline{s}})$ such that

$$\|f\|_{w^0;1} = \Big\|\Big(\sum_{\underline{n}\in V^{(m)}_{\underline{0};\underline{r},\underline{s}}} |a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{j>0}\sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} |a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p < \infty. \tag{3.34}$$

Then w^m is defined as the set of sequences $f \sim (a_{\underline{n}}, \underline{n} \in V^{(m)}_{\underline{0};\underline{r},\underline{s}} \cup \bigcup_{j>0} V_{j;\underline{r},\underline{s}})$ for which

$$\|f\|_{w^m;1} = \Big\|\Big(\sum_{\underline{n}\in V^{(m)}_{\underline{0};\underline{r},\underline{s}}} |a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{j>0}\sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} 2^{2jm}|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p < \infty. \tag{3.35}$$

The norm $\|\cdot\|_{b_m^{\alpha,q};1}$ is defined according to Definition 2.2 using the multiplier

$$\Big\|\Big(\sum_{\underline{n}\in V^{(m)}_{\underline{0};\underline{r},\underline{s}}} (2^{-m\mu}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{j>0}\sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.$$

That is, we set

$$\|f\|_{b_m^{\alpha,q};1} = \|\big(2^{\mu\alpha}S_\mu(f)\big)_{\mu\in\mathbb{Z}}\|_{\ell^q}, \tag{3.36}$$

where

$$S_\mu(f) = \Big\|\Big(\sum_{\underline{n}\in V^{(m)}_{\underline{0};\underline{r},\underline{s}}} (2^{-m\mu}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2 + \sum_{j>0}\sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.$$

Note that by (3.32) and (3.27) for $1 \le q < \infty$ we have

$$\|f\|_{b_m^{\alpha,q};1} \sim \Big(\int_0^\infty \Big(\frac{\omega_{m,p}(f,t)}{t^\alpha}\Big)^q \frac{dt}{t}\Big)^{1/q}, \tag{3.37}$$

with equivalence constants independent of $0 < \alpha < m$ and $1 \le q < \infty$, or more precisely

$$\|f\|_{b_m^{\alpha,q};1} \sim s(q\alpha)^{1/q}\|f - P_{m;\underline{r},\underline{s}}f\|_p + \Big(\int_0^1 \Big(\frac{\omega_{m,p}(f,t)}{t^\alpha}\Big)^q \frac{dt}{t}\Big)^{1/q}. \tag{3.38}$$

In case $q = \infty$ we have

$$\|f\|_{b_m^{\alpha,q};1} \sim \max\Big(\|f - P_{m;\underline{r},\underline{s}}f\|_p, \sup_{0<t\le 1} t^{-\alpha}\omega_{m,p}(f,t)\Big), \tag{3.39}$$

with equivalence constants independent of $0 < \alpha < m$.

Version 2. Now, both w^0 and w^m are defined as the sets of $f \sim (a_{\underline{n}}, \underline{n} \in \bigcup_{j\geq 0} V_{j,\underline{r},\underline{s}})$ for which

$$\|f\|_{w^0;2} = \Big\| \Big(\sum_{j\geq 0} \sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p < \infty, \tag{3.40}$$

respectively

$$\|f\|_{w^m;2} = \Big\| \Big(\sum_{j\geq 0} \sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} 2^{2mj} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p < \infty. \tag{3.41}$$

The norms of w^0 and w^m defined by (3.40) and (3.41) are just equivalent norms in $L^p[0,1]^d$ and $W^{m,p}[0,1]^d$ (see (3.20) and (3.24)).

The definition of $\|\cdot\|_{b_m^{\alpha,q};2}$ (in Definition 2.2) uses the multiplier

$$\Big\| \Big(\sum_{j\geq 0} \sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p.$$

That is, we set

$$\|f\|_{b_m^{\alpha,q};2} = \Big\| \Big(2^{\mu\alpha} \Big\| \Big(\sum_{j\geq 0} \sum_{\underline{n}\in V_{j;\underline{r},\underline{s}}} (2^{m(j-\mu)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p \Big)_{\mu\in\mathbb{Z}} \Big\|_{\ell^q}. \tag{3.42}$$

It follows from (3.32) that

$$\|f\|_{b_m^{\alpha,q},2} \sim s(q(m-\alpha))^{1/q} \|f\|_p + s(q\alpha)^{1/q} \|P_{m;\underline{r},\underline{s}} f\|_p + \Big(\int_0^1 \Big(\frac{\omega_{m,p}(f,t)}{t^\alpha} \Big)^q \frac{dt}{t} \Big)^{1/q}, \tag{3.43}$$

while for $q = \infty$,

$$\|f\|_{b_m^{\alpha,q},2} \sim \max\Big(\|f\|_p, \sup_{0<t\leq 1} t^{-\alpha} \omega_{m,p}(f,t) \Big), \tag{3.44}$$

with equivalence constants independent of $0 < \alpha < m$ and $1 \leq q \leq \infty$.

Note that in both cases $(i = 1, 2)$, the assumptions of Theorem 2.5 are satisfied, and we get asymptotic formulae for these norms. To summarize these considerations, we formulate the following:

THEOREM 3.6. *Fix $1 < p < \infty$ and $m \in \mathbb{N}$. Let $\underline{r} = (r_1, \dots, r_d)$ and $\underline{s} = (s_1, \dots, s_d)$ with $s_i \geq 0$ be such that $m + s_i \leq r_i - 1$ for all $i = 1, \dots, d$. Let $f \in L^p[0,1]^d$ be given by its expansion* (3.19) *with respect to the basis $\Psi_{\underline{r}}^{(\underline{s})}$.*

For $i = 1, 2$, let $\|\cdot\|_{w^0,i}$, $\|\cdot\|_{w^m,i}$ and $\|\cdot\|_{b_m^{\alpha,q},i}$ be given by (3.34)–(3.36) *and* (3.40)–(3.42). *Then*

$$\|f - P_{m;\underline{r},\underline{s}} f\|_p \sim \|f\|_{w^0,1}, \qquad \|f\|_p \sim \|f\|_{w^0,2},$$
$$\sum_{|\underline{l}|=m} \|D^{\underline{l}} f\|_p \sim \|f\|_{w^m,1}, \qquad \|f\|_{W^{m,p}} \sim \|f\|_{w^m,2},$$

and equivalent forms of $\|\cdot\|_{b_m^{\alpha,q},i}$ *given by* (3.37)–(3.39) *and* (3.43), (3.44), *with equivalence constants independent of* $0<\alpha<m$ *and* $1\le q\le\infty$.

