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Weighted embedding theorems for radial Besov and Triebel–Lizorkin spaces

by

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Abstract. We study the continuity and compactness of embeddings for radial Besov and Triebel–Lizorkin spaces with weights in the Muckenhoupt class A_∞. The main tool is a discretization in terms of an almost orthogonal wavelet expansion adapted to the radial situation.

1. Introduction. Weighted embedding theorems for smooth function spaces have beeen studied by many authors, mainly because they are a fundamental tool in the variational analysis of some nonlinear partial differential equations, for instance of degenerate or singular elliptic equations. It is therefore natural to study embedding results in the framework of Triebel–Lizorkin and Besov spaces, since these include many of the classical function spaces. In the unweighted case, a fundamental result in this context is the embedding theorem of Jawerth [15] and Franke [10], which generalizes the classical Sobolev embedding theorem.

Weighted Besov and Triebel–Lizorkin spaces have also been studied by many authors under different assumptions on the weights (see e.g. [2, 3, 17, 19, 18, 20, 29]). Embeddings of Besov and Triebel–Lizorkin spaces with Muckenhoupt's $\mathcal{A}_\infty$ weights were studied by Haroske and Skrzypczak [12, 13, 14] and Meyries and Veraar [24] (see also [23] for earlier work by the same authors in the case of power weights).

On the other hand, it is well known, since the pioneering works of Ni [25] and Strauss [33], that many embedding results can be improved when one considers subspaces of radial functions. More precisely, by restricting ourselves to the subspace of radial functions, we can recover, for instance,

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compactness properties of embeddings that are in general non-compact due to the action of some non-compact group of transformations such as the group of translations in $\mathbb{R}^n$ (see, e.g., [22]). Notice that compact embeddings are a fundamental feature for the success of variational methods in PDE. In the case of weighted embedding theorems one can also obtain a wider range of exponents for the admissible power weights in the radial situation (see e.g. [5]).

In the case of unweighted radial subspaces of Besov and Triebel–Lizorkin spaces, Sickel and Skrzypczak [30, 31] and Sickel, Skrzypczak and Vybiral [32] obtained compactness of the related embeddings and an extension of Strauss' radial lemma. Quantitative information in terms of entropy numbers for the embeddings was obtained by Kühn, Leopold, Sickel and Skrzypczak [21]. In those papers, the main tool is an atomic decomposition adapted to the radial situation.

Early results on embeddings for weighted radial Besov and Triebel–Lizorkin spaces can be found in Triebel's book [35, Section 6.5.2], where the weights considered are of the special forms $w_\alpha(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \in \mathbb{R}$, and $w^\beta(x) = e^{|x|^\beta}$ with $|x| \geq 1$ and $0 < \beta \leq 1$ (see also references therein). However, to the authors' knowledge, results on weighted radial Besov and Triebel–Lizorkin spaces for other important classes of weights, such as power weights or, more generally, Muckenhoupt weights, were still missing in the literature. The first two authors recently showed in [6] that the approach used by Meyries and Veraar [23] to obtain embedding theorems with power weights can be improved to obtain a better range of admissible exponents in the radial case.

In this work we consider embedding theorems for radial subspaces of Besov and Triebel–Lizorkin spaces with general $\mathcal{A}_\infty$ weights. It is important to stress that in the latter case the functions considered are radially symmetric, but the weights can be arbitrary. In the Triebel–Lizorkin case, we follow an argument by Meyries and Veraar [24] to derive the embeddings from the Besov case, but this time restrict ourselves not only to radially symmetric functions but also to radially symmetric $\mathcal{A}_\infty$ weights (see the discussion in Section 5). In both cases we obtain improved sufficient conditions for the continuity and compactness of the embeddings as compared with the non-radial case.

For our proof, instead of using the atomic decomposition for radial subspaces of Sickel and Skrzypczak [30], we shall closely follow the approach used by Haroske and Skrzypczak [12, 13] in the non-radial case, which is based on a discretization in terms of wavelet bases. To this end, we need a wavelet decomposition adapted to the radial situation, which we obtain by adapting arguments used by Epperson and Frazier [9] in the unweighted radial case. We remark that this is not a wavelet decomposition in the traditional sense, since the wavelets are localized near certain annuli instead of cubes. Hence, they have the advantage of being better adapted to the radial situation but have no translation structure and, more importantly, since they are not actual bases but rather frames, they do not characterize the (weighted) Besov and Triebel–Lizorkin spaces. In other words, they are useful to obtain sufficient conditions for the continuity and compactness of the embeddings, but cannot be used to prove sharpness of the conditions obtained. Unfortunately, as far we know, there are no known orthogonal wavelet decompositions for radial functions except in dimension three (see, e.g., [28, 4]).

The rest of the paper is as follows. In Section 2 we recall some definitions and known properties of Besov and Triebel–Lizorkin spaces. Section 3 is devoted to the construction of the wavelet bases and the representation of the weighted radial Besov and Triebel–Lizorkin spaces in terms of sequence spaces (Theorems 3.1 and 3.2). In Section 4 we prove our main theorem (Theorem 4.1) on sufficient conditions for the continuity and compactness of the embeddings for weighted radial Besov spaces and use it to analyze some important special examples. Finally, in Section 5 we obtain sufficient conditions for the continuity and compactness of the embeddings for Triebel–Lizorkin spaces with radial $\mathcal{A}_\infty$ weights (Theorem 5.1) and an example in this case.

2. Weighted Besov and Triebel–Lizorkin spaces. First we recall some necessary definitions. Classical references on Besov and Triebel–Lizorkin spaces are [26, 34]. For weighted versions see [3, 29].

Definition 2.1 (Construction of the Littlewood–Paley partition). Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be such that its Fourier transform $\widehat{\varphi}$ satisfies

$$0 \le \widehat{\varphi}(\xi) \le 1, \quad \xi \in \mathbb{R}^n, \quad \widehat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| \ge 3/2. \end{cases} \tag{2.1}$$

Let $\widehat{\varphi}_0 = \widehat{\varphi}$, $\widehat{\varphi}_1(\xi) = \widehat{\varphi}(\xi/2) - \widehat{\varphi}(\xi)$, and

$$\widehat{\varphi}_\mu(\xi) = \widehat{\varphi}_1(2^{-\mu+1}\xi) = \widehat{\varphi}(2^{-\mu}\xi) - \widehat{\varphi}(2^{-\mu+1}\xi), \quad \xi \in \mathbb{R}^n,\ \mu \ge 1.$$

Then

$$0 \le \widehat{\varphi}_\mu(\xi) \le 1, \quad \widehat{\varphi}_\mu(\xi) = 1 \quad \text{if } \tfrac{3}{2} \cdot 2^{\mu-1} \le |\xi| \le 2^\mu,$$
$$\operatorname{supp} \widehat{\varphi}_\mu \subset \{2^{\mu-1} \le |\xi| \le \tfrac{3}{2} \cdot 2^\mu\}.$$

Let Φ be the set of all sequences $(\varphi_\mu)_{\mu \ge 0}$ constructed in the above way from a function φ that satisfies (2.1).

