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On countable cofinality and decomposition of definable thin orderings

by

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Abstract. We prove that in some cases definable thin sets (including chains) of Borel partial orderings are necessarily countably cofinal. This includes the following cases: analytic thin sets, **ROD** thin sets in the Solovay model, and $\mathbf{\Sigma}^1_2$ thin sets under the assumption that $\omega_1^{\mathbf{L}[x]} < \omega_1$ for all reals x. We also prove that definable thin wellorderings admit partitions into definable chains in the Solovay model.

1. Introduction. Studies of maximal chains in partially ordered sets go back to as early as Hausdorff [7], where this issue appeared in connection with Du Bois Reymond's investigations of orders of infinity. Using the axiom of choice, Hausdorff proved the existence of maximal chains (which he called *pantachies*) in any partial ordering. On the other hand, Hausdorff clearly understood the difference between such a pure existence proof and an actual construction of a maximal chain—see e.g. [7, p. 110] or comments in [3]—which we would now describe as the existence of *definable* maximal chains.

The following theorem presents three cases in which all linear suborders (that is, chains), and even *thin* suborders (those containing no perfect pairwise incompatible subsets) of Borel PQOs are necessarily countably cofinal.

THEOREM 1. *If* $\preccurlyeq$ *is a Borel PQO on a* (*Borel*) *set* $D = \mathtt{dom}(\preccurlyeq) \subseteq \omega^\omega$, $X^* \subseteq D$, *and* $\preccurlyeq\restriction X^*$ *is a thin quasi-ordering then* $\langle X^*; \preccurlyeq\rangle$ *is countably cofinal in each of the following cases:*

(A) X^* *is a* $\mathbf{\Sigma}^1_1$ *set,*
(B) X^* *is a* **ROD** (real-ordinal definable) *set in the Solovay model,*
(C) X^* *is a* $\mathbf{\Sigma}^1_2$ *set, and* $\omega_1^{\mathbf{L}[r]} < \omega_1$ *for every real* r.

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Therefore, if, in addition, it is known that every countable set in $\langle D; \leq \rangle$ *has an upper bound, then in all three cases* X^* *has an upper bound.*

Moreover, if X^* *is a* $\mathbf{\Sigma}^1_1$ *set* (= Case (A)) *then there are no* $\preccurlyeq$*-chains* (*not necessarily cofinal chains*) *in* X^* *of uncountable cofinality.*

The additional condition in the theorem, of bounding of countable subsets, applies to many partial orders of interest, e.g., the eventual domination order or the *rate of growth order* on $\mathbb{R}^\omega$ (see a review in [12]). Needless to say, chains, gaps, and similar structures related to these or similar orderings have been subject of extensive studies, of which we mention [1, 2, 15, 19] where the definability aspect is considered.

Part (A) of Theorem 1 (together with the "moreover" claim at the end of the theorem) is proved in Section 3 by reduction to a result (Theorem 3) extending a theorem of [5] to the case of $\mathbf{\Sigma}^1_1$ suborders of a background Borel PQO as in (A). Part (B) is already known from [13] in the subcase of linear **ROD** suborders; we present an essentially simplified proof in Section 6. Part (C) is proved in Section 7 by reference to part (B) via a sequence of absoluteness arguments.

It is a challenging question to figure out whether claims (B) and (C) of Theorem 1 remain true in stronger forms similar to the "moreover" form of claim (A). The answer is pretty simple in the affirmative provided we consider only accordingly definable (but not necessarily cofinal) ω_1-sequences in the given set X^*—that is, **ROD** in claim (B) and $\mathbf{\Sigma}^1_2$ in claim (C).

The next theorem (our second main result) extends a classical decomposition theorem of [5] to the case of definable sets in the Solovay model.

Theorem 2 (in the Solovay model). *If* $\preccurlyeq$ *is an OD PQO on* ω^ω *then any OD* $\preccurlyeq$*-thin set* $X^* \subseteq \omega^\omega$ *is covered by the union of all OD* $\preccurlyeq$*-chains. The same is true for any definability class OD*(x)*, where* x *is a real.*

Note that conversely, *in the Solovay model, any set covered by the union of all OD* $\preccurlyeq$*-chains* $C \subseteq \omega^\omega$ *is thin*, as otherwise a perfect set X which witnesses the non-thinness would admit an OD wellordering, which is known to be impossible in the Solovay model.

The proof of Theorem 2 as given in Section 12 has a certain resemblance to the proof of Theorem 5.1 in [5], as regards its general combinatorial structure. Yet some modifications are necessary since OD sets in the Solovay model only partially resemble sets in Δ^1_1 and Σ^1_1. In particular we have to establish some properties of the OD forcing in Sections 8–11 rather different from the properties of the Gandy–Harrington forcing in [5], and also we prove a tricky compression lemma (Lemma 19).

2. Notation. We will mostly consider *non-strict* orderings. We list basic notation and terminology:

- PQO, *partial quasi-order*: reflexive ($x \le x$) and transitive in the domain;
- LQO, *linear quasi-order*: PQO and $x \le y \vee y \le x$ in the domain;
- LO, *linear order*: LQO and $x \le y \wedge y \le x \Rightarrow x = y$;
- *associated equivalence relation*: $x \approx y$ iff $x \le y \wedge y \le x$;
- *associated strict order*: $x < y$ iff $x \le y \wedge y \not\le x$.
- *LR* (*left-right*)*, resp. RL* (*right-left*)*, order preserving map*: any map $f : \langle X; \le\rangle \to \langle X'; \le'\rangle$ such that $x \le y \Rightarrow f(x) \le' f(y)$, resp. $x \le y \Leftarrow f(x) \le' f(y)$, for all $x, y \in \mathtt{dom}\, f$;
- *suborder*: a restriction of the given PQO to a subset of its domain;
- $<_{\mathtt{lex}}$, $\leqslant_{\mathtt{lex}}$: the lexicographical LOs on sets of the form 2^α, $\alpha \in \mathtt{Ord}$, resp. strict and non-strict.

Let $\langle P; \le\rangle$ be a background PQO. A subset $Q \subseteq P$ is:

- *cofinal in* P iff $\forall p \in P\ \exists q \in Q\ (p \le q)$;
- *countably cofinal* (*in itself*) iff there is a countable $Q' \subseteq Q$ cofinal in Q;
- *a chain* iff it consists of pairwise $\le$-comparable elements, i.e., LQO;
- *an antichain in* P iff it consists of pairwise $\le$-incomparable elements;
- *a thin set* iff it contains no perfect $\le$-antichains.

If E is an equivalence relation then let

$$[x]_{\mathsf{E}} = \{y \in \mathtt{dom}\ \mathsf{E} : x\ \mathsf{E}\ y\} \qquad (\text{the } \mathsf{E}\text{-}\textit{class} \text{ of an element } x \in \mathtt{dom}\ \mathsf{E}),$$

$$[X]_{\mathsf{E}} = \bigcup_{x\in X} [x]_{\mathsf{E}} \qquad (\text{the } \mathsf{E}\text{-}\textit{saturation} \text{ of a set } X \subseteq \mathtt{dom}\ \mathsf{E}).$$

3. Analytic thin subsets. We prove Theorem 1(A) (including the "moreover" claim) by reference to the following background result:

THEOREM 3 (see Section 13). *Let $\preccurlyeq$ be a Δ^1_1 PQO on ω^ω, $\approx$ be the associated equivalence relation, and $X^* \subseteq \omega^\omega$ be a Σ^1_1 $\preccurlyeq$-thin set. Then*

(I) *there is an ordinal $\alpha < \omega_1^{\mathrm{CK}}$ and a Δ^1_1 LR order preserving map $F : \langle \omega^\omega; \preccurlyeq\rangle \to \langle 2^\alpha; \leqslant_{\mathtt{lex}}\rangle$ satisfying the following additional requirement: if $x, y \in X^*$ then $x \not\approx y$ implies $F(x) \ne F(y)$;*

(II) *X^* is covered by the* (*countable*) *union of all Δ^1_1 $\preccurlyeq$-chains $C \subseteq \omega^\omega$.*

LR order preserving maps F satisfying the extra requirement of non-glued $\approx$-classes as in (I) were called *linearization maps* in [9].