If $f\in B_p^{\epsilon,q}[0,1]^d$ *for some* $\epsilon>0$ *and* $1\le q<\infty$ *then*

$$\lim_{\alpha\searrow 0}\alpha^{1/q}\|f\|_{b_m^{\alpha,q},i}=\left(\frac{1}{q(\ln 2-1)}\right)^{1/q}\|f\|_{w^0,i},$$

and for $f\in W^{m,p}[0,1]^d$,

$$\lim_{\alpha\nearrow m}(m-\alpha)^{1/q}\|f\|_{b_m^{\alpha,q},i}=\left(\frac{1}{q(\ln 2-1)}\right)^{1/q}\|f\|_{w^m,i}.$$

If $f\in B_p^{\epsilon,\infty}[0,1]^d$ *for some* $\epsilon>0$ *then*

$$\lim_{\alpha\searrow 0}\|f\|_{b_m^{\alpha,\infty},i}=\|f\|_{w^0,i},$$

and for $f\in W^{m,p}[0,1]^d$,

$$\lim_{\alpha\nearrow m}\|f\|_{b_m^{\alpha,\infty},i}=\|f\|_{w^m,i}.$$

Remark. It is observed in R. Arcangéli and J. J. Torrens [2] that the asymptotic behaviour of double integrals as in (1.1), (1.2) when $s\searrow 0$ is different in the two cases of $\mathbb{R}^d$ and Ω a bounded domain. That is, for a bounded domain Ω, the double integral in (1.1) is bounded when $s\searrow 0$. However, in that case, the double integral in question corresponds essentially to $(\int_0^{T_0}(\omega_{m,p}(f,t)/t^\alpha)^q\,dt/t)^{1/q}$ with some $T_0<\infty$. Note that our norms $\|\cdot\|_{b_m^{\alpha,q},i}$, $i=1,2$, contain some extra terms, which guarantee the asymptotic behaviour as in Theorem 3.6 when $s\searrow 0$.

3.3.3. *The dominating mixed smoothness* (*multiparameter*) *case.* We start the analysis of the mixed smoothness case with the following observation:

Proposition 3.7. *Fix* $1<p<\infty$, $\underline{m}=(m_1,\dots,m_d)$, $\underline{r}=(r_1,\dots,r_d)$ *and* $\underline{s}=(s_1,\dots,s_d)$ *with* $s_i\ge 0$ *and* $m_i+s_i\le r_i-1$. *Let* $\Psi_{\underline{r}}^{(\underline{s})}$ *be one of the tensor product bases described in Section* 3.3.1, *and let* $f\in L^p[0,1]^d$ *with* $f=\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}}a_{\underline{n}}\psi_{\underline{s};\underline{n}}$. *Then for* $\underline{\mu}=(\mu_1,\dots,\mu_d)\in\mathbb{Z}^d$, $\underline{t}_{\underline{\mu}}=(1/2^{\mu_1},\dots,1/2^{\mu_d})$ *and* $A\subset\mathcal{D}$, *we have*

$$\omega_{\underline{m},p,A}(f,\underline{t}_{\underline{\mu},A})\sim\Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}\prod_{i\in A}(2^{m_i(j_i-\mu_i)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p, \tag{3.45}$$

where

$$V_{\underline{j};\underline{r},\underline{s};\underline{m},A}=\{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}:n_i+r_i-2-s_i\ge m_i\ \textit{for}\ i\in A\}.$$

Moreover,

$$\sum_{A\subset\mathcal{D}}\prod_{i\in\mathcal{D}\setminus A}\left(1\wedge\frac{1}{2^{m_i\mu_i}}\right)\cdot\omega_{\underline{m},p,A}(f,\underline{t}_{\underline{\mu},A}) \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}}\prod_{i=1}^{d}(2^{m_i(j_i-\mu_i)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p. \tag{3.46}$$

The equivalence constants above do not depend on f or $\underline{\mu}$.

Proof. Once (3.45) is proved, formula (3.46) is a consequence of (3.45). The proof of (3.45) is by induction on $|A|$ and exploits (3.4). We can restrict our consideration to indices $\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}$, since for all $\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}\setminus V_{\underline{j};\underline{r},\underline{s};\underline{m},A}$, $B\subset A$ and $\underline{h}$ we have $\Delta_{\underline{h}}^{\underline{m}_{A\setminus B}}D^{\underline{m}_B}\psi_{\underline{s};\underline{n}}=0$.

First, let $|A|=1$, $A=\{i\}$. Let $f=\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}a_{\underline{n}}\psi_{\underline{s};\underline{n}}$. Then by (3.4),

$$\omega_{\underline{m},p,A}(f,\underline{t}_{\underline{\mu},A})\sim\inf\Big\{\|f-g\|_p+\frac{1}{2^{m_i\mu_i}}\|D^{m_i\underline{e}_i}g\|_p : g\in W^{m_i\underline{e}_i,p}[0,1]^d\Big\}.$$

Let $g\in W^{m_i\underline{e}_i,p}[0,1]^d$ with $g=\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}b_{\underline{n}}\psi_{\underline{s};\underline{n}}$. Then by (3.22),

$$\|f-g\|_p\sim\Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}|a_{\underline{n}}-b_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p,$$

$$\|D^{m_i\underline{e}_i}g\|_p\sim\Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}2^{2m_ij_i}|b_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.$$

Repeating essentially the argument from the proof of Proposition 2.2 we get

$$\inf\Big\{\|f-g\|_p+\frac{1}{2^{m_i\mu_i}}\|D^{m_i\underline{e}_i}g\|_p : g\in W^{m_i\underline{e}_i,p}[0,1]^d\Big\} \sim\Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}(2^{m_i(j_i-\mu_i)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p,$$

which is (3.45) for $A=\{i\}$.

To illustrate the inductive argument and simplify the notation, we show the argument for passing from $|A|=1$ to $|A|=2$. Let $A=\{i,k\}$. Set $A'=\{i\}$ and $A''=\{k\}$. Then by (3.4),

$$\begin{aligned}\omega_{\underline{m},p,A}&(f,\underline{t}_{\underline{\mu},A})\\ \sim\ \inf\Big\{&\|f-(g_i+g_k+g_{i,k})\|_p+\frac{1}{2^{m_i\mu_i}}\omega_{\underline{m},p,A''}(D^{m_i\underline{e}_i}g_i,\underline{t}_{\underline{\mu},A''})\\ &+\frac{1}{2^{m_k\mu_k}}\omega_{\underline{m},p,A'}(D^{m_k\underline{e}_k}g_k,\underline{t}_{\underline{\mu},A'})+\frac{1}{2^{m_i\mu_i+m_k\mu_k}}\|D^{m_i\underline{e}_i+m_k\underline{e}_k}g_{i,k}\|_p :\\ &g_i\in W^{m_i\underline{e}_i,p}[0,1]^d,\ g_k\in W^{m_k\underline{e}_k,p}[0,1]^d,\ g_{i,k}\in W^{m_i\underline{e}_i+m_k\underline{e}_k,p}[0,1]^d\Big\}.\end{aligned}$$

Let $g_i = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} b_{i,\underline{n}}\psi_{\underline{s};\underline{n}}$, $g_k = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} b_{k,\underline{n}}\psi_{\underline{s};\underline{n}}$ and $g_{i,k} = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} b_{i,k,\underline{n}}\psi_{\underline{s};\underline{n}}$. Then by (3.22),

$$\|f-(g_i+g_k+g_{i,k})\|_p \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} |a_{\underline{n}}-(b_{i,\underline{n}}+b_{k,\underline{n}}+b_{i,k,\underline{n}})|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p,$$

$$\|D^{m_i\underline{e}_i+m_k\underline{e}_k}g_{i,k}\|_p \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} 2^{2(m_ij_i+m_kj_k)}|b_{i,k,\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.$$

