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On automorphisms of the Banach space ℓ_∞/c_0

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Abstract. We investigate Banach space automorphisms $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ focusing on the possibility of representing their fragments of the form

$$T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$$

for $A, B \subseteq \mathbb{N}$ infinite by means of linear operators from $\ell_\infty(A)$ into $\ell_\infty(B)$, infinite $A\times B$-matrices, continuous maps from $B^* = \beta B \setminus B$ into A^*, or bijections from B to A. This leads to the analysis of general bounded linear operators on ℓ_∞/c_0. We present many examples, introduce and investigate several classes of operators, for some of them we obtain satisfactory representations and for others give examples showing that this is impossible. In particular, we show that there are automorphisms of ℓ_∞/c_0 which cannot be lifted to operators on ℓ_∞, and assuming OCA+MA we show that every automorphism T of ℓ_∞/c_0 with no fountains or with no funnels is locally induced by a bijection, i.e., $T_{B,A}$ is induced by a bijection from some infinite $B \subseteq \mathbb{N}$ to some infinite $A \subseteq \mathbb{N}$. This additional set-theoretic assumption is necessary as we show that the Continuum Hypothesis implies the existence of counterexamples of diverse flavours. However, many basic problems, some of which are listed in the last section, remain open.

Contents

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1. Introduction. The set-theoretic analysis of the Boolean algebra $\wp(\mathbb{N})/\mathrm{Fin}$ of subsets of the integers modulo finite sets, being far from concluded, has been quite successful. Some of the impressive results refer to the structure of automorphisms of the algebra which under CH can be quite pathologically complicated, as first observed by W. Rudin [39, Theorem 4.7], but consistently may all be trivial, that is, induced by an almost permutation of $\mathbb{N}$ (originally proved by S. Shelah [41]). During the development of the theory, the Proper Forcing Axiom [41] and the Open Colouring Axiom (introduced in [42] by S. Todorcevic) have been established as tools not only implying the triviality of all automorphisms but also excluding other pathological mappings, for example embeddings of a big class of quotients into $\wp(\mathbb{N})/\mathrm{Fin}$ ([46, 43, 19]).

These results also have a profound impact on more complex mathematical structures. For example, they directly imply that the question whether the only automorphisms of the Banach algebra ℓ_∞/c_0 are those induced by almost permutations of $\mathbb{N}$ is undecidable. Indirectly they recently inspired the research in C^*-algebras resulting in the undecidability of the structure of the automorphisms of the Calkin algebra of operators on the Hilbert space modulo the compact operators [20, 33].

The main focus of this paper is another natural question, namely, what is the impact of the combinatorics of $\wp(\mathbb{N})/\mathrm{Fin}$ on the automorphisms of ℓ_∞/c_0 considered as a Banach space ([1]), in particular if the Open Colouring Axiom (OCA) or the Proper Forcing Axiom (PFA) can be successfully used in this context. At the moment the situation seems similar to that of

([1]) Recall that the Banach space ℓ_∞/c_0 is canonically isometric to the Banach space $C(\mathbb{N}^*)$ of all continuous functions on the Stone space $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ of the Boolean algebra $\wp(\mathbb{N})/\mathrm{Fin}$. Hence, the link between ℓ_∞/c_0 and $\wp(\mathbb{N})/\mathrm{Fin}$ is canonical, however there are many more linear operators on ℓ_∞/c_0 than those induced by homomorphisms of $\wp(\mathbb{N})/\mathrm{Fin}$.

the early stage of the research on $\wp(\mathbb{N})/\mathrm{Fin}$ and $\mathbb{N}^*$: the usual axioms seem too weak to resolve many basic questions about the Banach space ℓ_∞/c_0 ([6, 5, 45, 26]), the Continuum Hypothesis provides some answers leaving a chaotic picture full of pathological objects obtained using transfinite induction [16, 7], and there is hope (based, for example, on [14]) that alternative axioms like OCA, OCA+MA, PFA etc., would provide an elegant structural theory of automorphisms of ℓ_∞/c_0. This hope is not only based on the case of $\wp(\mathbb{N})/\mathrm{Fin}$ but on some other cases as well [44, 31].

In order to explain our results we need to introduce some background and terminology. In the case of the Boolean algebra $\wp(\mathbb{N})/\mathrm{Fin}$ and one of its automorphisms h, the following conditions are equivalent for any two cofinite sets $A, B \subseteq \mathbb{N}$:

- There is an isomorphism $H : \wp(A) \to \wp(B)$ such that $[H(C)]_{\mathrm{Fin}} = h([C]_{\mathrm{Fin}})$ for all $C \subseteq A$ (h lifts to $\wp(\mathbb{N})$).
- There is an isomorphism $G : \mathrm{FinCofin}(A) \to \mathrm{FinCofin}(B)$ such that $[\bigcup\{G(n) : n \in C\}]_{\mathrm{Fin}} = h([C]_{\mathrm{Fin}})$ for all $C \subseteq A$ (h is induced by an almost automorphism of $\mathrm{FinCofin}(\mathbb{N})$).
- There is a bijection $\sigma : B \to A$ such that $[\{n \in B : \sigma(n) \in C\}]_{\mathrm{Fin}} = h([C]_{\mathrm{Fin}})$ for all $C \subseteq A$ (h is trivial).

Another special feature of liftings of automorphisms on $\wp(\mathbb{N})/\mathrm{Fin}$, i.e., homomorphisms of $\wp(\mathbb{N})$ satisfying the properties above, is that

- Every isomorphism from $\wp(A)$ onto $\wp(B)$ for $A, B \subseteq \mathbb{N}$ infinite is continuous with respect to the product topologies on $\{0,1\}^A$ and $\{0,1\}^B$.

Moreover, if we identify points of $\mathbb{N}^*$ with ultrafilters in $\wp(\mathbb{N})/\mathrm{Fin}$, the Stone duality gives:

- For every homomorphism h of $\wp(\mathbb{N})/\mathrm{Fin}$ there is a continuous map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\chi_{h([A]_{\mathrm{Fin}})^*} = \chi_{A^*} \circ \psi$ for every $A \subseteq \mathbb{N}$.

The corresponding notions for operators on ℓ_∞/c_0 are the following:

Definition 1.1. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator and $A, B \subseteq \mathbb{N}$ are cofinite, then we say that:

(1) T is *liftable* (*can be lifted*) if, and only if, there is a bounded linear $S : \ell_\infty(A) \to \ell_\infty(B)$ such that for all $f \in \ell_\infty(A)$ we have
$$T([f]_{c_0})|B = [S(f)]_{c_0(B)}.$$
(2) T is a *matrix operator* if, and only if, there is an operator $S : c_0(A) \to c_0(B)$ given by a real matrix $(b_{ij})_{i\in B,\, j\in A}$ such that for all $f \in \ell_\infty(A)$ we have
$$T([f]_{c_0})|B = \Big[\Big(\sum_{j\in A} b_{ij} f(j)\Big)_{i\in B}\Big]_{c_0(B)}.$$

(3) T is a *trivial operator* if, and only if, there is a *nonzero* $r \in \mathbb{R}$ and a bijection $\sigma : B \to A$ such that for all $f \in \ell_\infty(A)$ we have

$$T([f]_{c_0})|B = [rf \circ \sigma]_{c_0(B)}.$$

(4) T is *canonizable* (2) *along* $\psi : \mathbb{N}^* \to \mathbb{N}^*$ if, and only if, ψ is a *surjective* continuous mapping and there is a *nonzero* real r such that for all $f^* \in C(\mathbb{N}^*)$ we have

$$\hat{T}(f^*) = rf^* \circ \psi.$$

In contrast with the case of $\wp(\mathbb{N})/\mathrm{Fin}$ our results show that the relationships among these notions are far from equivalences:

trivial auto-morphism	$\substack{\Rightarrow \\ \nLeftarrow}$	automorphic matrix operator	$\Leftrightarrow$	automorphic liftable operator with a lifting continuous on B_{ℓ_∞}	$\substack{\Rightarrow \\ \nLeftarrow}$	automorphic liftable operator	$\substack{\Rightarrow \\ \nLeftarrow}$	auto-morphism

None of the implications or counterexamples to the reverse implications require additional set-theoretic axioms. The nontrivial parts of the above chart are:

- There are automorphisms of ℓ_∞/c_0 which are not liftable to a bounded linear operator on ℓ_∞ (4.17).
- There are automorphisms of ℓ_∞/c_0 which are liftable but are not matrix operators and none of their liftings are continuous on B_{ℓ_∞} in the product topology (4.14).
- Automorphisms of ℓ_∞/c_0 which have liftings to ℓ_∞ continuous in the product topology are exactly the automorphic matrix operators (2.14).

The question of canonizing globally all automorphisms other than trivial is outright excluded by the clear fact that there are matrices of isomorphisms on c_0 which are not matrices of almost permutations modulo c_0.

In the light of the above absolute results and the exclusion of the possibility of a global canonization or matricization, we will concentrate on "local" versions of the above properties of the operators in the sense that they hold in some sense for copies of ℓ_∞/c_0 of the form $\ell_\infty(A)/c_0(A)$ for an infinite $A \subseteq \mathbb{N}$. Since the above properties depend on the link between ℓ_∞/c_0 and $\mathbb{N}^*$ or $\mathbb{N}$, we choose the approach of Drewnowski and Roberts [16] which has functional-analytic motivations and applications:

(2) It would be reasonable to consider here also the possibility of having for all f^* in $C(\mathbb{N}^*)$ the condition $T(f^*) = gf^* \circ \psi$ for some continuous nonzero $g \in C(\mathbb{N}^*)$. However, on $\mathbb{N}^*$ all continuous functions are locally constant (since nonempty G_δ-sets have nonempty interior), so in the context of this paper it makes no sense to introduce such a property.

DEFINITION 1.2. Suppose that $A \subseteq \mathbb{N}$ is infinite. We define $P_A : \ell_\infty/c_0 \to \ell_\infty(A)/c_0(A)$ and $I_A : \ell_\infty(A)/c_0(A) \to \ell_\infty/c_0$ by

$$P_A([f]_{c_0}) = [f|A]_{c_0(A)} \quad \text{and} \quad I_A([g]_{c_0(A)}) = [g \cup 0_{\mathbb{N}\setminus A}]_{c_0}$$

for all $f \in \ell_\infty$ and all $g \in \ell_\infty(A)$. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator and $A, B \subseteq \mathbb{N}$ are infinite sets. The *localization* of T to (A, B) is the operator $T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$ given by

$$T_{B,A} = P_B \circ T \circ I_A.$$

It was proved by Drewnowski and Roberts [16] that for every operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ and every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that for all $[f]_{c_0(A_1)} \in \ell_\infty(A_1)/c_0(A_1)$ we have $T_{A_1,A_1}([f]_{c_0(A_1)}) = [rf]_{c_0(A_1)}$ for some real $r \in \mathbb{R}$. However this does not exclude the possibility of $T_{A_1,A_1} = 0$, which is actually quite common. Thus the focus of this paper is to obtain localizations which are isomorphic embeddings or isomorphisms, and the ultimate goal (not completely achieved) is to localize somewhere any automorphism to a canonical operator along a homeomorphism (which turned out to be impossible in ZFC by 6.5) and to a trivial automorphism under OCA+MA. However if one wants to iterate the use of several localization results (as in the case of [16]) it is useful to have right-local or left-local results and not just somewhere local results:

DEFINITION 1.3. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator. Let $\mathbb{P}$ be one of the properties "liftable", "matrix operator", "trivial", "canonizable". We say that:

(1) T is *somewhere* $\mathbb{P}$ if, and only if, there are infinite $A, B \subseteq \mathbb{N}$ such that $T_{B,A}$ has $\mathbb{P}$.
(2) T is *right-locally* $\mathbb{P}$ if, and only if, for every infinite $A \subseteq \mathbb{N}$ there are infinite $A_1 \subseteq A$ and $B \subseteq \mathbb{N}$ such that T_{B,A_1} has $\mathbb{P}$.
(3) T is *left-locally* $\mathbb{P}$ if, and only if, for every infinite $B \subseteq \mathbb{N}$ there are infinite $B_1 \subseteq B$ and $A \subseteq \mathbb{N}$ such that $T_{B_1,A}$ has $\mathbb{P}$.

To hope for isomorphic left-local properties one needs to assume that the image of T is large, for example that T is surjective. Similarly, for nontrivial right-local properties we need to assume that the kernel is small, for example that T is injective. In contrast to the global versions, the local versions of the notions from Definition 1.1 behave like the Boolean counterparts:

PROPOSITION 1.4. *Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is an automorphism. Then the following are equivalent:*

(1) *T is somewhere a liftable isomorphism.*
(2) *T is somewhere an isomorphic matrix operator.*

(3) *T is somewhere a liftable isomorphism with a lifting which is continuous in the product topology.*
(4) *T is somewhere trivial.*

Proof. The implication (1)⇒(2) follows from 4.10; the equivalence of (2) and (3) is 2.14; the implication (2)⇒(4) follows from 4.8; the fact that (4) implies (1) is clear. ∎

In fact the above equivalences hold (with the same proof) in the case of T being an isomorphic embedding ([3]) and for right-localizations which are isomorphic embeddings. However, a surjective operator can be globally liftable but nowhere a matrix operator (4.12), or can be globally a matrix operator but nowhere trivial (4.6). Another reason why the above local notions make sense is the following:

Proposition 1.5. *Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator and $A, B \subseteq \mathbb{N}$ are two infinite sets. Suppose that $T_{B,A}$ is canonical along a homeomorphism. Then T fixes a complemented copy of ℓ_∞/c_0 whose image under T is complemented in ℓ_∞/c_0.*

Proof. See [16, proof of Corollary 2.4]. ∎

In fact, the above proposition would also be true, with the same proof, if we weakened the hypothesis on B^* from clopen to closed subset of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$. But to make sure that A induces a subspace not just a quotient which is to be fixed, we must insist on A^* to be clopen. This approach in the context of other $C(K)$ spaces is quite fruitful for obtaining complemented copies of the entire $C(K)$ inside any isomorphic copy of the $C(K)$ (for example, for $C(K)$ with K metrizable see [32], for ℓ_∞ see [21], and for $C([0,\omega_1])$ see [23]); see also problems in Section 7.

One should note, however, that the notion of, e.g., somewhere trivial automorphism of ℓ_∞/c_0 has quite a different character than being a somewhere trivial automorphism of $\wp(\mathbb{N})/\mathrm{Fin}$; this is because the images of subspaces of the form $\{[f] \in \ell_\infty/c_0 : f|A = 0\}$ for $A \subseteq \mathbb{N}$ are usually not of the form $\{[f] \in \ell_\infty/c_0 : f|B = 0\}$ for $B \subseteq N$, even if $T_{B,A}$ is trivial. Also, trivialization or canonization of $T_{B,A}$ does not yield any information about $T^{-1}_{A,B}$ as in the case of automorphisms of $\wp(\mathbb{N})/\mathrm{Fin}$.

Having proven the equivalence of the local versions of the above notions one is left with deciding if automorphisms of ℓ_∞/c_0 are somewhere canonizable along homeomorphisms. If this happens, their local structure is similar to that of homeomorphisms of $\mathbb{N}^*$, i.e., assuming OCA+MA they would be trivial and, for example, under CH they are not.

([3]) By an *isomorphic embedding* we mean an operator which is an isomorphism onto its closed range. Sometimes these operators are called *bounded below*.

Canonization of automorphisms $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ (or corresponding $\hat{T} : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$) encounters, however, problems at least as difficult as understanding continuous maps defined on closed subsets of $\mathbb{N}^*$ with ranges in $\mathbb{N}^*$ (not only auto(homeo)morphisms of $\mathbb{N}^*$). To better understand why this is so, let us recall that bounded linear operators on $C(\mathbb{N}^*)$ can be represented as weak* continuous mappings $\tau : \mathbb{N}^* \to M(\mathbb{N}^*)$ (see [17, Theorem VI.7.1]), where $M(\mathbb{N}^*)$ denotes the Banach space of all Radon measures on $\mathbb{N}^*$ with the total variation norm identified by the Riesz representation theorem with the dual to $C(\mathbb{N}^*)$ with the weak* topology (see [40]). Often, points of $\mathbb{N}^*$ (identified with Dirac measures) are sent by this map to measures that do not have atoms, and if they have atoms they may have many of them giving rise to partial multivalued functions into $\mathbb{N}^*$. One obtains $\tau(x)$ as $T^\star(\delta_x)$ for each $x \in \mathbb{N}^*$, and the representation is given by

$$\hat{T}(f^*)(x) = \int f^*\, d\tau(x)$$

for every $f^* \in C(\mathbb{N}^*)$. The multifunctions, possibly of empty values, are given by

$$\varphi^T_\varepsilon(y) = \{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| \geq \varepsilon\}$$

for any $\varepsilon > 0$, or by $\varphi^T(y) = \bigcup_{\varepsilon>0} \varphi^T_\varepsilon$.

An equivalent condition for T being somewhere canonizable along a homeomorphism is the existence of infinite $A, B \subseteq \mathbb{N}$ and a homeomorphism $\psi : B^* \to A^*$ such that

$$T^\star(\delta_y)|A^* = r\delta_{\psi(y)}$$

for some nonzero $r \in \mathbb{R}$, which in particular means that $\varphi^T(y) \cap A^* = \{\psi(y)\}$, or in other words that ψ is a homeomorphic selection from φ^T. Right up front there could be two basic obstacles for the existence of such a selection, namely $\bigcup_{y\in B^*} \varphi^T(y)$ could have empty interior or $\{y \in B^* : \varphi^T(y) \neq \emptyset\}$ could have empty interior for an infinite $B \subseteq \mathbb{N}^*$. We call these obstacles (in stronger versions including nonatomic measures) fountains and funnels respectively and introduce two classes of operators (fountainless operators, Definition 3.13, and funnelless operators, Definition 3.18) for which by definition the above obstacles cannot arise, respectively, and we obtain some reasonable sufficient conditions for the canonization:

- Every automorphism on ℓ_∞/c_0 which is fountainless is left-locally canonizable along a quasi-open mapping (5.6).
- Every automorphism on ℓ_∞/c_0 which is funnelless is right-locally canonizable along a quasi-open mapping (5.8).

Here *quasi-open* means that the image of every open set has nonempty interior (3.20). The second result is in fact a consequence of a study by G. Plebanek [35], however the proof of the first takes a considerable part of this paper.

The possibility of obtaining these results is based on special properties of isomorphic embeddings and surjections. One ingredient is an improvement of a theorem of Cengiz ("P" in [8]) obtained by Plebanek [35, Theorem 3.3] which guarantees that the range of $\varphi^T_{\|T\|\|T^{-1}\|}$ covers $\mathbb{N}^*$ if T is an isomorphic embedding. However, in this result the set of y's where $\varphi^T(y)$ is nonempty could be nowhere dense, so we exclude this possibility by assuming that T has no funnels. On the other hand, we prove that if T is surjective, then either for each y the set $\varphi^T(y)$ is nonempty, or else there is an infinite $A \subseteq \mathbb{N}^*$ such that $\bigcup\{\varphi^T(y) : y \in A^*\}$ is nowhere dense, the second possibility being excluded if T has no fountains.

Then one is still left with the problem of reducing a quasi-open map to a homeomorphism between two clopen sets. The results of I. Farah [19] allow us to conclude that OCA+MA implies that a quasi-open mapping defined on a clopen subset of $\mathbb{N}^*$ and being onto a clopen subset of $\mathbb{N}^*$ is somewhere a homeomorphism, and so by results of Veličković [46] it is somewhere induced by a bijection between two infinite subsets of $\mathbb{N}$. Hence we obtain (6.4):

- (OCA+MA) Every fountainless automorphism of ℓ_∞/c_0 is left-locally trivial.
- (OCA+MA) Every funnelless automorphism of ℓ_∞/c_0 is right-locally trivial.

The Continuum Hypothesis shows that the above results are optimal in many directions. First, an obstacle to improving our above-mentioned ZFC selection results (5.6, 5.8) by replacing quasi-open with a homeomorphism between clopen sets is the following example:

- (CH) There is a fountainless and funnelless everywhere present isomorphic embedding globally canonizable along a quasi-open map which is nowhere canonizable along a homeomorphism (6.10).

Here *everywhere present* is a weak version of a surjective operator ($P_A \circ T \neq 0$ for any infinite $A \subseteq \mathbb{N}$; see 3.16). Automorphisms T have the property that $P_A \circ T$ is everywhere present and $T \circ I_A$ is an isomorphic embedding for any infinite $A \subseteq \mathbb{N}$. Moreover, we have the following:

- (CH) There is an automorphism of ℓ_∞/c_0 which is nowhere canonizable along a quasi-open map, in particular along a homeomorphism (6.5).

The above example is not a direct construction, but we have more concrete and slightly weaker examples (6.8) based on the existence in $\mathbb{N}^*$ of nowhere dense P-sets which are retracts of $\mathbb{N}^*$, due to van Douwen and van Mill [12]. It is not excluded by our results (see Section 7) that consistently all isomorphic embeddings on ℓ_∞/c_0 are funnelless, however there are ZFC surjective operators which are not fountainless (3.3). And, of course, under CH there are familiar nowhere trivial homeomorphisms of $\mathbb{N}^*$ which provide examples of globally canonizable operators which are nowhere liftable (6.9).

The are many basic problems concerning the automorphisms of ℓ_∞/c_0 left open; some of them are listed in Section 7. A breakthrough in developing the methods of direct applications of PFA in the space ℓ_∞/c_0 which was recently obtained by A. Dow [14] may be especially useful in attacking these problems.

Notation and conventions. Most of the terminology and notation follows the books [18, 22, 27]. By an operator we always mean a bounded linear operator,

- Fin: the ideal of finite subsets of $\mathbb{N}$.
- FinCofin: the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$.
- $[A] = [A]_{\mathrm{Fin}}$: the equivalence class of A with respect to Fin.
- $A =_* B$: $A \triangle B \in \mathrm{Fin}$.
- $A \subseteq_* B$: $A \setminus B \in \mathrm{Fin}$.
- $\beta\mathbb{N}$: the Čech–Stone compactification of the integers and the Stone space of $\wp(\mathbb{N})$.
- $\mathbb{N}^*$: the Čech–Stone remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ and the Stone space of $\wp(\mathbb{N})/\mathrm{Fin}$.
- βA: the clopen set in $\beta\mathbb{N}$ defined by $\{x \in \beta\mathbb{N} : A \in x\}$.
- A^*: the clopen set in $\mathbb{N}^*$ defined by $\beta A \setminus A$.
- βf: the element of $C(\beta\mathbb{N})$ which extends $f \in \ell_\infty$.
- f^*: the element of $C(\mathbb{N}^*)$ obtained by restricting βf to $\mathbb{N}^*$, where $f \in \ell_\infty$.
- $[f] = [f]_{c_0}$: the equivalence class of $f \in \ell_\infty$ with respect to c_0.