Any map F as in (I) sends any two $\preccurlyeq$-incomparable reals $x, y \in \omega^\omega$ onto a $<_{\mathtt{lex}}$-comparable pair of $F(x), F(y)$, that is, either $F(x) <_{\mathtt{lex}} F(y)$ or $F(y) <_{\mathtt{lex}} F(x)$. On the other hand, if the background set X^* is already a $\preccurlyeq$-chain then F has to be RL order preserving too, that is, $x \preccurlyeq y$ iff $F(x) \leqslant_{\mathtt{lex}} F(y)$ for all $x, y \in X^*$.

Proof of claim (A) of Theorem 1 modulo Theorem 3. First of all, assume that the given Borel order $\preccurlyeq$ is in fact lightface Δ^1_1, and the given set X^* is Σ^1_1. The case of $\Delta^1_1(p)$ and $\Sigma^1_1(p)$ with any fixed real parameter p is accordingly reducible to a corresponding version of Theorem 3.

Let, by Theorem 3(II), $X^* \subseteq \bigcup_n C_n$, where each C_n is a Δ^1_1 $\preccurlyeq$-chain, and let F and α be given by Theorem 3(I). To check that X^* is countably cofinal, it suffices to show that so is every set $X_n = X^* \cap C_n$. But X_n is a chain, so if it is *not* countably cofinal then there is a strictly $\prec$-increasing sequence $\{x_\alpha\}_{\alpha<\omega_1}$ of elements $x_\alpha \in X_n$. Then $\{F(x_\alpha)\}_{\alpha<\omega_1}$ is a strictly $<_{\mathtt{lex}}$-increasing sequence in 2^α, which is impossible.

Finally, if there is a $\preccurlyeq$-chain in X^* of uncountable cofinality then a similar argument leads to such a chain in $\langle 2^\alpha; \leqslant_{\mathtt{lex}}\rangle$, with the same contradiction. ∎

4. Remarks and corollaries. Claim (I) of Theorem 3 can be strengthened as follows:

(I′) *if there is no continuous* 1-1 *LR order preserving map* $F : \langle 2^\omega; \leq_0\rangle \to \langle X^*; \preccurlyeq\rangle$ *such that* $a \not\mathsf{E}_0\, b$ *implies that* $F(a)$ *and* $F(b)$ *are* $\preccurlyeq$*-incomparable, then there is an ordinal* $\alpha < \omega_1^{\mathrm{CK}}$ *and a* Δ^1_1 *LR order preserving map* $F : \langle \omega^\omega; \preccurlyeq\rangle \to \langle 2^\alpha; \leqslant_{\mathtt{lex}}\rangle$ *satisfying the following additional requirement*: *if* $x, y \in X^*$ *then* $x \not\approx y$ *implies* $F(x) \neq F(y)$.

Here $\leq_0$ is the PQO on 2^ω defined by: $x \leq_0 y$ iff $x \mathrel{\mathsf{E}_0} y$ and either $x = y$ or $x(k) < y(k)$, where k is the largest number with $x(k) \neq y(k)$ ([1]). The "if" premise in (I′) is an immediate consequence of the $\preccurlyeq$-thinness of X^* as in (I), and hence (I′) really strengthens (I) of Theorem 3.

Claim (I′) is an extension of [9, Theorem 3]; the latter corresponds to the case of Δ^1_1 sets X^*.

In the category of chains (rather than thin sets), the case of Σ^1_1 sets X^* in Theorem 1(A) is reducible to the case of Δ^1_1 sets simply because any Σ^1_1 chain X can be covered by a Δ^1_1 chain Y. We find such a set Y by means of the following two-step procedure ([2]). The set C of all elements that are $\preccurlyeq$-comparable with every element $x \in X$ is Π^1_1, and $X \subseteq C$ (as X is a chain). By the separation theorem, there is a Δ^1_1 set B such that $X \subseteq B \subseteq C$. Now, the set U of all elements in B that are comparable with every element in B is Π^1_1, and $X \subseteq B$. Once again, by separation, there is a Δ^1_1 set Y such that $X \subseteq Y \subseteq U$. By construction, U and Y are chains, as required.

Recall in passing the following well-known earlier result, originally due to H. Friedman, as mentioned in [6].

([1]) $<_0$ orders each E_0-class similarly to the (positive and negative) integers, except for the class $[\omega \times \{0\}]_{\mathsf{E}_0}$ ordered as ω and the class $[\omega \times \{1\}]_{\mathsf{E}_0}$ ordered as the inverse of ω.

([2]) For a different argument, based on a reflection principle, see [5, Corollary 1.5].

Corollary 4 (of Theorem 1(A)). *Every Borel LQO $\le$ is countably cofinal, and moreover there are no strictly increasing ω_1-sequences.* ■

The next immediate corollary says that maximal chains cannot be analytic if they are not countably cofinal.

Corollary 5. *If $\preccurlyeq$ is a Borel PQO, and every countable set $D \subseteq$ $\mathtt{dom}(\preccurlyeq)$ has a strict upper bound, then there are no $\mathbf{\Sigma}^1_1$ maximal $\preccurlyeq$-chains.* ■

Corollary 6 (Harrington and Shelah [6, 16]). *If $\preccurlyeq$ is a $\mathbf{\Pi}^1_1$ LQO on a Borel set then there are no strictly increasing ω_1-chains in $\preccurlyeq$.*

Proof. This result was first obtained by a direct and rather complicated argument. But fortunately there is a reduction to the Borel case.

Indeed, let $x \prec y$ iff $y \not\preccurlyeq x$, so in fact $R_0 = \prec$ is just the strict LQO associated with $\preccurlyeq$. As $R_0 \subseteq \preccurlyeq$, by separation there is a Borel set B_0 with $R_0 \subseteq B_0 \subseteq \preccurlyeq$. Let B'_0 be the relation of B_0-incomparability, and let R_1 be the PQO-hull of $B_0 \cup B'_0$. Thus R_1 is a LQO and $R_0 \subseteq B_0 \subseteq R_1 \subseteq \preccurlyeq$.

Once again, let B_1 be a Borel set such that $R_1 \subseteq B_1 \subseteq \preccurlyeq$. Define sets B'_1 and R_2 as above. And so on.

Finally, after ω steps, the union $R = \bigcup_n B_n = \bigcup_n R_n$ is a Borel LQO and $\prec \subseteq R \subseteq \preccurlyeq$. Any strictly $\preccurlyeq$-increasing chain is strictly R-increasing as well. It remains to apply Corollary 4. ■

5. Near-counterexamples for chains. The following examples show that, even in the particular case of chains instead of thin orderings, Theorem 1(A) is not true any more for different extensions of the domain of $\mathbf{\Sigma}^1_1$ suborders of Borel partial quasi-orders, such as 1) $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ linear quasi-orders (not necessarily suborders of Borel orderings), or 2) $\mathbf{\Delta}^1_2$ and $\mathbf{\Pi}^1_1$ suborders of Borel orderings. In each of these classes, a counterexample of cofinality ω_1 will be defined.