For φ as in the definition and $f \in \mathscr{S}'(\mathbb{R}^n)$ one sets

$$S_\mu f := \varphi_\mu * f = \mathscr{F}^{-1}[\widehat{\varphi}_\mu \widehat{f}],$$

which belongs to $C^\infty(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n)$. Since $\sum_{\mu\geq 0} \widehat{\varphi}_\mu(\xi) = 1$ for all $\xi \in \mathbb{R}^n$, we have $\sum_{\mu\geq 0} S_\mu f = f$ in the sense of distributions.

Given a *weight* w, that is, a non-negative locally integrable function on $\mathbb{R}^n$, and a real number $p \in [1,\infty]$, we denote by $L^p(\mathbb{R}^n, w)$ the weighted Lebesgue space of those measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|^p_{L^p(\mathbb{R}^n,w)} := \int_{\mathbb{R}^n} |f|^p w(x)\,dx < \infty$$

if $1 \leq p < \infty$, and $\|f\|_{L^\infty(\mathbb{R}^n,w)} = \|f\|_{L^\infty(\mathbb{R}^n)}$.

Let us recall that, for $1 < p < \infty$, the *Muckenhoupt class* $\mathcal{A}_p$ is the class of weights w for which the maximal Hardy–Littlewood operator is bounded from $L^p(\mathbb{R}^n, w)$ to itself, and that it can be characterized by the condition

$$\left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{1-p'}\right)^{p-1} \leq C$$

for all balls $B \subseteq \mathbb{R}^n$, where the constant C depends on w but is independent of B. On the other hand, we write $w \in \mathcal{A}_1$ if $Mw(x) \leq Cw(x)$ a.e., and we set $\mathcal{A}_\infty = \bigcup_{p\geq 1} \mathcal{A}_p$. We refer to [11] for a detailed account of these weights.

Given $p, q \in [1,\infty]$, $s \in \mathbb{R}$ and a weight $w \in \mathcal{A}_\infty$, following [2] we can define the weighted Besov and Triebel–Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ in the following way.

Definition 2.2. The (*inhomogeneous*) *Besov space* $B^s_{p,q}(\mathbb{R}^n, w)$ is defined as the space of all $f \in \mathscr{S}'(\mathbb{R}^n)$ for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n,w)} := \Big(\sum_{\mu\geq 0} 2^{q\mu s}\|S_\mu f\|^q_{L^p(\mathbb{R}^n,w)}\Big)^{1/q} < \infty$$

with the usual modifications for $q = \infty$.

Definition 2.3. Assume that $p < \infty$. The (*inhomogeneous*) *Triebel–Lizorkin* space $F^s_{p,q}(\mathbb{R}^n, w)$ is defined as the space of all $f \in \mathscr{S}'(\mathbb{R}^n)$ for which

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n,w)} := \Big\|\Big(\sum_{\mu\geq 0} 2^{q\mu s}|S_\mu f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n,w)} < \infty,$$

with the usual modifications for $q = \infty$.

Remark 2.1. (1) It can be proved that these definitions do not depend on the choice of φ in (2.1) (see e.g. [3]).

(2) The corresponding homogeneous spaces denoted by $\dot{B}^s_{p,q}(\mathbb{R}^n,w)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n,w)$ are defined in a similar way with the sum running over $\mathbb{Z}$ with appropriate modifications of the partition of unity. Observe that $\|f\|_{B^s_{p,q}(\mathbb{R}^n,w)} = 0$ if and only if $\operatorname{supp}\widehat{f} = \{0\}$, i.e., f is a polynomial. For this reason it is common practise to consider instead the quotient spaces $\dot{B}^s_{p,q}(\mathbb{R}^n,w)/\mathcal{P}$ and $\dot{F}^s_{p,q}(\mathbb{R}^n,w)/\mathcal{P}$ where $\mathcal{P}$ is the space of polynomials.

(3) If $w \equiv 1$, we write $B^s_{p,q}(\mathbb{R}^n)$ instead of $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n)$ instead of $F^s_{p,q}(\mathbb{R}^n,w)$.

The orthogonal group $O(n)$ acts on $\mathscr{S}(\mathbb{R}^n)$ by $O(n)\times\mathscr{S}(\mathbb{R}^n)\ni(\sigma,\phi)\mapsto\sigma\phi\in\mathscr{S}(\mathbb{R}^n)$ with $\sigma\phi(x):=\phi(\sigma^{-1}x)$. Then for any $f,\phi\in\mathscr{S}(\mathbb{R}^n)$ and $\sigma\in O(n)$ we have $(\sigma.f,\phi)_{L^2} = (f,\sigma^{-1}\phi)_{L^2}$. We thus define an action of $O(n)$ on $\mathscr{S}'(\mathbb{R}^n)$ by $O(n)\times\mathscr{S}'(\mathbb{R}^n)\ni(\sigma,f)\mapsto\sigma.f\in\mathscr{S}'(\mathbb{R}^n)$ with

$$(\sigma.f,\phi) := (f,\sigma^{-1}\phi) \qquad \text{for any } \phi\in\mathscr{S}(\mathbb{R}^n). \tag{2.2}$$

This motivates our next definition:

Definition 2.4. We say that a tempered distribution $f\in\mathscr{S}'(\mathbb{R}^n)$ is *radial* if $\sigma.f = f$ for any $\sigma\in O(n)$ where $\sigma.f$ is defined by (2.2).

The Besov and Triebel–Lizorkin spaces of radial distributions will be denoted by $RB^s_{p,q}(\mathbb{R}^n,w)$ and $RF^s_{p,q}(\mathbb{R}^n,w)$, respectively. The following embeddings between these spaces are elementary and follow from the corresponding non-radial situation (see [2, Theorem 2.6]).

Theorem 2.1. *Let $w\in A_\infty$.*

(1) *For all $1\le q_1\le q_2\le\infty$ and $s\in\mathbb{R}$ one has*

$$RB^s_{p,q_1}(\mathbb{R}^n,w)\hookrightarrow RB^s_{p,q_2}(\mathbb{R}^n,w), \qquad p\in[1,\infty];$$
$$RF^s_{p,q_1}(\mathbb{R}^n,w)\hookrightarrow RF^s_{p,q_2}(\mathbb{R}^n,w), \qquad p\in[1,\infty].$$

(2) *For all $q_1,q_2\in[1,\infty]$, $s\in\mathbb{R}$ and $\varepsilon>0$ one has*

$$RB^{s+\varepsilon}_{p,q_1}(\mathbb{R}^n,w)\hookrightarrow RB^s_{p,q_2}(\mathbb{R}^n,w), \qquad p\in[1,\infty];$$
$$RF^{s+\varepsilon}_{p,q_1}(\mathbb{R}^n,w)\hookrightarrow RF^s_{p,q_2}(\mathbb{R}^n,w), \qquad p\in[1,\infty].$$

(3) *For all $q\in[1,\infty]$, $s\in\mathbb{R}$ and $p\in[1,\infty)$ one has*

$$RB^s_{p,\min\{p,q\}}(\mathbb{R}^n,w)\hookrightarrow RF^s_{p,q}(\mathbb{R}^n,w)\hookrightarrow RB^s_{p,\max\{p,q\}}(\mathbb{R}^n,w).$$

We now state a weighted version due to [2] of the continuity of the Peetre maximal function originally defined in [27].