Since any element of $C(\mathbb{N}^*)$ or $C(\beta\mathbb{N})$ is of the form f^* or βf, respectively, for some $f \in \ell_\infty$, we may use this convention when talking about general elements of these spaces. However, not all continuous functions on $\mathbb{N}^*$ or linear operators on $C(\mathbb{N}^*)$ are induced by corresponding objects in $\mathbb{N}$ or ℓ_∞. So for the passage from an endomorphism h of $\wp(\mathbb{N})/\mathrm{Fin}$ to a continuous self-mapping on $\mathbb{N}^*$ or from a linear operator T on ℓ_∞/c_0 to a linear operator on $C(\mathbb{N}^*)$ we will use $\hat{h}$ and $\hat{T}$, respectively.

- $[T]$: the operator on ℓ_∞/c_0 induced by an operator $T : \ell_\infty \to \ell_\infty$ which preserves c_0 (i.e., $T[c_0] \subseteq c_0$), that is, $[T]([f]_{c_0}) = [T(f)]_{c_0}$ for any $f \in \ell_\infty$.
- βT: the operator on $C(\beta\mathbb{N})$ induced by an operator $T : \ell_\infty \to \ell_\infty$ which preserves c_0 (i.e., $T[c_0] \subseteq c_0$), that is, $\beta T(\beta f) = \beta(T(f))$ for any $f \in \ell_\infty$.
- T^*: the operator on $C(\mathbb{N}^*)$ induced by an operator $T : \ell_\infty \to \ell_\infty$ which preserves c_0 (i.e., $T[c_0] \subseteq c_0$), that is, $T^*(f^*) = (T(f))^*$ for any $f \in \ell_\infty$ (not to be confused with $T^\star$ which is explained below).

- $\hat{T}$: the operator from $C(\mathbb{N}^*)$ into itself which corresponds to $T : \ell_\infty/c_0 \to \ell_\infty/c_0$, i.e., $\hat{T}(f^*) = g^*$ where $[g] = T([f])$.
- $\hat{h}$: the continuous self-map of $\mathbb{N}^*$ which corresponds via the Stone duality to an endomorphism h of $\wp(\mathbb{N})/\mathrm{Fin}$, i.e., $\hat{h}(x) = h^{-1}[x]$ when we identify points of $\mathbb{N}^*$ with ultrafilters of $\wp(\mathbb{N})/\mathrm{Fin}$.
- T_ψ: the composition operator $T_\psi : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ which maps f to $f \circ \psi$ for continuous $\psi : \mathbb{N}^* \to \mathbb{N}^*$.

The remaining symbols often used are:

- $M(\mathbb{N}^*)$: the Banach space of Radon measures on $\mathbb{N}^*$ with the total variation norm, identified with the dual space to $C(\mathbb{N}^*)$ or the dual space to ℓ_∞/c_0 via the Riesz representation theorem.
- $T^\star$: the dual or adjoint operator of T, i.e., $T^\star(\mu)(f) = \mu(T(f))$; $T^\star$ acts on the space of Radon measures if T acts on a space of continuous functions.
- δ_x: the Dirac measure concentrated on x.
- $\mu|F$: the restriction of a measure $\mu \in M(\mathbb{N}^*)$ to a Borel subset $F \subseteq \mathbb{N}^*$, i.e., $\mu|F$ is an element of $M(\mathbb{N}^*)$ such that $(\mu|F)(G) = \mu(G \cap F)$ for any Borel $G \subseteq \mathbb{N}^*$.
- χ_a: the characteristic function of a.
- B_X: the unit ball of the Banach space X.
- $\varphi^T_\varepsilon(y)$: the set $\{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| > \varepsilon\}$, where T is an operator on $C(\mathbb{N}^*)$.
- $\varphi^T(y)$: the set $\bigcup_{\varepsilon>0} \varphi^T_\varepsilon(y)$, where T is an operator on $C(\mathbb{N}^*)$.

2. Operators on ℓ_∞ preserving c_0

2.1. Operators given by c_0-matrices. A linear operator R on ℓ_∞ which preserves c_0 (i.e., $R[c_0] \subseteq c_0$) defines an operator on c_0. In the case of the Boolean algebra $\wp(\mathbb{N})$, any Boolean automorphism preserves $\mathrm{FinCofin}(\mathbb{N})$ and its restriction to $\mathrm{FinCofin}(\mathbb{N})$ completely determines the automorphism. The analogous fact does not hold for linear automorphisms on ℓ_∞, e.g., there are distinct automorphisms of ℓ_∞ which do not move c_0 (2.15). However, the restrictions to c_0 of operators on ℓ_∞ which preserve c_0 will play an important role, and in some cases will determine the given operator.

An array of real numbers $\mathbb{B} = (b_{ij})_{i,j\in\mathbb{N}}$ will be called a *matrix*. For $f \in \ell_\infty$, by $\mathbb{B}f$ we will mean a sequence of formal sums $(\sum_{j\in\mathbb{N}} b_{i,j} f(j))_{i\in\mathbb{N}}$. A matrix $\mathbb{B}$ may represent a bounded linear operator T by letting $\mathbb{B}f = T(f)$ for each $f \in \ell_\infty, \ell_1, c_0$. As in linear algebra, we have $(b_{i,j})_{i\in\mathbb{N}} = \mathbb{B}\delta_j$, where $\delta_j \in \ell_\infty$ has all coordinates 0 except the jth which is 1. That is, if a matrix $\mathbb{B}$ represents a bounded operator T on $X \in \{\ell_\infty, \ell_1, c_0\}$, then its columns must be in X and the X-norms of the columns must be bounded.

As is well known, $\ell_\infty = \ell_1^\star = c_0^{\star\star}$ where $c_0 \subseteq \ell_\infty$ is the canonical inclusion of a Banach space into its bidual. The actions of the above spaces on their preduals are given by the infinite scalar product. By the definition of the adjoint $T^\star : X^\star \to X^\star$ of an operator $T : X \to X$, for $X \in \{c_0, \ell_1\}$, we obtain $(T^\star(\delta_i))(\delta_j) = T(\delta_j)(i)$, so $T^\star$ is given by the transpose matrix $\mathbb{B}^t$ of $\mathbb{B}$. It follows that every row of $\mathbb{B}$ must be in $X^\star$ if $\mathbb{B}$ represents an operator on $X \in \{\ell_1, c_0\}$ and that the dual norms of the rows must be bounded. Let us establish a transparent representation of operators on c_0:

PROPOSITION 2.1. *$R : c_0 \to c_0$ is a bounded linear operator if, and only if, there exists a matrix $\mathbb{B} = (b_{ij})_{i,j\in\mathbb{N}}$ such that:*

(1) *every row of $\mathbb{B}$ is in ℓ_1,*
(2) *if we write $b_i = (b_{ij})_j$, then $\{\|b_i\|_{\ell_1} : i \in \mathbb{N}\}$ is a bounded set,*
(3) *every column is in c_0,*

and such that for every $f \in c_0$ we have $R(f) = \mathbb{B}f$.

Proof. The necessity of the conditions follows from the above discussion. The sufficiency is checked directly. ∎

PROPOSITION 2.2. *Let $\mathbb{B} = (b_{ij})_{i,j\in\mathbb{N}}$ be a matrix. The condition $R(f) = \mathbb{B}f$ for every $f \in \ell_\infty$ defines a bounded linear operator $R : \ell_\infty \to \ell_\infty$ if, and only if, $b_i \in \ell_1$ for all $i \in \mathbb{N}$, and $\{\|b_i\|_{\ell_1} : i \in \mathbb{N}\}$ is a bounded set.*

Proof. The necessity follows from the above discussion and the fact that ℓ_1 is isometric to a subspace of ℓ_∞^*. The sufficiency is checked directly. ∎

DEFINITION 2.3.

(i) We say that a matrix is a c_0*-matrix* if it satisfies conditions (1)–(3) of Proposition 2.1.
(ii) We say that a bounded linear operator $R : \ell_\infty \to \ell_\infty$ *is given by a c_0-matrix* if there exists a c_0-matrix $\mathbb{B}$ such that $R(f) = \mathbb{B}f$ for every $f \in \ell_\infty$.

COROLLARY 2.4. *Suppose that $R : \ell_\infty \to \ell_\infty$ is a bounded linear operator which preserves c_0 and there is a matrix $\mathbb{B}$ which satisfies $R(f) = \mathbb{B}f$ for every $f \in \ell_\infty$. Then $\mathbb{B}$ is a c_0-matrix.*

Proof. If such an operator on ℓ_∞ were not given by a c_0-matrix, then by 2.2 and 2.1, some of the columns of the corresponding matrix would not be in c_0. Then the operator would not preserve c_0. ∎

PROPOSITION 2.5. *If a bounded linear operator $R : \ell_\infty \to \ell_\infty$ is given by a c_0-matrix, then $R = (R|c_0)^{\star\star}$.*

Proof. Let $\mathbb{B}$ be a c_0-matrix and $T : c_0 \to c_0$ the operator it defines whose double adjoint is represented by the double transpose of $\mathbb{B}$, so $T^{\star\star} = R$. As

$T^{\star\star}|X = T$ for any operator T and any Banach space X, the proposition follows. ∎

2.2. Falling and weakly compact operators. Let us recall the following characterization of weakly compact operators on c_0:

THEOREM 2.6. *Let $R : c_0 \to c_0$ be a bounded linear operator and let $\mathbb{B} = (b_{ij})_{i,j\in\mathbb{N}}$ be the corresponding matrix. The following are equivalent:*

(1) *R is weakly compact,*
(2) *$R^{\star\star}[\ell_\infty] \subseteq c_0$,*
(3) *$\|b_i\|_{\ell_1} \to 0$, where $b_i = (b_{ij})_{j\in\mathbb{N}}$.*

Proof. The equivalence of (1) and (2) is well known (see [10, Chapter 3, exercise 3]).

If $\|b_i\|_{\ell_1} \not\to 0$, as the columns of $\mathbb{B}$ are in c_0, one can recursively choose pairwise disjoint finite sets $F_n \subseteq \mathbb{N}$ such that $\sum_{j\in F_n} |b_{i_n,j}| > \varepsilon$ for some $\varepsilon > 0$ and some strictly increasing sequence $(i_n)_{n\in N}$ of elements of $\mathbb{N}$. By the Grothendieck–Dieudonné characterization of weakly compact sets in the dual to a $C(K)$ space, this implies that $\{R^\star(\delta_{i_n}) : n \in \mathbb{N}\}$ is not weakly compact, which by the Gantmacher theorem shows that R is not weakly compact, so (1) implies (3).

On the other hand, (3) implies that the image of the unit ball of c_0 under R is included in $\prod_{i\in\mathbb{N}}[-\|b_i\|_{\ell_1}, \|b_i\|_{\ell_1}] \subseteq c_0$. This set is norm compact, and hence R is a compact operator, so in particular weakly compact (it is well known that all weakly compact operators on c_0 are compact). ∎

PROPOSITION 2.7. *Let $R : \ell_\infty \to \ell_\infty$ be an operator given by a c_0-matrix. Then R is weakly compact if, and only if, $R[\ell_\infty] \subseteq c_0$.*

Proof. If R is weakly compact, then $R|c_0$ must be as well, so by 2.6 we get $(R|c_0)^{\star\star}[\ell_\infty] \subseteq c_0$ but by 2.5 we have $(R|c_0)^{\star\star} = R$. In the other direction, use the fact that an operator defined on a Grothendieck Banach space into a separable Banach space is weakly compact [9, Theorem 1(v)]. ∎

DEFINITION 2.8. A c_0-matrix operator $R : \ell_\infty \to \ell_\infty$ is called *falling* if, and only if, for every $\varepsilon > 0$ there is a partition $A_0, \ldots, A_{k-1}$ of $\mathbb{N}$ such that $\sum_{j\in A_m} |b_{ij}| < \varepsilon$ for all $m < k$ and all $i \in \mathbb{N}$ sufficiently large.

PROPOSITION 2.9. *Every operator on ℓ_∞ which is given by a c_0-matrix and is weakly compact is falling.*

Proof. Use Theorem 2.6. ∎

PROPOSITION 2.10. *There is a falling, nonweakly compact operator on ℓ_∞ given by a c_0-matrix.*

Proof. Let $R : \ell_\infty \to \ell_\infty$ be given by the matrix $b_{ij} = 1/(i + 1)$ if $j \le i$, and $b_{ij} = 0$ otherwise, for all $i \in \mathbb{N}$. By 2.6 this is not a weakly compact operator. Given $k \in \mathbb{N}$, if we consider $A_m = \{lk + m : l \in \mathbb{N}\}$ for $m < k$, then $\sum_{j\in A_m} |b_{ij}| \le \left(\frac{i+1}{k}\right)\left(\frac{1}{i+1}\right) = 1/k$, so the operator is falling. ∎

2.3. Antimatrix operators. The behaviour opposite to operators given by a c_0-matrix is the subject of the following:

DEFINITION 2.11. A bounded linear operator $R : \ell_\infty \to \ell_\infty$ will be called an *antimatrix* operator if, and only if, $R[c_0] = \{0\}$.

Using the isometry between ℓ_∞ and $C(\beta\mathbb{N})$, an operator R on ℓ_∞ can be associated with an operator βR on $C(\beta\mathbb{N})$, and these operators can be associated with weak* continuous functions from $\beta\mathbb{N}$ into the Radon measures $M(\beta\mathbb{N})$ on $\beta\mathbb{N}$ (see [17, Theorem VI.7.1]). Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, such functions are determined by their values on $\{\delta_n : n \in \mathbb{N}\}$. The following characterizations will be useful later on:

LEMMA 2.12. *Suppose $R : \ell_\infty \to \ell_\infty$ is a bounded linear operator such that $R[c_0] \subseteq c_0$. Then:*

(a) *R is given by a c_0-matrix if, and only if, $R^\star(\delta_n)$ is concentrated on $\mathbb{N}$ for all $n \in \mathbb{N}$, that is, $R^\star(\delta_n) \in \ell_1$ for all $n \in \mathbb{N}$.*
(b) *R is an antimatrix operator if, and only if, $R^\star(\delta_n)$ is concentrated on $\mathbb{N}^*$ for all $n \in \mathbb{N}$.*

Proof. (a) Assume R is given by a c_0-matrix $(b_{ij})_{i,j\in\mathbb{N}}$. Then for every $f \in \ell_\infty$ we have $R^\star(\delta_n)(f) = R(f)(n) = b_n(f)$, where b_n is the nth row of $(b_{ij})_{i,j\in\mathbb{N}}$. Hence $R^\star(\delta_n) = b_n$, so by definition of c_0-matrix, $b_n \in \ell_1$.

Conversely, assume $R^\star(\delta_n) \in \ell_1$. Let M be the matrix with nth row $R^\star(\delta_n)$. Then R is induced by M. Moreover, since $R[c_0] \subseteq c_0$, we know that M is a c_0-matrix.

(b) Suppose $R^\star(\delta_n)$ is not concentrated on $\mathbb{N}^*$ for some $n \in \mathbb{N}$. Then there exists an $m \in \mathbb{N}$ such that $R^\star(\delta_n)(\{m\}) \neq 0$. Hence $R(\chi_{\{m\}})(n) = R^\star(\delta_n)(\chi_{\{m\}}) \neq 0$. Therefore, $\chi_{\{m\}} \in c_0$ witnesses that $R[c_0] \neq \{0\}$, so R is not an antimatrix operator.

Conversely, assume $R^\star(\delta_n)$ is concentrated on $\mathbb{N}^*$ for every $n \in \mathbb{N}$. Fix $f \in c_0$. Then for every $n \in \mathbb{N}$ we have $R(f)(n) = R^\star(\delta_n)(\beta f) = \int \beta f \, dR^\star(\delta_n) = \int_{\mathbb{N}^*} \beta f \, dR^\star(\delta_n) = 0$, because $\beta f|\mathbb{N}^* = 0$. ∎

Thus a typical example of an antimatrix operator is one given by $R(f) = ((\beta f(x_i))_{i\in\mathbb{N}})$ where $(x_i)_{i\in\mathbb{N}}$ is any sequence of nonprincipal ultrafilters.

PROPOSITION 2.13. *If $R : \ell_\infty \to \ell_\infty$ is such that $R[c_0] \subseteq c_0$, then $R = S_0 + S_1$, where S_0 is given by a c_0-matrix and S_1 is an antimatrix operator.*

Proof. As $R[c_0] \subseteq c_0$, there is a matrix $(b_{ij})_{i,j\in\mathbb{N}}$ which satisfies 2.1. Define S_0 as multiplication by this matrix, i.e. $S_0 = (R|c_0)^{\star\star}$ by 2.5. Now $S_1 = R - S_0$ is an antimatrix, so we obtain the desired decomposition. ∎

2.4. Product topology continuity of operators. The importance of operators on ℓ_∞ given by c_0-matrices is expressed in the following theorem which exploits the fact that ℓ_∞ is the bidual space of c_0.

THEOREM 2.14. *Let $R : \ell_\infty \to \ell_\infty$ be a bounded linear operator. The following are equivalent:*

(1) $R = (R|c_0)^{\star\star}$.
(2) *R is given by a c_0-matrix.*
(3) *$R|B_{\ell_\infty} : (B_{\ell_\infty}, \tau_p) \to (\ell_\infty, \tau_p)$ is continuous and $R[c_0] \subseteq c_0$.*

Proof. Assume (1). Then $R[c_0] \subseteq c_0$, so by Proposition 2.1 there is a matrix $\mathbb{B}$ such that $R(f) = \mathbb{B}f$ for all $f \in c_0$. Hence the double adjoint of $R|c_0$ is given by the double transpose of $\mathbb{B}$, i.e., by $\mathbb{B}$, so we deduce (2).

(2) implies (1) because $(R|c_0)^{\star\star}$ is given by the same matrix as $R|c_0$ (the double transpose).

It is clear that the restriction to the unit ball of an operator R on ℓ_∞ which is given by a c_0-matrix $\mathbb{B}$ is continuous in the product topology, because to make $|R(f)(i)-R(g)(i)| = |\sum_{j\in\mathbb{N}} b_{i,j}(f(j)-g(j))|$ small it is enough to choose $f, g \in B_{\ell_\infty}$ so that $|f(j) - g(j)|$ is small for finitely many $j \in \mathbb{N}$, as $\sum_{j\in\mathbb{N}} |b_{i,j}| < \infty$ by Proposition 2.1.

If R is not given by a c_0-matrix, then by Proposition 2.12 there is $i \in \mathbb{N}$ such that $R^\star(\delta_i) = \mu + \nu$ where μ is concentrated on $\mathbb{N}$, ν is concentrated on $\mathbb{N}^*$ and $\nu \neq 0$. Let $f \in B_{\ell_\infty}$ be such that $\int \beta f\, d\nu \neq 0$. Consider $f_n = f \cdot \chi_{\mathbb{N}\setminus\{0,\dots,n\}} \in B_{\ell_\infty}$ and notice that this sequence converges to 0 in the product topology. Note that at the same time $\int \beta f_n\, d\nu = \int \beta f\, d\nu \neq 0$. Then

$$R(f_n)(i) = \int \beta f_n\, d(\nu + \mu) = \int_{\mathbb{N}} \beta f_n\, d\mu + \int_{\mathbb{N}^*} \beta f_n\, d\mu$$

does not tend to 0 as $n \to \infty$ since $\int_{\mathbb{N}} \beta f_n\, d\mu \to 0$. ∎

Thus, nonzero antimatrix operators are discontinuous in the product topology. Such discontinuities are not, however, incompatible with being an automorphism or having a nice behaviour on c_0.

THEOREM 2.15. *There are discontinuous automorphisms of ℓ_∞ preserving c_0. There are different automorphisms of ℓ_∞ which agree on c_0. They can be the identity on c_0.*

Proof. Let $(A_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets. Let x_i be any nonprincipal ultrafilter such that $A_i \in x_i$ for all $i \in \mathbb{N}$. For a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ define

$$R_\sigma(f)(n) = f(n) - \beta f(x_i) + \beta f(x_{\sigma(i)}),$$

where $i \in \mathbb{N}$ is such that $n \in A_i$. First note that $R_{\sigma^{-1}} \circ R_\sigma = R_\sigma \circ R_{\sigma^{-1}} = \mathrm{Id}$, and so R_σ is an automorphism. One verifies that $R_\sigma|c_0$ is the identity for any permutation σ, in particular $R_\sigma - \mathrm{Id} \neq 0$ is antimatrix for any permutation σ different from the identity and hence R_σ is discontinuous by 2.14. ∎

In this proof we decompose ℓ_∞ as a direct sum $X \oplus Y$, both factors necessarily isomorphic to ℓ_∞: the first consists of the functions constant on each set A_i and the second of the functions equal to zero at each point x_i. Since the second factor contains c_0, the automorphisms of the first factor induce automorphisms of ℓ_∞ which do not move c_0. This lack of continuity is also present in homomorphisms of $\wp(\mathbb{N})$ [20, 3.2.3] but not in its automorphisms.

3. Operators on ℓ_∞/c_0

3.1. Ideals of operators on ℓ_∞/c_0. As usual, by a [left, right] ideal we will mean a collection $\mathcal{I}$ of operators such that $T+S \in \mathcal{I}$ whenever $T, S \in \mathcal{I}$ and $S \circ R, R \circ S \in \mathcal{I}$ [$R \circ S \in \mathcal{I}$, $S \circ R \in \mathcal{I}$] whenever $S \in \mathcal{I}$ and R is any operator on ℓ_∞/c_0. We say that an operator T on ℓ_∞/c_0 *factors through* ℓ_∞ if, and only if, there are operators $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$ such that $T = R_2 \circ R_1$. It is clear that the operators which factor through ℓ_∞ form a two-sided ideal. Also it is well known that the weakly compact operators form a two-sided proper ideal [17, VI.4.5]. We introduce another ideal:

Definition 3.1. A bounded linear operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is *locally null* if, and only if, for every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that $T \circ I_{A_1} = 0$.

Locally null should really read right-locally null, but left-locally null is just null, so there is no need of using the word "right".

Proposition 3.2. *The locally null operators form a proper left ideal containing all weakly compact operators and all operators which factor through ℓ_∞.*

Proof. It is clear that the locally null operators form a proper left ideal.