Example 1 ($\mathbf{\Sigma}^1_1$ LQO). Consider a recursive coding of sets of rationals by reals. Let Q_x be the set coded by a real x. Let X_α be the set of all reals x such that the maximal well-ordered initial segment of Q_x has the order type α. We define $x \le y$ iff $\exists \alpha\ \exists \beta\ (x \in X_\alpha \wedge y \in X_\beta \wedge \alpha \le \beta)$. Then $\le$ is a Σ^1_1 LQO on ω^ω of cofinality ω_1. Note that the associated strict order, $x < y$ iff $x \le y$ but not $y \le x$, is then more complicated than just $\mathbf{\Sigma}^1_1$, so this example does not contradict Corollary 6.

Example 2 ($\mathbf{\Pi}^1_1$ LQO). Let $D \subseteq \omega^\omega$ be the Π^1_1 set of codes of (countable) ordinals. Then the relation "$x \le y$ iff $x, y \in D \wedge |x| \le |y|$" is a Π^1_1 LQO of cofinality ω_1. Note that $\le$ is defined on a non-Borel Π^1_1 set D, and there is no $\mathbf{\Pi}^1_1$ LQO of cofinality ω_1 but defined on a Borel set—by exactly the same argument as in Remark 6.

Example 3 ($\mathbf{\Pi}^1_1$ LO). To sharpen Example 2, define $x \le y$ iff $x, y \in D$ and $\big(|x| < |y| \vee (|x| = |y| \wedge x <_{\mathtt{lex}} y)\big)$; this is a $\varPi^1_1$ LO of cofinality ω_1.

Example 4 ($\mathbf{\Delta}^1_2$ suborders). Let $\le$ be the eventual domination order on ω^ω. Assuming the axiom of constructibility $\mathbf{V} = \mathbf{L}$, one can define a strictly $\le$-increasing $\varDelta^1_2$ ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω.

Example 5 ($\mathbf{\Pi}^1_1$ suborders). Define a PQO $\le$ on $(\omega \smallsetminus \{0\})^\omega$ so that $x \le y$ iff either $x = y$ or $\lim_{n\to\infty} y(n)/x(n) = \infty$; the "or" option defines the associated strict order $<$. Assuming $\mathbf{V} = \mathbf{L}$, define a strictly increasing $\varDelta^1_2$ ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω. By the $\varPi^1_1$ uniformization theorem, there is a $\varPi^1_1$ set $\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha<\omega_1} \subseteq \omega^\omega \times 2^\omega$. Let $z_\alpha(n) = 3^{x_\alpha(n)} \cdot 2^{y_\alpha(n)}$ for all n. Then the ω_1-sequence $\{z_\alpha\}_{\alpha<\omega_1}$ is $\varPi^1_1$ and strictly increasing: indeed, factors of the form $2^{y_\alpha(n)}$ are equal to 1 or 2 whenever $\alpha \in 2^\omega$.

6. Definable thin suborders in the Solovay model

Proof of Theorem 1(B). Arguing in the Solovay model (a model of **ZFC** defined in [17], in which all **ROD** sets of reals are Lebesgue measurable), we assume that $\preccurlyeq$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, $X^* \subseteq D$ is a **ROD** (real-ordinal definable) set, and the set X^* is $\preccurlyeq$-thin.

Let $\rho < \omega_1$ be such that $\preccurlyeq$ is a relation in the Borel class $\mathbf{\Sigma}^0_\rho$.

We will prove that the restricted ordering $\langle X^*; \preccurlyeq\rangle$ *is countably cofinal*, i.e., contains a countable cofinal subset (not necessarily a chain).

It is known that, in the Solovay model, any **ROD** set in ω^ω is a union of a **ROD** ω_1-sequence of analytic sets. Thus there is a $\subseteq$-increasing **ROD** sequence $\{X_\alpha\}_{\alpha<\omega_1}$ of $\mathbf{\Sigma}^1_1$ sets X_α such that $X^* = \bigcup_{\alpha<\omega_1} X_\alpha$. Let $r \in \omega^\omega$ be a real parameter such that in fact the sequence $\{X_\alpha\}_{\alpha<\omega_1}$ is $\mathrm{OD}(r)$.

As the sets X_α are countably $\preccurlyeq$-cofinal by claim (A) of Theorem 1, it suffices to prove that one of the sets X_α is cofinal in X^*.

Suppose otherwise. Then the sets $D_\alpha = \{z \in D : \exists x \in X_\alpha\ (z \preccurlyeq x)\}$ contain $\aleph_1$ different sets and form an $\mathrm{OD}(r)$ sequence. We claim that every set D_α belongs to the same class $\mathbf{\Sigma}^0_\rho$ as the given Borel order $\preccurlyeq$. Indeed, let $\{x_n : n \in \omega\}$ be any countable cofinal set in X_α. Then the set $D_\alpha = \{z \in D : \exists n\ (z \preccurlyeq x_n)\}$ is $\mathbf{\Sigma}^0_\rho$ for obvious reasons, and hence the Borel class $\mathbf{\Sigma}^0_\rho$ contains $\aleph_1$ pairwise different sets in $\mathrm{OD}(r)$ for one and the same $r \in \omega^\omega$. But this contradicts a result of Stern [18]. ∎

7. $\mathbf{\Sigma}^1_2$ thin suborders of Borel PQOs

Proof of Theorem 1(C). Assume that $\omega_1^{\mathbf{L}[r]} < \omega_1$ for every real r, $\preccurlyeq$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, and $X^* \subseteq D$ is a $\preccurlyeq$-thin $\mathbf{\Sigma}^1_2$ set.

We will prove that the ordering $\langle X^*; \preccurlyeq\rangle$ *is countably cofinal.*

Pick a real r such that X^* is $\Sigma^1_2(r)$ and $\preccurlyeq$ is $\Delta^1_1(r)$. To prepare for an absoluteness argument, fix canonical formulas,

$$\varphi(\cdot,\cdot) \text{ of type } \Sigma^1_2, \quad \sigma(\cdot,\cdot,\cdot) \text{ of type } \Sigma^1_1, \quad \pi(\cdot,\cdot,\cdot) \text{ of type } \Pi^1_1,$$

which define X^* and $\preccurlyeq$ in the set universe $\mathbf{V}$, so that it is true in $\mathbf{V}$ that

$$x \preccurlyeq y \ \Leftrightarrow\ \sigma(r,x,y) \ \Leftrightarrow\ \pi(r,x,y) \quad \text{and} \quad x \in X^* \ \Leftrightarrow\ \varphi(r,x)$$

for all $x, y \in \omega^\omega$. We let $X_\varphi = \{x \in \omega^\omega : \varphi(r,x)\}$ and

$$x \leq_{\sigma\pi} y \ \Leftrightarrow\ \sigma(r,x,y) \ \Leftrightarrow\ \pi(r,x,y),$$

so that $X_\varphi = X^*$ and $\leq_{\sigma\pi}$ is $\preccurlyeq$ in $\mathbf{V}$, but X_φ and $\leq_{\sigma\pi}$ can be defined in any transitive universe containing r and containing all ordinals (to preserve the equivalence of the formulas σ and π).

Let $\mathbf{WO}$ be the canonical Π^1_1 set of codes of (countable) ordinals, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w.

Let $X_\varphi = \bigcup_{\alpha<\omega_1} X_\varphi(\alpha)$ be a usual representation of X_φ as an increasing union of $\mathbf{\Sigma}^1_1$ sets. Thus to define $X_\varphi(\alpha)$ fix a $\Pi^1_1(r)$ set $P \subseteq (\omega^\omega)^2$ with $X^* = \{x : \exists y\ P(x,y)\}$, fix a canonical $\Pi^1_1(r)$ norm $f : P \to \omega_1$, and let

$$P_\alpha = \{\langle x,y\rangle : f(x,y) < \alpha\} \quad \text{and} \quad X_\varphi(\alpha) = \{x : \exists y\ (\langle x,y\rangle \in P_\alpha)\}.$$

Under our assumptions, the ordinal $\Omega = \omega_1$ is inaccessible in $\mathbf{L}[r]$. Let $\mathscr{P} = \mathrm{Coll}({<}\Omega,\omega) \in \mathbf{L}[r]$ be the corresponding Levy collapse forcing. Consider a $\mathscr{P}$-generic extension $\mathbf{V}[G]$ of the universe. Then $\mathbf{L}[r][G]$ is a Solovay-model generic extension of $\mathbf{L}[r]$. The plan is to compare the models $\mathbf{V}$ and $\mathbf{L}[r][G]$. Note that $\mathbf{L}[r]$ is their common part, $\mathbf{V}[G]$ is their common extension, and the three models have the same cardinal $\omega_1^{\mathbf{V}} = \omega_1^{\mathbf{L}[r][G]} = \omega_1^{\mathbf{V}[G]} = \Omega > \omega_1^{\mathbf{L}[r]}$.