Let $a>0$ and $\{\phi_\mu\}_{\mu\ge0}$ be a sequence of functions in $\mathscr{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\widehat{\phi_\mu}\subset\{2^{\mu-a}\le|\xi|\le2^{\mu+a}\},$$
$$|D^\alpha\widehat{\phi_\mu}(\xi)|\le C_n2^{-\mu|\alpha|} \qquad \text{for all } \mu\ge0,\ \alpha\in\mathbb{N}^d,\ \xi\in\mathbb{R}^n.$$

This holds e.g. if $\widehat{\phi_\mu}(\xi) = \widehat{\phi}_1(2^{-\mu}\xi)$. For a given $\lambda > 0$ the *Peetre maximal functions* of $f \in \mathscr{S}'(\mathbb{R}^n)$ are

$$\phi^*_{\mu,\lambda} f(x) = \phi^*_\mu f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\phi_\mu * f(x-y)|}{(1+2^\mu|y|)^\lambda}, \qquad x \in \mathbb{R}^n,\ \mu \ge 0. \tag{2.3}$$

THEOREM 2.2 ([2, Section 5]). *Let* $r_0 = \inf\{r : w \in A_r\}$.

(i) *If* $\lambda > \max\{nr_0/p, n/q\}$ *then*

$$\Big\| \Big(\sum_{\mu\ge 0} [2^{\mu s}\phi^*_\mu f(x)]^q\Big)^{1/q}\Big\|_{L_p(\mathbb{R}^n,w)} \le C\|f\|_{F^s_{p,q}(\mathbb{R}^n,w)} \quad \textit{for all } f \in \mathscr{S}'(\mathbb{R}^n). \tag{2.4}$$

(ii) *If* $\lambda > nr_0/p$ *then*

$$\Big(\sum_{\mu\ge 0} [2^{\mu s}\|\phi^*_\mu f\|_{L_p(\mathbb{R}^n,w)}]^q\Big)^{1/q} \le C\|f\|_{B^s_{p,q}(\mathbb{R}^n,w)} \quad \textit{for all } f \in \mathscr{S}'(\mathbb{R}^n).$$

3. Construction of radial wavelets for weighted Besov and Triebel–Lizorkin spaces. In this section we develop a suitable wavelet decomposition adapted to the weighted radial situation. Our starting point is the construction of radial wavelets of Epperson and Frazier [9].

Let $\Phi, \Psi, \varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ be radial functions such that

$$\operatorname{supp}\widehat{\Phi}, \operatorname{supp}\widehat{\Psi} \subset \{|\xi| \le 1\}, \quad |\widehat{\Phi}(\xi)|, |\widehat{\Psi}(\xi)| \ge c > 0 \quad \text{if } |\xi| \le 5/6,$$
$$\operatorname{supp}\widehat{\varphi}, \widehat{\psi} \subset \{1/4 < |\xi| < 1\}, \quad |\widehat{\varphi}|, |\widehat{\psi}| \ge c > 0 \quad \text{if } 3/10 \le |\xi| \le 5/6,$$

and

$$\overline{\widehat{\Phi}}(\xi)\widehat{\Psi}(\xi) + \sum_{\mu\ge 1} \overline{\widehat{\varphi_\mu}}(\xi)\widehat{\psi_\mu}(\xi) = 1 \quad \text{for } \xi \ne 0.$$

where $\varphi_\mu(x) = 2^{\mu n}\varphi(2^\mu x)$ and $\psi_\mu(x) = 2^{\mu n}\psi(2^\mu x)$. We then define a family $(\varphi_{\mu k})_{\mu\ge 0,\, k\ge 1}$ of functions by

$$\varphi_{\mu k} = \begin{cases} \left(\dfrac{2^{(\mu(n-2)+1)}}{j^n_{\nu,k} J^2_{\nu+1}(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \varphi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}} & \text{for } \mu \ge 1, \\[2ex] \left(\dfrac{2}{j^n_{\nu,k} J^2_{\nu+1}(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \Phi * d\sigma_{2^{-\mu} j_{\nu,k}} & \text{for } \mu = 0, \end{cases}$$

where $d\sigma_t$ denotes the (unnormalized) surface Lebesgue measure on the sphere of radius t in $\mathbb{R}^n$, ω_{n-1} the surface of the unit sphere, and

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots$$

denote the positive zeros of the Bessel function J_ν with $\nu = (n-2)/2$. We define the functions $(\psi_{\mu k})_{\mu\geq 0,\, k\geq 1}$ in a similar way. Then the Epperson–Frazier wavelet expansion for a radial distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ is given by

$$f = \sum_{\mu\geq 0}\sum_{k\geq 1}\langle f, \varphi_{\mu,k}\rangle \psi_{\mu,k}.$$

Epperson and Frazier were able to characterize the membership of f in (unweighted) Besov or Triebel–Lizorkin spaces in terms of the wavelet coefficients $\langle f, \varphi_{\mu,k}\rangle$. Our purpose in this section is to show that analogous results hold for the weighted version of these spaces when the weight belongs to the $\mathcal{A}_\infty$ class.

We consider the annuli $A_{\mu,k}$, $\mu \geq 0$, $k \geq 1$, defined by

$$A_{\mu,k} = \{x \in \mathbb{R}^n : 2^{-\mu} j_{\nu,k-1} \leq |x| \leq 2^{-\mu} j_{\nu,k}\} \quad \text{with } j_{\nu,0} = 0,$$

and denote by $\chi_{\mu,k} := |A_{\mu,k}|^{-1/2}\chi_{A_{\mu,k}}$ its L^2-normalized characteristic function. Given $p, q \in [1,\infty]$, $s \in \mathbb{R}$ and a weight $w \in \mathcal{A}_\infty$ we let $b^s_{p,q}(w)$ and $f^s_{p,q}(w)$ be the spaces of sequences $\lambda := (\lambda_{\mu,k})_{\mu,k}$ of complex numbers such that

$$\|\lambda\|_{b^s_{p,q}(w)} := \Big(\sum_{\mu\geq 0}\Big\|\sum_{k\geq 1} 2^{\mu s}|\lambda_{\mu,k}|\chi_{\mu,k}\Big\|^q_{L_p(\mathbb{R}^n,w)}\Big)^{1/q} < \infty$$

and

$$\|\lambda\|_{f^s_{p,q}(w)} := \Big\|\Big(\sum_{\mu\geq 0}\sum_{k\geq 1}[2^{\mu s}|\lambda_{\mu,k}|\chi_{\mu,k}]^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n,w)} < \infty$$

respectively, with the usual modifications if $q = \infty$.

Our first result is the following:

Theorem 3.1. *Let $p, q \in [1,\infty]$ and $w \in \mathcal{A}_\infty$. Then the operators*

$$S : RF^s_{p,q}(\mathbb{R}^n, w) \ni f \mapsto (\langle f, \varphi_{\mu,k}\rangle)_{\mu,k} \in f^s_{p,q}(w)$$

and

$$T : f^s_{p,q}(w) \ni \lambda \mapsto \sum_{\mu\geq 0}\sum_{k\geq 1}\lambda_{\mu,k}\psi_{\mu,k} \in RF^s_{p,q}(\mathbb{R}^n, w)$$

are bounded, and the composition $T \circ S$ is the identity on $RF^s_{p,q}(\mathbb{R}^n, w)$. In particular, $\|f\|_{RF^s_{p,q}(w)} \simeq \|S(f)\|_{f^s_{p,q}(w)}$.

Remark 3.1. The same type of result holds for homogeneous spaces with the usual modification, namely, with sum over $\mu \in \mathbb{Z}$ and with Φ and Ψ suppressed.