Let us prove that every weakly compact operator on ℓ_∞/c_0 is locally null. We will use the fact that an operator T on a $C(K)$ space is weakly compact if, and only if, $\|T(f_n)\| \to 0$ whenever $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ is a bounded pairwise disjoint sequence (i.e., $f_n \cdot f_m = 0$ for $n \neq m$); see [11, Corollary VI.2.17].

Let $\{A_\xi : \xi < \omega_1\}$ be a family of almost disjoint infinite subsets of an infinite $A \subseteq \mathbb{N}$. By the weak compactness of T, the set of $\alpha \in \omega_1$ such that $T \circ I_{A_\alpha} \neq 0$ must be at most countable, so take α outside this set.

Now let $T = R_2 \circ R_1$ where $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$. Let $\mu_n = R_1^\star(\delta_n)$. Let $A \subseteq \mathbb{N}$ be infinite. As the supports of μ_n's are c.c.c. and there are continuum many pairwise disjoint clopen subsets of A^*, there is an infinite $A_1 \subseteq A$ such that $|\mu_n|(A_1^*) = 0$ for every $n \in \mathbb{N}$. It follows that $R_1 \circ I_{A_1} = 0$, which completes the proof. ∎

Proposition 3.3. *There is a locally null operator on ℓ_∞/c_0 which factors through ℓ_∞ and is surjective. There is no surjective weakly compact operator.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a discrete subset of $\mathbb{N}^*$. Define $R : \ell_\infty/c_0 \to \ell_\infty$ by $R([f]_{c_0}) = (f^*(x_n))_{n\in\mathbb{N}}$. It is well known that the closure of $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{N}^*$ is homeomorphic to $\beta\mathbb{N}$. So by the Tietze extension theorem, R is onto ℓ_∞. Furthermore, $Q \circ R$ is surjective, where $Q : \ell_\infty \to \ell_\infty/c_0$ is the quotient map. As no clopen subset A^* of $\mathbb{N}^*$ is separable, below every infinite A there is an infinite $A_1 \subseteq A$ such that no x_n belongs to A_1^*. Then $R \circ I_{A_1} = 0$, which proves that R is locally null, and hence so is $Q \circ R$.

Weakly compact operators on an infinite-dimensional $C(K)$ cannot be surjective because weakly compact subsets of an infinite-dimensional Banach space have empty interior if the space is not reflexive. So countable unions of them are of the first Baire category, and in particular, images of balls cannot cover an infinite-dimensional Banach space $C(K)$. ∎

See 6.7 for more on the ideal of locally null operators under CH.

3.2. Local behaviour of functions associated with the adjoint operator. In general, for a bounded linear operator T acting on the Banach space $C(K)$ the function which sends $x \in K$ to $\|T^\star(\delta_x)\|$ is lower semicontinuous (see, e.g., [35, Lemma 2.1]) and may be quite discontinuous.

Proposition 3.4. *Suppose $F \subseteq \mathbb{N}^*$ is a nowhere dense retract of $\mathbb{N}^*$. There is a bounded linear operator T on $C(\mathbb{N}^*)$ such that the function $\alpha : \mathbb{N}^* \to \mathbb{R}$ defined by*

$$\alpha(y) = \|T^\star(\delta_y)\| \quad \text{for } y \in \mathbb{N}^*$$

is discontinuous at every point of F.

Proof. Define T by $T(f) = f - f \circ r$, where $r : \mathbb{N}^* \to F$ is the retraction onto F. Then $T(f)(y) = f(y) - f(r(y))$, so $T^\star(\delta_y) = \delta_y - \delta_{r(y)}$. Hence $\alpha = 2\chi_{\mathbb{N}^*\setminus F}$. Since F is nowhere dense, the set of discontinuities of α is F. ∎

By [35, Lemma 4.1], for every lower semicontinuous function $\mathbb{N}^* \to \mathbb{R}$, every $\varepsilon > 0$ and every open $U \subseteq \mathbb{N}^*$ there is an open $V \subseteq U$ such that the function's oscillation on V is smaller than ε. Hence, since nonempty G_δ-sets in $\mathbb{N}^*$ have nonempty interior, there is a dense open subset of $\mathbb{N}^*$ where the function which sends $y \in \mathbb{N}^*$ to $\|T^\star(\delta_y)\|$ is locally constant. In the case of $\mathbb{N}^*$ we have not only the local stabilization of the values of $\|T^\star(\delta_y)\|$ but the local stabilization of Hahn decompositions for the measures $T^\star(\delta_y)$:

Lemma 3.5. *Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is bounded linear and $B \subseteq \mathbb{N}$ is infinite. Then there are an infinite $B_1 \subseteq_* B$, a real number s, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets such that for every $y \in B_1^*$ we have:*

(i) $s = \|T^\star(\delta_y)\|$,
(ii) *if* $T^\star(\delta_y) = \mu^+ - \mu^-$ *is the Jordan decomposition of the measure, then* $\mu^-(C_n^*) < 1/(4(n+1))$ *and* $\mu^+(D_n^*) < 1/(4(n+1))$.

Proof. We construct by induction a $\subseteq_*$-decreasing sequence of infinite sets $(A_n)_{n\in\mathbb{N}}$, $y_n \in A_n^*$, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets such that for every $n \in \mathbb{N}$:

(1) $\sup\{\|T^\star(\delta_y)\| : y \in A_n^*\} - \|T^\star(\delta_{y_n})\| < 1/(6(n+1))$,
(2) $\|T^\star(\delta_{y_n})\| - T^\star(\delta_{y_n})(C_n^*) + T^\star(\delta_{y_n})(D_n^*) < 2/(6(n+1))$,
(3) for all $y \in A_{n+1}^*$ we have $|T^\star(\delta_y)(C_n^*) - T^\star(\delta_{y_n})(C_n^*)| < 1/(6(n+1))$ and $|T^\star(\delta_y)(D_n^*) - T^\star(\delta_{y_n})(D_n^*)| < 1/(6(n+1))$.

This is arranged as follows. Set $A_0 = B$ and assume we have constructed A_n. Take $y_n \in A_n^*$ such that $\|T^\star(\delta_{y_n})\| > \sup\{\|T^\star(\delta_y)\| : y \in A_n^*\} - 1/(6(n+1))$. Take a Hahn decomposition $\mathbb{N}^* = H_n^+ \cup H_n^-$ for the measure $T^\star(\delta_{y_n})$. By regularity, we may choose an infinite $C_n \subseteq \mathbb{N}$ with $|T^\star(\delta_{y_n})(H_n^+) - T^\star(\delta_{y_n})(C_n^*)| < 1/(6(n+1))$. If we set $D_n = \mathbb{N}\setminus C_n$, we obtain $|T^\star(\delta_{y_n})(H_n^-) - T^\star(\delta_{y_n})(D_n^*)| < 1/(6(n+1))$. Therefore,

$$\begin{aligned}\|T^\star(\delta_{y_n})\| &= T^\star(\delta_{y_n})(H_n^+) - T^\star(\delta_{y_n})(H_n^-)\\ &< T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) + 2/(6(n+1)),\end{aligned}$$

and so (2) holds.

By the weak* continuity of $T^\star$, the set of points which satisfy the condition in (3) is an open neighbourhood of y_n, so we may take $A_{n+1} \subseteq_* A_n$ satisfying (3). This ends the induction.

Notice that $|T^\star(\delta_{y_n})(C_n^*)| \le \|T\|$ for every $n \in \mathbb{N}$, and so there exists a convergent subsequence of $(T^\star(\delta_{y_n})(C_n^*))_{n\in\mathbb{N}}$. The same is true for the D_n's, and so we may assume that both of these sequences converge. Let us define $s^+ = \lim_{n\to\infty} T^\star(\delta_{y_n})(C_n^*)$ and $s^- = \lim_{n\to\infty} T^\star(\delta_{y_n})(D_n^*)$. Now let $B_1 \subseteq \mathbb{N}$ be infinite such that $B_1 \subseteq_* A_n$ for all $n \in \mathbb{N}$. We will show that for every $y \in B_1^*$ we have $\|T^\star(\delta_y)\| = s$, where $s = s^+ - s^-$.

Indeed, let us fix $y \in B_1^*$. Notice that from (3) we obtain

$$s = \lim_{n\to\infty}(T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*)) = \lim_{n\to\infty} T^\star(\delta_y)(\chi_{C_n^*} - \chi_{D_n^*}).$$

Therefore, $s \le \|T^\star(\delta_y)\|$. By (1) and (2), for every $n \in \mathbb{N}$,

$$\begin{aligned}&T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) > \|T^\star(\delta_{y_n})\| - 2/(6(n+1))\\ &\qquad > \sup\{\|T^\star(\delta_z)\| : z \in A_n^*\} - 1/(2(n+1)) \ge \|T^\star(\delta_y)\| - 1/(2(n+1)).\end{aligned}$$

Hence, $s = \lim_{n\to\infty} T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_y)(D_n^*) \ge \|T^\star(\delta_y)\|$. This proves the first statement of the lemma.

To check (ii) let us fix $y \in B_1^*$ and the Jordan decomposition $T^\star(\delta_y) = \mu^+ - \mu^-$. By going to a subsequence if necessary, we may assume that $|s - (T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*))| < 1/(6(n+1))$ for every $n \in \mathbb{N}$.

Observe that since $C_n^* \cup D_n^* = \mathbb{N}^*$, we have

$$\begin{aligned}
&T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*)\\
&\qquad = |T^\star(\delta_y)|(C_n^*) + |T^\star(\delta_y)|(D_n^*) - 2\mu^-(C_n^*) - 2\mu^+(D_n^*)\\
&\qquad = \|T^\star(\delta_y)\| - 2\mu^-(C_n^*) - 2\mu^+(D_n^*)
\end{aligned}$$

for every $n \in \mathbb{N}$. Then by (3) we obtain

$$\begin{aligned}
2(\mu^-(C_n^*) + \mu^+(D_n^*)) &= s - (T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*))\\
&\le s - (T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) - 2/(6(n+1)))\\
&< 1/(6(n+1)) + 2/(6(n+1)) = 1/(2(n+1)).
\end{aligned}$$

Since both $\mu^-(C_n^*)$ and $\mu^+(D_n^*)$ are nonnegative, they are both strictly less than $1/(4(n+1))$, and (ii) is proved. ∎

Corollary 3.6. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be bounded linear and $B \subseteq \mathbb{N}$ be infinite. There are an infinite $B_1 \subseteq_* B$ and a Borel partition $\mathbb{N}^* = X \cup Y$ such that X and Y form a Hahn decomposition of $T^\star(\delta_y)$ for every $y \in B_1^*$.*

Proof. Let $B_1 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ be as in 3.5. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers such that $1/(4(n_k+1)) < 1/2^k$ for all $k \in \mathbb{N}$.

Let $F_i = \bigcap_{k\ge i} C_{n_k}^*$, $X = \bigcup_{i\in\mathbb{N}} F_i$ and $Y = \mathbb{N}^* \setminus X$. Fix $y \in B_1^*$ and let $T^\star(\delta_y) = \mu^+ - \mu^-$ be the Jordan decomposition of the measure. Since $F_i \subseteq F_{i+1}$ and $F_i \subseteq C_{n_i}^*$ for every $i \in \mathbb{N}$, by 3.5 we have

$$\mu^-(F_{i_0}) \le \mu^-(F_i) \le \mu^-(C_{n_i}) < 1/(4(n_i+1))$$

for every $i_0 \in \mathbb{N}$ and every $i \ge i_0$. Therefore, $\mu^-(F_{i_0}) = 0$ for every $i_0 \in \mathbb{N}$, and so $\mu^-(X) = 0$.

On the other hand, we have $Y \subseteq \mathbb{N}^* \setminus F_i = \bigcup_{k\ge i} D_{n_k}^*$ for every $i \in \mathbb{N}$. Therefore, $\mu^+(Y) \le \sum_{k\ge i} \mu^+(D_{n_k}^*) < \sum_{k\ge i} 1/(4(n_k+1)) < \sum_{k\ge i} 1/2^k$ for every $i \in \mathbb{N}$. It follows that $\mu^+(Y) = 0$. ∎

Corollary 3.7. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a bounded linear operator and $B \subseteq \mathbb{N}$ is infinite. Then there is an infinite $B_1 \subseteq B$ such that the functions which map $y \in B_1^*$ to the positive part, to the negative part, and to the total variation measure of the measure $T^\star(\delta_y)$, respectively, are all weak* continuous. In particular, T is left-locally a regular operator.*

Proof. Let $B_1 \subseteq B$ and s be as in 3.5. If $s = 0$, then the first part of the corollary is trivially true and $P_{B_1} \circ T = 0$ is positive. Otherwise, consider the operator $s^{-1}T$. We have $\|(s^{-1}T)^\star(\delta_y)\| = 1$ for all $y \in B_1^*$. Now apply [34, second part of Lemma 2.2], which says that, on the dual sphere, sending the measure μ to its total variation $|\mu|$ is weak* continuous. Since the positive and the negative parts of μ can be obtained from μ and $|\mu|$, and $|s\mu| = s|\mu|$

for nonnegative s, using the weak* continuity of $T^\star$ we deduce the first part of the corollary.

For the second part we will define two positive operators T^+ and T^- such that $T^+ - T^- = P_{B_1} \circ T$. For every $y \in B_1^*$ we see that $(P_{B_1} \circ T)^\star(\delta_y) = T^\star(P_{B_1}^\star(\delta_y)) = T^\star(\delta_y)$. So for every $y \in B_1^*$ define

$$T^+(f)(y) = \int f \, d(T^\star(\delta_y))^+, \qquad T^-(f)(x) = \int f \, d(T^\star(\delta_y))^-.$$

It is clear that $P_{B_1} \circ T = T^+ - T^-$. The linearity of T^+ and T^- follows from general properties of the integral. To see that they are bounded, notice that for every $f \in C(\mathbb{N}^*)$ with $\|f\| \le 1$ we know that $\|T^+(f)\| \le \sup_{y \in B_1^*}(T^\star(\delta_y))^+(\mathbb{N}^*)$. If $\{(T^\star(\delta_y))^+(\mathbb{N}^*) : y \in B_1^*\}$ were unbounded, the set $F_n = \{y \in B_1^* : (T^\star(\delta_y))^+(\mathbb{N}^*) \ge n\}$ would be nonempty for every $n \in \mathbb{N}$. But by the weak* continuity of the map $y \mapsto (T^\star(\delta_y))^+$, each F_n is closed. As the F_n's form a decreasing chain of nonempty closed sets, there is $y \in \bigcap_{n \in \mathbb{N}} F_n$, a contradiction. The same argument shows that T^- is bounded. ∎

Definition 3.8. Let X and Y be topological spaces. A function $\varphi : X \to \wp(Y)$ is called *upper semicontinuous* if for every open set $V \subseteq Y$ the set $\{x \in X : \varphi(x) \subseteq V\}$ is open in X.

Our main interest in multifunctions is related to the following:

Definition 3.9. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator and $\varepsilon > 0$. We define

$$\varphi_\varepsilon^T(y) = \{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| \ge \varepsilon\}, \qquad \varphi^T(y) = \bigcup_{\varepsilon > 0} \varphi_\varepsilon^T(y).$$

Proposition 3.10. *There is a bounded linear $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ such that $\varphi_{1/2}^T$ is not upper semicontinuous.*

Proof. Consider T from the proof of 3.4. We have $\varphi_{1/2}(y) = \emptyset$ if $y \in F$, and $\varphi_{1/2}(y) = \{r(y), y\}$ if $y \in \mathbb{N}^* \setminus F$. So taking $V = \mathbb{N}^* \setminus F$ we deduce that $\{y : \varphi_{1/2}(y) \subseteq V\} = F$, which is not open, but closed nowhere dense. ∎

However we obtain left-local upper semicontinuity:

Lemma 3.11. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $B \subseteq \mathbb{N}$ be infinite. Then there exists $B_1 \subseteq_* B$ such that $\varphi_\varepsilon^T | B_1^*$ is upper semicontinuous for every $\varepsilon > 0$.*

Proof. Let $B_1 \subseteq_* B$ and C_n, D_n be as in 3.5. Fix $y \in B_1^*$ and an open $V \subseteq \mathbb{N}^*$ such that $\varphi_\varepsilon^T(y) \subseteq V$. Then for every $x \in \mathbb{N}^* \setminus V$ we find a clopen neighbourhood U_x of x as follows. First notice that $|T^\star(\delta_y)(\{x\})| < \varepsilon$. Let $N_x \in \mathbb{N}$ be such that $|T^\star(\delta_y)(\{x\})| < \varepsilon - 1/N_x$. Now, using the regularity of the measure $T^\star(\delta_y)$, find U_x such that $|T^\star(\delta_y)|(U_x) < \varepsilon - 1/N_x$. We may assume that U_x is included in either $C^*_{N_x}$ or $D^*_{N_x}$.

Since $\mathbb{N}^* \setminus V$ is compact, we may find $x_0, \ldots, x_{k-1} \in \mathbb{N}^* \setminus V$ such that $\mathbb{N}^* \setminus V \subseteq \bigcup_{i<k} U_{x_i}$. Using the weak* continuity of $T^\star$, we now find a clopen neighbourhood of y, say E^*, which we may assume to be included in B^*_1, such that for every $z \in E^*$ we have

$$|T^\star(\delta_z)(U_{x_i})| < \varepsilon - 1/N_{x_i} \quad \text{for each } i < k.$$

This is possible because $|T^\star(\delta_y)(U_x)| \leq |T^\star(\delta_y)|(U_x) < \varepsilon - 1/N_x$.

We claim that for every $z \in E^*$ and every $x \in \bigcup_{i<k} U_{x_i}$ we have $|T^\star(\delta_z)(\{x\})| < \varepsilon$, that is, $\varphi^T_\varepsilon(z) \subseteq V$. Indeed, fix $z \in E^*$ and $x \in U_{x_i}$. Let $T^\star(\delta_z) = \mu^+_z - \mu^-_z$ be the Jordan decomposition of the measure. Notice that $|T^\star(\delta_z)|(U_{x_j}) \leq |T^\star(\delta_z)(U_{x_j})| + 2\mu^-_z(U_{x_j})$ and $|T^\star(\delta_z)|(U_{x_j}) \leq |T^\star(\delta_z)(U_{x_j})| + 2\mu^+_z(U_{x_j})$. So if $U_{x_i} \subseteq C^*_{N_{x_i}}$, since $\mu^-_z(U_{x_i}) \leq \mu^-_z(C^*_{N_{x_i}}) < 1/(4(N_{x_i}+1))$ we have

$$|T^\star(\delta_z)(\{x\})| \leq |T^\star(\delta_z)(U_{x_j})| + 2\mu^-_z(U_{x_j}) < \varepsilon - 1/N_{x_j} + 1/(2(N_{x_j}+1)) < \varepsilon.$$

If $U_{x_i} \subseteq D^*_{N_{x_i}}$, we use the fact that $\mu^+_z(U_{x_i}) \leq \mu^+_z(D^*_{N_{x_i}}) < 1/(4(N_{x_j}+1))$ to obtain the same result. ∎

3.3. Fountains and funnels. The property of being locally null can be expressed using a topological property of $T^\star$.

Proposition 3.12. *A bounded linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is locally null if, and only if, there is a nowhere dense set $F \subseteq \mathbb{N}^*$ such that $T^\star(\delta_y)$ is concentrated on F for every $y \in \mathbb{N}^*$.*

Proof. Suppose T is locally null. If we set $D = \bigcup\{A^* : T \circ I_A = 0\}$, then D is an open dense set. Suppose $|T^\star(\delta_y)|(D) > \varepsilon$ for some $y \in \mathbb{N}^*$ and some $\varepsilon > 0$. By the regularity of the measure we may find a compact $G \subseteq D$ such that $|T^\star(\delta_y)|(G) > \varepsilon$. We may further find finitely many $A_0, \ldots, A_{n-1} \subseteq \mathbb{N}$ such that $T \circ I_{A_i} = 0$ for all $i < n$ and $\sum_{i<n} |T^\star(\delta_y)|(A^*_i) > \varepsilon$. Choose $i < n$ such that $|T^\star(\delta_y)|(A^*_i) > \varepsilon/n$ and a function f with support included in A^*_i such that $T^\star(\delta_y)(f) > \varepsilon/n$. Then $T(f)(y) \neq 0$, which contradicts the hypothesis. Therefore, $T^\star(\delta_y)$ is concentrated on $F = \mathbb{N}^* \setminus D$ for all $y \in \mathbb{N}^*$.

Conversely, suppose F is a nowhere dense set such that for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ is concentrated on F. Given an infinite $A \subseteq \mathbb{N}$, take $A_1 \subseteq_* A$ infinite with $A^*_1 \cap F = \emptyset$. Then $|T^\star(\delta_y)|(A^*_1) = 0$ and it follows that $T \circ I_{A_1} = 0$. ∎

As in the previous proposition, many results in the following parts of the paper will show the important role played by nowhere dense sets of $\mathbb{N}^*$ in the context of operators on $C(\mathbb{N}^*)$. It is this fact that leads to the definitions of fountains, funnels, and fountainless and funnelless operators:

DEFINITION 3.13. A bounded linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is called *fountainless* or *without fountains* if, and only if, for every nowhere dense set $F \subseteq \mathbb{N}^*$ the set

$$G = \{y \in \mathbb{N}^* : T^\star(\delta_y) \text{ is nonzero and concentrated on } F\}$$

is nowhere dense. A *fountain* for T is a pair (F, U) with $F \subseteq \mathbb{N}^*$ nowhere dense and $U \subseteq \mathbb{N}^*$ open such that all the measures $T^\star(\delta_y)$ for $y \in U$ are concentrated on F.

LEMMA 3.14. *Let T be fountainless and let $B \subseteq \mathbb{N}$ be infinite. If $P_B \circ T$ is locally null, then $P_B \circ T = 0$.*

Proof. By 3.12 there is a nowhere dense $F \subseteq \mathbb{N}^*$ such that for every $y \in B^*$ the measure $(P_B \circ T)^\star(\delta_y)$, which is equal to $T^\star(\delta_y)$, is concentrated on F. By 3.13 the set $G = \{y \in B^* : T^\star(\delta_y) \neq 0\}$ is nowhere dense. But this means that for every $f \in C(\mathbb{N}^*)$ we have $T(f)(x) = 0$ if $x \in B^* \setminus G$. Since $B^* \setminus G$ is dense in B^* we conclude that $P_B \circ T = 0$. ■

COROLLARY 3.15. *If T is locally null and has no fountains, then $T = 0$.*

Proof. Set $B = \mathbb{N}$ in 3.14. ■

DEFINITION 3.16. We say that a bounded linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is *everywhere present* if, and only if, $P_B \circ T \neq 0$ for every infinite $B \subseteq \mathbb{N}$.