Lemma 7. *It is true both in $\mathbf{V}[G]$ and $\mathbf{L}[r][G]$ that if $\alpha < \Omega$ then the set $X_\varphi(\alpha)$ is $\leq_{\sigma\pi}$-thin* ([3]).

Proof. The thinness of $X_\varphi(\alpha)$ is a Π^1_3 statement with parameters r and any real which codes α. This makes the step $\mathbf{V}[G] \to \mathbf{L}[r][G]$ trivial by Shoenfield absoluteness, and allows us to concentrate on $\mathbf{V}[G]$.

Suppose towards a contradiction that there is a perfect tree $T \in \mathbf{V}[G]$ with $T \subseteq \omega^{<\omega}$ such that the perfect set $[T] = \{x \in \omega^\omega : \forall n\,(x{\restriction}n \in T)\}$ satisfies

(1) $[T] \subseteq X_\varphi(\alpha)$ and $[T]$ is a $\leq_{\sigma\pi}$-antichain

in $\mathbf{V}[G]$. There exist an ordinal $\gamma < \Omega$ and a $\mathbf{Coll}(\omega,\gamma)$-generic map $F \in \mathbf{V}[G]$ such that already $T \in \mathbf{V}[F]$, so that $T = t[F]$, where $t \in \mathbf{V}$ with $t \subseteq \mathbf{Coll}(\omega,\gamma) \times \omega^{<\omega}$ is a $\mathbf{Coll}(\omega,\gamma)$-name.

([3]) The absolutenes of the thinness of the whole set $X_\varphi = \bigcup_{\alpha<\Omega} X_\varphi(\alpha)$ is not asserted!

Note that (1) is still true in $\mathbf{V}[F] \subseteq \mathbf{V}[G]$ by Shoenfield absoluteness, and moreover (1) is true in $\mathbf{L}[z][F]$, where a real $z \in \omega^\omega \cap \mathbf{V}$ codes all of α, γ, r, t. Therefore there is a condition $s \subset F$ (a finite string of ordinals $\xi < \gamma$) which $\mathbf{Coll}(\omega, \gamma)$-forces (1) (with T replaced by the name t) over $\mathbf{L}[z]$.

By the assumptions of Theorem 1(C), there is a map $F' \in \mathbf{V}$, $\mathbf{Coll}(\omega, \gamma)$-generic over $\mathbf{L}[z]$ and satisfying $s \subset F'$. Then the tree $T' = t[F']$ belongs to $\mathbf{L}[z][F'] \subseteq \mathbf{V}$ and satisfies (1) in $\mathbf{L}[z][F']$, hence in $\mathbf{V}$ as well by Shoenfield absoluteness. But this contradicts the choice of $X^* = X_\varphi$. ■

We continue the proof of Theorem 1(C). It follows from the lemma that all orderings $\langle X_\varphi(\alpha); \leq_{\sigma\pi}\rangle$, $\alpha < \Omega$, are countably cofinal in $\mathbf{L}[r][G]$ by Theorem 1(A). However, $\mathbf{L}[r][G]$ is a Solovay-model type extension of $\mathbf{L}[r]$. Therefore (see the argument in Section 6) it is true in $\mathbf{L}[r][G]$ that the whole ordering $\langle X_\varphi; \leq_{\sigma\pi}\rangle$ is countably cofinal, hence there is an ordinal $\alpha < \Omega = \omega_1^{\mathbf{L}[r][G]}$ such that the sentence

(2) the subset $X_\varphi(\alpha)$ is $\leq_{\sigma\pi}$-cofinal in the whole set X_φ

is true in $\mathbf{L}[r][G]$. However, (2) can be expressed by a Π^1_2 formula with r and an arbitrary code $w \in \mathbf{WO} \cap \mathbf{L}[r][G]$ such that $|w| = \alpha$ as the only parameters. It follows, by Shoenfield absoluteness, that (2) is true in $\mathbf{V}[G]$ as well.

Then by exactly the same absoluteness argument, (2) is true in $\mathbf{V}$, too. Thus it is true in $\mathbf{V}$ that $X_\varphi(\alpha)$, a $\mathbf{\Sigma}^1_1$ set, is cofinal in the whole set $X^* = X_\varphi$. But $X_\varphi(\alpha)$ itself is countably cofinal by Theorem 1(A). ■

8. The Solovay model and OD forcing. Here we begin the proof of Theorem 2. We emulate the proof in [5, Theorem 5.1], changing the Gandy–Harrington forcing $\mathbf{P}$ to the OD forcing $\mathbb{P}$. There is only a partial similarity between the two forcing notions, so we will both enjoy some simplifications and suffer from some complications.

We start with a brief review of the Solovay model. Let Ω be an ordinal. Let Ω-SM be the following hypothesis:

Ω-SM: $\Omega = \omega_1$, Ω is strongly inaccessible in $\mathbf{L}$, the constructible universe, and the whole universe $\mathbf{V}$ is a generic extension of $\mathbf{L}$ via the Levy collapse forcing $\mathbf{Coll}(\omega, {<}\Omega)$, as in [17].

Assuming Ω-SM, let $\mathbb{P}$ be the set of all *non-empty* OD sets $Y \subseteq \omega^\omega$. We consider $\mathbb{P}$ as a forcing notion (smaller sets are stronger). A set $D \subseteq \mathbb{P}$ is *dense* iff for every $Y \in \mathbb{P}$ there exists $Z \in D$ with $Z \subseteq Y$, and *open dense* iff in addition $Y \in D \Rightarrow X \in D$ holds whenever the sets $Y \subseteq X$ belong to $\mathbb{P}$.

A set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic iff 1) if $X, Y \in G$ then there is a set $Z \in G$ with $Z \subseteq X \cap Y$, and 2) if $D \subseteq \mathbb{P}$ is OD and dense then $G \cap D \neq \emptyset$.

Given an OD equivalence relation E on ω^ω, a *reduced product* forcing notion $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ consists of all sets of the form $X \times Y$, where $X, Y \in \mathbb{P}$ and $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \emptyset$. For instance $X \times X$ belongs to $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ whenever $X \in \mathbb{P}$. The notions of sets dense and open dense in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$, and of $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic sets, are similar to the case of $\mathbb{P}$.

This version of genericity can be viewed as genericity over OD. We will see below that sets generic in this sense do exist under Ω-SM.

A condition $X \times Y$ in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ is *saturated* iff $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$. To prove the next lemma let $X' = X \cap [Y]_{\mathsf{E}}$ and $Y' = Y \cap [X]_{\mathsf{E}}$.