Next, by (3.21),

$$D^{m_i\underline{e}_i}g_i = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} 2^{m_ij_i}b_{i,\underline{n}}\psi_{\underline{s}+m_i\underline{e}_i;\underline{n}},$$

$$D^{m_k\underline{e}_k}g_k = \sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} 2^{m_kj_k}b_{k,\underline{n}}\psi_{\underline{s}+m_k\underline{e}_k;\underline{n}}.$$

Note that

$$V_{\underline{j};\underline{r},\underline{s};\underline{m},A} = V_{\underline{j};\underline{r},\underline{s}+m_i\underline{e}_i;\underline{m},A''} = V_{\underline{j};\underline{r},\underline{s}+m_k\underline{e}_k;\underline{m},A'}.$$

Therefore, since $|A'|=|A''|=1$, by the already proved part, applied to the basis $\Psi_{\underline{r}}^{(\underline{s}+m_k\underline{e}_k)}$ and $\underline{m}'=(m_1',\dots,m_d')$ with $m_l'=m_l$ for $l\neq k$ and $m_k'=0$, we get

$$\begin{aligned}
&\omega_{\underline{m},p,A'}(D^{m_k\underline{e}_k}g_k,\underline{t}_{\underline{\mu},A'})\\
&\quad = \omega_{\underline{m}',p,A'}(D^{m_k\underline{e}_k}g_k,\underline{t}_{\underline{\mu},A'})\\
&\quad \sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} (2^{m_i(j_i-\mu_i)}\wedge 1)^2 2^{2m_kj_k}|b_{k,\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.
\end{aligned}$$

Applying a similar argument to $\Psi_{\underline{r}}^{(\underline{s}+m_i\underline{e}_i)}$ and $\underline{m}''=(m_1'',\dots,m_d'')$ with $m_l''=m_l$ for $l\neq i$ and $m_i''=0$, we get

$$\begin{aligned}
\omega_{\underline{m},p,A''}(D^{m_i\underline{e}_i}g_i,\underline{t}_{\underline{\mu},A''}) &= \omega_{\underline{m}'',p,A''}(D^{m_i\underline{e}_i}g_i,\underline{t}_{\underline{\mu},A''})\\
&\sim \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} (2^{m_k(j_k-\mu_k)}\wedge 1)^2 2^{2m_ij_i}|b_{i,\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.
\end{aligned}$$

With these equivalences in hand, we proceed as in the proof of Proposition 2.7 to get (3.45) for $A=\{i,k\}$.

The general case follows by induction on $|A|$. ∎

Now, we present two versions of asymptotic formulae for mixed smoothness spaces $B_p^{\underline{\alpha},q}[0,1]^d$: the first version describes the asymptotics of $\|\cdot\|_{B_p^{\underline{\alpha},q}}$, while the second describes the asymptotics of a piece of the norms defined by (3.2) and (3.3) with fixed $A \subset \mathcal{D}$ (see Theorems 3.8 and 3.9 below). This will be done by applying Theorem 2.10.

First, let us discuss the regularity assumptions on f needed to apply Theorem 2.10 in this setting. Clearly, the minimal assumption should be $f \in W^{\underline{m}_F,p}[0,1]^d$, but we also need some condition on regularity of f in directions in F^c or in $A \setminus F$. For example, the direct interpretation of the assumptions of Theorem 2.10 in the setting of Theorem 3.8 (see below) would be the following: $f \in L^p[0,1]^d$ is given via its expansion (3.19) and there is $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$ with $\epsilon_i > 0$ such that

$$\sum_{\underline{j} \in \mathbb{N}_0^d} \sum_{n \in V_{\underline{j};\underline{r},\underline{s}}} 2^{\underline{j}\cdot(\underline{m}_F + \underline{\epsilon}_{F^c})} a_{\underline{n}} \psi_{\underline{s};\underline{n}} \in L^p[0,1]^d. \tag{3.47}$$

If the vector $\underline{m}_F + \underline{\epsilon}_{F^c}$ has some non-integer entries, we do not have an interpretation of this condition in terms of derivatives of f and/or asymptotics of moduli of smoothness of f. However, we use the Comment following Theorem 2.10 to present assumptions for Theorems 3.8 and 3.9 directly in terms of f and its derivatives. The spaces which we describe below combine Sobolev type regularity in directions in F and Besov type regularity in directions in E, where $E, F \subset \mathcal{D}$ with $E \cap F = \emptyset$.

For this, fix $\emptyset \neq E \subset \mathcal{D}$. For $\underline{0} < \underline{\alpha} < \underline{m}$ and $1 \le q \le \infty$, let $B^{\underline{\alpha},q}_{p,E}[0,1]^d$ be the space with the norm defined by formulae analogous to (3.2) and (3.3), but using only $A \subset E$, that is,

$$\|f\|_{B^{\underline{\alpha},q}_{p,E}} = \sum_{A \subset E} \bigg(\int\limits_{(0,\infty)^d} \bigg(\frac{\omega_{\underline{m},p,A}(f, \underline{t}_A)}{\underline{t}_A^{\underline{\alpha}_A}} \bigg)^q \prod_{i \notin A} u(t_i) \, \frac{d\underline{t}_A}{\underline{t}_A^{\underline{1}_A}} \, d\underline{t}_{A^c} \bigg)^{1/q},$$

$$\|f\|_{B^{\underline{\alpha},\infty}_{p,E}} = \sup_{\underline{t} \in (0,\infty)^d,\, A \subset E} \underline{t}_A^{-\underline{\alpha}_A} \omega_{\underline{m},p,A}(f, \underline{t}_A).$$

For $E = \emptyset$ we mean $B^{\underline{\alpha},q}_{p,E} = L^p[0,1]^d$. Next, for $F \subset \mathcal{D}$ such that $F \cap E = \emptyset$ and $\underline{k} \in \mathbb{N}_0^d$ set

$$W^{\underline{k}}_F B^{\underline{\alpha},q}_{p,E}[0,1]^d = \{ f \in L^p[0,1]^d : D^{\underline{l}} f \in B^{\underline{\alpha},q}_{p,E}[0,1]^d \text{ for each } \underline{0} \le \underline{l} \le \underline{k}_F \},$$

with the norm

$$\|f\|_{W^{\underline{k}}_F B^{\underline{\alpha},q}_{p,E}} = \sum_{G \subset F} \|D^{\underline{k}_G} f\|_{B^{\underline{\alpha},q}_{p,E}}.$$

Let f be given by the series (3.19). Observe that by (3.45) and (3.21), for each $A \subset E$ and $\underline{k}$ such that $\underline{k}_F + \underline{s}_F \le \underline{r}_F - \underline{1}_F$ (and clearly $\underline{m}_A + \underline{s}_A \le \underline{r}_A - \underline{1}_A$)

we have

$$\sum_{G\subset F} \omega_{\underline{m},p,A}(D^{\underline{k}_G} f, \underline{t}_{\underline{\mu}_A})$$
$$\sim \Big\| \Big(\sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} \prod_{i\in A} (2^{m_i(j_i-\mu_i)} \wedge 1)^2 2^{2\underline{k}_F\cdot \underline{j}_F} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p,$$

and consequently

$$\|f\|_{W_F^{\underline{k}} B_{p,E}^{\underline{\alpha},q}} \sim \sum_{A\subset E} \|(2^{\underline{\alpha}_A\cdot \underline{\mu}_A} S_{\underline{k}_F,\underline{\mu}_A}(f))_{\underline{\mu}_A\in\mathbb{Z}^{|A|}}\|_{\ell^q},$$

where

$$S_{\underline{k}_F,\underline{\mu}_A}(f) = \Big\| \Big(\sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} \prod_{i\in A} (2^{m_i(j_i-\mu_i)} \wedge 1)^2 2^{2\underline{k}_F\cdot \underline{j}_F} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p.$$

Observe that for $E=\emptyset$ we have $W_F^{\underline{k}} B_{p,E}^{\underline{\alpha},q}[0,1]^d = W^{\underline{k}_F,p}[0,1]^d$, and for $F=\emptyset$ and $E=\mathcal{D}$ we have $W_F^{\underline{k}} B_{p,E}^{\underline{\alpha},q}[0,1]^d = B_p^{\underline{\alpha},q}[0,1]^d$.