In the following lemma we obtain a kind of left dual to an improvement of a theorem of Cengiz ("P" in [8]) obtained by Plebanek [35, Theorem 3.3] which implies that if T is an isomorphic embedding then every $x \in \mathbb{N}^*$ is in $\varphi^T(y)$ for some $y \in \mathbb{N}^*$.

LEMMA 3.17. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an everywhere present fountainless operator. Then for every infinite $B \subseteq \mathbb{N}$ there exists an infinite $B_1 \subseteq_* B$ such that $\varphi^T(y) \neq \emptyset$ for every $y \in B_1^*$.*

Proof. Given an infinite $B \subseteq \mathbb{N}$, let $B_1 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ be as in 3.5. Suppose that $y_0 \in B_1^*$ is such that $\varphi^T(y_0) = \emptyset$. For every $n \in \mathbb{N}$ we find an open covering of $\mathbb{N}^*$ as follows. Given $x \in \mathbb{N}^*$, find by the regularity of the measure $T^\star(\delta_{y_0})$ a clopen neighbourhood of x, say U_x, such that $|T^\star(\delta_{y_0})|(U_x) < 1/(2(n+1))$ and U_x is included in either C_n^* or D_n^*.

By the compactness of $\mathbb{N}^*$ we obtain for each $n \in \mathbb{N}$ an open covering $\{U_{n,i} : i < j_n\}$ of $\mathbb{N}^*$ such that for each $i < j_n$ we have

(1) $|T^\star(\delta_{y_0})(U_{n,i})| \leq |T^\star(\delta_{y_0})|(U_{n,i}) < 1/(2(n+1))$, and
(2) either $U_{n,i} \subseteq C_n^*$ or $U_{n,i} \subseteq D_n^*$.

By the weak* continuity of $T^\star$ there are open neighbourhoods V_n of y_0 such that $|T^\star(\delta_y)(U_{n,i})| < 1/(2(n+1))$ for all $y \in V_n$ and all $i < j_n$. Let V^* be a clopen subset of $\bigcap_{n\in\mathbb{N}} V_n \cap B_1^*$, and consider the family $\mathcal{A} \subseteq \wp(\mathbb{N})$ of those sets A such that for each $n \in \mathbb{N}$ we have $A^* \subseteq U_{n,i_n}$ for some $i_n < j_n$. We claim that $|T^\star(\delta_y)|(A^*) = 0$ for every $y \in V^*$ and every $A \in \mathcal{A}$.

Indeed, fix $y \in V^*$, $A \in \mathcal{A}$ and $n \in \mathbb{N}$. We will show that $|T^\star(\delta_y)|(A^*) < 1/(n+1)$. Let $T^\star(\delta_y) = \mu^+ - \mu^-$ be the Jordan decomposition. By 3.5 we have $\mu^-(C_n^*) < 1/(4(n+1))$ and $\mu^+(D_n^*) < 1/(4(n+1))$. Assume without loss of generality that $U_{n,i_n} \subseteq C_n^*$. Then

$$\begin{aligned}|T^\star(\delta_y)|(A^*) &\le |T^\star(\delta_y)|(U_{n,i_n}) \le T^\star(\delta_y)(U_{n,i_n}) + 2\mu^-(U_{n,i_n})\\ &\le |T^\star(\delta_y)(U_{n,i_n})| + 2\mu^-(C_n^*) < 1/(2(n+1)) + 2/(4(n+1)) = 1/(n+1).\end{aligned}$$

So the claim is proved.

Notice that this implies that $(P_V \circ T)(f) = 0$ for every $f \in C(\mathbb{N}^*)$ whose support is included in A^* for some $A \in \mathcal{A}$. Therefore, if $\mathcal{A}$ is a dense family, by 3.14 we would have $(P_V \circ T)(g) = 0$ for all $g \in C(\mathbb{N}^*)$; but this would contradict the hypothesis that T is everywhere present.

We prove that $\mathcal{A}$ is a dense family. For a fixed infinite $E_0 \subseteq \mathbb{N}$, we may define by induction a $\subseteq_*$-decreasing sequence (E_n) of infinite sets by choosing $\emptyset \ne E_{n+1}^* \subseteq E_n^* \cap U_{n,i_n}$ for some $i_n < j_n$ (this is possible because $\{U_{n,i} : i < j_n\}$ is an open covering of $\mathbb{N}^*$ for each $n \in \mathbb{N}$). Take A such that $A \subseteq_* E_n$ for all $n \in \mathbb{N}$. It is clear that $A \subseteq_* E_0$ and $A \in \mathcal{A}$. ∎

Let us introduce a dual notion to a fountain:

Definition 3.18. A bounded linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is called *funnelless* or *without funnels* if, and only if, for every nowhere dense set $F \subseteq \mathbb{N}^*$ there is a nowhere dense $G \subseteq \mathbb{N}^*$ such that for all $y \in F$ the measure $T^\star(\delta_y)$ is concentrated on G. A *funnel* for T is a pair (U, F) with $F \subseteq \mathbb{N}^*$ nowhere dense and $U \subseteq \mathbb{N}^*$ open such that there is no proper closed subset of U where all the measures $T^\star(\delta_y)|U$ for $y \in F$ are concentrated.

3.4. Operators induced by continuous maps and nonatomic operators

Definition 3.19. Suppose that $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous map. Then $T_\psi : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is given for every $f \in C(\mathbb{N}^*)$ by $T_\psi(f) = f \circ \psi$.

Definition 3.20. A continuous map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is called *quasi-open* if, and only if, the image of every nonempty open set under ψ has nonempty interior.

Proposition 3.21. *Suppose that $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous map. Then T_ψ is fountainless if, and only if, ψ is quasi-open.*

Proof. Notice that for every $y \in \mathbb{N}^*$ we have $T^\star_\psi(\delta_y) = \delta_{\psi(y)}$. Also for every subset $X \subseteq \mathbb{N}^*$ the following holds:

$$|\delta_{\psi(y)}|(X) \neq 0 \quad \text{iff} \quad \psi(y) \in X \quad \text{iff} \quad y \in \psi^{-1}[X].$$

Therefore, if ψ is quasi-open and $F \subseteq \mathbb{N}^*$ is nowhere dense, we see that $\{y \in \mathbb{N}^* : |T^\star_\psi(\delta_y)|(\mathbb{N}^* \setminus F) = 0\} = \psi^{-1}[F]$ is nowhere dense, so T_ψ is fountainless. On the other hand, if T_ψ is fountainless, let U be open. If $\psi[U]$ were nowhere dense, then $\{y \in \mathbb{N}^* : |T^\star_\psi(\delta_y)|(\mathbb{N}^* \setminus \psi[U]) = 0\} = \psi^{-1}[\psi[U]]$ would be nowhere dense, which contradicts the fact that $U \subseteq \psi^{-1}[\psi[U]]$. ∎

PROPOSITION 3.22. *Let $\psi : \mathbb{N}^* \to \mathbb{N}^*$ be continuous. Then T_ψ is funnelless if, and only if, ψ sends nowhere dense sets into nowhere dense sets.*

Proof. Let T_ψ be funnelless and $F \subseteq \mathbb{N}^*$ be nowhere dense. Let $G \subseteq \mathbb{N}^*$ be nowhere dense such that $T^\star_\psi(\delta_y)$ is concentrated on G for every $y \in F$. As in the proof of 3.21, we have $F \subseteq \{y \in \mathbb{N}^* : |T^\star_\psi(\delta_y)|(\mathbb{N}^* \setminus G) = 0\} = \psi^{-1}[G]$.

Now suppose ψ sends nowhere dense sets into nowhere dense sets and let $F \subseteq \mathbb{N}^*$ be nowhere dense. Then we have $F \subseteq \psi^{-1}[\psi[F]] = \{y \in \mathbb{N}^* : |T^\star_\psi(\delta_y)|(\mathbb{N}^* \setminus \psi[F]) = 0\}$. That is, $T^\star_\psi(\delta_y)$ is concentrated on $\psi[F]$ for every $y \in F$. ∎

DEFINITION 3.23. A bounded linear operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is *nonatomic* if, and only if, for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ is nonatomic.

PROPOSITION 3.24. *A positive nonatomic operator on ℓ_∞/c_0 is locally null.*

Proof. Since for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ has no atoms, by the regularity of $T^\star(\delta_y)$ and the compactness of $\mathbb{N}^*$ we may find for each $n \in \mathbb{N}$ a finite open covering $(U_i(y,n))_{i<j(y,n)}$ of $\mathbb{N}^*$ by clopen sets such that $|T^\star(\delta_y)|(U_i(y,n)) < 1/(2(n+1))$ for all $i < j(y,n)$.

Now by the weak* continuity of $T^\star$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of y such that for all $z \in V_n(y)$ we have

$$|T^\star(\delta_z)|(U_i(y,n)) = |T^\star(\delta_z)(U_i(y,n))| < 1/(2(n+1))$$

for all $i < j(y,n)$. The equality follows from the positivity of T.

So $\{V_n(y) : y \in \mathbb{N}^*\}$ is an open covering of $\mathbb{N}^*$ for each $n \in \mathbb{N}$. By the compactness of $\mathbb{N}^*$, for each $n \in \mathbb{N}$ take $y_0(n), \ldots, y_{m(n)-1}(n) \in \mathbb{N}^*$ such that

$$\mathbb{N}^* \subseteq \bigcup_{l<m(n)} V_n(y_l).$$

Now consider the family $\mathcal{A}$ of those sets $A \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l, n)$ such that A^* is included in $U_i(y_l, n)$. As in the proof of 3.17, it is easy to see that $\mathcal{A}$ is dense and that for every $z \in \mathbb{N}^*$ and every $A \in \mathcal{A}$ we have $|T^\star(\delta_z)|(A^*) = 0$. Therefore, if $f \in C(\mathbb{N}^*)$ is A^*-supported, we have $T(f) = 0$, as required. ∎

4. Operators on ℓ_∞/c_0 and operators on ℓ_∞

4.1. Operators induced by operators on ℓ_∞

Definition 4.1. Suppose that $R : \ell_\infty \to \ell_\infty$ is a bounded linear operator which preserves c_0. Then $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator defined by

$$[R]([f]_{c_0}) = [R(f)]_{c_0}$$

for every $f \in \ell_\infty$. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a bounded linear operator, then a *lifting* $R : \ell_\infty \to \ell_\infty$ is any bounded linear operator such that $[R] = T$.

Note that our terminology is slightly different from the one used in the literature concerning the trivialization of endomorphisms of $\wp(\mathbb{N})/\mathrm{Fin}$. This is due to the fact that we do not use nonlinear liftings of linear operators.

Lemma 4.2. *Let $R_0, R_1 : \ell_\infty \to \ell_\infty$ be bounded linear operators which preserve c_0. Then:*

(1) $[R_0 + \alpha R_1] = [R_0] + \alpha[R_1]$ *for every real* α.
(2) $[R_1 \circ R_0] = [R_1] \circ [R_0]$.

Proof. Fix $f \in \ell_\infty$. Then

$$\begin{aligned}[R_0 + \alpha R_1]([f]_{c_0}) &= [(R_0 + \alpha R_1)(f)]_{c_0} = [R_0(f)]_{c_0} + \alpha[R_1(f)]_{c_0} \\ &= ([R_0] + \alpha[R_1])([f]_{c_0})\end{aligned}$$

and $[R_1] \circ [R_0]([f]_{c_0}) = [R_1]([R_0(f)]_{c_0}) = [R_1(R_0(f))]_{c_0} = [R_1 \circ R_0]([f]_{c_0})$. ■

Proposition 4.3. *Let $R_0, R_1 : \ell_\infty \to \ell_\infty$ be bounded linear operators which preserve c_0. Then:*

(1) *If* $[R_0] = 0$, *then R is weakly compact.*
(2) *If* $[R_0] = [R_1]$, *then $R_0 - R_1$ is weakly compact.*

Proof. $[R_0] = 0$ implies the image of R_0 is included in c_0. But ℓ_∞ is a Grothendieck space, so all operators from it into separable spaces are weakly compact [9, Theorem 1]. For part (2) apply 4.2 and part (1) to $R_0 - R_1$. ■

So there could be many liftings of the same operator but they all differ by a weakly compact perturbation. When we look at ℓ_∞/c_0 as $C(\mathbb{N}^*)$, then liftings correspond to extensions.

Lemma 4.4. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be liftable to $R : C(\beta\mathbb{N}) \to C(\beta\mathbb{N})$. Then for every $y \in \mathbb{N}^*$ we have $R^\star(\delta_y)|\mathbb{N}^* = T^\star(\delta_y)$.*

Proof. If $Q : C(\beta\mathbb{N}) \to C(\mathbb{N}^*)$ is the restriction map, then the dual of the lifting relation $T \circ Q = Q \circ R$ is $Q^\star \circ T^\star = R^\star \circ Q^\star$. Furthermore, $Q^\star$ acts on measures on $\mathbb{N}^*$ by extending them to $\beta\mathbb{N}$ with $\mathbb{N}$ having measure zero. So for all $y \in \mathbb{N}^*$ we have $T^\star(\delta_y) = (Q^\star \circ T^\star)(\delta_y)|\mathbb{N}^* = (R^\star \circ Q^\star)(\delta_y)|\mathbb{N}^* = R^\star(\delta_y)|\mathbb{N}^*$. ■

4.2. Local properties of liftable operators on ℓ_∞/c_0

PROPOSITION 4.5. *If $R : C(\beta\mathbb{N}) \to C(\beta\mathbb{N})$ is a positive falling operator, then the operator $[R] : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is nonatomic and locally null.*

Proof. By Definition 2.8, given $\varepsilon > 0$ we have a cofinite set $B \subseteq \mathbb{N}$ and a partition $\{A_1, \ldots, A_k\}$ of $\mathbb{N}$ such that $R^\star(\delta_i)(\beta A_m) = |R^\star(\delta_i)|(\beta A_m) < \varepsilon$ for all $m \le k$ and $i \in B$. As any δ_y, for $y \in \mathbb{N}^*$, is in the weak* closure of $\{\delta_n : n \in B\}$, it follows by the weak* continuity of $R^\star$ that $R^\star(\delta_y)(\beta A_m) < \varepsilon$ for all $y \in \mathbb{N}^*$ and $m \le k$. But by 4.4 and the positivity of R we have $[R]^\star(\delta_y)(A_m^*) = R^\star(\delta_y)(A_m^*) \le R^\star(\delta_y)(\beta A_m) < \varepsilon$ for all $y \in \mathbb{N}^*$ and $m \le k$. As $\{A_1^*, \ldots, A_k^*\}$ is a partition of $\mathbb{N}^*$, $[R]^\star(\delta_y)$ is nonatomic for every $y \in \mathbb{N}^*$. By 3.24, $[R]$ is locally null. ∎

COROLLARY 4.6. *There is a matrix operator T which has fountains and is such that whenever $T \circ I_A \neq 0$, the operator $T \circ I_A$ is not canonizable along any continuous map. In particular, T is nowhere trivial.*

Proof. The operator R from 2.10 is a nonweakly compact, positive, falling operator on ℓ_∞. Its range is not included in c_0 by 2.7 (actually, the characteristic function of a subset of $\mathbb{N}$ of positive density is sent to an element not in c_0). So $T = [R] \neq 0$. On the other hand, by 4.5 we know that T is locally null, so by 3.15 it follows that T has fountains.

Now, by 4.5 we see that $(T \circ I_A)^\star(\delta_y) = T^\star(\delta_y)|A^*$ is nonatomic or zero for every $y \in \mathbb{N}^*$, so the second part of the corollary follows. ∎

PROPOSITION 4.7. *If $R : \ell_\infty \to \ell_\infty$ is an antimatrix operator, then the operator $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ factors through ℓ_∞, and so it is locally null.*

Proof. Let $\mu_n = R^\star(\delta_n)$. Since R is antimatrix (Definition 2.11) we may consider μ_n as a measure on $\mathbb{N}^*$. Let $S : \ell_\infty/c_0 \to \ell_\infty$ be given by $S([f]_{c_0}) = (\mu_n(\beta f))_{n\in\mathbb{N}^*}$ for every $f \in \ell_\infty$, and $Q : \ell_\infty \to \ell_\infty/c_0$ be the quotient map. Then S is well defined since the measures μ_n are null on $\mathbb{N}$. For every $f \in \ell_\infty$ we have

$$(Q \circ S)([f]_{c_0}) = [(\mu_n(\beta f))_{n\in\mathbb{N}}]_{c_0} = [R(f)]_{c_0} = [R]([f]_{c_0}),$$

so $Q \circ S$ is $[R]$. To conclude that $[R]$ is locally null use 3.2. ∎

THEOREM 4.8. *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a matrix operator which is an isomorphic embedding, then it is right-locally trivial.*

Proof. Let $R : \ell_\infty \to \ell_\infty$ be given by a c_0-matrix and $[R] = T$. Let $(b_{ij})_{i,j\in\mathbb{N}}$ be the matrix corresponding to R. Let $M > 0$ be such that $\|T([f]_{c_0})\| \ge M\|[f]_{c_0}\|$ for every $f \in \ell_\infty \setminus c_0$. Notice that this condition is equivalent to the statement that $\limsup_{n\to\infty} |R(f)(n)| \ge M$ for every $f \in \ell_\infty$ such that $\limsup_{n\to\infty} |f(n)| = 1$. Fix an infinite $\tilde{A} \subseteq \mathbb{N}$.

CLAIM 1. $\lim_{i\to\infty} \max\{|b_{ij}| : j \in \tilde{A}\} \neq 0$.

Assume otherwise. We will construct an $f \in \ell_\infty$ with $\limsup_{n\to\infty} |f(n)| = 1$ and $\limsup_{n\to\infty} |R(f)(n)| < M$. Let $m_i = \min\{k \in \mathbb{N} : \sum_{j\geq k} |b_{ij}| < 1/(i+1)\}$ for every $i \in \mathbb{N}$. We shall construct by induction two strictly increasing sequences of integers $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$ with $j_n \in \tilde{A}$ for every $n \in \mathbb{N}$. Let $i_0 = 0$ and $j_0 = \min \tilde{A}$. If we have constructed i_l, j_l for $l \leq n$, take $i_{n+1} > i_n$ such that $\max\{|b_{ij}| : j \in \tilde{A}\} < 1/(n+2)^2$ for every $i \geq i_{n+1}$; take $j_{n+1} \in \tilde{A}$ such that $j_{n+1} > \max\{m_l : l < i_{n+1}\}$ and $j_{n+1} > j_n$.

Now let f be the characteristic function of $\{j_n : n \in \mathbb{N}\}$ and let $N \in \mathbb{N}$ be such that $N/(N+1)^2 < M/4$ and $1/N < M/4$. Fix $k \geq i_N$. Then $k \geq N$, and also $i_n \leq k < i_{n+1}$ for some $n \geq N$. Now:

$$\begin{aligned}|R(f)(k)| &= \Big|\sum_{j\in\mathbb{N}} b_{kj} f(j)\Big| \leq \sum_{j<m_k} |b_{kj} f(j)| + \sum_{j\geq m_k} |b_{kj}| \\ &\leq \sum_{j<m_k} |b_{kj} f(j)| + \frac{1}{k} \leq \sum_{l<n} |b_{kj_l}| + \frac{1}{N} \quad \text{(because } j_{n+1} > m_k) \\ &\leq n \cdot \max\{|b_{kj}| : j \in \tilde{A}\} + \frac{1}{N} \leq \frac{n}{(n+1)^2} + \frac{1}{N} < \frac{M}{2} \quad \text{(because } k \geq i_n).\end{aligned}$$

This contradicts the definition of M, and so the claim is proved.

Let $\delta > 0$ and let $B_0 \subseteq \mathbb{N}$ be infinite such that $\max\{|b_{ij}| : j \in \tilde{A}\} > \delta$ for every $i \in B_0$. We shall construct by induction three strictly increasing sequences of integers, $(i_n)_{n\in\mathbb{N}}$, $(j_n)_{n\in\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$:

(1) $|b_{i_n j_n}| > \delta$,
(2) $j_n \in \tilde{A}$,
(3) $k_n \leq j_n < k_{n+1}$,
(4) $\sum_{j<k_n} |b_{i_n j}| < 1/(2(n+1))$,
(5) $\sum_{j\geq k_{n+1}} |b_{i_n j}| < 1/(2(n+1))$.

Let $k_0 = 0$ and $i_0 = \min B_0$. Let $j_0 \in \tilde{A}$ be such that $|b_{i_0 j_0}| > \delta$. Let $k_1 > j_0$ be such that $\sum_{j\geq k_1} |b_{i_0 j}| < 1$. Assume we have constructed i_l, j_l and k_{l+1}, satisfying (1)–(5) for every $l \leq n$. Let N be such that $\sum_{j<k_{n+1}} |b_{ij}| < \min\{\delta, 1/(2(n+2))\}$ for every $i \geq N$ (it exists because $(b_{ij})_{i,j\in\mathbb{N}}$ is a c_0-matrix). Let $i_{n+1} \in B_0 \setminus N$. Let $j_{n+1} \in \tilde{A}$ be such that $|b_{i_{n+1} j_{n+1}}| > \delta$ (it exists because $i_{n+1} \in B_0$). Notice that $j_{n+1} \geq k_{n+1}$ because $|b_{i_{n+1} j}| < \delta$, for every $j < k_{n+1}$. Let $k_{n+2} > j_{n+1}$ be such that $\sum_{j\geq k_{n+2}} |b_{i_{n+1} j}| < 1/(2(n+2))$. This ends the inductive construction.

Now, $\delta < |b_{i_n j_n}| \leq \sup\{|b_{ij}| : i, j \in \mathbb{N}\}$ for every $n \in \mathbb{N}$. Therefore, by going to a subsequence we may assume that $b_{i_n j_n}$ converges to some r with $|r| \geq \delta$. Let $A = \{j_n : n \in \mathbb{N}\}$ and $B = \{i_n : n \in \mathbb{N}\}$. Let $\sigma : B \to A$ be given by $\sigma(i_n) = j_n$ for each $n \in \mathbb{N}$.