Lemma 8. *If $X \times Y$ is a condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ then there is a stronger saturated subcondition $X' \times Y'$ in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$.* ■

Proposition 9 ([8, Lemmas 14, 16]). *Assume Ω-SM. If a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic then the intersection $\bigcap G = \{x[G]\}$ consists of a single real $x[G]$, called $\mathbb{P}$-generic—its name will be $\dot{\boldsymbol{x}}$.*

Given an OD equivalence relation E on ω^ω, if $G \subseteq \mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ is $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic then the intersection $\bigcap G = \{\langle x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]\rangle\}$ consists of a single pair of reals $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$, called an $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic pair—their names will be $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$; either of $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$ is separately $\mathbb{P}$-generic. ■

As the set $\mathbb{P}$ is definitely uncountable, the existence of $\mathbb{P}$-generic sets does not immediately follow from Ω-SM by a cardinality argument. Yet fortunately $\mathbb{P}$ is *locally countable*, in a sense.

Definition 10 (assuming Ω-SM). A set $X \in \mathrm{OD}$ is *OD-1st-countable* if the set $\mathscr{P}_{\mathsf{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD}$ of all OD subsets of X is at most countable. (But we do not require $\mathscr{P}_{\mathsf{OD}}(X)$ to be necessarily OD-countable.)

For instance, assuming Ω-SM, the set $X = \omega^\omega \cap \mathrm{OD} = \omega^\omega \cap \mathbf{L}$ of all OD reals is OD-1st-countable. Indeed $\mathscr{P}_{\mathsf{OD}}(X) = \mathscr{P}(X) \cap \mathbf{L}$, and hence $\mathscr{P}_{\mathsf{OD}}(X)$ admits an OD bijection onto the ordinal $\omega_2^{\mathbf{L}} < \omega_1 = \Omega$.

Lemma 11 (assuming Ω-SM). *If a set $X \in$ OD is OD-1st-countable then the set $\mathscr{P}_{\mathsf{OD}}(X)$ is OD-1st-countable.*

Proof. There is an ordinal $\lambda < \omega_1 = \Omega$ and an OD bijection $b : \lambda \to \mathscr{P}_{\mathsf{OD}}(X)$. Any OD set $Y \subseteq \lambda$ belongs to $\mathbf{L}$, hence the OD power set $\mathscr{P}_{\mathsf{OD}}(\lambda) = \mathscr{P}(\lambda) \cap \mathbf{L}$ belongs to $\mathbf{L}$ and $\mathtt{card}(\mathscr{P}_{\mathsf{OD}}(\lambda)) \le \lambda^+ < \Omega$ in $\mathbf{L}$. We conclude that $\mathscr{P}_{\mathsf{OD}}(\lambda)$ is countable. It follows that $\mathscr{P}_{\mathsf{OD}}(\mathscr{P}_{\mathsf{OD}}(X))$ is countable, as required. ■

Lemma 12 (assuming Ω-SM). *If $\lambda < \Omega$ then the set* Coh_λ *of all elements $f \in \lambda^\omega$ that are $\mathbf{Coll}(\omega, \lambda)$-generic over $\mathbf{L}$ is OD-1st-countable.*

Proof. If $Y \subseteq \mathrm{Coh}_\lambda$ is OD and $x \in Y$ then "$\check{x} \in \check{Y}$" is $\mathbf{Coll}(\omega, \lambda)$-forced over $\mathbf{L}$. It follows that there is a set $S \subseteq \lambda^{<\omega} = \mathbf{Coll}(\omega, \lambda)$, $S \in \mathbf{L}$, such that $Y = \mathrm{Coh}_\lambda \cap \bigcup_{t \in S} \mathscr{N}_t$, where $\mathscr{N}_t = \{x \in \lambda^{<\omega} : t \subset x\}$, is a Baire interval in $\lambda^{<\omega}$. But the collection of all such sets S belongs to $\mathbf{L}$ and has cardinality λ^+ in $\mathbf{L}$, hence is countable under Ω-SM. ∎

Let $\mathbb{P}^*$ be the set of all OD-1st-countable sets $X \in \mathbb{P}$. We also define

$$\mathbb{P}^* \times_{\mathsf{E}} \mathbb{P}^* = \{X \times Y \in \mathbb{P} \times_{\mathsf{E}} \mathbb{P} : X, Y \in \mathbb{P}^*\}.$$

Lemma 13 (assuming Ω-SM).

(i) *The set $\mathbb{P}^*$ is dense in $\mathbb{P}$, that is, if $X \in \mathbb{P}$ then there is a condition $Y \in \mathbb{P}^*$ such that $Y \subseteq X$.*

(ii) *If E is an OD equivalence relation on ω^ω then the set $\mathbb{P}^* \times_{\mathsf{E}} \mathbb{P}^*$ is dense in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ and any $X \times Y$ in $\mathbb{P}^* \times_{\mathsf{E}} \mathbb{P}^*$ is OD-1st-countable.*

Proof. (i) Let $x \in X \in \mathbb{P}$. It follows from Ω-SM that there is an ordinal $\lambda < \omega_1 = \Omega$, an element $f \in \mathrm{Coh}_\lambda$, and an OD map $H : \lambda^\omega \to \omega^\omega$, such that $x = H(f)$. The set $P = \{f' \in \mathrm{Coh}_\lambda : H(f') \in X\}$ is then OD and nonempty (contains f), and hence so is its image $Y = \{H(f') : f' \in P\} \subseteq X$ (contains x). Finally, $Y \in \mathbb{P}^*$ by Lemma 12.

(ii) Let $X \times Y$ be a condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$. By Lemma 8 there is a saturated subcondition $X' \times Y' \subseteq X \times Y$. By (i), let $X'' \subseteq X'$ be a condition in $\mathbb{P}^*$, and $Y'' = Y' \cap [X'']_{\mathsf{E}}$. Similarly, let $Y''' \subseteq Y''$ be a condition in $\mathbb{P}^*$, and $X''' = X'' \cap [Y''']_{\mathsf{E}}$. Then $X''' \times Y'''$ belongs to $\mathbb{P}^* \times_{\mathsf{E}} \mathbb{P}^*$. ∎

Corollary 14 (assuming Ω-SM). *If $X \in \mathbb{P}$ then there exists a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ containing X. If $X \times Y$ is a condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ then there exists a $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic set $G \subseteq \mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ containing $X \times Y$.*

Proof. By Lemma 13, assume that $X \in \mathbb{P}^*$. Then the set $\mathbb{P}_{\subseteq X}$ of stronger conditions contains only countably many OD subsets by Lemma 11. ∎

9. The OD forcing relation. The forcing notion $\mathbb{P}$ will play the same role below as the Gandy–Harrington forcing in [5]. There is a notable technical difference: under Ω-SM, OD-generic sets exist in the ground Solovay-model universe by Corollary 14. Another notable difference is connected with the forcing relation.

Definition 15 (assuming Ω-SM). Let $\varphi(x)$ be an $\mathtt{Ord}$-*formula*, that is, a formula with ordinals as parameters.

A condition $X \in \mathbb{P}$ is said to $\mathbb{P}$-*force* $\varphi(\dot{\boldsymbol{x}})$ if $\varphi(x)$ is true (in the Solovay-model set universe considered) for any $\mathbb{P}$-generic real x.

If E is an OD equivalence relation on ω^ω then a condition $X \times Y$ in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ is said to $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-*force* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}})$ if $\varphi(x, y)$ is true for any $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic pair $\langle x, y \rangle$.

Lemma 16 (assuming Ω-SM). *Given an $\mathtt{Ord}$-formula $\varphi(x)$ and a $\mathbb{P}$-generic real x, if $\varphi(x)$ is true* (in the Solovay-model universe considered) *then there is a condition $X \in \mathbb{P}$ which contains x and $\mathbb{P}$-forces $\varphi(\dot{\boldsymbol{x}})$.*

Let E *be an OD equivalence relation on* ω^ω. *Given an* $\mathtt{Ord}$*-formula* $\varphi(x,y)$ *and a* $(\mathbb{P}\times_{\mathsf{E}}\mathbb{P})$*-generic pair* $\langle x,y\rangle$, *if* $\varphi(x,y)$ *is true then there is a condition in* $\mathbb{P}\times_{\mathsf{E}}\mathbb{P}$ *which contains* $\langle x,y\rangle$ *and* $(\mathbb{P}\times_{\mathsf{E}}\mathbb{P})$*-forces* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}},\dot{\boldsymbol{x}}_{\mathtt{ri}})$.