We are ready to present the asymptotic behaviour of mixed smoothness Besov norms.

Version 1. First we consider spaces with the norm equivalent to the norm in $B_p^{\underline{\alpha},q}[0,1]^d$. For $f \sim (a_{\underline{n}}, \underline{n} \in \bigcup_{\underline{j}\in\mathbb{N}_0^d} V_{\underline{j};\underline{r},\underline{s}})$ we set

$$\|f\|_{w^{\underline{\alpha}}} = \Big\| \Big(\sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} 2^{2\underline{j}\cdot\underline{\alpha}} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p. \tag{3.48}$$

The norm in $b_{\underline{m}}^{\underline{\alpha},q}$ uses the multiplier

$$\Big\| \Big(\sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} \prod_{i=1}^d (2^{m_i(j_i-\mu_i)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p.$$

That is, we set

$$\|f\|_{b_{\underline{m}}^{\underline{\alpha},q}} = \|(2^{\underline{\mu}\cdot\underline{\alpha}} S_{\underline{\mu}}(f))_{\underline{\mu}\in\mathbb{Z}^d}\|_{\ell^q}, \tag{3.49}$$

where

$$S_{\underline{\mu}}(f) = \Big\| \Big(\sum_{\underline{j}\in\mathbb{N}_0^d} \sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s}}} \prod_{i=1}^d (2^{m_i(j_i-\mu_i)} \wedge 1)^2 |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p.$$

It follows from Proposition 3.7 that for $1 \le q < \infty$,

$$\|f\|_{b_{\underline{m}}^{\underline{\alpha},q}} \sim \tag{3.50}$$
$$\sum_{A\subset\mathcal{D}} \prod_{i\in\mathcal{D}\setminus A} \max\big(s(q(m_i-\alpha_i)), s(q\alpha_i)\big)^{1/q} \Big(\int_{(0,\infty)^{|A|}} \Big(\frac{\omega_{\underline{m},p,A}(f,\underline{t}_A)}{\underline{t}_A^{\underline{\alpha}_A}} \Big)^q \frac{d\underline{t}_A}{\underline{t}_A^{\underline{1}_A}} \Big)^{1/q},$$

while for $q = \infty$,

$$\|f\|_{b^{\underline{\alpha},q}_{\underline{m}}} \sim \sup_{\underline{t} \in (0,\infty)^d} \max_{A \subset \mathcal{D}} \underline{t}_A^{-\underline{\alpha}_A} \omega_{\underline{m},p,A}(f, \underline{t}_A). \tag{3.51}$$

The equivalence constants above do not depend on $\underline{0} < \underline{\alpha} < \underline{m}$ or $1 \le q \le \infty$.

In this setting, we get the following result:

THEOREM 3.8. *Let $1 < p < \infty$, $d \in \mathbb{N}$, $\underline{r} = (r_1, \ldots, r_d)$, $\underline{s} = (s_1, \ldots, s_d)$ with $s_i \ge 0$ and $\underline{m} = (m_1, \ldots, m_d)$ with $m_i + s_i \le r_i - 1$. Let $f \in L^p[0,1]^d$ be given via its expansion* (3.19) *with respect to the basis $\Psi^{(\underline{s})}_{\underline{r}}$. Let $F \subset \mathcal{D}$. Let $\|\cdot\|_{w^{\underline{m}_F}}$ and $\|\cdot\|_{b^{\underline{\alpha},q}_{\underline{m}}}$ be given by* (3.48) *and* (3.49). *Then*

$$\|f\|_{w^{\underline{m}_F}} \sim \|f\|_{W^{\underline{m}_F,p}},$$

and the equivalent form of $\|f\|_{b^{\underline{\alpha},q}_{\underline{m}}}$ is given by (3.50), (3.51) *with equivalence constants independent of $\underline{0} < \underline{\alpha} < \underline{m}$ and $1 \le q \le \infty$.*

Assume that $f \in W^{\underline{m}}_F B^{\underline{\epsilon},q}_{p,F^c}[0,1]^d$ for some $\underline{\epsilon} > \underline{0}$. Then for $1 \le q < \infty$,

$$\lim_{\underline{\alpha}_F \to \underline{m}_F} \prod_{i \in F} (m_i - \alpha_i)^{1/q} \prod_{i \in \mathcal{D} \setminus F} \alpha_i^{1/q} \|f\|_{b^{\underline{\alpha},q}_{\underline{m}}} = \left(\frac{1}{q(\ln 2 - 1)} \right)^{d/q} \|f\|_{w^{\underline{m}_F}},$$

while for $q = \infty$,

$$\lim_{\underline{\alpha}_F \to \underline{m}_F} \|f\|_{b^{\underline{\alpha},\infty}_{\underline{m}}} = \|f\|_{w^{\underline{m}_F}},$$

and the convergence $\underline{\alpha} \to \underline{m}_F$ is understood as $\alpha_i \nearrow m_i$ for $i \in F$ and $\alpha_i \searrow 0$ for $i \in \mathcal{D} \setminus F$.

Proof. The spaces $w^{\underline{m}}$ and $b^{\underline{\alpha},q}_{\underline{m}}$ satisfy the assumptions of Theorem 2.10. In the setting of Theorem 3.8, the condition $f \in W^{\underline{m}}_F B^{\underline{\epsilon},q}_{p,F^c}[0,1]^d$ means that the corresponding sequence of coefficients belongs to the space $b^{\underline{\epsilon}_{F^c},q}_{\underline{m},\underline{m},F}$, as described in the Comment following Theorem 2.10. As explained there, Theorem 2.10 applies under this assumption and the corresponding asymptotic formulae follow. ∎

Version 2. Now, we describe spaces $w^{\underline{\alpha}}_A$ and $b^{\underline{\alpha},q}_{\underline{m},A}$ which correspond to one of the terms in the norm in $B^{\underline{\alpha},q}_p[0,1]^d$ as defined by (3.2), (3.3), namely to $\left(\int_{(0,\infty)^{|A|}} \left(\frac{\omega_{\underline{m},p,A}(f,\underline{t}_A)}{\underline{t}_A^{\underline{\alpha}_A}} \right)^q \frac{d\underline{t}_A}{\underline{t}_A^{\underline{1}_A}} \right)^{1/q}$ with fixed $A \subset \mathcal{D}$.