Proof of Theorem 3.1. The case $w \equiv 1$ corresponds to [9, Theorems 2.1 and 2.2]. Since the proof in the general case is a modification of those results, we just sketch it, indicating where changes are needed. These mainly concern the continuity of the Peetre maximal function and of the Hardy–Littlewood maximal function for sequences of functions.

Concerning the continuity of S, as in the proof of [9, Theorem 2.1] we have, for any $\mu \geq 0$ and $\lambda > 0$,

$$\sum_{k\geq 1} (2^{\mu s}|\langle f, \varphi_{\mu,k}\rangle|\chi_{\mu,k}(x))^q \leq C_\lambda 2^{\mu s q}|\varphi_\mu^* f(x)|^q \quad \text{a.e.}$$

where φ_μ^* is the Peetre maximal function as defined in (2.3) for $\lambda > 0$. According to Theorem 2.2 we obtain, taking λ large enough,

$$\|S(f)\|_{f^s_{p,q}(w)} \leq C\Big\|\Big(\sum_{\mu\geq 0} 2^{\mu s q}|\varphi_\mu^* f(x)|^q\Big)^{1/q}\Big\|_{L_p(\mathbb{R}^n,w)} \leq C\|f\|_{RF^s_{p,q}(w)}.$$

For the continuity of T, fix $\lambda \in f^s_{p,q}(w)$ and let $f = \sum_{\mu\geq 0}\sum_{k\geq 1}\lambda_{\mu,k}\psi_{\mu,k}$. Then for any $\eta \in (0,1]$ such that $p/\eta, q/\eta > 1$ we see as in [9] that

$$\begin{aligned}\|f\|_{RF^s_{p,q}(w)} &= \Big\|\Big(\sum_{\mu\geq 0}(2^{\mu s}|\varphi_\mu * f|)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n,w)} \\ &\leq C\Big\|\Big(\sum_{\mu\geq 0}\Big(M\Big(\sum_{k\geq 1}(2^{\mu s}|\lambda_{\mu,k}|\chi_{\mu,k})^\eta\Big)\Big)^{q/\eta}\Big)^{\eta/q}\Big\|^{1/\eta}_{L^{p/\eta}(\mathbb{R}^n,w)},\end{aligned}$$

where M is the Hardy–Littlewood maximal function. According to [1, Theorem 3.1] or [16, Theorem 1], the vector-valued maximal function between weighted spaces

$$M : L^\alpha(\ell_\beta, w) \ni (f_\mu)_\mu \mapsto (Mf_\mu)_\mu \in L^\alpha(\ell_\beta, w)$$

is continuous when the weight w belongs to the $\mathcal{A}_\alpha$ class with $1 < \alpha, \beta < \infty$. Here $L^\alpha(\ell^\beta)$ denotes the space of sequences of locally integrable functions $(f_\mu)_\mu$ such that

$$\|(f_\mu)_\mu\|^\alpha_{L^\alpha(\ell_\beta,w)} := \int_{\mathbb{R}^n}\Big(\sum_\mu |f_\mu|^\beta\Big)^{\alpha/\beta} w\,dx < \infty.$$

Since $w \in \mathcal{A}_p$, taking η small enough to have $p/\eta > r_0 := \inf\{r : w \in \mathcal{A}_r\}$ we find that $w \in \mathcal{A}_{p/\eta}$. It follows that $M : L^{p/\eta}(\ell_{q/\eta}, w) \to L^{p/\eta}(\ell_{q/\eta}, w)$ is continuous. We thus obtain

$$\|f\|_{RF^s_{p,q}(w)} \leq C\Big\|\Big(\sum_{\mu\geq 0}\Big(\sum_{k\geq 1}(2^{\mu s}|\lambda_{\mu,k}|\chi_{\mu,k})^\eta\Big)^{q/\eta}\Big)^{\eta/q}\Big\|^{1/\eta}_{L_{p/\eta}(\mathbb{R}^n,w)}.$$

Since for given μ the annuli $A_{\mu,k}$, $k \geq 1$, are essentially disjoint, we obtain

$$\|f\|_{RF^s_{p,q}(w)} \leq C\Big\|\Big(\sum_{\mu\geq 0}\sum_{k\geq 1}(2^{\mu s}|\lambda_{\mu,k}|\chi_{\mu,k})^q\Big)^{1/q}\Big\|_{L_p(\mathbb{R}^n,w)} = C\|\lambda\|_{f^s_{p,q}(w)}. \blacksquare$$

The analogous statement for weighted Besov spaces reads as follows:

THEOREM 3.2. *Let $p, q \in [1,\infty]$ and $w \in \mathcal{A}_\infty$. Then the operators*

$$S : RB^s_{p,q}(\mathbb{R}^n, w) \ni f \mapsto (\langle f, \varphi_{\mu,k}\rangle)_{\mu,k} \in b^s_{p,q}(w)$$

and

$$T : b^s_{p,q}(w) \ni \lambda \mapsto \sum_{\mu\ge 0}\sum_{k\ge 1} \lambda_{\mu,k}\psi_{\mu,k} \in RB^s_{p,q}(\mathbb{R}^n, w)$$

are bounded, and the composition $T \circ S$ is the identity on $RB^s_{p,q}(\mathbb{R}^n, w)$. In particular, $\|f\|_{RB^s_{p,q}(w)} \simeq \|S(f)\|_{b^s_{p,q}(w)}$. The same result also holds for the homogeneous version of these spaces.

Proof. The unweighted case $w = 1$ corresponds to [9, Theorems 5.1 and 5.2].

For the continuity of S, as in the proof of the previous theorem, we obtain

$$\sum_{k\ge 1} 2^{\mu s}|(\langle f, \varphi_{\mu,k}\rangle)|\chi_{\mu,k}(x) \le C2^{\mu s}|\varphi^*_\mu f(x)| \quad \text{a.e.}$$

where φ^*_μ is the Peetre maximal function for a given $\lambda > 0$. Taking λ large enough and using Theorem 2.2 we get

$$\|S(f)\|_{b^s_{p,q}(w)} \le C\Big(\sum_{\mu\ge 0} 2^{\mu s q}\|\varphi^*_\mu f\|^q_{L^p(\mathbb{R}^n,w)}\Big)^{1/q} \le C\|f\|_{RB^s_{p,q}(w)}.$$

For the continuity of T, fix $\lambda \in b^s_{p,q}(w)$ and let $f = \sum_{\mu\ge 0}\sum_{k\ge 1}\lambda_{\mu,k}\psi_{\mu,k}$. Then, arguing much as in the Triebel–Lizorkin case we see that for any $\mu \ge 0$,

$$\begin{aligned}\|\varphi_\mu * f\|_{L^p(\mathbb{R}^n,w)} &\le C\sum_{\nu=\mu-1}^{\mu+1}\Big\|\Big(M\Big(\sum_{k\ge 1}|\lambda_{\nu,k}|^\eta\chi^\eta_{\nu,k}\Big)\Big)^{1/\eta}\Big\|_{L^p(\mathbb{R}^n,w)}\\ &= C\sum_{\nu=\mu-1}^{\mu+1}\Big\|M\Big(\sum_{k\ge 1}|\lambda_{\nu,k}|^\eta\chi^\eta_{\nu,k}\Big)\Big\|^{1/\eta}_{L^{p/\eta}(\mathbb{R}^n,w)}.\end{aligned}$$