Claim 2. *$(P_B \circ T \circ I_A)([f]_{c_0(A)}) = [rf \circ \sigma]_{c_0(B)}$ for every $f \in \ell_\infty(A)$.*

We need to show that $\lim_{n\to\infty} |R(f)(i_n) - rf(\sigma(i_n))| = 0$ for every f in $\ell_\infty(A)$. So fix $f \in \ell_\infty(A)$ and $\varepsilon > 0$. Let M' be such that $\|T^\star(\delta_n)\| \le M'$ for every $n \in \mathbb{N}$ (it exists by definition of c_0-matrix). Let N_0 be such that $|b_{i_n j_n} - r| < \varepsilon/(3\|f\|)$ for all $n \ge N_0$, and let N_1 be such that $1/(N_1+1) < \varepsilon/(3\|f\|)$. Then, for every $n \ge N_0 + N_1$ we have

$$\begin{aligned}
&|R(f)(i_n) - rf(\sigma(i_n))| = \Big|\sum_{j\in\mathbb{N}} b_{i_n j} f(j) - rf(j_n)\Big| \\
&\le \sum_{j<k_n} |b_{i_n j} f(j)| + \sum_{\substack{k_n \le j < k_{n+1}\\ j \ne j_n}} |b_{i_n j} f(j)| \\
&\quad + \sum_{j \ge k_{n+1}} |b_{i_n j} f(j)| + |b_{i_n j_n} f(j_n) - rf(j_n)| \\
&< \|f\|/(2(n+1)) + 0 + \|f\|/(2(n+1)) + \|f\|\,|b_{i_n j_n} - r| < \varepsilon/3 + \varepsilon/3 < \varepsilon. \ \blacksquare
\end{aligned}$$

Corollary 4.9. *Every liftable isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally trivial.*

Proof. Since T is liftable, there exist $R_0, R_1 : \ell_\infty \to \ell_\infty$ an antimatrix operator and one given by a c_0-matrix, respectively, such that $T = [R_0 + R_1] = [R_0] + [R_1]$. Fix an infinite $A \subseteq \mathbb{N}$. By 4.7, take an infinite $A_0 \subseteq A$ such that $T \circ I_{A_0} = [R_1] \circ I_{A_0}$. Then $[R_1] \circ I_{A_0}$ is a matrix operator which is an isomorphic embedding, so by 4.8 there exist infinite $A_1 \subseteq A_0$ and $B \subseteq \mathbb{N}$ such that $P_B \circ T \circ I_{A_1} = P_B \circ [R_1] \circ I_{A_1}$ is trivial. ∎

Corollary 4.10. *Every liftable isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally an isomorphic matrix operator.*

Proof. By 4.9 a trivial operator is an isomorphic matrix operator. ∎

Corollary 4.11. *Let $\mathbb{P}$ be one of the following properties: isomorphically liftable, isomorphically matrix, trivial, canonizable along ψ. Suppose that $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ is locally null. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally $\mathbb{P}$ [left-locally $\mathbb{P}$, somewhere $\mathbb{P}$], then $S + T$ is right-locally $\mathbb{P}$ [left-locally $\mathbb{P}$, somewhere $\mathbb{P}$].*

Proof. First we will note that if the localization $T_{B,A}$ of T to (A, B) has $\mathbb{P}$, then for every infinite $A' \subseteq A$ there is an infinite $B' \subseteq B$ such that the localization $T_{B',A'}$ of T to (A', B') has $\mathbb{P}$.

In the case where $T_{B,A}$ is isomorphically liftable, by 4.9 it is enough to notice that a trivial operator is isomorphically liftable. Similarly, if $T_{B,A}$ is isomorphically matrix, by 4.8 it is enough to notice that a trivial operator is isomorphically matrix.

If $T_{B,A}$ is trivial, it is enough to take $B' = \sigma^{-1}[A']$, where $\sigma : B \to A$ is the bijection witnessing the triviality of $T_{B,A}$. Similarly, if $T_{B,A}$ is canonizable along ψ, we take $B' \subseteq B$ such that $(B')^* = \psi^{-1}[(A')^*]$.

Now, given a localization $T_{B,A}$ with property $\mathbb{P}$, take an infinite $A' \subseteq A$ such that $S \circ I_{A'} = 0$. By the above, there exists $B' \subseteq B$ such that $T_{B',A'} = (S+T)_{B',A'}$ has $\mathbb{P}$. ∎

If we do not assume that the operator is bounded below, then there is no hope of obtaining local trivialization anywhere:

Proposition 4.12. *There is a surjective bounded linear operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which is globally liftable but is nowhere a nonzero matrix operator.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a discrete sequence of nonprincipal ultrafilters and consider the typical antimatrix operator $R : \ell_\infty \to \ell_\infty$ given by $R(f) = ((\beta f)(x_n))_{n\in\mathbb{N}}$. Let $T = [R]$. By 3.3 we know that T is surjective. Suppose for some infinite $A, B \subseteq \mathbb{N}$ there is $S : \ell_\infty(A) \to \ell_\infty(B)$ given by a c_0-matrix and such that $[S] = T_{B,A}$. Let us denote by $R_{B,A}$ the operator which maps $f \in \ell_\infty(A)$ into $R(f \cup 0_{\mathbb{N}\setminus A})|B$. By 4.3 we deduce that $S - R_{B,A}$ is weakly compact, and since R is an antimatrix operator we have $R|c_0 = 0$, so $S|c_0(A)$ is weakly compact. Therefore, by 2.6 and 2.5, the image of S is included in $c_0(B)$ and so $T_{B,A} = [S] = 0$. ∎

Proposition 4.13. *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a liftable operator such that for each $A, B \subseteq \mathbb{N}$ the operator $T_{B,A}$ is either not everywhere present (in B^*) or fountainless, then T is a matrix operator.*

Proof. Suppose $R : \ell_\infty \to \ell_\infty$ is a lifting of T. Decompose $R = R_1 + R_2$ as in Proposition 2.13, where R_1 is given by a c_0-matrix and R_2 is antimatrix. More concretely, by Lemma 2.12, $R_1(f)(n) = \int f\, d\lambda_n$ and $R_2(f)(n) = \int f\, d\nu_n$ for each $f \in \ell_\infty$ and each $n \in \mathbb{N}$, where $\lambda_n = \mu_n|\mathbb{N}$ and $\nu_n = \mu_n|\mathbb{N}^*$ for $\mu_n = R^\star(\delta_n)$. We will show that $[R_2] = 0$.

Given $A \subseteq \mathbb{N}$, consider the operator $\tilde{P}_A : \ell_\infty \to \ell_\infty$ given by $\tilde{P}_A(f) = \chi_A \cdot f$. Choose any infinite $A \subseteq \mathbb{N}$. We will prove that $(\tilde{P}_{A'} \circ R_2)[\ell_\infty] \subseteq c_0$ for some infinite $A' \subseteq A$. This will show that $R_2[\ell_\infty] \subseteq c_0$, and consequently $[R_2] = 0$. Let F be the closure of the carriers of all measures ν_n for $n \in \mathbb{N}$, that is,

$$F = \mathbb{N}^* \setminus \bigcup\{C^* : |\nu_n|(C^*) = 0 \text{ for all } n \in \mathbb{N}\}.$$

Then F is nowhere dense because it is a closed c.c.c. set while nonempty open sets in $\mathbb{N}^*$ are not c.c.c. Note that

$$R_2 \circ \tilde{P}_B = R_2 \tag{1}$$

whenever $F \subseteq B^*$. This is since $R_2^*(\delta_n) = \nu_n = \tilde{P}_B^*(\nu_n) = (\tilde{P}_B^* \circ R_2^*)(\delta_n) = (R_2 \circ \tilde{P}_B)^*(\delta_n)$ for all $n \in \mathbb{N}$ as $\tilde{P}_B^*(\mu) = \mu|B^*$ for all $\mu \in M(\beta\mathbb{N})$.

Now, knowing from Proposition 2.1 that the columns of the matrix of R_1 are in c_0 and its rows, λ_n, may be considered as elements of ℓ_1, choose an infinite $A_1 \subseteq A$ and pairwise disjoint and finite $F_i \subseteq \mathbb{N}$ for $i \in A_1$ such that $\sum_{j\in\mathbb{N}\setminus F_i} |\lambda_i(\{j\})| < 1/i$ for each $i \in A_1$. It follows that

$$\tilde{P}_{A'} \circ R_1 \circ \tilde{P}_{\mathbb{N}\setminus F_{A'}}[\ell_\infty] \subseteq c_0 \tag{2}$$

for any $A' \subseteq A_1$ where $F_{A'} = \bigcup_{i\in A'} F_i$. Finally, we would like to choose an infinite $A' \subseteq A_1$ such that

$$F \subseteq (\mathbb{N} \setminus F_{A'})^*, \tag{3}$$

which by the definition of F means that $|\nu_n|(F^*_{A'}) = 0$ for every $n \in \mathbb{N}$. This is done as usual, by considering an uncountable almost disjoint family $(A_\xi : \xi < \omega_1)$ of infinite subsets of A_1 and the corresponding F_{A_ξ}'s. Notice that the F_{A_ξ}'s are also almost disjoint, so the $F^*_{A_\xi}$'s are pairwise disjoint and one of them must be null with respect to all measures ν_n. Choose $A' = A_\xi$ for this ξ. From (1)–(3), after passing to ℓ_∞/c_0 it follows that

$$[\tilde{P}_{A'}] \circ [R] \circ [\tilde{P}_{\mathbb{N}\setminus F_{A'}}] = [\tilde{P}_{A'}] \circ [R_2].$$

By the hypothesis, either $[\tilde{P}_{A'}] \circ [R_2]$ is not everywhere present (in $(A')^*$) or it is fountainless. If the first possibility holds, by going to a subset of A' we may assume that $[\tilde{P}_{A'}] \circ [R_2] = 0$. If the second possibility holds, notice that $[\tilde{P}_{A'}] \circ [R_2]$ is locally null by Proposition 4.7 and recall that Corollary 3.15 says that locally null fountainless operators are zero. This is what we wanted to prove to conclude that $T = [R_1]$. ∎

4.3. Lifting operators on ℓ_∞/c_0. In the case of the Boolean algebra $\wp(\mathbb{N})/\mathrm{Fin}$, any endomorphism which can be lifted to a homomorphism of $\wp(\mathbb{N})$ is induced by a homomorphism of $\mathrm{FinCofin}(\mathbb{N})$. However, in the case of ℓ_∞/c_0, just as for ℓ_∞ (2.15), there exist automorphisms which are not determined by their values on c_0:

Proposition 4.14. *There are liftable operators such that all their liftings are discontinuous and are not induced by their action on c_0, i.e., are not matrix operators. Moreover, such operators can be automorphisms of ℓ_∞/c_0.*

Proof. Let $(A_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets. For each $i \in \mathbb{N}$, let x_i be any nonprincipal ultrafilter such that $A_i \in x_i$. For a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ consider the automorphism $R_\sigma : \ell_\infty \to \ell_\infty$ from the proof of 2.15 which is given by

$$R_\sigma(f)(n) = f(n) - \beta f(x_i) + \beta f(x_{\sigma(i)}),$$

where $i \in \mathbb{N}$ is such that $n \in A_i$. Recall that $R_\sigma \circ R_{\sigma^{-1}} = \mathrm{Id}_{\ell_\infty}$, so by 4.2 we have $[R_\sigma] \circ [R_{\sigma^{-1}}] = [\mathrm{Id}_{\ell_\infty}] = \mathrm{Id}_{\ell_\infty/c_0}$. It follows that the operators $[R_\sigma]$ are automorphisms of ℓ_∞/c_0.

Now suppose that $S : \ell_\infty \to \ell_\infty$ is a continuous lifting of $[R_\sigma]$. By 2.14 the operator S is given by a c_0-matrix, and by 4.3 we know that $S - R_\sigma$ is a weakly compact operator into c_0. Note that $R_\sigma|c_0 = \mathrm{Id}_{c_0}$, therefore $S|c_0 = \mathrm{Id}_{c_0} + W$, where $W : c_0 \to c_0$ is the restriction of $S - R_\sigma$ to c_0 and thus it is weakly compact. By 2.5 we have

$$S = (S|c_0)^{\star\star} = \mathrm{Id}_{c_0}^{\star\star} + W^{\star\star} = \mathrm{Id}_{\ell_\infty} + W^{\star\star},$$

and so $S^\star = \mathrm{Id}_{M(\beta\mathbb{N}^*)} + U$, where U is weakly compact by the Gantmacher theorem. Hence, $\mathrm{Id}_{M(\beta\mathbb{N}^*)} + U - R_\sigma^\star$ is a weakly compact operator, and so is $\mathrm{Id}_{M(\beta\mathbb{N}^*)} - R_\sigma^\star$. We will show that this is impossible as the bounded sequence $(\delta_{x_i})_{i\in\mathbb{N}}$ of measures is not mapped onto a relatively weakly compact set.

A simple calculation gives $R_\sigma^\star(\delta_x) = \delta_x - \delta_{x_i} + \delta_{x_{\sigma(i)}}$ if $x \in A_i^*$. It follows that $R_\sigma^\star(\delta_{x_i}) = \delta_{x_{\sigma(i)}}$ for every $i \in \mathbb{N}$. So $(\mathrm{Id}_{M(\beta\mathbb{N}^*)} - R_\sigma^\star)(\delta_{x_i}) = \delta_{x_i} - \delta_{x_{\sigma(i)}}$, which by the Dieudonné–Grothendieck theorem implies that $\mathrm{Id}_{M(\beta\mathbb{N}^*)} - R_\sigma^\star$ is not weakly compact unless σ moves only finitely many $i \in \mathbb{N}$, as the sequence $(x_i)_{i\in\mathbb{N}}$ is discrete. ∎

Unlike in the case of the algebra $\wp(\mathbb{N})/\mathrm{Fin}$, nonliftable automorphisms of ℓ_∞/c_0 exist in ZFC. To prove this we need:

LEMMA 4.15. *Suppose $R : \ell_\infty \to \ell_\infty$ is a c_0 preserving operator. If R is not weakly compact, then $[R]$ is not weakly compact either.*

Proof. If R is not weakly compact, then there is an infinite $A \subseteq \mathbb{N}$ such that R restricted to $\ell_\infty^0(A) = \{f \in \ell_\infty : f|(\mathbb{N} \setminus A) = 0\}$ is an isomorphism onto its range (see [37, Prop. 1.2] and [11, Corollary VI.2.17]). Consider $X = R^{-1}[c_0]$, a closed subspace of ℓ_∞ containing c_0. Note that $X \cap \ell_\infty^0(A)$ is separable as $R[X \cap \ell_\infty^0(A)] \subseteq c_0$ and R is an isomorphism on $\ell_\infty^0(A)$. By the standard argument using the Stone–Weierstrass theorem with respect to simple functions one can find a countable Boolean algebra $\mathfrak{B}$ of subsets of A such that $X \cap \ell_\infty^0(A)$ is included in the closure of the span of $\{\chi_B : B \in \mathfrak{B}\}$.

Let $(D_\xi)_{\xi<\omega_1}$ be a family of pairwise almost disjoint infinite subsets of A. For each $\xi < \omega_1$ take $x \in D_\xi^*$ and let E_ξ be infinite such that $E_\xi^* \subseteq \bigcap\{B^* : B \in \mathfrak{B} \cap x\} \cap \bigcap\{\mathbb{N}^* \setminus B^* : B \in \mathfrak{B} \setminus x\}$ (it exists since nonempty G_δ-sets in $\mathbb{N}^*$ have nonempty interior, and because $\mathfrak{B}$ is countable). Now take $u_\xi, v_\xi \in E_\xi^*$ distinct. It follows that no element of $\mathfrak{B}$ separates any of the pairs (u_ξ, v_ξ). Therefore, $\beta f(u_n) = \beta f(v_n)$ for every $f \in X \cap \ell_\infty^0(A)$.

For every $\xi < \omega_1$ choose $g_\xi \in \ell_\infty^0(A)$ with support in D_ξ such that $\|g_\xi\| = 1$, $g_\xi(u_\xi) = 1$ and $g_\xi(v_\xi) = -1$. Note that $R(g_\xi) \notin c_0$ for all $\xi < \omega$. This implies that $\|[R(g_\xi)]_{c_0}\| > 0$ for all $\xi < \omega_1$, so there is $n \in \mathbb{N}$ such that for infinitely many $\xi < \omega_1$ we have $\|[R(g_\xi)]_{c_0}\| > 1/n$. As the g_ξ^* are pairwise disjoint, $[R]$ is not weakly compact by [11, Corollary VI.2.17]. ∎

PROPOSITION 4.16. *Every weakly compact operator on ℓ_∞/c_0 with nonseparable range is nonliftable. Such operators exist.*

Proof. Suppose that $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ is weakly compact with nonseparable range and $R : \ell_\infty \to \ell_\infty$ is such that $[R] = S$ and R preserves c_0. Then R must be weakly compact itself by 4.15. In particular, the image of the unit ball under R is weakly compact. Since weakly compact subsets of ℓ_∞ are norm separable [38, Corollary 4.6] the image of R is separable. But this implies that the image of $[R] = S$ is separable as well, contradicting the hypothesis.

Now we construct a weakly compact operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ with nonseparable range which is weakly compact. The construction is based on the fact that ℓ_∞/c_0 contains an isometric copy of $\ell_2(2^\omega)$. This follows from the result of Avilés [1] which states that the unit ball in $\ell_2(2^\omega)$ with the weak topology (equivalently weak* topology) is a continuous image of $A(2^\omega)^{\mathbb{N}}$, where $A(2^\omega)$ is the one-point compactification of the discrete space of size 2^ω. On the other hand, by [4, Theorem 2.5 and Example 5.3] we know that $A(2^\omega)^{\mathbb{N}}$ is a continuous image of $\mathbb{N}^*$. Hence $C(B_{\ell_2(2^\omega)})$ embeds isometrically into $C(\mathbb{N}^*)$ and so does $\ell_2(2^\omega)$. Let then $S_1 : \ell_2(2^\omega) \to \ell_\infty/c_0$ be an isomorphism onto its range.

To complete the construction, it is enough to take a surjective operator $S_2 : \ell_\infty/c_0 \to \ell_2(2^\omega)$ and consider $S = S_1 \circ S_2$. This is because any operator into a reflexive Banach space is weakly compact [17, Corollary VI.4.3] and such operators form a two-sided ideal [17, Theorem VI.4.5].

The existence of such a surjective operator follows from the complementation of ℓ_∞ in ℓ_∞/c_0 and the existence of a surjective operator $T : \ell_\infty \to \ell_2(2^\omega)$, which was proved in [36, Proposition 3.4 and Remark 2 below it]. The proof of the latter result is based on a construction of an isomorphic copy of $\ell_2(2^\omega)$ inside $\ell_\infty^\star$ [36, Proposition 3.4]. Once we have an isomorphic embedding $T : \ell_2(2^\omega) \to \ell_\infty^\star$, we consider $T^\star \circ J : \ell_\infty \to \ell_\infty^{\star\star} \to \ell_2(2^\omega)^\star$, where $J : \ell_\infty \to \ell_\infty^{\star\star}$ is the canonical embedding. We observe that $(T^\star \circ J)^\star = J^\star \circ T^{\star\star} = T$ using the reflexivity of $\ell_2(2^\omega)$ to identify it with $\ell_2(2^\omega)^{\star\star}$. But T is one-to-one with closed range, so $T^\star \circ J$ must be onto, as required. ■

THEOREM 4.17. *There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which cannot be lifted to a linear operator on ℓ_∞.*

Proof. Consider $T_1 = \mathrm{Id} + S$ where S is any weakly compact operator on ℓ_∞/c_0 from the previous proposition. Since S is strictly singular, T_1 is a Fredholm operator of index 0 (see [28, Proposition 2.c.10]), i.e., its kernel is finite-dimensional of dimension n and its range is of the same finite codimension n. Since finite-dimensional subspaces of Banach spaces are complemented, we can write

$$T_1 : \mathrm{Ker}(T_1) \oplus X \to \mathrm{Range}(T_1) \oplus Y$$

where Y is of finite dimension n and X of finite codimension n. Let U :

$\mathrm{Ker}(T_1) \to Y$ be an isomorphism and define $T : \mathrm{Ker}(T_1) \oplus X \to \mathrm{Range}(T_1) \oplus Y$ by $T(z, x) = (T_1(x), U(z))$. It follows that $T = T_1 + U = \mathrm{Id} + S + U$. Having null kernel and being surjective, T is an automorphism of ℓ_∞/c_0. Now let us show that T cannot be lifted to an operator on ℓ_∞/c_0. Indeed, $S + U$ is weakly compact with nonseparable range as the sum of an operator with this property and a finite rank operator, so it cannot be lifted by 4.16. As the sum of two liftable operators is liftable and Id is liftable, it follows that T cannot be lifted. ∎

PROPOSITION 4.18. *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is canonizable along a homeomorphism $\psi : \mathbb{N}^* \to \mathbb{N}^*$ and ψ is a nontrivial homeomorphism* (*i.e., it is not induced by a bijection of two coinfinite subsets of* $\mathbb{N}$), *then T is not liftable.*

Proof. By multiplying we may assume that $\hat{T} = T_\psi$, that is, the constant r of Definition 1.1(4) is 1. Suppose $R : \ell_\infty \to \ell_\infty$ is a lifting of T. Note that as ψ is a homeomorphism, the operators $(T_\psi)_{B,A}$ are fountainless or not everywhere present for all infinite $A, B \subseteq \mathbb{N}$. Indeed, if $\psi[B^*] \subseteq A^*$, then for $x \in B^*$ we have $((T_\psi)_{B,A})^\star(\delta_x) = I_A^\star \circ T_\psi^\star \circ P_B^\star(\delta_x) = \delta_{\psi(x)}$, and so $(T_\psi)_{B,A}$ is fountainless. Otherwise, there is an infinite $B_1 \subseteq B$ such that $\psi[B_1^*] \cap A^* = \emptyset$, and so for every $f \in \ell_\infty(A)/c_0(A)$ we have $P_{B_1} \circ (T_\psi)_{B,A}(f) = P_{B_1} \circ P_B \circ T_\psi \circ I_A(f) = (f \circ \psi)|B_1 = 0$ because $\psi^{-1}[A^*] \cap B_1 = \emptyset$; that is, $T_{B,A}$ is not everywhere present. It follows from Proposition 4.13 that R is a matrix operator.

For an automorphism $h : \wp(\mathbb{N})/\mathrm{Fin} \to \wp(\mathbb{N})/\mathrm{Fin}$, call a function $H : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ a *weak lifting* of h whenever $[H(A)]_{\mathrm{Fin}} = h([A]_{\mathrm{Fin}})$ for all $A \subseteq \mathbb{N}$ (i.e., H does not have to be a homomorphism). It was proved by Veličković [46, Theorem 1.2] that if an automorphism of $\wp(\mathbb{N})/\mathrm{Fin}$ has a weak lifting (simply "lifting" in the terminology of [46]) which is Borel with respect to the product topology in $\{0, 1\}^{\mathbb{N}}$, then it is trivial. Let h_ψ be the automorphism of $\wp(\mathbb{N})/\mathrm{Fin}$ corresponding to ψ via the Stone duality. We will construct a Borel weak lifting for h_ψ proving that ψ is trivial.