Fix $\underline{m}$ and A. The space under consideration consists of $f \sim (a_{\underline{n}}, \underline{n} \in \bigcup_{\underline{j} \in \mathbb{N}_0^d} V_{\underline{j};\underline{r},\underline{s};\underline{m},A})$. Then we set

$$\|f\|_{w^{\underline{\alpha}}_A} = \Big\| \Big(\sum_{\underline{j} \in \mathbb{N}_0^d} \sum_{\underline{n} \in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}} 2^{2\underline{j}_A \cdot \underline{\alpha}_A} |a_{\underline{n}}|^2 \kappa_{\underline{n}}^2 \Big)^{1/2} \Big\|_p. \tag{3.52}$$

Note that in fact we now deal with spaces with $|A|$ parameters (instead of d parameters). The grouping of the indices for the multiparameter model from Section 2.2 is in fact into

$$\tilde{V}_{\underline{\xi}} = \bigcup_{\underline{j}\in\mathbb{N}_0^d:\underline{j}_A=\underline{\xi}} V_{\underline{j};\underline{r},\underline{s};\underline{m},A} \quad \text{for } \underline{\xi}\in\mathbb{N}_0^{|A|}.$$

It follows from (3.29) and (3.22) that for $F\subset A$ we have

$$\|f\|_{w_A^{\underline{m}_F}} \sim \|D^{\underline{m}_F}(f-P_{\underline{m}_A;\underline{r},\underline{s}}f)\|_p. \tag{3.53}$$

Next, the norm in $b^{\underline{\alpha},q}_{\underline{m},A}$ uses the multiplier

$$\Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}\prod_{i\in A}(2^{m_i(j_i-\mu_i)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p.$$

That is, we set

$$\|f\|_{b^{\underline{\alpha},q}_{\underline{m},A}} = \|(2^{\underline{\mu}_A\cdot\underline{\alpha}_A}S_{\underline{\mu}_A}(f))_{\underline{\mu}_A\in\mathbb{Z}^{|A|}}\|_{\ell^q}, \tag{3.54}$$

where

$$S_{\underline{\mu}_A}(f) = \Big\|\Big(\sum_{\underline{j}\in\mathbb{N}_0^d}\sum_{\underline{n}\in V_{\underline{j};\underline{r},\underline{s};\underline{m},A}}\prod_{i\in A}(2^{m_i(j_i-\mu_i)}\wedge 1)^2|a_{\underline{n}}|^2\kappa_{\underline{n}}^2\Big)^{1/2}\Big\|_p\Big).$$

It follows from Proposition 3.7 that for $1\le q<\infty$,

$$\|f\|_{b^{\underline{\alpha},q}_{\underline{m},A}} \sim \bigg(\int_{(0,\infty)^{|A|}}\bigg(\frac{\omega_{\underline{m},p,A}(f,\underline{t}_A)}{\underline{t}_A^{\underline{\alpha}_A}}\bigg)^q\frac{d\underline{t}_A}{\underline{t}_A^{\underline{1}_A}}\bigg)^{1/q}, \tag{3.55}$$

while for $q=\infty$,

$$\|f\|_{b^{\underline{\alpha},q}_{\underline{m},A}} \sim \sup_{\underline{t}_A\in(0,\infty)^{|A|}} \underline{t}_A^{-\underline{\alpha}_A\cdot\underline{t}_A}\omega_{\underline{m},p,A}(f,\underline{t}_A), \tag{3.56}$$

with equivalence constants independent of $\underline{0}<\underline{\alpha}<\underline{m}$ and $1\le q\le\infty$. Note that the norms in (3.54)–(3.56) depend on $\underline{\alpha}$ only via $\underline{\alpha}_A$.

In this setting, we get the following result:

THEOREM 3.9. *Let $1<p<\infty$, $d\in\mathbb{N}$, $\underline{r}=(r_1,\ldots,r_d)$, $\underline{s}=(s_1,\ldots,s_d)$ with $s_i\ge 0$ and $\underline{m}=(m_1,\ldots,m_d)$ with $m_i+s_i\le r_i-1$. Let $f\in L^p[0,1]^d$ be given via its expansion* (3.19) *with respect to the basis $\Psi_{\underline{r}}^{(\underline{s})}$. Fix $\emptyset\ne A\subset\mathcal{D}$, and let $\|\cdot\|_{w_A^{\underline{m}_F}}$ for $F\subset A$ and $\|\cdot\|_{b^{\underline{\alpha},q}_{\underline{m},A}}$ be given by* (3.52), (3.54). *Then for $F\subset A$,*

$$\|f\|_{w_A^{\underline{m}_F}} \sim \|D^{\underline{m}_F}(f-P_{\underline{m}_A;\underline{r},\underline{s}}f)\|_p,$$

and an equivalent form of $\|\cdot\|_{b^{\underline{\alpha},q}_{\underline{m},A}}$ given by (3.55), (3.56), *with equivalence constants independent of $\underline{0}<\underline{\alpha}<\underline{m}$ and $1\le q\le\infty$.*

Assume that $f - P_{\underline{m}_A;\underline{r},\underline{s}} f \in W_F^{\underline{m}} B_{p,A\setminus F}^{\epsilon,q}[0,1]^d$ *for some* $\underline{\epsilon} > \underline{0}$. *Then for* $1 \le q < \infty$,

$$\lim_{\underline{\alpha}_A \to \underline{m}_F} \prod_{i\in F} (m_i - \alpha_i)^{1/q} \prod_{i\in A\setminus F} \alpha_i^{1/q} \|f\|_{b_{\underline{m},A}^{\underline{\alpha},q}} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{|A|/q} \|f\|_{w_A^{\underline{m}_F}},$$

while for $q = \infty$,

$$\lim_{\underline{\alpha}_A \to \underline{m}_F} \|f\|_{b_{\underline{m},A}^{\underline{\alpha},\infty}} = \|f\|_{w_A^{\underline{m}_F}},$$

and the convergence $\underline{\alpha}_A \to \underline{m}_F$ *is understood as* $\alpha_i \nearrow m_i$ *for* $i \in F$ *and* $\alpha_i \searrow 0$ *for* $i \in A \setminus F$.

Proof. The spaces $w_A^{\underline{\alpha}}$ and $b_{\underline{m},A}^{\underline{\alpha},q}$ satisfy the assumptions of Theorem 2.10. In the setting of Theorem 3.9, the condition $f - P_{\underline{m}_A;\underline{r},\underline{s}} f \in W_F^{\underline{m}} B_{p,A\setminus F}^{\epsilon,q}[0,1]^d$ means that the corresponding sequence of coefficients belongs to $b_{\underline{m},A,\underline{m},F}^{\underline{\epsilon}_{A\setminus F},q}$, as described in the Comment following Theorem 2.10. As explained there, Theorem 2.10 applies under this assumption and the corresponding asymptotic formulae follow. ∎

3.4. Besov type spaces corresponding to Ditzian–Totik moduli of smoothness. Here we present yet another application of the results of Section 2, namely to Besov type spaces defined in terms of Ditzian–Totik moduli of smoothness. Z. Ditzian and V. Totik [13] introduced moduli of smoothness with variable step in order to characterize the order of approximation of functions on the interval $[0,1]$ by algebraic polynomials in the norm of $L^p[0,1]$. The definition is as follows: given $f : [0,1] \to \mathbb{R}$ and a step-weight function $\varphi(x) = \sqrt{x(1-x)}$, and $h > 0$, define $\overline{\Delta}_{h\varphi(x)}^m f(x)$, the symmetric difference of f at x with step $h\varphi(x)$, by the formula

$$\overline{\Delta}_{h\varphi(x)}^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x + (m/2 - j)h\varphi(x)).$$