Since $w \in \mathcal{A}_\infty$, setting as before $r_0 := \inf\{r : w \in \mathcal{A}_r\}$ and taking η small enough to have $r_0 < p/\eta$ we find that $w \in \mathcal{A}_{p/\eta}$, so the maximal operator $M : L^{p/\eta}(\mathbb{R}^n, w) \to L^{p/\eta}(\mathbb{R}^n, w)$ is continuous. Then

$$\begin{aligned}\Big\|M\Big(\sum_{k\ge 1}|\lambda_{\nu,k}|^\eta\chi^\eta_{\nu,k}\Big)\Big\|^{1/\eta}_{L^{p/\eta}(\mathbb{R}^n,w)} &\le C\Big\|\sum_{k\ge 1}|\lambda_{\nu,k}|^\eta\chi^\eta_{\nu,k}\Big\|^{1/\eta}_{L^{p/\eta}(\mathbb{R}^n,w)}\\ &= C\Big\|\sum_{k\ge 1}|\lambda_{\nu,k}|\chi_{\nu,k}\Big\|_{L^p(\mathbb{R}^n,w)},\end{aligned}$$

where we have used the fact that for given ν, the annuli $A_{\nu,k}$ are essentially disjoint. We deduce that

$$\begin{aligned}\|f\|^q_{RB^s_{p,q}(w)} &= \sum_{\mu\ge 0} 2^{\mu sq}\|\varphi_\mu * f\|^q_{L^p(\mathbb{R}^n,w)}\\ &\le C\sum_{\mu\ge 0} 2^{\mu sq}\sum_{\nu=\mu-1}^{\mu+1}\Big\|\sum_{k\ge 1}|\lambda_{\nu,k}|\chi_{\nu,k}\Big\|^q_{L^p(\mathbb{R}^n,w)}\\ &\le C\sum_{\mu\ge 0} 2^{\mu sq}\Big\|\sum_{k\ge 1}|\lambda_{\nu,k}|\chi_{\nu,k}\Big\|^q_{L^p(\mathbb{R}^n,w)} = C\|\lambda\|^q_{b^s_{p,q}(w)}. \ \blacksquare\end{aligned}$$

4. Continuous and compact embeddings of weighted radial Besov spaces. In this section we use Theorem 3.2 to obtain sufficient conditions for the continuity and compactness of embeddings of weighted radial Besov spaces, and apply these results to some relevant examples.

THEOREM 4.1. *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$ and w_1, w_2 be $\mathcal{A}_\infty$-weights. There is a continuous embedding $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$ provided that*

$$\left\{2^{-\mu(s_1-s_2)}\left\|\left\{\frac{w^2_{\mu k}}{w^1_{\mu k}}\right\}_k\right\|_{\ell_{p^*}}\right\}_\mu \in \ell_{q^*} \tag{4.1}$$

where

$$w^1_{\mu k} = \|\chi_{\mu k}\|_{L^{p_1}(\mathbb{R}^n,w_1)}, \qquad w^2_{\mu k} = \|\chi_{\mu k}\|_{L^{p_2}(\mathbb{R}^n,w_2)},$$

and

$$\frac{1}{p^*} := \left(\frac{1}{p_2}-\frac{1}{p_1}\right)_+, \qquad \frac{1}{q^*} := \left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+.$$

The embedding is compact provided that (4.1) *holds and moreover*

$$\lim_{\mu\to\infty} 2^{\mu(s_2-s_1)}\left\|\left\{\frac{w^2_{\mu k}}{w^1_{\mu k}}\right\}_k\right\|_{\ell_{p^*}} = 0 \qquad \text{if } q^* = \infty,$$

$$\lim_{|k|\to\infty}\frac{w^1_{\mu k}}{w^2_{\mu k}} = \infty \qquad \text{for all } \mu \ge 0 \text{ if } p^* = \infty.$$

Proof. By Theorem 3.2 it suffices to study the embedding of the corresponding sequence spaces

$$b^{s_1}_{p_1,q_1}(w_1) \to b^{s_2}_{p_2,q_2}(w_2),$$

that is, using the notation of [19, Section 3],

$$\ell_{q_1}(2^{\mu s_1}\ell_{p_1}(w_1)) \to \ell_{q_2}(2^{\mu s_2}\ell_{p_2}(w_2)).$$

Notice that the continuity of this embedding is equivalent to the continuity of the embedding

$$\ell_{q_1}(2^{\mu(s_1-s_2)}\ell_{p_1}(w_1/w_2)) \to \ell_{q_2}(\ell_{p_2}).$$

Indeed,

$$\|\lambda\|_{\ell_{q_2}(2^{\mu s_2}\ell_{p_2}(w_2))} = \|\tilde{\lambda}\|_{\ell_{q_2}(\ell_{p_2})} \quad \text{with} \quad \tilde{\lambda}_{\mu k} = \lambda_{\mu k} w^2_{\mu k} 2^{\mu s_2}.$$

We can rewrite this embedding using the notation of [19] as

$$\ell_{q_1}(\beta_\mu \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \quad \text{with} \quad \beta_\mu = 2^{\mu(s_1-s_2)},\ w = (w_{\mu k})_{\mu k},\ w_{\mu k} = \frac{w^1_{\mu k}}{w^2_{\mu k}}.$$

According to [19, Theorem 3.1], this embedding is continuous if and only if

$$(\beta_\mu^{-1}\|(w^{-1}_{\mu k})_k\|_{\ell_{p^*}})_\mu \in \ell_{q^*},$$

which proves that $RB^{s_1}_{p_1,q_1}(w_1) \subseteq RB^{s_2}_{p_2,q_2}(w_2)$ if (4.1) holds.

This embedding is compact if moreover

$$\lim_{\mu\to\infty} \beta_\mu^{-1}\|(w^{-1}_{\mu k})_k\|_{\ell_{p^*}} = 0 \qquad \text{if } q^* = \infty,$$

$$\lim_{|k|\to\infty} w_{\mu k} = \infty \qquad \text{for all } \mu \geq 0 \text{ if } p^* = \infty. \ \blacksquare$$

As an application we now consider the case $w_1(x) = |x|^{\gamma_1}$, $w_2(x) = |x|^{\gamma_2}$ with $\gamma_1, \gamma_2 > -n$, so that w_1, w_2 are $\mathcal{A}_\infty$-weights. In order to simplify the statement of the following examples we introduce

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}. \tag{4.2}$$

Example 4.1. *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$ and $\gamma_1, \gamma_2 > -n$. There is a continuous embedding $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, |x|^{\gamma_1}) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n, |x|^{\gamma_2})$ provided that*

$$\begin{cases} \frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} \geq (n-1)\big(\frac{1}{p_2} - \frac{1}{p_1}\big) & \text{if } p^* = \infty, \\ \frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} > \frac{n}{p^*} & \text{if } p^* < \infty, \end{cases} \qquad \begin{cases} \delta \geq \frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} & \text{if } q^* = \infty, \\ \delta > \frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} & \text{if } q^* < \infty, \end{cases}$$

where δ is as in (4.2). *This embedding is compact provided that moreover*

$$\frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} > (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) \quad \text{if } p^* = \infty, \qquad \delta > \frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} \quad \text{if } q^* = \infty.$$