Define $H(A) = \{n \in \mathbb{N} : R(\chi_A)(n) \geq 1/2\}$ for every $A \subseteq \mathbb{N}$, and let H_ψ be any weak lifting of h_ψ. Note that $T_\psi[\chi_{A^*}] = \chi_{\psi^{-1}[A^*]} = \chi_{(H_\psi(A))^*}$, so $[R(\chi_A)]_{c_0} = [\chi_{H_\psi(A)}]_{c_0}$. In particular, $H(A) \setminus H_\psi(A)$ and $H_\psi(A) \setminus H(A)$ are both finite, and so H is also a weak lifting of h_ψ.

Now let us check that H is Borel when $\wp(\mathbb{N})$ is considered as $\{0, 1\}^{\mathbb{N}}$ with the product topology. We will use Theorem 2.14(3) which says that R is continuous on the unit ball B_{ℓ_∞}. For every $n \in \mathbb{N}$ the set $H^{-1}[\{B \subseteq \mathbb{N} : n \in B\}] = \{A \subseteq \mathbb{N} : R(\chi_A)(n) \geq 1/2\}$ is closed, by the continuity of R. Similarly, $H^{-1}[\{B \subseteq \mathbb{N} : n \notin B\}]$ is open for every $n \in \mathbb{N}$. Hence H is Borel and so ψ is trivial, which completes the proof of the proposition. ∎

5. Canonizing operators acting along a quasi-open mapping. In [16] it was proved that for a bounded linear operator T on ℓ_∞/c_0 and an infinite $A \subseteq \mathbb{N}$ there is a real $r \in \mathbb{R}$ and an infinite $B \subseteq A$ such that

$$T(f)|B^* = rf$$

for every B-supported f. This shows, for example, that if P_1 and P_2 are complementary projections on ℓ_∞/c_0, then at least one of them canonizes as above for a nonzero r, in other words we obtain a local canonization along the identity on B^*. However, a disadvantage of this result is that in general we cannot guarantee that the constant r is nonzero. If one works with an automorphism, this kind of result is of no use. For example, consider an infinite and coinfinite set $D \subseteq \mathbb{N}$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma[D] = \mathbb{N} \setminus D$, $\sigma[\mathbb{N} \setminus D] = D$ and σ^2 is the identity. Define an automorphism T of ℓ_∞/c_0 by $T([f]_{c_0}) = [f \circ \sigma]_{c_0}$. The above result gives an infinite $B \subseteq D$ such that $T([f]_{c_0})|B = 0[f]_{c_0}$ for every B-supported $f \in \ell_\infty$, which loses much information. Therefore in this section we embark on finding a surjective $\psi : B^* \to A^*$ along which T may canonize with r nonzero as required in Definition 1.1. Note that a potential obstacle to finding such a canonization would be if $\bigcup \varphi^T[B^*]$ were nowhere dense. Actually, we have examples such that $\bigcup \varphi^T[\mathbb{N}^*]$ is nowhere dense and T is surjective (3.3, 3.12). So it is natural to assume that the surjections we consider are fountainless and that embeddings are funnelless. Under these assumptions we obtain a quasi-open ψ such that $T^\star(\delta_x)(\{\psi(x)\}) \neq 0$ locally, which is sufficient for the canonization by the following:

THEOREM 5.1. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\tilde{A} \subseteq \mathbb{N}$ be infinite. If $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a quasi-open continuous function such that $\tilde{A}^* \subseteq \psi[\mathbb{N}^*]$, then there exist $r \in \mathbb{R}$ and clopen sets $A^* \subseteq \tilde{A}^*$ and $B^* = \psi^{-1}[A^*]$ such that $T_{B,A}(f^*) = r(f^* \circ \psi)|B^*$ for every $f \in \ell_\infty(A)$.*

Proof. Fix $\tilde{A}$ and ψ as above.

CLAIM. *There exists an infinite $A \subseteq \tilde{A}$ and a clopen $E^* \subseteq \psi^{-1}[A^*]$ such that for every $y \in E^*$ there exists $r_y \in \mathbb{R}$ satisfying*

$$T^\star(\delta_y)|A^* = r_y \delta_{\psi(y)}.$$

Suppose this does not hold. We will recursively construct sequences $(A_\xi)_{\xi<\omega_1}$, $(D_\xi)_{\xi<\omega_1}$ and $(E_\xi)_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$, and a sequence $(a_\xi)_{\xi<\omega_1}$ of nonzero reals, such that:

(1) $A_\eta \subseteq_* A_\xi \subseteq_* \tilde{A}$ and $D_\xi \subseteq_* A_\xi \setminus A_{\xi+1}$ for all $\xi < \eta < \omega_1$,
(2) $E_\eta \subseteq_* E_\xi$ for all $\xi < \eta < \omega_1$,
(3) $E^*_\xi \subseteq \psi^{-1}[A^*_\xi]$ for all $\xi < \omega_1$,
(4) either $T^\star(\delta_y)(D^*_\xi) > a_\xi > 0$ for all $y \in E^*_{\xi+1}$, or $T^\star(\delta_y)(D^*_\xi) < a_\xi < 0$ for all $y \in E^*_{\xi+1}$.

Let $A_0 = \tilde{A}$ and $E_0^* = \psi^{-1}[A_0^*]$. Let $\eta < \omega_1$ and suppose we have constructed A_ξ, D_ξ, E_ξ and a_ξ satisfying (1)–(4) for all $\xi < \eta$. If η is a limit ordinal, take an infinite E such that $E \subseteq_* E_\xi$ for all $\xi < \eta$. By hypothesis there exists a clopen $A_\eta^* \subseteq \psi[E^*]$. Set $E_\eta^* = \psi^{-1}[A_\eta^*] \cap E^*$. Now we may suppose we have A_ξ and E_ξ for all $\xi \leq \eta$, and D_ξ and a_ξ for all $\xi < \eta$.

Take an infinite A'_η such that $(A'_\eta)^* \subseteq \psi[E_\eta^*]$. By our assumption, there exist $y \in \psi^{-1}[(A'_\eta)^*] \cap E_\eta^*$ and $X \subseteq (A'_\eta)^* \setminus \{\psi(y)\}$ such that $T^\star(\delta_y)(X) \neq 0$. By the regularity of $T^\star(\delta_y)$, there exists an infinite $D_\eta \subseteq_* A'_\eta$ such that $\psi(y) \notin D_\eta^*$ and $T^\star(\delta_y)(D_\eta^*) \neq 0$. Let a_η be such that either $0 < a_\eta < T^\star(\delta_y)(D_\eta^*)$ or $0 > a_\eta > T^\star(\delta_y)(D_\eta^*)$.

By the weak* continuity of $T^\star$, there exists a clopen neighbourhood V of y such that either $T^\star(\delta_z)(D_\eta^*) > a_\eta$ for all $z \in V$, or $T^\star(\delta_z)(D_\eta^*) < a_\eta$ for all $z \in V$. Finally, choose $A_{\eta+1} = A'_\eta \setminus D_\eta$ and $E_{\eta+1}^* = \psi^{-1}[A_{\eta+1}^*] \cap V \cap E_\eta^*$ (notice that $y \in E_{\eta+1}^*$). This ends the construction.

Since $|a_\xi| > 0$ for all $\xi < \omega_1$, there must exist $n \in \mathbb{N}$ and an infinite $I \subseteq \omega_1$ such that $|a_\xi| > 1/n$ for all $\xi \in I$. Hence, we may choose $\xi_0, \ldots, \xi_{k-1} \in I$, for some $k \in \mathbb{N}$, such that $a_{\xi_0}, \ldots, a_{\xi_{k-1}}$ are all of the same sign and $|\sum_{i<k} a_{\xi_i}| > \|T^\star\|$. Assume $\xi_0 \geq \xi_i$ for $i < k$. Take $y \in E_{\xi_0+1}^*$. Then, since the $D_{\xi_i}^*$ are pairwise disjoint and since $y \in E_{\xi_i+1}^*$, for every $i < k$, we have

$$\Big|T^\star(\delta_y)\Big(\bigcup_{i<k} D_{\xi_i}^*\Big)\Big| = \Big|\sum_{i<k} T^\star(\delta_y)(D_{\xi_i}^*)\Big| > \Big|\sum_{i<k} a_{\xi_i}\Big| > \|T^\star\|.$$

This contradiction proves the claim.

Therefore, for every A-supported $f \in \ell_\infty$ and every $y \in E^*$ we have $T(f)(y) = T^\star(\delta_y)(f) = r_y f(\psi(y))$. In particular, $T(\chi_{A^*})(y) = r_y$ for every $y \in E^*$. This means that the function $y \mapsto r_y$ with domain E^* is continuous. Then, since nonempty G_δ-sets in $\mathbb{N}^*$ have nonempty interior, the function must be constant on some clopen $B^* \subseteq E^*$. This means that for some $r \in \mathbb{R}$ we have $T_{B,A}(f) = r(f \circ \psi)|B$ for every A-supported $f \in \ell_\infty$. By going to a subset of A we may choose A and B so that $\psi^{-1}[A^*] = B^*$. ■

THEOREM 5.2. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\tilde{A} \subseteq \mathbb{N}$ be infinite and $F \subseteq \mathbb{N}^*$ be closed. If $\psi : F \to \mathbb{N}^*$ is an irreducible continuous function, then there exist $r \in \mathbb{R}$ and an infinite $A \subseteq \tilde{A}$ such that $T(f^*)|\psi^{-1}[A^*] = r(f^* \circ \psi)|\psi^{-1}[A^*]$ for every A-supported $f \in \ell_\infty$.*

Proof. The reasoning is similar to that for 5.1, so we will skip identical parts. The main difference is that nonempty G_δ's of F do not need to have nonempty interior. However, an irreducible surjection corresponds in the Stone duality to a Boolean monomorphism onto a dense subalgebra, which implies that the appropriate G_δ's have nonempty interior. Fix $\tilde{A}$, F and ψ as above.

Claim. *There exists an infinite $A \subseteq \tilde{A}$ such that for every $y \in \psi^{-1}[A^*]$ there exists $r_y \in \mathbb{R}$ satisfying*

$$T^\star(\delta_y)|A^* = r_y \delta_{\psi(y)}.$$

Suppose this does not hold. We will recursively construct sequences $(A_\xi)_{\xi<\omega_1}$ and $(D_\xi)_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$, and a sequence $(a_\xi)_{\xi<\omega_1}$ of nonzero reals such that:

(1) $A_\eta \subseteq_* A_\xi \subseteq_* \tilde{A}$ and $D_\xi \subseteq_* A_\xi \setminus A_{\xi+1}$ for all $\xi < \eta < \omega_1$,
(2) either $T^\star(\delta_y)(D^*_\xi) > a_\xi > 0$ for all $y \in \psi^{-1}[A^*_{\xi+1}]$, or $T^\star(\delta_y)(D^*_\xi) < a_\xi < 0$ for all $y \in \psi^{-1}[A^*_{\xi+1}]$.

Let $A_0 = \tilde{A}$. Let $\eta < \omega_1$ and suppose we have constructed A_ξ, D_ξ and a_ξ satisfying (1)–(2) for all $\xi < \eta$. If η is a limit ordinal, take an infinite A_η such that $A_\eta \subseteq_* A_\xi$ for every $\xi < \eta$. Now we may suppose we have A_ξ for all $\xi \le \eta$, and D_ξ and a_ξ for all $\xi < \eta$.

By our assumption, there exist $y \in \psi^{-1}[A^*_\eta]$ and $X \subseteq A^*_\eta \setminus \{\psi(y)\}$ such that $T^\star(\delta_y)(X) \neq 0$. By the regularity of $T^\star(\delta_y)$, there exists an infinite $D_\eta \subseteq_* A_\eta$ satisfying $\psi(y) \notin D^*_\eta$ and $T^\star(\delta_y)(D^*_\eta) \neq 0$. Let a_η be such that either $0 < a_\eta < T^\star(\delta_y)(D^*_\eta)$ or $0 > a_\eta > T^\star(\delta_y)(D^*_\eta)$.

By the weak* continuity of $T^\star$, there exists a clopen neighbourhood V of y in F such that either $T^\star(\delta_z)(D^*_\eta) > a_\eta$ for all $z \in V$, or $T^\star(\delta_z)(D^*_\eta) < a_\eta$ for all $z \in V$. We may assume that V is included in $\psi^{-1}[A^*_\eta \setminus D^*_\eta]$ as $y \in V \cap \psi^{-1}[A^*_\eta \setminus D^*_\eta]$. Using the irreducibility of ψ and its correspondence to a dense subalgebra via the Stone duality, we find an infinite $A_{\eta+1} \subseteq \mathbb{N}$ such that $\psi^{-1}[A^*_{\eta+1}] \subseteq V \subseteq \psi^{-1}[A^*_\eta \setminus D^*_\eta]$. In particular, $A_{\eta+1} \subseteq_* A_\eta \setminus D_\eta$ (note that y may not belong to $\psi^{-1}[A^*_{\eta+1}]$). This ends the construction. We finish the proof of the claim as in Theorem 5.1.

Therefore, for every A-supported $f \in \ell_\infty$ and every $y \in \psi^{-1}[A^*]$ we have $T(f^*)(y) = T^\star(\delta_y)(f) = r_y f(\psi(y))$. In particular, $T(\chi_{A^*})(y) = r_y$ for every $y \in \psi^{-1}[A^*]$. This means that the function $y \mapsto r_y$ with domain $\psi^{-1}[A^*]$ is continuous. Then, it is easy to see by the irreducibility of ψ that the function must be constant on some clopen set of F of the form $\psi^{-1}[B^*]$ for an infinite $B \subseteq A$. This means that for some $r \in \mathbb{R}$ we have $T(f) = r(f \circ \psi)|\psi^{-1}[B]$ for every B-supported $f \in \ell_\infty$. ∎

5.1. Left-local canonization of fountainless operators

Lemma 5.3. *Let $B \subseteq \mathbb{N}$ be infinite, and $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be fountainless and everywhere present. Then $F = \bigcup\{\varphi^T(y) : y \in B^*\}$ has nonempty interior.*

Proof. Suppose F is nowhere dense. Take $B_1 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ as in 3.5. With a view to applying 3.14, we will find a dense family $\mathcal{A}$ such that

$T(f^*)|B_1^* = 0$ whenever the support of f is included in an element of $\mathcal{A}$. This would contradict the fact that T is everywhere present.

Fix a nonempty clopen $U \subseteq \mathbb{N}^*$ disjoint from F. Notice that for every $y \in B_1^*$ the measure $T^\star(\delta_y)$ has no atoms in U. Therefore, by the regularity of $T^\star(\delta_y)$ and the compactness of U, for each $n \in \mathbb{N}$ we may find an open covering $(U_i(y,n))_{i<j(y,n)}$ of U by clopen sets such that $|T^\star(\delta_y)|(U_i(y,n)) < 1/(2(n+1))$ for all $i < j(y,n)$. We may further assume that either $U_i(y,n) \subseteq C_n^*$ or $U_i(y,n) \subseteq D_n^*$, for each $n \in \mathbb{N}$ and each $i < j(y,n)$.

Now by the weak* continuity of $T^\star$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of y such that for all $z \in V_n(y)$ we have

$$|T^\star(\delta_z)(U_i(y,n))| < 1/(2(n+1)) \tag{1}$$

for all $i < j(y,n)$.

So for each $n \in \mathbb{N}$ we have an open covering $\{V_n(y) : y \in B_1^*\}$ of B_1^*. As B_1^* is compact, for each $n \in \mathbb{N}$ take $y_0(n), \ldots, y_{m(n)-1}(n) \in B_1^*$ such that

$$B_1^* \subseteq \bigcup_{l<m(n)} V_n(y_l).$$

Now consider the family $\mathcal{A}_U$ of those sets $E \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l, n)$ such that E^* is included in $U_i(y_l, n)$. We claim that if $E \in \mathcal{A}_U$, then for every E^*-supported $f^* \in C(\mathbb{N}^*)$ we have $T(f^*)|B_1^* = 0$.

Fix $E \in \mathcal{A}_U$ and $y \in B_1^*$. We show that for every $n \in \mathbb{N}$ we have $|T^\star(\delta_y)|(E^*) < 1/(n+1)$. Let $T^\star(\delta_y) = \mu^+ - \mu^-$ be the Jordan decomposition. By 3.5 we have $\mu^-(C_n^*) < 1/(4(n+1))$ and $\mu^+(D_n^*) < 1/(4(n+1))$. By construction there exists $l < m(n)$ such that $y \in V_n(y_l)$, and by the definition of $\mathcal{A}_U$ there exists $i < j(y_l, n)$ such that $E^* \subseteq U_i(y_l, n)$. We may assume without loss of generality that $U_i(y_l, n) \subseteq C_n^*$. From this and from (1) above we obtain

$$\begin{aligned}|T^\star(\delta_y)|(E^*) &\le |T^\star(\delta_y)|(U_i(y_l,n)) = T^\star(\delta_y)(U_i(y_l,n)) + 2\mu^-(U_i(y_l,n))\\ &\le |T^\star(\delta_y)(U_i(y_l,n))| + 2\mu^-(C_n^*) < 1/(2(n+1)) + 2/(4(n+1)) = 1/(n+1).\end{aligned}$$

Therefore, $|T^\star(\delta_y)|(E^*) = 0$ for every $y \in B_1^*$. So if the support of $f^* \in C(\mathbb{N}^*)$ is included in E^*, we see that $0 = \int f^*\, dT^\star(\delta_y) = T^\star(\delta_y)(f^*) = T(f^*)(y)$ for every $y \in B_1^*$, as claimed.

Note that it is enough to show that $\mathcal{A}_U$ is dense under U, as in that case $\mathcal{A} = \bigcup\{\mathcal{A}_U : U \subseteq \mathbb{N}^* \setminus F,\ U \text{ clopen}\}$ is the dense family we are after.

To see that $\mathcal{A}_U$ is dense under U, fix an infinite $E_{0,0} \subseteq \mathbb{N}$ such that $E_{0,0}^* \subseteq U$. We define by induction a $\subseteq_*$-decreasing sequence of infinite sets, $\{E_{n,l} : n \in \mathbb{N},\ l < m(n)\}$, ordered lexicographically. Suppose (n', l') is successor of (n, l). Since $(U_i(y_l, n))_{i<j(y_l,n)}$ is an open covering of U, we may choose $i < j(y_l, n)$ such that $E_{n,l}^* \cap U_i(y_l, n) \neq \emptyset$. Take $\emptyset \neq E_{n',l'}^* \subseteq E_{n,l}^* \cap U_i(y_l, n)$.

Finally, if we take an infinite E such that $E \subseteq_* E_{n,l}$ for every $n \in \mathbb{N}$ and $l < m(n)$, then it is clear that $E \subseteq_* E_{0,0}$ and $E \in \mathcal{A}_U$. ∎

LEMMA 5.4. *Let $B \subseteq \mathbb{N}$ be infinite. Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is such that $\varphi^T(y) \neq \emptyset$ for each $y \in B^*$ and $\bigcup\{\varphi^T(y) : y \in V\}$ has nonempty interior for every open $V \subseteq \mathbb{N}^*$. Then there is an infinite $B_1 \subseteq_* B$ and $\varepsilon > 0$ such that:*

(1) *$\varphi^T_\varepsilon(y) \neq \emptyset$ for each $y \in B_1^*$, and*
(2) *$\bigcup\{\varphi^T_\varepsilon(y) : y \in D^*\}$ has nonempty interior for all infinite $D \subseteq_* B_1$.*

Proof. Suppose that (1) fails for all infinite $B' \subseteq_* B$ and all $\varepsilon > 0$. Let $B_0 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ be given by Lemma 3.5. We will construct by induction a $\subseteq_*$-decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $\varphi^T_{1/(n+1)}(y) = \emptyset$ for every $y \in B^*_{n+1}$. If we then take any $y \in \bigcap_{n\in\mathbb{N}} B_n^*$, we will have $\varphi^T(y) = \emptyset$, which contradicts our hypothesis.

Assume we have already constructed B_n. Since we are assuming that (1) fails, there exists $y \in B_n^*$ such that $\varphi^T_{1/2(n+1)}(y) = \emptyset$. This means that $|T^\star(\delta_y)|(\{x\}) = |T^\star(\delta_y)(\{x\})| < 1/(2(n+1))$ for every $x \in \mathbb{N}^*$.

By the regularity of the measure $T^\star(\delta_y)$ and by the compactness of $\mathbb{N}^*$, we may cover $\mathbb{N}^*$ by finitely many clopen $(U_i)_{i<k}$ such that $|T^\star(\delta_y)|(U_i) < 1/(2(n+1))$ for each $i < k$. We may further assume that each U_i is included in either C_n^* or D_n^*. Since $T^\star$ is weak* continuous, we may find an open neighbourhood of y, say V, such that for every $z \in V$ we have $|T^\star(\delta_z)(U_i)| < 1/(2(n+1))$ for each $i < k$. Take B_{n+1} such that $y \in B^*_{n+1} \subseteq B_n^* \cap V$. We claim that $\varphi^T_{1/(n+1)}(z) = \emptyset$ for every $z \in B^*_{n+1}$.

Fix any $z \in B^*_{n+1}$ and let $T^\star(\delta_z) = \mu^+ - \mu^-$ be the Jordan decomposition. Recall that by Lemma 3.5 we have $\mu^-(C_n^*) < 1/(4(n+1))$ and $\mu^+(D_n^*) < 1/(4(n+1))$. Now take any $x \in \mathbb{N}^*$. Let $i < k$ be such that $x \in U_i$ and assume without loss of generality that $U_i \subseteq C_n^*$. Then

$$|T^\star(\delta_z)(\{x\})| \leq |T^\star(\delta_z)|(U_i) \leq |T^\star(\delta_z)(U_i)| + 2\mu^-(C_n^*) < 1/(n+1),$$

and the claim is proved. This finishes the proof of the first part, so let us assume that $\varepsilon_0 > 0$ and $B_0 \subseteq_* B$ are such that (1) holds.