Proof. To prove the first claim, set $X=\{x'\in\omega^\omega:\varphi(x')\}$. But this argument does not work for $\mathbb{P}\times_{\mathsf{E}}\mathbb{P}$. To fix the problem, we propose a longer argument which works in both cases—but we present it in the case of $\mathbb{P}$, which is slightly simpler.

Formally the forcing notion $\mathbb{P}$ does not belong to $\mathbf{L}$. But it is order-isomorphic to a certain forcing notion $P\in\mathbf{L}$, namely, the set P of *codes* (4) of OD sets in $\mathbb{P}$. The order between the codes in P, which reflects the relation $\subseteq$ between the OD sets themselves, is expressible in $\mathbf{L}$, too. Furthermore, dense OD sets in $\mathbb{P}$ correspond to dense sets in the coded forcing P in $\mathbf{L}$.

Now, let x be $\mathbb{P}$-generic and suppose $\varphi(x)$ is true. It is a known property of the Solovay model that there is an $\mathtt{Ord}$-formula $\psi(x)$ such that $\varphi(x)$ iff $\mathbf{L}[x]\models\psi(x)$. Let $g\subseteq P$ be the set of all codes of conditions $X\in\mathbb{P}$ such that $x\in X$. Then g is P-generic over $\mathbf{L}$ by the choice of x, and x is the corresponding generic object, hence there is a condition $p\in g$ which P-forces $\psi(\dot{\boldsymbol{x}})$ over $\mathbf{L}$. Let $X\in\mathbb{P}$ be the OD set coded by p, so $x\in X$. To prove that X OD-forces $\varphi(\dot{\boldsymbol{x}})$, let $x'\in X$ be a $\mathbb{P}$-generic real. Let $g'\subseteq P$ be the P-generic set of all codes of conditions $Y\in\mathbb{P}$ such that $x'\in Y$. Then $p\in g'$, hence $\psi(x')$ holds in $\mathbf{L}[x']$, by the choice of p. Then $\varphi(x')$ holds (in the Solovay-model set universe) by the choice of ψ, as required. ∎

Corollary 17 (assuming Ω-SM). *Given an* $\mathtt{Ord}$*-formula* $\varphi(x)$, *if* $X\in\mathbb{P}$ *does not* $\mathbb{P}$*-force* $\varphi(\dot{\boldsymbol{x}})$ *then there is a condition* $Y\in\mathbb{P}$ *with* $Y\subseteq X$, *which* $\mathbb{P}$*-forces* $\neg\varphi(\dot{\boldsymbol{x}})$, *and similarly for* $\mathbb{P}\times_{\mathsf{E}}\mathbb{P}$. ∎

10. Adding a perfect antichain. The next result will be pretty important.

Lemma 18 (assuming Ω-SM). *Let* $\preccurlyeq$ *be an OD PQO on* ω^ω, *and* E_R *be an OD equivalence relation on* ω^ω *for any* $R\in\mathbb{P}$ *such that if* $R\subseteq R'$ *then* $x\mathrel{\mathsf{E}_R}y$ *implies* $x\mathrel{\mathsf{E}_{R'}}y$. *Suppose that* $X^*\in\mathbb{P}$, *and if* $R\in\mathbb{P}$ *with* $R\subseteq X^*$ *then* $R\times R$ *does* ***not*** $(\mathbb{P}\times_{\mathsf{E}_R}\mathbb{P})$*-force that* $\dot{\boldsymbol{x}}_{\mathtt{le}}$ *and* $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ *are* $\preccurlyeq$*-comparable. Then* X^* *is not* $\preccurlyeq$*-thin.*

Proof (follows [5, 2.9]). Let T be the set of all finite trees $t\subseteq 2^{<\omega}$. If $t\in T$ then let $M(t)$ be the set of all $\subset$-maximal elements of t.

Let Φ be the set of systems $\varphi=\{X_u\}_{u\in t}$ of sets $X_u\in\mathbb{P}^*$ such that $t\in T$ and the following conditions are satisfied:

(i) $X_\Lambda\subseteq X^*$ (where Λ is the empty string);

(4) A *code* of an OD set X is a finite sequence of logical symbols and ordinals which correspond to a definition in the form $X=\{x\in\mathbf{V}_\alpha:\mathbf{V}_\alpha\models\varphi(x)\}$.

(ii) if $u \subset v \in t$ then $X_v \subseteq X_u$;

(iii) if $u^\wedge 0$ and $u^\wedge 1$ belong to t then $X_{u^\wedge 0} \times X_{u^\wedge 1}$ belongs to $\mathbb{P}^* \times_{\mathsf{E}_{X_u}} \mathbb{P}^*$ and $(\mathbb{P} \times_{\mathsf{E}_{X_u}} \mathbb{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}$ is $\preccurlyeq$-incomparable to $\dot{\boldsymbol{x}}_{\mathtt{ri}}$;

(iv) compatibility: there is a sequence $\{x_u\}_{u \in M(t)}$ of points $x_u \in X_u$ such that if $u, v \in M(t)$ then $x_u \mathrel{\mathsf{E}_{X_{u \wedge v}}} x_v$, where $u \wedge v$ is the largest string $w \in 2^{<\omega}$ such that $w \subset u$ and $w \subset v$—it easily follows that then $X_u \times X_v$ is a condition in $\mathbb{P} \times_{\mathsf{E}_{X_{u \wedge v}}} \mathbb{P}$.

Say that a system $\{X_u\}_{u \in t} \in \varPhi$ is *saturated* if in addition

(v) for any $v \in M(t)$ and $x \in X_v$ there is a sequence $\{x_u\}_{u \in M(t)}$ as in (iv) such that $x_v = x$.

Say that a system $\{X'_u\}_{u \in t'} \in \varPhi$: 1) *weakly extends* another system $\varphi = \{X_u\}_{u \in t}$ if $t \subseteq t'$, $X_u = X'_u$ for all $u \in t \smallsetminus M(t)$, and $X'_u \subseteq X_u$ for all $u \in M(t)$; and 2) *properly extends* $\{X_u\}_{u \in t}$ if $t \subseteq t'$ and $X_u = X'_u$ for all $u \in t$. Thus a weak extension not just adds new sets to a given system φ but also shrinks old sets of the top layer $\varphi = \{X_u\}_{u \in M(t)}$ of φ. ■

Claim 18.1. *For any system $\varphi = \{X_u\}_{u \in t} \in \varPhi$ there is a **saturated** system $\{X'_u\}_{u \in t}$ in $\varPhi$ (with the same domain t) which weakly extends φ.*

Proof. If $u \in M(t)$ then simply let X'_u be the set of all points $x \in X_u$ such that $x = x_u$ for some sequence $\{x_u\}_{u \in M(t)}$ as in (iv). ■

Claim 18.2. *For any saturated system $\varphi = \{X_u\}_{u \in t} \in \varPhi$, if $u \in M(t)$ then there are sets $X_{u^\wedge 0}, X_{u^\wedge 1}$ such that the system φ extended by those sets still belongs to $\varPhi$ and properly extends φ.*

Proof. As $X_u \in \mathbb{P}$ and $X_u \subseteq X^*$, the condition $X_u \times X_u$ does *not* $(\mathbb{P} \times_{\mathsf{E}_{X_u}} \mathbb{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable. By Corollary 17, pick a stronger condition $U \times V \subseteq X_u \times X_u$ in $\mathbb{P} \times_{\mathsf{E}_{X_u}} \mathbb{P}$ which $(\mathbb{P} \times_{\mathsf{E}_{X_u}} \mathbb{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-incomparable. By Lemmas 13 and 8 we may assume that $U \times V$ belongs to $\mathbb{P}^* \times_{\mathsf{E}_{X_u}} \mathbb{P}^*$ and is E_{X_u}-saturated, so that $[U]_{\mathsf{E}_{X_u}} = [V]_{\mathsf{E}_{X_u}}$. We assert that the sets $X_{u^\wedge 0} = U$ and $X_{u^\wedge 1} = V$ prove the claim. It is enough to check (iv) for the extended system.