Proof. Since $|x| \sim k2^{-\mu}$ for $x \in A_{\mu k}$, we see that for $i = 1, 2$,

$$w^i_{\mu k} = \|\chi_{\mu k}\|_{L^{p_i}(|x|^{\gamma_i})} \sim |A_{\mu k}|^{-1/2}((k2^{-\mu})^{\gamma_i}|A_{\mu k}|)^{1/p_i}.$$

Moreover $|A_{\mu k}| \sim k^{n-1}2^{-\mu n}$. Hence

$$\frac{w^2_{\mu k}}{w^1_{\mu k}} \sim 2^{\mu(\frac{n+\gamma_1}{p_1} - \frac{n+\gamma_2}{p_2})} k^{\frac{\gamma_2}{p_2} - \frac{\gamma_1}{p_1} + (n-1)(\frac{1}{p_2} - \frac{1}{p_1})}.$$

Then if e.g. $p^*, q^* < \infty$ then (4.1) reads

$$\sum_k k^{p^*(\frac{\gamma_2}{p_2} - \frac{\gamma_1}{p_1} + (n-1)(\frac{1}{p_2} - \frac{1}{p_1}))} < \infty \quad \text{and} \quad \sum_\mu 2^{\mu q^*(\frac{\gamma_1}{p_1} - \frac{\gamma_2}{p_2} - \delta)} < \infty,$$

i.e.

$$p^*\left(\frac{\gamma_2}{p_2}-\frac{\gamma_1}{p_1}+(n-1)\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)<-1, \qquad q^*\left(\frac{\gamma_1}{p_1}-\frac{\gamma_2}{p_2}-\delta\right)<0.$$

Recalling the definition of p^*, q^* gives the statement.

Concerning compactness, we have

$$2^{\mu(s_2-s_1)}\left\{\sum_k\left(\frac{w^2_{\mu k}}{w^1_{\mu k}}\right)^{p^*}\right\}^{1/p^*} \sim 2^{\mu(\frac{\gamma_1}{p_1}-\frac{\gamma_2}{p_2}-\delta)}\left\{\sum_k k^{p^*(\frac{\gamma_2}{p_2}-\frac{\gamma_1}{p_1}+(n-1)(\frac{1}{p_2}-\frac{1}{p_1}))}\right\}^{1/p^*}$$

where the sum on the right hand side is finite. ∎

Remark 4.1. (1) It is immediate from the above example that one has an improvement with respect to the non-radial case [12, Proposition 2.8]. Indeed, if $p^* = \infty$ (that is, $p_1 \le p_2$) we can have $\gamma_1/p_1 - \gamma_2/p_2 < 0$, in which case δ may be negative as well, while in the non-radial case both values must be non-negative.

(2) An alternative proof of the continuity part of the above example can be found in [6, Theorem 12]. For the corresponding non-radial case see [23, Theorem 1.1].

Our next examples concern weights of purely polynomial growth. Let

$$w_{\alpha,\beta} = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1 \end{cases} \qquad \text{with } \alpha, \beta > -n.$$

Example 4.2. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 < \infty$, $0 < p_2 \le \infty$ and $0 < q_1, q_2 \le \infty$. Then there is a continuous embedding $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ provided*

$$\begin{cases} \frac{\beta}{p_1} \ge (n-1)\left(\frac{1}{p_2}-\frac{1}{p_1}\right) & \text{if } p^* = \infty, \\ \frac{\beta}{p_1} > \frac{n}{p^*} & \text{if } p^* < \infty, \end{cases}$$

and one of the following conditions is satisfied:

$$\begin{cases} \delta \ge \max\left(\frac{\alpha}{p_1}, (n-1)\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right) & \text{if } q^* = \infty,\ p^* = \infty, \\ \delta > \max\left(\frac{\alpha}{p_1}, (n-1)\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right) & \text{if } q^* < \infty,\ p^* = \infty, \\ \delta \ge \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right) & \text{if } q^* = \infty,\ p^* < \infty,\ \frac{n}{p^*} \ne \frac{\alpha}{p_1}, \\ \delta > \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right) & \text{otherwise.} \end{cases}$$

Moreover the embedding $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ is compact provided that

$$\begin{cases} \frac{\beta}{p_1} > (n-1)\left(\frac{1}{p_2}-\frac{1}{p_1}\right) & \text{if } p^* = \infty, \\ \frac{\beta}{p_1} > \frac{n}{p^*} & \text{if } p^* < \infty, \end{cases}$$

and

$$\begin{cases} \delta > \max\left(\frac{\alpha}{p_1}, (n-1)(\frac{1}{p_2} - \frac{1}{p_1})\right) & \text{if } p^* = \infty, \\ \delta > \max\left(\frac{\alpha}{p_1}, \frac{n}{p^*}\right) & \text{if } p^* < \infty. \end{cases}$$

Proof. Consider first the Besov case. We have

$$\frac{w^2_{\mu k}}{w^1_{\mu k}} \sim k^{(n-1)(\frac{1}{p_2}-\frac{1}{p_1})}\, 2^{-\mu n(\frac{1}{p_2}-\frac{1}{p_1})} \times \begin{cases} k^{-\alpha/p_1}\, 2^{\mu\alpha/p_1} & \text{if } k \le 2^\mu, \\ k^{-\beta/p_1}\, 2^{\mu\beta/p_1} & \text{if } k > 2^\mu. \end{cases}$$

Then if e.g. $p^* = \infty, q^* < \infty$, (4.1) reads

$$\sum_\mu 2^{\mu q^*[(s_2-s_1)-n(\frac{1}{p_2}-\frac{1}{p_1})+\frac{\alpha}{p_1}]} \Big(\sup_{k\le 2^\mu} k^{(n-1)(\frac{1}{p_2}-\frac{1}{p_1})-\frac{\alpha}{p_1}}\Big)^{q^*} < \infty$$

and

$$\sum_\mu 2^{\mu q^*[(s_2-s_1)-n(\frac{1}{p_2}-\frac{1}{p_1})+\frac{\beta}{p_1}]} \Big(\sup_{k> 2^\mu} k^{(n-1)(\frac{1}{p_2}-\frac{1}{p_1})-\frac{\beta}{p_1}}\Big)^{q^*} < \infty,$$

which gives the statement. As for compactness, we need

$$\lim_{|k|\to\infty} k^{(n-1)(\frac{1}{p_2}-\frac{1}{p_1})-\frac{\beta}{p_1}} = 0.$$

The remaining cases are analogous. ∎

The generalization to the following two-weighted embeddings is straightforward:

EXAMPLE 4.3. *Let* $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 < \infty$, $0 < p_2 \le \infty$ *and* $0 < q_1, q_2 \le \infty$. *Then there is a continuous embedding* $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})$ *provided*

$$\begin{cases} \frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} \ge (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) & \text{if } p^* = \infty, \\ \frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > \frac{n}{p^*} & \text{if } p^* < \infty, \end{cases}$$

and one of the following conditions is satisfied:

$$\begin{cases} \delta \ge \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right) & \text{if } q^* = \infty,\ p^* = \infty, \\ \delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right), & \text{if } q^* < \infty,\ p^* = \infty, \\ \delta \ge \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right) & \text{if } q^* = \infty,\ p^* < \infty,\ \frac{n}{p^*} \ne \frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \\ \delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right) & \text{otherwise}, \end{cases}$$

where δ *is as in* (4.2).