To prove the second part, assume further that for $B' \subseteq_* B_0$ and $\varepsilon > 0$ there exists an infinite $D \subseteq_* B_0$ with $\bigcup\{\varphi^T_\varepsilon(y) : y \in D^*\}$ nowhere dense. We may then find a $\subseteq_*$-descending sequence $(D_n)_{n\in\mathbb{N}}$ of infinite sets such that $D_n \subseteq_* B$ and $\bigcup\{\varphi^T_{1/n}(y) : y \in D_n^*\}$ is nowhere dense for every $n \in \mathbb{N}$. Let $V \subseteq \bigcap_{n\in\mathbb{N}} D_n^*$ be nonempty open. Then, since $\varphi^T(y) = \bigcup_{n\in\mathbb{N}} \varphi^T_{1/n}(y)$, we have

$$\bigcup\{\varphi^T(y) : y \in V\} = \bigcup_{n\in\mathbb{N}} \bigcup\{\varphi^T_{1/n}(y) : y \in V\} \subseteq \bigcup_{n\in\mathbb{N}} \bigcup\{\varphi^T_{1/n}(y) : y \in D_n^*\},$$

which is nowhere dense since a countable union of nowhere dense sets in $\mathbb{N}^*$ is nowhere dense. This contradicts the hypothesis of the lemma. Therefore, there exist an infinite $B_1 \subseteq_* B_0$ and $\varepsilon_1 > 0$ which satisfy (2).

The lemma holds for B_1 and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. ■

LEMMA 5.5. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is fountainless and everywhere present. Then for every infinite $B \subseteq \mathbb{N}$ there is an infinite $B_1 \subseteq_* B$ and a continuous quasi-open $\psi : B_1^* \to \mathbb{N}^*$ such that $T^\star(\delta_y)(\{\psi(y)\}) \neq 0$ for all $y \in B_1^*$.*

Proof. Since T is fountainless and everywhere present, by 3.17 and 5.3 we know that the hypotheses of 5.4 are satisfied. So find $\varepsilon > 0$ and an infinite $B_0 \subseteq_* B$ such that $\varphi_\varepsilon^T(y) \neq \emptyset$ for every $y \in B_0$, and $\bigcup\{\varphi_\varepsilon^T(y) : y \in D^*\}$ has nonempty interior for every infinite $D \subseteq_* B_0$. We may also assume that there exist $C_n, D_n \subseteq \mathbb{N}$ for every $n \in \mathbb{N}$ such that the statement in 3.5 holds for B_0 and C_n, D_n.

CLAIM 1. *There exists an infinite $B_0' \subseteq_* B_0$ and a finite collection $(V_i)_{i<k}$ of almost disjoint infinite subsets of $\mathbb{N}$ such that for every $z \in (B_0')^*$ we have $\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k} V_i^*$ and $|\varphi_\varepsilon^T(z) \cap V_i^*| = 1$ for each $i < k$.*

We will recursively construct a $\subseteq_*$-descending sequence (A_n) of subsets of $\mathbb{N}$, $y_n \in A_n^*$, finite collections $(V_{n,i})_{i<k_n}$ of almost disjoint infinite subsets of $\mathbb{N}$ and open intervals $I_i^n \subseteq \mathbb{R}$, $i < k_n$, such that for every n we have:

(1) $\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k_n} V_{n,i}^*$ for all $z \in A_{n+1}^*$,
(2) for every $i < k_{n+1}$ there exists $j < k_n$ such that $V_{n+1,i} \subseteq_* V_{n,j}$,
(3) $|T^\star(\delta_z)(V_{n,i})| \in I_i^n$ for all $z \in A_{n+1}^*$ and all $i < k_n$,
(4) $\operatorname{length}(I_i^n) = \varepsilon(2^{n+1}k_n)^{-1}$ for all $i < k_n$.

We begin by noticing that for every $y \in \mathbb{N}^*$ the number of elements of $\varphi_\varepsilon^T(y)$ is finite, as it must be bounded by $\|T\|/\varepsilon$. Let $A_0 = B_0$, fix any $y_0 \in A_0^*$ and let $\{x_i^0 : i < k_0\}$ be an enumeration of $\varphi_\varepsilon^T(y_0)$ (note that $k_0 \geq 1$). Let $N_0 \in \mathbb{N}$ be such that $1/N_0 < \varepsilon/(8k_0)$. By the regularity of the measure $T^\star(\delta_{y_0})$, we find for each $i < k_0$ a clopen neighbourhood $V_{0,i}^*$ of x_i^0 such that

$$|T^\star(\delta_{y_0})|(V_{0,i}^*) < |T^\star(\delta_{y_0})(\{x_i^0\})| + \varepsilon/(8k_0).$$

We may assume that the $V_{0,i}$'s are almost disjoint and that each of them is almost included in either C_{N_0} or D_{N_0}.

For each $i < k_0$ we define $I_i^0 \subseteq \mathbb{R}$ to be the open interval with centre $|T^\star(\delta_{y_0})(\{x_i^0\})|$ and radius $\varepsilon/(4k_0)$. By the above, $|T^\star(\delta_{y_0})|(V_{0,i}^*)$ lies in I_i^0, and as we shall see, so does $|T^\star(\delta_{y_0})(V_{0,i}^*)|$. Indeed, take $i < k_0$ and assume without loss of generality that $V_{0,i}^* \subseteq C_{N_0}^*$. If $T^\star(\delta_{y_0}) = \mu^* - \mu^-$ is the Jordan decomposition, then

$$|T^\star(\delta_{y_0})|(V_{0,i}^*) \leq T^\star(\delta_{y_0})(V_{0,i}^*) + 2\mu^-(V_{0,i}^*) \leq |T^\star(\delta_{y_0})(V_{0,i}^*)| + 2\mu^-(C_{N_0}^*).$$

From this it follows that $|T^\star(\delta_{y_0})|(V^*_{0,i}) - |T^\star(\delta_{y_0})(V^*_{0,i})| < 1/(2(N_0+1)) < (\varepsilon/8k_0)$, and so $|T^\star(\delta_{y_0})(V^*_{0,i})| \in I^0_i$.

By the upper semicontinuity of φ^T_ε (Lemma 3.11) and the weak* continuity of $T^\star$, we now find a clopen neighbourhood of y_0, say A^*_1, which we may assume to be included in A^*_0, such that for every $z \in A^*_1$ we have $\varphi^T_\varepsilon(z) \subseteq \bigcup_{i<k_0} V^*_{0,i}$ and $|T^\star(\delta_z)(V^*_{0,i})| \in I^0_i$ for each $i < k_0$. If $|\varphi^T_\varepsilon(z) \cap V^*_{0,i}| = 1$ for all $z \in A^*_1$ and $i < k_0$, then the recursion stops and the claim is proved. Otherwise, choose one $z \in A^*_1$ for which the above fails, call it y_1, and repeat the procedure to obtain open intervals I^1_i with centre $|T^\star(\delta_{y_1})(\{x^1_i\})|$ and radius $\varepsilon/(2^3k_1)$, and clopen sets $V^*_{1,i}$ such that both $|T^\star(\delta_{y_1})|(V^*_{1,i})$ and $|T^\star(\delta_{y_1})(V^*_{1,i})|$ lie inside I^1_i, for each $i < k_1$. Notice that we may take each set $V_{1,i}$ as a subset of one of the $V_{0,j}$. Then, by the same argument using the upper semicontinuity of φ^T_ε and the weak* continuity of $T^\star$, we obtain an infinite $A_2 \subseteq_* A_1$ such that for every $z \in A^*_2$ we have $\varphi^T_\varepsilon(z) \subseteq \bigcup_{i<k_1} V^*_{1,i}$ and $|T^\star(\delta_z)(V^*_{1,i})| \in I^1_i$, for each $i < k_1$.

We claim that this process stops after finitely many steps. First notice that the failure to stop at step n is due to one of two reasons:

(a) there is $y_{n+1} \in A^*_{n+1}$ such that $|\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i}| \geq 2$ for some $i < k_n$,
(b) there is $y_{n+1} \in A^*_{n+1}$ such that $\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i} = \emptyset$ for some $i < k_n$.

Notice also that once condition (a) fails, it continues to fail in subsequent steps. So we may assume that we first only check (a), and if it does not occur we check (b).

By condition (2) in the construction, we see that every time (a) occurs there exists $i < k_0$ such that $|\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{0,i}| \geq 2$. So for each $i < k_0$ consider $m_i \in \mathbb{N}$ such that $m_i \cdot \varepsilon \geq T^\star(\delta_{y_0})(\{x^0_i\})$ and suppose (a) has occurred at $n = \sum_{i<k_0} m_i$ many steps. Suppose that still (a) happens once more. Then there exist $i_0 < k_0$ and $n_0 < \cdots < n_{m_{i_0}} = n$ such that $|\varphi^T_\varepsilon(y_{n_j+1}) \cap V^*_{0,i_0}| \geq 2$ for every $j \leq m_{i_0}$. Hence, if $x^{n+1}_0, x^{n+1}_1 \in \varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i_n}$ for certain $i_n < k_n$, then $|T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0\}) + \varepsilon \leq |T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0, x^{n+1}_1\}) \leq |T^\star(\delta_{y_{n+1}})|(V^*_{n,i_n})$ and we obtain

$$\begin{aligned}
|T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0\}) &< \sup I^n_{i_n} - \varepsilon = |T^\star(\delta_{y_n})|(\{x^n_{i_n}\}) + \varepsilon(2^{n+2}k_n)^{-1} - \varepsilon \\
&< \cdots < |T^\star(\delta_{y_0})|(\{x^0_{i_0}\}) + \sum_{j<n+1} \varepsilon(2^{j+2}k_j)^{-1} - (m_{i_0}+1)\cdot\varepsilon \\
&\leq (|T^\star(\delta_{y_0})|(\{x^0_{i_0}\}) - m_{i_0}\cdot\varepsilon) + \Big(\varepsilon \sum_{j<n+1} 1/2^{j+2} - \varepsilon\Big) < 0 - \varepsilon/2.
\end{aligned}$$

This implies that (a) can occur at most at $\sum_{i<k_0} m_i$ steps.

Now assume that n_0 is such that (a) does not hold at step n, for all $n \geq n_0$. Suppose the recursion does not stop at step $n \geq n_0$. Assume without loss of generality that $\varphi_\varepsilon^T(y_{n+1}) \cap V_{n,0}^* = \emptyset$. Therefore $\varphi_\varepsilon^T(y_{n+1}) \subseteq \bigcup_{0<i<k_n} V_{n,i}^*$. Since we also have $|T^\star(\delta_{y_{n+1}})|(V_{n,i}^*) \in I_i^n$, for each $i < k_n$, we obtain

$$\begin{aligned}
|T^\star(\delta_{y_{n+1}})|(\{x_i^{n+1} : i < k_{n+1}\})& \\
&\le \sum_{i<k_n} |T^\star(\delta_{y_{n+1}})|(V_{n,i}^*) - |T^\star(\delta_{y_{n+1}})|(V_{n,0}^*) \\
&< \sum_{i<k_n} \sup I_i^n - \inf I_0^n \\
&\le \sum_{i<k_n} |T^\star(\delta_{y_n})(\{x_i^n\})| + k_n\varepsilon/(2^{n+2}k_n) - (\varepsilon - \varepsilon/(2k_n)) \\
&\le |T^\star(\delta_{y_n})|(\{x_i^n : i < k_n\}) - \varepsilon/2.
\end{aligned}$$

The last inequality holds because $k_n \ge 1$. Since $|T^\star(\delta_{y_n})|(\{x_i^n : i < k_n\}) \ge \varepsilon$ for every n, we conclude that (b) cannot occur for infnitely any $n \in \mathbb{N}$. Hence the recursion must stop after finitely many steps.

Claim 2. *There exists $i_0 < k$ and an infinite $B_1 \subseteq_* B_0'$ such that $V_{i_0}^* \cap \bigcup\{\varphi_\varepsilon^T(y) : y \in D^*\}$ has nonempty interior for every infinite $D \subseteq_* B_1$.*

Suppose this is not the case. Then we may find a sequence of infinite sets $B_0' = A_0 \supseteq_* A_1 \supseteq_* \cdots \supseteq A_k$ such that $V_i^* \cap \bigcup\{\varphi_\varepsilon^T(y) : y \in A_{i+1}^*\}$ is nowhere dense for $i < k$. By Claim 1, we know that $\bigcup\{\varphi_\varepsilon^T(y) : y \in A_k^*\} \subseteq \bigcup_{i<k} V_i^*$, and so $\bigcup\{\varphi_\varepsilon^T(y) : y \in A_k^*\} = \bigcup_{i<k} V_i^* \cap \bigcup\{\varphi_\varepsilon^T(y) : y \in A_k^*\}$ is also nowhere dense. But this contradicts the choice of B_0.

Now we define $\psi : B_1^* \to \mathbb{N}^*$ by $\{\psi(y)\} = V_{i_0}^* \cap \varphi_\varepsilon^T(y)$. It is clear that ψ is quasi-open by Claim 2, so showing that ψ is continuous completes the proof. Let $U \subseteq V_{i_0}^*$ be an open set. As $V_{i_0}^*$ is clopen and φ_ε^T is upper semicontinuous, we conclude that $\psi^{-1}[U] = B_1^* \cap \{y \in \mathbb{N}^* : \varphi_\varepsilon^T(y) \subseteq U \cup \mathbb{N}^* \setminus V_{i_0}^*\}$ is open. ∎

Theorem 5.6. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a fountainless, everywhere present operator. Then T is left-locally canonizable along a quasi-open map. Consequently, every fountainless nonzero operator is somewhere canonizable along a quasi-open map.*

Proof. Fix an infinite $B \subseteq \mathbb{N}$ and find using 5.5 an infinite $B_1 \subseteq B$ and a quasi-open $\psi : B_1^* \to \mathbb{N}^*$ such that $T^\star(\delta_y)(\{\psi(y)\}) \neq 0$ for all $y \in B_1^*$. Now use 5.1 to find clopen sets $A^* \subseteq \psi[B_1^*]$ and $B_2^* \subseteq \psi^{-1}[A^*]$ and a real $r \in \mathbb{R}$ such that $T_{B_2,A}(f^*) = r(f^* \circ \psi)|B_2$ for every $f \in \ell_\infty(A)$. It follows that $T^\star(\delta_y)|A^* = r\delta_{\psi(y)}$ for each $y \in B_2^*$, and so $0 \neq T^\star(\delta_y)(\{\psi(y)\}) = r$. Hence $T_{B_2,A}$ is canonizable along ψ. As for every nonzero operator T there is an infinite $B \subseteq \mathbb{N}$ such that $P_B \circ T$ is everywhere present, we obtain the last part of the theorem. ∎

5.2. Right-local canonization of funnelless automorphisms. The main result of this section is a consequence of our generalization 5.1 of the Drewnowski–Roberts canonization lemma and the following result of Plebanek which is implicitly proved in [35, Theorem 6.1].

THEOREM 5.7. *Let $T : C(K) \to C(K)$ be an automorphism. There is a π-base $\mathcal{U}$ of K such that for every $U \in \mathcal{U}$ there is a closed $F \subseteq K$ and a continuous surjection $\psi : F \to \overline{U}$ with $|T^\star(\delta_y)|(\{\psi(y)\}) \neq 0$ for all $y \in F$.*

THEOREM 5.8. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a funnelless automorphism. Then T is right-locally canonizable along a quasi-open map.*

Proof. Fix an infinite $A \subseteq \mathbb{N}$. Let $\mathcal{U}$ be as in 5.7 for T and find $U \in U$ such that $U \subseteq A^*$. Let F and ψ be as in 5.7 for U. Let $A_1 \subseteq \mathbb{N}$ be infinite such that $A_1^* \subseteq U$ and set $F_1 = \psi^{-1}[A_1^*] \subseteq F$. Let $F_2 \subseteq F_1$ be closed such that $\psi|F_2$ is irreducible and onto A_1^* and hence quasi-open (relative to the subspace topology on F_2).

As T is funnelless (Definition 3.18) and A_1^* is open, F_2 cannot be nowhere dense, so let B^* be a nonempty clopen set included in F_2. Now $\psi : B^* \to A_1^*$ is a continuous, quasi-open map (as a restriction of a quasi-open map to a clopen subset of F_2) satisfying $T^\star(\delta_y)(\{\psi(y)\}) \neq 0$ for each y in B^*. Therefore, we can apply 5.1 to obtain an infinite $A_2 \subseteq A_1$, a clopen $B_1^* \subseteq \psi^{-1}[A_2^*]$ and a real $r \in \mathbb{R}$ such that $T_{B_1,A_2}(f^*) = r(f^* \circ \psi)|B_1^*$ for every $f \in \ell_\infty(A_2)$. In particular, $T^\star(\delta_y)(E^*) = T_{B_1,A_2}(\chi_{E^*})(y) = r(\chi_{E^*} \circ \psi)(y) = r\delta_{\psi(y)}(E^*)$ for every infinite $E \subseteq A_2$ and every $y \in B_1^*$. It follows that for each $y \in B_1^*$ we have $T^\star(\delta_y)|A_2^* = r\delta_{\psi(y)}$, and so $0 \neq T^\star(\delta_y)(\{\psi(y)\}) = r$. Having $r \neq 0$, we conclude that T_{B_1,A_2} is canonizable along ψ. ■

6. The impact of combinatorics on the canonization and trivialization of operators on ℓ_∞/c_0. As expected, based on the study of $\mathbb{N}^*$ (e.g., [41, 43, 46, 13, 15, 19]), the impact of additional set-theoretic assumptions on the structure of operators on ℓ_∞/c_0 is also decisive.

6.1. Canonization and trivialization of operators on ℓ_∞/c_0 under OCA+MA. Recall from [19] that an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ is called *c.c.c. over* Fin if, and only if, there are no uncountable almost disjoint families of $\mathcal{I}$-positive sets. Dually, a closed subset $F \subseteq \mathbb{N}^*$ is called c.c.c. over Fin if $A_\xi^* \cap F = \emptyset$ for some $\xi < \omega_1$ whenever $\{A_\xi : \xi < \omega_1\}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. The following theorem by I. Farah [19, 3.3.3 and 3.8.1] will be crucial in this subsection:

THEOREM 6.1 ([19]). (OCA+MA) *Let $h : \wp(\mathbb{N})/\mathrm{Fin} \to \wp(\mathbb{N})/\mathrm{Fin}$ be a homomorphism. Then there is an infinite $B \subseteq \mathbb{N}$, a function $\sigma : B \to \mathbb{N}$ and a homomorphism $h_2 : \wp(\mathbb{N})/\mathrm{Fin} \to \wp(\mathbb{N} \setminus B)/\mathrm{Fin}$ such that $h(A) = [\sigma^{-1}[A]] \cup h_2([A])$ for every $A \subseteq \mathbb{N}$, and* $\mathrm{Ker}(h_2)$ *is c.c.c. over* Fin.

The following is a topological reformulation of the above theorem:

THEOREM 6.2 ([13, Corollary 7]). (OCA+MA) *Suppose $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous mapping. Then there exist an infinite $B \subseteq \mathbb{N}$ and a function $\sigma : B \to \mathbb{N}$ such that $\psi(x) = \sigma^*(x)$ for all $x \in B^*$ and $F = \psi[(\mathbb{N} \setminus B)^*]$ is a nowhere dense closed c.c.c. over* Fin *set.*

Proof. By the Stone duality every continuous $\psi : \mathbb{N}^* \to \mathbb{N}^*$ corresponds to the homomorphism $h : \wp(\mathbb{N})/\mathrm{Fin} \to \wp(\mathbb{N})/\mathrm{Fin}$ given by $h([A]) = [D]$, where $D^* = \psi^{-1}[A^*]$, for every $A \subseteq \mathbb{N}$. Let $B \subseteq \mathbb{N}$, $\sigma : B \to \mathbb{N}$ and h_2 be as in 6.1 for the homomorphism h. For infinite $A \subseteq \mathbb{N}$ we also have $h([A]) \cap [B] = [\sigma^{-1}[A]]$ by 6.1, and so $\psi^{-1}[A^*] \cap B^* = (\sigma^{-1}[A])^*$. Therefore, $\sigma^* = \psi|B^*$. For infinite $A \subseteq \mathbb{N}$ we also have $h([A]) \cap [\mathbb{N} \setminus B] = h_2([A])$ by 6.1, so for every $x \in (\mathbb{N} \setminus B)^*$ we get $\psi(x) = h_2^{-1}[\{[A] : A \in x\}]$ by the Stone duality. The set $F = \psi[(\mathbb{N} \setminus B)^*]$ is closed, and for every $x \in (\mathbb{N} \setminus B)^*$ the set $h_2^{-1}[\{[A] : A \in x\}]$ is disjoint from $\mathrm{Ker}(h_2)$, which is c.c.c. over Fin. Therefore, F is c.c.c. over Fin, and such sets are nowhere dense. ∎

It turns out that, under OCA+MA, quasi-open maps $\psi : B^* \to \mathbb{N}^*$ can be reduced to bijections between subsets of $\mathbb{N}$.

LEMMA 6.3. (OCA+MA) *Let $B \subseteq \mathbb{N}$ be infinite. Suppose that $\psi : B^* \to \mathbb{N}^*$ is a continuous quasi-open mapping. Then there are infinite $B_1 \subseteq B$, $A \subseteq \mathbb{N}$ and a bijection $\sigma : B_1 \to A$ such that $\psi|B_1^* = \sigma^*$. In particular, $\psi|B_1^*$ is a homeomorphism.*

Proof. By 6.2 there exist $B_0 \subseteq B$ and a function $\sigma : B_0 \to \mathbb{N}$ such that $\psi(x) = \sigma^*(x)$ for all $x \in B_0^*$. As ψ is quasi-open, there is an infinite $E \subseteq \mathbb{N}$ such that $E^* \subseteq \psi[B_0^*] = \sigma^*[B_0^*]$. Therefore, $E \subseteq^* \sigma[B_0]$ and hence the image of σ is infinite, and so there is an infinite $B_1 \subseteq B_0$ such that $\sigma|B_1$ is a bijection onto its image A. ∎

THEOREM 6.4. (OCA+MA) *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a fountainless everywhere present operator, then it is left-locally canonizable and so left-locally trivial. Consequently, every fountainless operator is somewhere trivial. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a funnelless automorphism, then it is right-locally canonizable and so right-locally trivial.*

Proof. Apply 5.6 and 5.8 to obtain left-local and right-local canonization along a quasi-open mapping, respectively. Now use 6.3 to conclude that this mapping is somewhere induced by a bijection. ∎

6.2. Operators on ℓ_∞/c_0 under CH. The Continuum Hypothesis is a strong tool allowing transfinite induction constructions in $\wp(\mathbb{N})/\mathrm{Fin}$ which induce objects in ℓ_∞/c_0. Actually, a considerable part of this strength is included in a powerful consequence of Parovichenko's theorem: if X is a zero-dimensional, locally compact, σ-compact, noncompact Hausdorff space of weight at most continuum, then $X^* = \beta X \setminus X$ is homeomorphic to $\mathbb{N}^*$ [29, 1.2.6]. In this section we will often be using this result combined with the universal property of βX for locally compact X, namely that every continuous function on X into a compact space extends to βX.