Then the corresponding modulus of smoothness is defined as

$$\omega_{m,p,\varphi}(f,t) = \sup_{0<h\le t} \|\overline{\Delta}_{h\varphi(\cdot)}^m f(\cdot)\|_p.$$

This modulus of smoothness is equivalent to the following K-functional (see [13] or [12]):

FACT 3.10. *Let* $1 \le p \le \infty$ *and* $m \in \mathbb{N}$. *Then for* $f \in L^p[0,1]$ *with* $1 \le p < \infty$, *or* $f \in C[0,1]$, $p = \infty$ *and* $0 < t \le 1/(2m)$,

$$\omega_{m,p,\varphi}(f,t) \sim K_{m,p,\varphi}(f,t),$$

where

$$K_{m,p,\varphi}(f,t) = \inf\{\|f - g\|_p + t^m \|g^{(m)} \cdot \varphi^m\|_p : g^{(m-1)} \in AC_{\mathrm{loc}}\},$$

and AC_{loc} denotes the set of functions absolutely continuous on $(0,1)$*. The equivalence constants above do not depend on f or t.*

We would like to apply an analysis similar to that in the preceding sections to Besov type spaces defined in terms of Ditzian–Totik moduli of smoothness, that is, to spaces of functions for which

$$\Big(\int_0^{1/(2m)} \Big(\frac{\omega_{m,p,\varphi}(f,t)}{t^\alpha}\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty,$$

or in case $q=\infty$,

$$\sup_{0<t\le 1/(2m)} t^{-\alpha}\omega_{m,p,\varphi}(f,t) < \infty.$$

For this, as before, we would like to apply Fact 3.10 to express $\omega_{m,p,\varphi}(f,1/2^n)$ as a norm of some multiplier on the expansion of f with respect to the basis $\Psi_r^{(s)}$, i.e. on the expansion of f given by (3.15). This requires that the bases under consideration (i.e. the bases $\Psi_r^{(s)}$ discussed in Section 3.3.1) are unconditional bases in the weighted L^p space on $[0,1]$ with weight φ^{mp}, i.e. in the space

$$L^p_{\varphi^{mp}}[0,1] = \Big\{g : \|g\|^p_{L^p_{\varphi^{mp}}} = \int_0^1 |g(x)|^p \varphi(x)^{mp}\,dx < \infty\Big\}.$$

This can be achieved if we know that the weight under consideration, $\varphi(x)^{mp} = x^{mp/2}(1-x)^{mp/2}$, belongs to the Muckenhoupt class A_p on $[0,1]$. This restricts the range of parameters: we need $-1 < mp/2 < p-1$. This condition can be satisfied only in case $m=1$ and $p>2$, and from now on, we work under this restriction.

For comparison, let us recall that wavelet systems on $\mathbb{R}$ or on $\mathbb{R}^d$ are unconditional bases in $L^p_w(\mathbb{R}^d)$ with $1<p<\infty$ when w belongs to the Muckenhoupt class $A_p(\mathbb{R}^d)$ (see e.g. [23, 15, 1]). Below, we work out in detail the case of the bases discussed in Section 3.3.1 and the particular weight $\varphi(x)^p = x^{p/2}(1-x)^{p/2}$ on $[0,1]$, which we need for our application.

We will use various facts concerning the boundedness of Calderón–Zygmund operators or Hardy–Littlewood maximal function on weighted L^p spaces with A_p weight on $[0,1]$. References for these facts are e.g. [16] or [17].

To apply Fact 3.10 (for $m=1$ and $2<p<\infty$, as explained above), we need to express $\|g'\cdot\varphi\|_p$ for $g\in L^p[0,1]$ in terms of the coefficients of the expansion of g with respect to the basis $\Psi_r^{(s)}$ with $s\ge 0$ and $s+2\le r-1$ (one of the univariate bases discussed in Section 3.3.1, cf. (3.15)). We are interested in the behaviour of g', and the function $\psi_{s,-r+s+2}$ is constant on $[0,1]$, so without loss of generality we assume that $(g,\psi_{-s,-r+s+2})=0$. Thus, let $g=\sum_{j=0}^\infty\sum_{n\in U_j}(g,\psi_{-s,n})\psi_{s,n}$, where U_0 means $U_{0,r-(s+1)}$. First, observe that

$$\sum_{j=0}^{\infty}\sum_{n\in U_j}|\psi_{-s-1,n}(x)|\,|\psi_{s+1,n}(y)|\le\frac{C}{|x-y|},$$

$$\sum_{j=0}^{\infty}\sum_{n\in U_j}|\psi_{-s-1,n}(x)|\,|\psi'_{s+1,n}(y)|\le\frac{C}{|x-y|^2},$$

$$\sum_{j=0}^{\infty}\sum_{n\in U_j}|\psi'_{-s-1,n}(x)|\,|\psi_{s+1,n}(y)|\le\frac{C}{|x-y|^2}.$$

The proof of the above inequalities is routine, using the exponential decay of the functions under consideration and of their derivatives (see Section 3.3.1), and the inequality $\sum_{k\in\mathbb{Z}}\theta^{|u-k|+|v-k|}\le C\theta^{|u-v|/2}$ for all $0<\theta<1$, where $C>0$ is a constant depending on θ, but not on $u,v\in\mathbb{R}$. As $\Psi_r^{(s+1)},\Psi_r^{(-s-1)}$ are Riesz bases in $L^2[0,1]$, each operator $T_{s+1,M,\epsilon}$ with kernel

$$K_{M,\epsilon}(x,y)=\sum_{j=0}^{M}\sum_{n\in U_j}\epsilon_n\psi_{-s-1,n}(x)\psi_{s+1,n}(y),$$

$\epsilon=\{\epsilon_n\}$ with $\epsilon_n=\pm1$, is a Calderón–Zygmund operator on $[0,1]$, and the parameters of those Calderón–Zygmund operators are bounded independently of M,ϵ. Since $\varphi^p\in A_p$, we infer that each $T_{s+1,M,\epsilon}$ is bounded on $L^p_{\varphi^p}[0,1]$, uniformly in M,ϵ: there is $C>0$, independent of M and ϵ, such that for each $f\in L^p_{\varphi^p}[0,1]$,

$$\Big(\int_0^1|T_{s+1,M,\epsilon}f(x)|^p\varphi(x)^p\,dx\Big)^{1/p}\le C\Big(\int_0^1|f(x)|^p\varphi(x)^p\,dx\Big)^{1/p}.$$