Moreover the embedding $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha_1,\beta_1}) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_{\alpha_2,\beta_2})$, $p_1, p_2 \in (0,\infty)$, *is compact provided that*

$$\begin{cases} \frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) & \text{if } p^* = \infty, \\ \frac{\beta_1}{p_1} - \frac{\beta_2}{p_2} > \frac{n}{p^*} & \text{if } p^* < \infty, \end{cases}$$

and one of the following conditions is satisfied:

$$\begin{cases} \delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, (n-1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right) & \textit{if } p^* = \infty, \\ \delta > \max\left(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}, \frac{n}{p^*}\right) & \textit{if } p^* < \infty. \end{cases}$$

5. Continuous and compact embeddings of weighted radial Triebel–Lizorkin spaces. Our next result concers embeddings for Triebel–Lizorkin spaces with radial $\mathcal{A}_\infty$ weights. We will follow the approach in [24], which is based on a Gagliardo–Nirenberg type inequality and two lemmas on products of Muckenhoupt weights that we recall for the reader's convenience.

PROPOSITION 5.1 ([23, Proposition 5.1]). *Let $q, q_0, q_1 \in [1,\infty]$ and $\theta \in (0,1)$. Let $p, p_0, p_1 \in (1,\infty)$ and $-\infty < s_0 < s_1 < \infty$ satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad s = (1-\theta)s_0 + \theta s_1.$$

Let further $w, w_0, w_1 \in \mathcal{A}_\infty$ be such that $w = w_0^{(1-\theta)p/p_0} w_1^{\theta p/p_1}$. Then there exists a constant C such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ one has

$$\|f\|_{F^s_{p,q}(w)} \le C \|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(w_0)} \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}(w_1)}.$$

LEMMA 5.1 ([24, Lemma 3.1]). *Let $1 < p < \infty$ and $w_1, w_2 \in \mathcal{A}_p$. Then there is $\eta_0 > 0$ such that for all $\varepsilon, \delta \in [0, \eta_0)$ one has $w_1^{-\varepsilon} w_2^{1+\delta} \in \mathcal{A}_p$.*

LEMMA 5.2 ([24, Lemma 3.2]). *Let $w_1, w_2 \in \mathcal{A}_\infty$. Then there are $\eta_0 > 0$ and a constant $C > 0$ such that for all $\varepsilon, \delta \in (0, \eta_0)$ and all cubes $Q \subset \mathbb{R}^n$ we have*

$$\int_Q w_1^{-\varepsilon} w_2^{1+\delta}\, dx \le C|Q|^{\varepsilon-\delta} \Big(\int_Q w_1\, dx\Big)^{-\varepsilon} \Big(\int_Q w_2\, dx\Big)^{1+\delta}.$$

Since our functions are supported on annuli instead of cubes, we will need another auxiliary lemma on the behavior of products of radial Muckenhoupt weights over these sets. To this end, we first recall the following characterization of radial $\mathcal{A}_p$ weights given by Duoandikoetxea et al. [8]:

LEMMA 5.3 ([8, Theorem 3.2]). *Let $w_0 : (0,\infty) \to [0,\infty]$ and $w_n(x) = w_0(|x|)$ for $x \in \mathbb{R}^n$. Then w_n is in $\mathcal{A}_p(\mathbb{R}^n)$ if and only if $\delta_n w_0$ is in $\mathcal{A}_p(0,\infty)$, where $\delta_n w_0(t) = w_0(t^{1/n})$.*

LEMMA 5.4. *Let $w_1, w_2 \in \mathcal{A}_\infty$, $w_1(x) = \tilde{w}_1(|x|)$, $w_2(x) = \tilde{w}_2(|x|)$ for all $x \in \mathbb{R}^n$. Then there exists $\eta_0 > 0$ such that for all $\varepsilon \in (0, \eta_0)$ and any annulus $D_{ab} = \{x \in \mathbb{R}^n : a \le |x| \le b\}$ with $a, b \in \mathbb{R}_+$,*

$$\int_{D_{ab}} w_1^{-\varepsilon} w_2^{1+\varepsilon}\, dx \le C \Big(\int_{D_{ab}} w_1\, dx\Big)^{-\varepsilon} \Big(\int_{D_{ab}} w_2\, dx\Big)^{1+\varepsilon}.$$

Proof. Fix $p > 1$ such that $w_1, w_2 \in \mathcal{A}_p$, let η_0 be as in Lemma 5.2 and $\varepsilon \in (0, \eta_0)$. Taking polar coordinates we obtain

$$\begin{aligned}\int_{D_{ab}} w_1^{-\varepsilon} w_2^{1+\varepsilon}\, dx &= \omega_{n-1} \int_a^b \tilde{w}_1^{-\varepsilon}\, \tilde{w}_2^{1+\varepsilon} r^{n-1}\, dr \\ &= \omega_{n-1} \int_{a^n}^{b^n} (\delta_n \tilde{w}_1)^{-\varepsilon}\, (\delta_n \tilde{w}_2)^{1+\varepsilon}\, \frac{dr}{n} \\ &\le C\omega_{n-1} \Big(\int_a^b \delta_n \tilde{w}_1\, dr \Big)^{-\varepsilon} \Big(\int_a^b \delta_n \tilde{w}_2\, dr \Big)^{1+\varepsilon},\end{aligned}$$

where the last bound follows from Lemma 5.2, and we have used the fact that $\delta_n \tilde{w}_1, \delta_n \tilde{w}_2 \in \mathcal{A}_p(0, \infty)$ by Lemma 5.3. Changing variables again, we obtain the desired bound. ∎

Now we are ready to prove our result for Triebel–Lizorkin spaces:

THEOREM 5.1. *Let $1 < p_1 \le p_2 < \infty$, $s_1 > s_2$, $q_1, q_2 \in [1, \infty]$ and let w_1, w_2 be radially symmetric $\mathcal{A}_\infty$-weights. There is a continuous embedding $RF^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \to RF^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$ provided that*

$$\sup_{\mu,k} \Big\{ 2^{-\mu(s_1-s_2)} \frac{w^2_{\mu k}}{w^1_{\mu k}} \Big\} < \infty \tag{5.1}$$

where

$$w^1_{\mu k} = \|\chi_{\mu k}\|_{L^{p_1}(\mathbb{R}^n, w_1)}, \qquad w^2_{\mu k} = \|\chi_{\mu k}\|_{L^{p_2}(\mathbb{R}^n, w_2)}.$$

The embedding is compact provided that

$$\lim_{|k|\to\infty} \frac{w^1_{\mu k}}{w^2_{\mu k}} = \infty \qquad \textit{for all } \mu \ge 0,$$

$$\lim_{\mu\to\infty} 2^{-\mu(s_1-s_2)} \sup_k \frac{w^2_{\mu k}}{w^1_{\mu k}} = 0.$$

Proof. The proof is in two steps: proving the continuity of the embedding and then the compactness.

For the first part, we follow closely the approach in [24], which we outline for the reader's convenience. Note that it suffices to prove the continuity of the embedding $RF^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow RF^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$ with $q_2 \le \min\{p_1, p_2\}$, since then the general case follows by using Theorem 2.1(1).