THEOREM 6.5. (CH) *There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which is nowhere canonizable along a quasi-open map on an open set, in particular along a homeomorphism.*

Proof. Let K be an uncountable compact zero-dimensional metrizable space with countably many isolated points $\{x_m : m \in \mathbb{N}\}$ which form a dense open subspace of K. By the classical classification of separable spaces of continuous functions there is an isomorphism $S : C(2^{\mathbb{N}}) \to C(K)$.

Let $X = \mathbb{N}\times K$ and $Y = \mathbb{N}\times 2^{\mathbb{N}}$. Note that X, Y satisfy the hypothesis of the topological consequence of Parovichenko's theorem [29, 1.2.6] mentioned above, hence there are homeomorphisms $\pi : \mathbb{N}^* \to X^*$ and $\rho : Y^* \to \mathbb{N}^*$. Define $\tilde{\tau} : K \to \|S\|B_{C(2^{\mathbb{N}})^\star}$ by $\tilde{\tau}(x) = S^\star(\delta_x)$ for each $x \in K$, where the dual ball is considered with the weak* topology and identified with the Radon measures on $2^{\mathbb{N}}$. Define $\tau : X \to \|S\|B_{C(\beta Y)^\star}$ by letting $\tau(n, x)$ be the measure on βY which is zero on the complement of $\{n\}\times 2^{\mathbb{N}}$ and is equal to the measure $\tilde{\tau}(x)$ on $\{n\} \times 2^{\mathbb{N}}$. By the universal property of βX there is an extension $\beta\tau : \beta X \to \|S\|B_{C(\beta Y)^\star}$

CLAIM. *For each $t \in X^*$ the measure $\beta\tau(t)$ is concentrated on Y^*.*

Fix $t \in X^*$. Note that for every $n \in \mathbb{N}$ the set $\beta X \setminus \{k \in \mathbb{N} : k \le n\} \times K$ is a neighbourhood of t. Also, for $k > n$, if $x \in \{k\} \times K$, then $\tau(x)(U) = 0$ for every Borel subset U of $\{n\} \times 2^{\mathbb{N}}$. This completes the proof of the claim by the weak* continuity of $\beta\tau$.

Now we can define $T : C(Y^*) \to C(X^*)$ by $T(f)(t) = \int f\, d(\beta\tau(t))$ for every $t \in X^*$. It is a well defined bounded linear operator by [17, Theorem VI.7.1]. We will show that $T_\pi \circ T \circ T_\rho$ is an automorphism of $\mathbb{N}^*$ which is nowhere canonizable along a quasi-open map. For the former we need to prove that T is an isomorphism, and for the latter we need to prove that for any nonempty clopen sets $U \subseteq X^*$, $O \subseteq Y^*$ there is no quasi-open $\phi : U \to O$ such that $(\beta\tau(t))|O = r\delta_{\phi(t)}$ for every t in U and an $r \in \mathbb{R} \setminus \{0\}$.

To prove that T is an isomorphism, note that one can define $R : C(\beta Y) \to C(\beta X)$ by $R(f)(x) = \int f\, d(\beta\tau(x))$ for every $x \in X$, and that $C(\beta Y)$ can be identified with the ℓ_∞-sum of $C(2^{\mathbb{N}})$ while $C(\beta X)$ can be identified with the ℓ_∞-sum of $C(K)$. Furthermore, R sends the subspace corresponding to the c_0-sum of $C(2^{\mathbb{N}})$ into the subspace corresponding to the c_0-sum of $C(K)$ since the original S is an isomorphism and R is the ℓ_∞-sum of the operator S. It follows that T is induced by R modulo the subspaces corresponding to the c_0-sums. Moreover, one can note, using the fact that S is bounded below, that elements outside the subspace corresponding to the c_0-sum of $C(K)$ are sent by R onto elements outside the subspace corresponding to the c_0-sum of $C(K)$. It follows that T is nonzero on every nonzero element, i.e., is injective. The surjectivity of T follows from the surjectivity of R, which in turn follows from the surjectivity of S.

Now let us prove that T is nowhere canonizable along a quasi-open mapping. Fix clopen subsets U and O of X^* and Y^* respectively, and suppose ϕ is as above and quasi-open. Fix a clopen $V \subseteq \phi[U]$. Let U' be a clopen subset of X such that $\beta U' \cap X^* = U$. The set E of integers n such that $U_n = U' \cap (\{n\} \times K) \neq \emptyset$ must be infinite. Since the isolated points $\{x_m : m \in \mathbb{N}\}$ are dense in K, we may assume, by going to a subset of U, that $U_n = \{x_{k_n}\}$ for all $n \in E$ and some $k_n \in \mathbb{N}$. Therefore,

$$U' = \bigcup_{n\in E} \{n\} \times \{x_{k_n}\}.$$

Let $V_n = V' \cap \{n\} \times 2^{\mathbb{N}}$ for $n \in E$, where $\beta V' \cap Y^* = V$. Let $W_n \subseteq V_n$ be nonempty clopen such that $\tilde{\tau}(x_{k_n})|W_n$ has its total variation less than $|r|/2$, which can be found since $2^{\mathbb{N}}$ has no isolated points. Consider

$$W = \bigcup_{n\in E} \{n\} \times W_n.$$

Then $|\beta\tau(n, x_{k_n})(W')| < |r|/2$ for any $W' \subseteq W$ and any $n \in E$. By the weak* continuity of $\beta\tau$ we have $|\beta\tau(t)(W')| \leq |r|/2$ for any $t \in U$, but this shows that $\beta\tau(t)$ is not $r\delta_{\phi(t)}$, as required. ∎

One example using the methods as above due to E. van Douwen and J. van Mill is a nowhere dense retract of $\mathbb{N}^*$ which is homeomorphic to $\mathbb{N}^*$ and which is a P-set (see [29, 1.4.3 and 1.8.1]). We will require the following:

Lemma 6.6. (CH) *Let $F \subseteq \mathbb{N}^*$ be a nowhere dense P-set. The space $\{f \in C(\mathbb{N}^*) : f|F = 0\}$ is isomorphic to $C(\mathbb{N}^*)$.*

Proof. Fix a P-point $p \in \mathbb{N}^*$, which exists under CH by the results of [39]. Let $(A^*_\alpha : \alpha < \omega_1)$ and $(B^*_\alpha : \alpha < \omega_1)$ be sequences of strictly increasing clopen sets such that $\bigcap_{\alpha<\omega_1}(\mathbb{N}^* \setminus A^*_\alpha) = F$ and $\bigcap_{\alpha<\omega_1}(\mathbb{N}^* \setminus B^*_\alpha) = \{p\}$ (they exist because F is a P-set and p is a P-point).

Using the standard argument we recursively construct one-to-one, onto functions $\sigma_\alpha : B_\alpha \to A_\alpha$ such that $\sigma_\alpha =^* \sigma_\beta|B_\alpha$ for all $\alpha < \beta < \omega_1$. Let $\psi_\beta = \sigma^*_\beta : B^*_\beta \to A^*_\beta$, which is the corresponding homeomorphism.

Note that if $f \in C(\mathbb{N}^*)$ is such that $f|F = 0$, then for every $n \in \mathbb{N}$ there exists $\alpha < \omega_1$ such that $\mathbb{N}^* \setminus A^*_\alpha \subseteq f^{-1}[\{t \in \mathbb{R} : |t| < 1/(n+1)\}]$. Therefore, for each such f there exists $\alpha < \omega_1$ such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. So define

$$S : \{f \in C(\mathbb{N}^*) : f|F = 0\} \to \{f \in C(\mathbb{N}^*) : f(p) = 0\}$$

by setting $S(f) = (f \circ \psi_\alpha) \cup 0_{\mathbb{N}\setminus B^*_\alpha}$, where α is any countable ordinal such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. It is well defined because the homeomorphisms extend each other, and it is clearly a linear isometry. Now it is enough to note that $\{f \in C(\mathbb{N}^*) : f(p) = 0\}$ is isomorphic to $C(\mathbb{N}^*)$. To see this, notice that this space is a hyperplane, and recall that all hyperplanes are isomorphic to each other in any Banach space (see [18, exercises 2.6 and 2.7]). In the case of $C(\mathbb{N}^*)$ we have

$$C(\mathbb{N}^*) \sim C(\mathbb{N}^*) \oplus \ell_\infty \sim C(\mathbb{N}^*) \oplus \ell_\infty \oplus \mathbb{R} \sim C(\mathbb{N}^*) \oplus \mathbb{R}$$

and so all hyperplanes are isomorphic to the entire $C(\mathbb{N}^*)$. ■

Proposition 6.7. (CH) *The collection of locally null operators is not a right ideal. Moreover, the right ideal generated by locally null operators is improper.*

Proof. Let $F \subsetneq \mathbb{N}^*$ be a nowhere dense retract of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$, $\psi_1 : \mathbb{N}^* \to F$ the witnessing retraction and $\psi_2 : \mathbb{N}^* \to F$ the homeomorphism. Then $\psi_2^{-1} \circ \psi_1$ is a well defined continuous map from $\mathbb{N}^*$ onto itself, and so $T_{\psi_2^{-1}\circ\psi_1}$ is a well defined operator from $C(\mathbb{N}^*)$ into itself. Note that T_{ψ_2} is locally null because F is nowhere dense, and hence $T_{\psi_2}(f^*) = f^* \circ \psi_2$ is zero for every $f \in \ell_\infty$ such that $f^*|F$ is zero. But for every $f \in \ell_\infty$ we have

$$T_{\psi_2} \circ T_{\psi_2^{-1}\circ\psi_1}(f^*) = f^* \circ \psi_2^{-1} \circ \psi_1 \circ \psi_2 = f^*,$$

because $\mathrm{Im}(\psi_2) = F$ and $\psi_1|F = \mathrm{Id}_F$. This means that $T_{\psi_2} \circ T_{\psi_2^{-1}\circ\psi_1} = \mathrm{Id}$, which is not locally null. Moreover, for any operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ we have $S = (T_{\psi_2} \circ T_{\psi_2^{-1}\circ\psi_1}) \circ S$, which is in the right ideal generated by locally null operators. ■

Nowhere dense P-sets homeomorphic to $\mathbb{N}^*$ which are retracts also give more concrete (compared to 6.5) examples of automorphisms failing canonizability as in 6.4.

Example 6.8. (CH) There is an automorphism T of ℓ_∞/c_0 with the following properties:

(1) T is not fountainless,
(2) T is not left-locally canonizable along any continuous map,
(3) T^{-1} is not funnelless,
(4) T^{-1} is not right-locally canonizable along any continuous map.

Proof. Let F be a nowhere dense retract of $\mathbb{N}^*$ which is a P-set and is homeomorphic to $\mathbb{N}^*$. Let $\psi_1 : \mathbb{N}^* \to F$ be the witnessing retraction. We will need one more property of F, namely that ψ_1 is not one-to-one when restricted to any nonempty clopen set. This can be obtained by modifying the construction of [29, 1.4.3] by replacing $W(\omega_1+1)$ with the "zero-dimensional long line", i.e., the space K obtained by gluing Cantor sets inside every ordinal interval $[\alpha, \alpha+1)$ for $\alpha < \omega_1$; this leads to a nonmetrizable subspace K of the long line which contains $W(\omega_1+1)$ and has no isolated points. One takes $\tilde{\pi} : K \to K$ which collapses the entire K to the point ω_1, $X = \mathbb{N} \times K$, and $\pi : X \to X$ given by $\pi(n,x) = (n, \tilde{\pi}(x))$. As in [29, 1.4.3], one proves that $\beta\pi[X^*] \subseteq X^*$ and $\psi_1 = \beta\pi|X^*$ is the required retraction. The proof that ψ_1 is not one-to-one when restricted to any clopen set is similar to the reasoning from the proof of [30, Theorem 2.1]: if $U \subseteq X^*$ clopen, it is of the form $\beta U' \cap X^*$ where

$$U' = \bigcup_{n\in E} \{n\} \times U_n$$

for some infinite $E \subseteq \mathbb{N}$ and nonempty clopen sets $U_n \subseteq K$ (consider χ_U and the relation of X to βX). But these nonempty open sets have at least two points x_n, y_n as K has no isolated points. Of course $\pi(n, x_n) = (n, \omega_1) = \pi(n, y_n)$. Consider $x = \lim_{n\in u} x_n$ and $y = \lim_{n\in u} y_n$ (u is a nonprincipal ultrafilter in $\wp(\mathbb{N})$), which can be easily separated, so $x \neq y$ and $x, y \in U$. However $\psi_1(x) = \lim_{n\in u} \pi(n, x_n) = \lim_{n\in u} \pi(n, y_n) = \psi_1(y)$.

We can decompose $C(\mathbb{N}^*) = X \oplus Y$ where

$$X = \{g \circ \psi_1 : g \in C(F)\}, \qquad Y = \{f \in C(\mathbb{N}^*) : f|F = 0\}.$$

The first summand is isometric to $C(F)$ (the isometry is defined by restricting to F), which in turn is isometric to $C(\mathbb{N}^*)$ because of the homeomorphism between F and $\mathbb{N}^*$. By Lemma 6.6, the second summand is also isomorphic to $C(\mathbb{N}^*)$.

Fix an infinite, coinfinite $A \subseteq \mathbb{N}$. Let $S : Y \to C(\mathbb{N}^* \setminus A^*)$ be an isomorphism, and $\psi_2 : A^* \to F$ a homeomorphism. Finally, let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be the isomorphism defined by

$$T = I_A \circ T_{\psi_2} + I_{\mathbb{N}^*\setminus A^*} \circ S \circ (\mathrm{Id} - T_{\psi_1}).$$

That is, roughly speaking, T sends X to $C(A^*)$ and Y to $C(\mathbb{N}^* \setminus A^*)$. For $y \in A^*$ we have $T^\star(\delta_y) = (I_A \circ T_{\psi_2})^\star(\delta_y) = \delta_{\psi_2(y)}$, i.e., $T^\star(\delta_y)$ is concentrated on F, so (F, A) is a fountain for T. This also implies that T cannot be canonized along a homeomorphism onto a clopen set below A^* because F is nowhere dense. On the other hand,

$$T^{-1} = T_{\psi_1} \circ T_{\psi_2^{-1}} \circ P_A + S^{-1} \circ P_{\mathbb{N}\setminus A}.$$

So $(T^{-1})^\star(\delta_x)|A^* = \delta_{\psi_2^{-1}(\psi_1(x))}$ for every $x \in \mathbb{N}^*$. In particular (A^*, F) is a funnel for T^{-1}, and T^{-1} cannot be canonized on a pair (A_0, B_0) for infinite $A_0 \subseteq A$, $B_0 \subseteq \mathbb{N}$ because $\psi_2^{-1} \circ \psi_1$ is not one-to-one on any clopen set $B_0 \subseteq \mathbb{N}$ by the choice of ψ_1. ∎

THEOREM 6.9. (CH) *There is an automorphism of ℓ_∞/c_0 with no fountains and no funnels which is nowhere trivial.*

Proof. Let $\psi : \mathbb{N}^* \to \mathbb{N}^*$ be a nowhere trivial homeomorphism of $\mathbb{N}^*$. Its existence is a folklore result and its first construction is implicitly included in [39]. By Propositions 3.21 and 3.22, T_ψ has neither fountains nor funnels. It is not locally trivial because ψ is not trivial on any clopen set. ∎

THEOREM 6.10. (CH) *There is a quasi-open surjective map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that the images of nowhere dense sets under ψ are nowhere dense and ψ is not a bijection when restricted to any clopen set. Therefore, T_ψ is an everywhere present isomorphic embedding of ℓ_∞/c_0 into itself with no fountains and with no funnels which is nowhere canonizable along a homeomorphism.*

Proof. It is enough to construct a quasi-open irreducible surjection $\psi : \mathbb{N}^* \to \mathbb{N}^*$ which is not a homeomorphism when restricted to any clopen set, and to consider T_ψ as in 3.21 and 3.22, since irreducible maps send nowhere dense sets onto nowhere dense sets. Let $\tilde{\phi} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be an irreducible surjection which is not a bijection when restricted to any clopen subset of $2^{\mathbb{N}}$ (e.g., obtained via the Stone duality by taking a dense atomless subalgebra of the free countable algebra which is proper below any element). Consider $X = \mathbb{N} \times 2^{\mathbb{N}}$ and $\phi : X \to X$ given by $\phi(n, x) = (n, \tilde{\phi}(x))$. By a topological consequence of Parovichenko's theorem (see [29, Theorem 1.2.6]), $X^* = \beta X \setminus X$ is homeomorphic to $\mathbb{N}^*$. Moreover, $\beta\phi : \beta X \to \beta X$ sends X^* into X^*.

To check that $\psi = \beta\phi|X^*$ is irreducible take any clopen $U \subseteq X^*$, which must be of the form $\beta U' \cap X^*$, where

$$U' = \bigcup_{n\in E} \{n\} \times U_n$$

for some infinite $E \subseteq \mathbb{N}$ and nonempty clopen sets $U_n \subsetneq K$ (consider χ_U and the relation of X to βX). By the irreducibility of $\tilde{\phi}$, there are clopen $V_n \subseteq 2^{\mathbb{N}}$ such that $\tilde{\phi}[2^{\mathbb{N}} \setminus U_n] \cap V_n = \emptyset$. So

$$\beta\phi[U] \cap \beta\Big(\bigcup_{n\in E} \{n\} \times V_n\Big) = \emptyset,$$

which completes the proof of the irreducibility of ψ. The proof that ψ is not one-to-one when restricted to any clopen set is similar to the argument from the proof of Example 6.8. ∎

A similar example to the above is constructed in [30, proof of 2.1], but it does not have the property of preserving nowhere dense sets.

7. Open problems and final remarks. This section should not be considered as a full list of urgent open problems concerning the Banach space ℓ_∞/c_0, for example we do not touch problems related to the primariness of

ℓ_∞/c_0 (see [16, 14, 24]), subspaces of ℓ_∞/c_0 (see [5, 25, 45, 26]), ℓ_∞-sums (see [16, 14, 6]) or extensions of operators on ℓ_∞/c_0 (see [7, 2]).

Problem 7.1. *Is it consistent* (*does it follow from* PFA *or* OCA+MA) *that every automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ can be lifted modulo a locally null operator? That is, is every such operator of the form $T = [R] + S$, where $R : \ell_\infty \to \ell_\infty$ and S is locally null?*

This is related to the fact that our ZFC nonliftable operator (see 4.17) is of the above form. A ZFC possibility of somewhere canonizing every isomorphic embedding is excluded by 6.10 or 6.5. As under PFA or OCA+MA canonization along a homeomorphism gives trivialization, we may ask:

Problem 7.2. *Is it consistent* (*does it follow from* PFA *or* OCA+MA) *that every isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is somewhere trivial?*

Problem 7.3. *Is it true in ZFC that for every isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ there is an infinite A, a closed $F \subseteq \mathbb{N}^*$ and a homeomorphism $\psi : F \to A^*$ such that $T(f^*)|F = r(f^* \circ \psi)$ for A-supported f's and some nonzero $r \in \mathbb{R}$?*

The solution of the above problem would solve Problem 7.9.

Problem 7.4. *Is it consistent* (*does it follow from* PFA *or* OCA+MA) *that every automorphism of ℓ_∞/c_0 is somewhere trivial?*

So we ask if the hypothesis in 6.4 of T having no funnels or no fountains is needed under PFA or OCA+MA. The only examples we have of such phenomena are for automorphisms under CH (6.8).

Note that by Plebanek's result 5.7 and by going to a subset of F using 5.2, we may assume that there is an infinite $A \subseteq \mathbb{N}$, a closed F and a continuous $\psi : F \to A^*$ such that $T(f^*)|F = f^* \circ \psi$ for every A-supported f. If we knew that (A^*, F) is not a funnel, i.e., that F is not nowhere dense, we could use Farah's result 6.2 as in the proof of 6.4 to obtain somewhere triviality of T. However, we have not been able to prove a similar reduction for fountains, which could be more useful in the context of applying Farah's result, as then F would be a continuous image of A^* which is a copy of $\mathbb{N}^*$, so the domain of ψ is as required in 6.2.

One strategy for proving that under OCA+MA automorphisms do not have funnels or fountains is to use the result of I. Farah 6.2 directly to prove that the sets F appearing in potential funnels or fountains cannot be nowhere dense. There are several related natural questions which we were unable to solve.

Problem 7.5. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an automorphism* (*an isomorphic embedding*), *$A \subseteq \mathbb{N}$ is infinite, $F \subseteq \mathbb{N}^*$ is closed nowhere dense and $\psi : F \to A^*$ is continuous irreducible such that $T(f^*)|F = f^* \circ \psi$*

for each A-supported $f \in \ell_\infty$. Is it consistent (*under* OCA+MA *or* PFA) *that F cannot be c.c.c. over* Fin*?*

PROBLEM 7.6. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an automorphism, $F \subseteq \mathbb{N}^*$ is closed nowhere dense and $A \subseteq \mathbb{N}$ is infinite such that (F, A^*) is a fountain for T or (A^*, F) is a funnel for T. Is it true or consistent* (*under* OCA+MA *or* PFA) *that F cannot be c.c.c. over* Fin*?*

PROBLEM 7.7. *Suppose that $F \subseteq \mathbb{N}^*$ is nowhere dense closed not c.c.c* (*or even homeomorphic to* $\mathbb{N}^*$) *and has the linear extension property. Is it consistent* (*under* OCA+MA *or* PFA) *that F cannot be c.c.c. over* Fin*?*

A set $F \subseteq \mathbb{N}^*$ has the *linear extension property* if there is a bounded linear operator $T : C(F) \to C(\mathbb{N}^*)$ such that $T(f)|F = f$ for each $f \in C(F)$. The existence of such an operator is a weak version of the existence of a retraction from $\mathbb{N}^*$ onto F. A. Dow [14] developed new methods (which may be quite useful in the above context) proving that PFA implies that cozero sets do not have the linear extension property.

The last couple of problems is related to possible applications of canonizations of embeddings.

PROBLEM 7.8. *Is it consistent that every copy of ℓ_∞/c_0 inside ℓ_∞/c_0 is complemented?*

PROBLEM 7.9. *Is it true or consistent that every copy of ℓ_∞/c_0 inside ℓ_∞/c_0 contains a further copy of ℓ_∞/c_0 which is complemented in the entire space?*

Under CH, examples of uncomplemented copies of ℓ_∞/c_0 inside ℓ_∞/c_0 were constructed in [7]. They can also be obtained under CH from a superspace of ℓ_∞/c_0 obtained in [3] in which ℓ_∞/c_0 is not complemented.

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