Moreover, for $\epsilon=\{1\}$, i.e. with $\epsilon_n=1$, and $f\in C[0,1]$, we have $f=\lim_{M\to\infty}T_{s+1,M,\{1\}}f$, with uniform convergence (see [7]). As $\int_0^1\varphi(x)^p\,dx<\infty$, this implies $T_{s+1,M,\{1\}}f\to f$ as $M\to\infty$ in $L^p_{\varphi^p}[0,1]$ as well. Since continuous functions are dense in $L^p_{\varphi^p}[0,1]$, it follows that $\Psi_r^{(s+1)}$ is an unconditional basis in $L^p_{\varphi^p}[0,1]$. The biorthogonal functional to $\psi_{s+1,n}$ is now $\psi_{-s-1,n}/\varphi^p\in L^{p'}_{\varphi^p}[0,1]$, so the corresponding coefficient in the expansion of $f\in L^p_{\varphi^p}[0,1]$ is

$$\int_0^1 f(x)\frac{\psi_{-s-1,n}(x)}{\varphi^p(x)}\varphi(x)^p\,dx=\int_0^1 f(x)\psi_{-s-1,n}(x)\,dx=(f,\psi_{-s-1,n}).$$

Thus we have $f=\sum_{j=0}^{\infty}\sum_{n\in U_j}(f,\psi_{-s-1,n})\psi_{s+1,n}$, with the series unconditionally convergent in $L^p_{\varphi^p}[0,1]$. Therefore, applying the standard argument with Khintchine's inequality we find that

$$\Big(\int_0^1 |f(x)|^p \varphi(x)^p\,dx\Big)^{1/p} \sim \Big(\int_0^1 \Big(\sum_{j=0}^{\infty}\sum_{n\in U_j} |(f,\psi_{-s-1,n})|^2|\psi_{s+1,n}(x)|^2\Big)^{p/2}\varphi(x)^p\,dx\Big)^{1/p}.$$

Next, if $f = g'$ with $g \in L^p[0,1]$ then $(g',\psi_{-s-1,n}) = 2^j(g,\psi_{-s,n})$, where $n \in U_j$. Therefore

$$\Big(\int_0^1 |g'(x)\varphi(x)^p|^p\,dx\Big)^{1/p} \sim \Big(\int_0^1 \Big(\sum_{j=0}^{\infty}\sum_{n\in U_j} 2^{2j}|(g,\psi_{-s,n})|^2|\psi_{s+1,n}(x)|^2\Big)^{p/2}\varphi(x)^p\,dx\Big)^{1/p}.$$

This means that

$$\|g'\cdot\varphi\|_p \sim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{n\in U_j} 2^{2j}|(g,\psi_{-s,n})|^2|\psi_{s+1,n}(x)|^2\Big)^{1/2}\varphi\Big\|_p. \tag{3.57}$$

Recall the notation $\kappa_n = 2^{(j-1)/2}\chi_{[(k-1)/2^{j-1},k/2^{j-1}]}$ for $n = 2^{j-1}+k$ (and $\kappa_n = \chi_{[0,1]}$ for $n \in U_0$). Further, recall that (cf. [9])

$$|\psi_{s+1,n}(x)| \le C\mathcal{M}\kappa_n(x), \quad \kappa_n(x) \le \mathcal{M}\psi_{s+1,n}(x),$$

where $\mathcal{M}f$ denotes the Hardy–Littlewood maximal function of f (with respect to the Lebesgue measure on $[0,1]$). Since $\varphi^p \in A_p$, applying the vector valued weighted Fefferman–Stein inequality (see e.g. [17]) we find

$$\|g'\cdot\varphi\|_p \sim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{n\in U_j} 2^{2j}|(g,\psi_{-s,n})|^2|\kappa_n|^2\varphi^2\Big)^{1/2}\Big\|_p. \tag{3.58}$$

Setting

$$z_n = \begin{cases} \big(\frac{k}{2^{j-1}}\big)^{1/2} & \text{for } n = 2^{j-1}+k \text{ with } 1 \le k \le 2^{j-2},\\ \big(\frac{2^{j-1}-k+1}{2^{j-1}}\big)^{1/2} & \text{for } n = 2^{j-1}+k \text{ with } 2^{j-2}+1 \le k \le 2^{j-1},\end{cases}$$

we find that

$$\varphi(x) \sim z_n \quad \text{for } x \in \Big[\frac{k-1}{2^{j-1}},\frac{k}{2^{j-1}}\Big],\ n = 2^{j-1}+k,\ 2 \le k \le 2^{j-1}-1.$$

In case $n = 2^{j-1}+1$ we have

$$\varphi(x) \sim z_n \quad \text{for } x \in \Big[\frac{1}{2^j},\frac{1}{2^{j-1}}\Big], \qquad \varphi(x) \le Cz_n \quad \text{for } x \in \Big[0,\frac{1}{2^{j-1}}\Big],$$

and denoting $\tilde{\kappa}_n = 2^{(j-1)/2}\chi_{[1/2^j,1/2^{j-1}]}$,

$$\kappa_n \le C\mathcal{M}\tilde{\kappa}_n, \quad \tilde{\kappa}_n \le \kappa_n.$$

Similarly, in case $n = 2^j$ we have

$$\varphi(x) \sim z_n \text{ for } x \in \left[1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}\right], \qquad \varphi(x) \le Cz_n \text{ for } x \in \left[1 - \frac{1}{2^{j-1}}, 1\right],$$

and denoting $\tilde{\kappa}_n = 2^{(j-1)/2}\chi_{[1-1/2^{j-1},1-1/2^j]}$,

$$\kappa_n \le C\mathcal{M}\tilde{\kappa}_n, \quad \tilde{\kappa}_n \le \kappa_n.$$

Similarly, for $n \in U_0$ we define $z_n = 1$ and $\tilde{\kappa}_n = \chi_{[1/4,3/4]}$. Combining these estimates with (3.58), and applying again the Fefferman–Stein vector valued maximal inequality (this time with respect to the Lebesgue measure on $[0,1]$) we get

$$\|g' \cdot \varphi\|_p \sim \Big\| \Big(\sum_{j=0}^{\infty} \sum_{n\in U_j} 2^{2j} |(g, \psi_{-s,n})|^2 z_n^2 |\kappa_n|^2 \Big)^{1/2} \Big\|_p. \tag{3.59}$$

For later convenience take $\mu_n \in \mathbb{N} \cup \{0\}$ such that $1/2^{\mu_n+1} < z_n \le 1/2^{\mu_n}$. Note that for $n \in U_j$ we have $1/2^{(j-1)/2} \le z_n \le 1/2^{1/2}$, hence $0 \le \mu_n \le j/2$, and consequently $j - \mu_n \ge 0$. Then (3.59) takes the following form:

PROPOSITION 3.11. *Let $2 < p < \infty$ and $0 \le s \le r - 3$. Let $g \in L^p[0,1]$ with $g = \sum_{n\in U_{0,r-s}} (g, \psi_{-s,n})\psi_{s,n} + \sum_{j=1}^{\infty} \sum_{n\in U_j} (g, \psi_{-s,n})\psi_{s,n}$ be such that $g' \cdot \varphi \in L^p[0,1]$. Then*

$$\|g' \cdot \varphi\|_p \sim \Big\| \Big(\sum_{n\in U_{0,r-(s+1)}} |(g, \psi_{-s,n})|^2 |\kappa_n|^2 + \sum_{j=1}^{\infty} \sum_{n\in U_j} 2^{2(j-\mu_n)} |(g, \psi_{-s,n})|^2 |\kappa_n|^2 \Big)^{1/2} \Big\|_p, \tag{3.60}$$

where $\mu_n \in \mathbb{N}_0$ are defined by the following rules: $\mu_n = 0$ if $n \in U_{0;r-(s+1)}$ or $n \in U_1$, and for $n \in U_j$ with $j \ge 2$,

$$\frac{1}{2^{\mu_n+1}} < \left(\frac{k}{2^{j-1}}\right)^{1/2} \le \frac{1}{2^{\mu_n}} \textit{ for } n = 2^{j-1} + k,\ 1 \le k \le 2^{j-2},$$

$$\frac{1}{2^{\mu_n+1}} < \left(\frac{2^{j-1} - k + 1}{2^{j-1}}\right)^{1/2} \le \frac{1}{2^{\mu_n}} \textit{ for } n = 2^{j-1} + k,\ 2^{j-2} + 1 \le k \le 2^{j-1}.$$