Since $p_1 \le p_2 < \infty$, there exists $\theta_0 \in [0, 1)$ such that $1/p_2 - (1 - \theta_0/p_1) = 0$ (in fact, $\theta_0 = 1 - p_1/p_2$). For $\theta \in (\theta_0, 1)$, let

$$\varepsilon = \frac{\frac{1-\theta}{p_1}}{\frac{1}{p_2} - \frac{1-\theta}{p_1}} > 0,$$

which clearly tends to zero as $\theta \to 1$, and let v, r, t be defined by the identities

$$v = w_1^{-\varepsilon} w_2^{1+\varepsilon}, \qquad \frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{r}, \qquad s_2 = (1-\theta)s_1 + \theta t.$$

Then one can check that $w_2 = w_1^{(1-\theta)p_2/p_1} v^{p_2\theta/r}$, $r \in [p_2, \infty)$ and $t < s_2 < s_1$. Hence, by Proposition 5.1,

$$\|f\|_{RF^{s_2}_{p_2,q_2}(\mathbb{R}^n,w_2)} \le C\|f\|^{1-\theta}_{RF^{s_1}_{p_1,q_1}(\mathbb{R}^n,w_1)} \|f\|^{\theta}_{RF^{t}_{r,r}(\mathbb{R}^n,v)}. \tag{5.2}$$

Now, since $RB^s_{p,p} = RF^s_{p,p}$ and $r \ge p_2$, by Theorem 4.1, the embedding

$$\|f\|_{RF^{t}_{r,r}(\mathbb{R}^n,v)} \le C\|f\|_{RF^{s_2}_{p_2,p_2}(\mathbb{R}^n,w_2)} \tag{5.3}$$

holds provided that

$$\sup_{k,\mu} 2^{-\mu(s_2-t)} \Big(\int_{A_{\mu k}} v\Big)^{1/r} \Big(\int_{A_{\mu k}} w_2\Big)^{-1/p_2} < \infty. \tag{5.4}$$

But, by Lemma 5.4,

$$\int_{A_{\mu k}} v \le C\Big(\int_{A_{\mu k}} w_1\Big)^{-\varepsilon} \Big(\int_{A_{\mu k}} w_2\Big)^{1+\varepsilon},$$

whence, by inserting this bound into (5.4) and noting that $s_2 - t = (s_1 - s_2)\frac{1-\theta}{\theta}$, $\frac{\varepsilon}{r} = \frac{1}{p_1}\frac{1-\theta}{\theta}$, and $\frac{1+\varepsilon}{r} = \frac{1}{\theta p_2}$, the desired embedding (5.3) finally follows from condition (5.1).

Inserting (5.3) into (5.2) gives

$$\|f\|_{RF^{s_2}_{p_2,q_2}(\mathbb{R}^n,w_2)} \le C\|f\|^{1-\theta}_{RF^{s_1}_{p_1,q_1}(\mathbb{R}^n,w_1)} \|f\|^{\theta}_{RF^{s_2}_{p_2,p_2}(\mathbb{R}^n,w_2)}.$$

Since $q_2 \le p_2$ by the above assumption, we may replace $RF^{s_2}_{p_2,p_2}$ on the right hand side by $RF^{s_2}_{p_2,q_2}$, and divide by $\|f\|^{\theta}_{RF^{s_2}_{p_2,q_2}(\mathbb{R}^n,w_2)}$ to obtain

$$\|f\|_{RF^{s_2}_{p_2,q_2}(\mathbb{R}^n,w_2)} \le C\|f\|_{RF^{s_1}_{p_1,q_1}(\mathbb{R}^n,w_1)}.$$

Notice that, in principle, this bound holds in $RF^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \cap RF^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$, but it can be extended by density to $RF^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1)$ (see [24, proof of Theorem 1.2]).

It remains to prove that the embedding is compact. To this end, let $(f_k)_{k\in\mathbb{N}}$ be such that $\|f_k\|_{RF^{s_1}_{p_1,q_1}(w_1)} \le C$. Then, by the embedding we have already proved, $(f_k)_{k\in\mathbb{N}}$ is also bounded in $RF^{s_2}_{p_2,q_2}(w_2) = RB^{s_2}_{p_2,q_2}(w_2)$ with $q_2 \le \min\{p_1, p_2\}$, and by Theorem 2.1, in $RB^{s_2}_{p_2,p_2}(w_2)$. Since, under our hypotheses, the embedding $RB^{s_2}_{p_2,p_2}(w_2) \hookrightarrow RB^{t}_{r,r}(v)$ is compact by Theorem 4.1, we infer that $f_k \to f$ in $B^t_{r,r} = F^t_{r,r}$. Then, in view of (5.2),

$$\|f_k - f\|_{RF^{s_2}_{p_2,q_2}(w_2)} \le \|f_k - f\|^{1-\theta}_{RF^{s_1}_{p_1,q_1}(w_1)} \|f_k - f\|^{\theta}_{RF^{t}_{r,r}(v)} \to 0,$$

which proves our statement. ■

Examples for the same weights considered in the Besov case can be obtained in an analogous manner. We leave the proofs to the reader.

An interesting special case of the inhomogeneous Triebel–Lizorkin spaces is given by the Bessel potential spaces. In [5] the first two authors proved the following result (with a more elementary argument).

EXAMPLE 5.1 ([5, Theorems 6.4 and 7.2]). *Let* $1 < p < \infty$, $0 < s \le n/p$, $p \le q < \infty$ *and denote*

$$p_c^* := \frac{p(n+c)}{n-sp}.$$

Then we have a continuous embedding

$$H^{s,p}_{\mathrm{rad}}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c dx)$$

provided that

$$\begin{aligned} -sp \le c &\le \frac{(n-1)(q-p)}{p} \quad \text{and} \quad p \le q \le p_c^* \quad \text{if } sp < n,\\ -n < c &\le \frac{(n-1)(q-p)}{p} \quad \text{and} \quad p \le q < \infty \quad \text{if } sp = n. \end{aligned}$$

Morevover, the embedding is compact provided that

$$\begin{aligned} -sp < c &< \frac{(n-1)(q-p)}{p} \quad \text{and} \quad p \le q < p_c^* \quad \text{if } sp < n,\\ -n < c &< \frac{(n-1)(q-p)}{p} \quad \text{and} \quad p \le q < \infty \quad \text{if } sp = n. \end{aligned}$$

Proof. To see this result as a special case of the embeddings in Theorem 5.1, notice that $H^{s,p}_{\mathrm{rad}} = RF^s_{p,2}$ and $L^q_{\mathrm{rad}}(|x|^c) = RF^0_{q,2}(|x|^c)$ provided $|x|^c \in A_q$ (that is, $-n < c < n(q-1)$). Hence, this case corresponds to $w_1 = 1$, $w_2 = |x|^c$, $p_1 = p$, $q_1 = 2$, $p_2 = q$, and $q_2 = 2$. Using the computations in Example 4.1, we then have

$$\frac{w^2_{\mu k}}{w^1_{\mu k}} = k^{(n-1)(1/q-1/p)+c/q} 2^{-\mu\{n(1/q-1/p)+c/q\}}.$$

The result follows from this expression and the fact that $p_c^* \ge p$ iff $c \ge -sp$. ∎

A different proof of the previous example for $p = 2$ was also given in [7] by the first two authors jointly with R. Durán, where that result was used to analyze the existence of radial solutions of a weighted elliptic system with hamiltonian structure in $\mathbb{R}^n$.

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