Fix any $x \in X_{u^\wedge 0} = U$. Then $x \in X_u$, hence, as the given system is saturated, there is a sequence $\{x_v\}_{v \in M(t)}$ of points $x_v \in X_v$ as in (iv) such that $x_u = x$. On the other hand, as $[U]_{\mathsf{E}_{X_u}} = [V]_{\mathsf{E}_{X_u}}$, there is a point $y \in V = X_{u^\wedge 1}$ such that $x \mathrel{\mathsf{E}_{X_u}} y$. Set $x_{u^\wedge 0} = x$ and $x_{u^\wedge 1} = y$. ■

If E is an OD equivalence relation and $X \times Y \in \mathbb{P}^* \times_{\mathsf{E}} \mathbb{P}^*$ then the set $\mathscr{D}(\mathsf{E}, X, Y)$ of all sets, *open dense in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ below $X \times Y$* ([5]), is countable by Lemma 13; fix an enumeration $\mathscr{D}(\mathsf{E}, X, Y) = \{D_n(\mathsf{E}, X, Y) : n \in \omega\}$ such that $D_n(\mathsf{E}, X, Y) \subseteq D_m(\mathsf{E}, X, Y)$ whenever $m < n$.

([5]) That is, open dense subsets of the restricted forcing $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})_{\subseteq X \times Y} = \{X' \times Y' \in \mathbb{P} \times_{\mathsf{E}} \mathbb{P} : X' \subseteq X \wedge Y' \subseteq Y\}$.

Claim 18.3. *Let $n \in \omega$ and $\varphi = \{X_u\}_{u \in 2^{\le n}} \in \varPhi$. Then there is a system $\varphi' = \{X'_u\}_{u \in 2^{\le n+1}} \in \varPhi$ which weakly extends φ and satisfies the following additional genericity requirement*:

($*$) *if strings $u \ne v$ belong to 2^{n+1} and $w = u \wedge v$ (defined as in* (iv)) *then the condition $X_u \times X_v$ belongs to $D_n(\mathsf{E}_{X_w}, X_{w^\wedge 0}, X_{w^\wedge 1})$.*

Proof. We first extend φ by one layer of sets $X'_{u^\wedge i}$, $u \in 2^n$ and $i = 0, 1$, obtained by consecutive 2^n splitting operations as in Claim 18.2, followed by the saturating reduction as in Claim 18.1. This way we get a saturated system $\eta = \{Y_u\}_{u \in 2^{\le n+1}} \in \varPhi$ which weakly extends φ.

To fulfill ($*$), let us shrink the sets in the top layer $\{Y_u\}_{u \in 2^{n+1}}$ of η.

Consider any pair of strings $u \ne v$ in 2^{n+1}. Let $w = u \wedge u$, so that $k = \mathtt{dom}\, w < n$, $w \subset u$, $w \subset v$, and $u(k) \ne v(k)$; let, say, $u(k) = 0$, $v(k) = 1$. The condition $Y_{w^\wedge 0} \times Y_{w^\wedge 1}$ belongs to $\mathbb{P}^* \times_{\mathsf{E}_{Y_w}} \mathbb{P}^*$ by (iii), while $Y_u \times Y_v$ belongs to $\mathbb{P} \times_{\mathsf{E}_{Y_w}} \mathbb{P}$ by (iv) and satisfies $Y_u \subseteq Y_{w^\wedge 0}$ and $Y_v \subseteq Y_{w^\wedge 1}$ by (ii). By density, there is a subcondition $Z_u \times Z_v \subseteq Y_u \times Y_v$ in $D_n(\mathsf{E}_{Y_w}, Y_{w^\wedge 0}, Y_{w^\wedge 1})$; in particular, $Z_u \times Z_v$ still belongs to $\mathbb{P} \times_{\mathsf{E}_{Y_w}} \mathbb{P}$. In addition to Z_u and Z_v, we let $Z_s = Y_s$ for any $s \in 2^{n+1} \smallsetminus \{u, v\}$. Then $\psi = \{Z_s\}_{s \in 2^{\le n+1}}$ is still a system in $\varPhi$. By Claim 18.1, there is a saturated system $\psi' = \{Z'_s\}_{s \in 2^{\le n+1}} \in \varPhi$ such that $Z'_s = Z_s = Y_s$ for all $u \in 2^{\le n}$, and $Z'_s \subseteq Z_s$ for all $s \in 2^{n+1}$. Then $Z'_u \subseteq Z_u$ and $Z'_v \subseteq Z_v$—so that $Z'_u \times Z'_v \in D_n(\mathsf{E}_{Y_w}, Y_{w^\wedge 0}, Y_{w^\wedge 1})$.

Iterating this shrinking construction $2^n(2^n - 1)$ times (the number of pairs $s \ne t$ in 2^n), we get a required system φ'. ∎

Claim 18.3 allows us to define, by induction, sets $X_u \subseteq X'_u \subseteq X^*$ in $\mathbb{P}^*$ ($u \in 2^{<\omega}$) and systems $\varphi_n = \{X_u\}_{u \in 2^{<n}} \cup \{X'_u\}_{u \in 2^n}$, such that, for any n:

(1) φ_n is a saturated system in $\varPhi$, weakly extended by φ_{n+1}, and
(2) condition ($*$) of Claim 18.3 holds.

We will show that this leads to a required perfect set. Let $a \ne b$ be reals in 2^ω, and $w = a \wedge b$, so that $w \subset a$, $w \subset b$, and $a(k) \ne b(k)$, where $k = \mathtt{dom}\, w$; let, say, $a(k) = 0$, $b(k) = 1$. Then the sequence of sets $X_{a \restriction m} \times X_{b \restriction m}$, $m > k$, is $(\mathbb{P} \times_{\mathsf{E}_{X_w}} \mathbb{P})$-generic by (10), so that $\bigcap_{m>k}(X_{a \restriction m} \times X_{b \restriction m})$ consists of a single pair of reals $\langle x_a, x_b \rangle$ by Proposition 9. Moreover, x_a, x_b are $\preccurlyeq$-incomparable by (iii). Finally the diameters of X_n uniformly converge to 0 as $n \to \infty$ by (10), and hence the map $a \mapsto x_a$ is continuous. Thus $P = \{x_a : a \in 2^\omega\}$ is a perfect $\preccurlyeq$-antichain in X^*. ∎$_{\text{Lemma 18}}$

11. Compression lemma. Let $\varTheta = \varOmega^+$, the cardinal successor of $\varOmega$ in both $\mathbf{L}$, the ground model, and its $\mathbf{Coll}(\omega, {<}\varOmega)$-generic extension postulated by $\varOmega$-SM to be the set universe; in the latter, $\varOmega = \omega_1$ and $\varTheta = \omega_2$.

Lemma 19 (Compression lemma). *Assume that $\Omega \le \vartheta \le \Theta$ and $X \subseteq 2^{\Theta}$ is the image of ω^ω via an OD map. Then there is an OD antichain $A(X) \subseteq 2^{<\Omega}$ and an OD isomorphism $f : \langle X; \leqslant_{\mathtt{lex}}\rangle \to \langle A(X); \leqslant_{\mathtt{lex}}\rangle$* (6).