The equivalence constants in (3.60) *do not depend on g.*

Next, we get the following:

PROPOSITION 3.12. *Let $2 < p < \infty$ and $0 \le s \le r-3$. For $f \in L^p[0,1]$ with $f = \sum_{n\in U_{0;r-s}} c_n\psi_{s,n} + \sum_{j=1}^{\infty}\sum_{n\in U_j} c_n\psi_{s,n}$ we have, for $l \in \mathbb{N}$,*

$$(3.61)\qquad \omega_{1,p,\varphi}(f,1/2^l) \sim \Big\|\Big(\sum_{n\in U_{0;r-(s+1)}} 2^{-2l}|c_n|^2|\kappa_n|^2 + \sum_{j=1}^{\infty}\sum_{n\in U_j}(2^{j-l-\mu_n}\wedge 1)^2|c_n|^2|\kappa_n|^2\Big)^{1/2}\Big\|_p,$$

where the exponents μ_n are as in Proposition 3.11. The equivalence constants do not depend on f or l.

Proof. With Proposition 3.11, Fact 3.10 and equivalence (3.16) in hand, we proceed as in the proof of Proposition 3.5 or 2.2, but with another grouping of indices, i.e. according to the exponent $j-\mu_n$. That is, for $k \in \mathbb{N}\cup\{0\}$ we set

$$V_k^* = \bigcup_{j\ge 0}\{n \in U_j : j-\mu_n = k\}.$$

Since for each $n \in U_j$, the exponent $j-\mu_n$ is a non-negative integer, it follows that

$$\bigcup_{k\ge 0} V_k^* = \bigcup_{j\ge 0} U_j.$$

We skip the technical details. ∎

Now we are ready to introduce spaces $\tilde{w}^\alpha$ and $\tilde{b}_1^{\alpha,q}$: for $f \sim (c_n, n \ge -r+s+3)$ and $0 \le \alpha \le 1$,

$$(3.62)\qquad \|f\|_{\tilde{w}^\alpha} = \Big\|\Big(\sum_{n\in U_{0;r-(s+1)}} |c_n|^2|\kappa_n|^2 + \sum_{j=1}^{\infty}\sum_{n\in U_j} 2^{2\alpha(j-\mu_n)}|c_n|^2|\kappa_n|^2\Big)^{1/2}\Big\|_p,$$

and for $0 < \alpha < 1$ and $1 \le q \le \infty$,

$$(3.63)\qquad \|f\|_{\tilde{b}_1^{\alpha,q}} = \|(2^{l\alpha}S_l(f))_{l\in\mathbb{Z}}\|_{\ell^q},$$

where

$$S_l(f) = \Big\|\Big(\sum_{n\in U_{0;r-(s+1)}} (2^{-l}\wedge 1)^2|c_n|^2|\kappa_n|^2 + \sum_{j=1}^{\infty}\sum_{n\in U_j}(2^{j-l-\mu_n}\wedge 1)^2|c_n|^2|\kappa_n|^2\Big)^{1/2}\Big\|_p.$$

The main result of this section is the following:

THEOREM 3.13. *Let $2 < p < \infty$ and $0 \le s \le r-3$. Let $\tilde{w}^\alpha$ and $\tilde{b}_1^{\alpha,q}$ be given by (3.62) and (3.63), respectively. Then for $f \in L^p[0,1]$ with expansion (3.15) with respect to the basis $\Psi_r^{(s)}$ we have*

$$(3.64)\qquad \|f\|_{\tilde{w}^0} \sim \|f - P_{1;r,s}f\|_p, \qquad \|f\|_{\tilde{w}^1} \sim \|f'\cdot\varphi\|_p,$$

and for $1 \le q < \infty$,

$$\|f\|_{\tilde{b}_1^{\alpha,q}} \sim s(\alpha q)^{1/q}\|f - P_{1;r,s}f\|_p + \left(\int_0^{1/2}\left(\frac{\omega_{1,p,\varphi}(f,t)}{t^\alpha}\right)^q \frac{dt}{t}\right)^{1/q}, \tag{3.65}$$

while for $q = \infty$,

$$\|f\|_{\tilde{b}_1^{\alpha,\infty}} \sim \max\Big(\|f - P_{1;r,s}f\|_p, \sup_{0<t\le 1/2} t^{-\alpha}\omega_{1,p,\varphi}(f,t)\Big), \tag{3.66}$$

with equivalence constants independent of $0 < \alpha < 1$ *and* $1 \le q \le \infty$.

In this setting, let $1 \le q < \infty$, *and let* f *be such that there is* $\alpha > 0$ *for which the right-hand side of* (3.65) *is finite. Then*

$$\lim_{\alpha\searrow 0} \alpha^{1/q}\|f\|_{\tilde{b}_1^{\alpha,q}} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q}\|f\|_{\tilde{w}^0}. \tag{3.67}$$

For f *such that* $\|f' \cdot \varphi\|_p < \infty$ *we have*

$$\lim_{\alpha\nearrow 1}(1-\alpha)^{1/q}\|f\|_{\tilde{b}_1^{\alpha,q}} = \left(\frac{1}{q(\ln 2 - 1)}\right)^{1/q}\|f\|_{\tilde{w}^1}. \tag{3.68}$$

In case $q = \infty$, *if* f *is such that there is* $\alpha > 0$ *for which the right-hand side of* (3.66) *is finite, then*

$$\lim_{\alpha\searrow 0}\|f\|_{\tilde{b}_1^{\alpha,\infty}} = \|f\|_{\tilde{w}^0}, \tag{3.69}$$

and for f *such that* $\|f' \cdot \varphi\|_p < \infty$ *we have*

$$\lim_{\alpha\nearrow 1}\|f\|_{\tilde{b}_1^{\alpha,\infty}} = \|f\|_{\tilde{w}^1}. \tag{3.70}$$

Proof. Inequalities (3.64) are in fact a reformulation of (3.16) and Proposition 3.11. Equivalences (3.65) and (3.66) are direct consequences of Proposition 3.12.

To get the asymptotic formulae (3.67)–(3.70), we use the same grouping of indices as in the proof of Proposition 3.12, i.e. for $k \in \mathbb{N} \cup \{0\}$ we take

$$V_k^* = \bigcup_{j\ge 0}\{n \in U_j : j - \mu_n = k\}.$$

Now, the asymptotic formulae (3.67) and (3.68) follow by applying Theorem 2.6 with the splitting $\bigcup_{k\ge 0} V_k^*$. ■

Acknowledgements. This research was supported by Polish NCN grant N N201 607840.

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Anna Kamont
Institute of Mathematics
Polish Academy of Sciences
Branch in Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: A.Kamont@impan.gda.pl