Proof. If $\vartheta = \Theta$ then, as $\mathtt{card}\, X \le \mathtt{card}\, \omega^\omega = \Omega$, there is $\vartheta < \Theta$ such that $x{\restriction}\vartheta \ne y{\restriction}\vartheta$ whenever $x \ne y$ belong to X—this reduces the case $\vartheta = \Theta$ to the case $\Omega \le \vartheta < \Theta$. We prove the latter by induction on ϑ.

The non-trivial step is when $\mathtt{cof}\,\lambda = \Omega$, so let $\vartheta = \bigcup_{\alpha<\Omega} \vartheta_\alpha$ for an increasing OD sequence of ordinals ϑ_α. Let $I_\alpha = [\vartheta_\alpha, \vartheta_{\alpha+1})$. Then, by the induction hypothesis, for any $\alpha < \Omega$ the set $X_\alpha = \{S{\restriction}I_\alpha : S \in X\} \subseteq 2^{I_\alpha}$ is $<_{\text{lex}}$-order-isomorphic to an antichain $A_\alpha \subseteq 2^{<\Omega}$ via an OD isomorphism i_α, and the map which sends α to A_α and i_α is OD. It follows that the map which sends each $S \in X$ to the concatenation of all sequences $i_\alpha(x{\restriction}I_\alpha)$ is an OD $<_{\text{lex}}$-order isomorphism X onto an antichain in 2^{Ω}. Therefore it suffices to prove the lemma for $\vartheta = \Omega$. Thus let $X \subseteq 2^{\Omega}$.

First of all, note that each sequence $S \in X$ is ROD. Lemma 7 in [8] shows that, in this case, we have $S \in \mathbf{L}[S{\restriction}\eta]$ for an ordinal $\eta < \Omega$. Let $\eta(S)$ be the least such ordinal, and $h(S) = S{\restriction}\eta(S)$, so that $h(S)$ is a countable initial segment of S and $S \in \mathbf{L}[h(S)]$. Note that h is still OD.

Consider the set $U = \mathtt{ran}\, h = \{h(S) : S \in X\} \subseteq 2^{<\Omega}$. We can assume that every sequence $u \in U$ has a limit length. Then $U = \bigcup_{\gamma<\Omega} U_\gamma$, where $U_\gamma = U \cap 2^{\omega\gamma}$ ($\omega\gamma$ is the the γth limit ordinal). For $u \in U_\gamma$, let $\gamma_u = \gamma$.

If $u \in U$ then by construction the set $X_u = \{S \in X : h(S) = u\}$ is OD(u) and satisfies $X_u \subseteq \mathbf{L}[u]$. Therefore, it follows from the known properties of the Solovay model that X_u belongs to $\mathbf{L}[u]$ and is of cardinality $\le \Omega$ in $\mathbf{L}[u]$. Fix an enumeration $X_u = \{S_u(\alpha) : \gamma_u \le \alpha < \Omega\}$ for all $u \in U$. We can assume that the map $\alpha, u \mapsto S_u(\alpha)$ is OD. If $u \in U$ and $\gamma_u \le \alpha < \Omega$, then define a shorter sequence, $s_u(\alpha) \in 3^{\omega\alpha+1}$, as follows:

- (i) $s_u(\alpha)(\xi + 1) = S_u(\alpha)(\xi)$ for any $\xi < \omega\alpha$.
- (ii) $s_u(\alpha)(\omega\alpha) = 1$.
- (iii) Let $\delta < \alpha$. If $S_u(\alpha){\restriction}\omega\delta = S_v(\delta){\restriction}\omega\delta$ for some $v \in U$ (equal to or different from u) then $s_u(\alpha)(\omega\delta) = 0$ whenever $S_u(\alpha) <_{\mathtt{lex}} S_v(\delta)$, and $s_u(\alpha)(\omega\delta) = 2$ whenever $S_v(\delta) <_{\text{lex}} S_u(\alpha)$.
- (iv) Otherwise (i.e., if there is no such v), $s_u(\alpha)(\omega\delta) = 1$.

To demonstrate that (iii) is consistent, we show that $S_{u'}(\delta){\restriction}\omega\delta = S_{u''}(\delta){\restriction}\omega\delta$ implies $u' = u''$. Indeed, as by definition $u' \subset S_{u'}(\delta)$ and $u'' \subset S_{u''}(\delta)$, u' and u'' must be $\subseteq$-compatible, say $u' \subseteq u''$. Now, by definition, $S_{u''}(\delta)$ is in $\mathbf{L}[u'']$, therefore in $\mathbf{L}[S_{u'}(\delta)]$ because $u'' \subseteq S_{u''}(\delta){\restriction}\omega\delta = S_{u'}(\delta){\restriction}\omega\delta$, and finally in $\mathbf{L}[u']$, which shows that $u' = u''$ as $S_{u''}(\delta) \in X_{u''}$.

(6) Proved in [10, Theorem 31]. We present a slightly simplified proof to make the exposition self-contained. Note that any antichain $A \subseteq 2^{<\Omega}$ is linearly ordered by $\leqslant_{\mathtt{lex}}$!

We are going to prove that the map $S_u(\alpha) \mapsto s_u(\alpha)$ is a $<_{\mathtt{lex}}$-order isomorphism, so that $S_v(\beta) <_{\mathtt{lex}} S_u(\alpha)$ implies $s_v(\beta) <_{\mathtt{lex}} s_u(\alpha)$.

We first observe that $s_v(\beta)$ and $s_u(\alpha)$ are $\subseteq$-incomparable. Indeed, assume that $\beta < \alpha$. If $S_u(\alpha)\restriction\omega\beta \neq S_v(\beta)\restriction\omega\beta$ then clearly $s_v(\beta) \not\subseteq s_u(\alpha)$ by (i). If $S_u(\alpha)\restriction\omega\beta = S_v(\beta)\restriction\omega\beta$ then $s_u(\alpha)(\omega\beta) = 0$ or 2 by (iii), while $s_v(\beta)(\omega\beta)=1$ by (ii). Thus all $s_u(\alpha)$ are mutually $\subseteq$-incomparable, so that it suffices to show that conversely $s_v(\beta) <_{\text{lex}} s_u(\alpha)$ implies $S_v(\beta) <_{\text{lex}} S_u(\alpha)$. Let ζ be the least ordinal such that $s_v(\beta)(\zeta) < s_u(\alpha)(\zeta)$; then $s_u(\alpha)\restriction\zeta = s_v(\beta)\restriction\zeta$ and $\zeta \leq \mathtt{min}\{\omega\alpha, \omega\beta\}$.

The case when $\zeta = \xi + 1$ is clear: then by definition $S_u(\alpha)\restriction\xi = S_v(\beta)\restriction\xi$ while $S_v(\beta)(\xi) < S_u(\alpha)(\xi)$, so suppose that $\zeta = \omega\delta$, where $\delta \leq \mathtt{min}\{\alpha, \beta\}$. Then obviously $S_u(\alpha)\restriction\omega\delta = S_v(\beta)\restriction\omega\delta$. Assume that one of the ordinals α, β is equal to δ, say $\beta = \delta$. Then $s_v(\beta)(\omega\delta) = 1$ while $s_u(\alpha)(\omega\delta)$ is computed by (iii). Now, as $s_v(\beta)(\omega\delta) < s_u(\alpha)(\omega\delta)$, we conclude that $s_u(\alpha)(\omega\delta) = 2$, hence $S_v(\beta) <_{\text{lex}} S_u(\alpha)$, as required. Assume now that $\delta < \mathtt{min}\{\alpha, \beta\}$. Then clearly α and β appear in one and the same class (iii) or (i) with respect to δ. However this cannot be (iv) because $s_v(\beta)(\omega\delta) \neq s_u(\alpha)(\omega\delta)$. Hence we are in (iii), so that, for some (unique) $w \in U$,

$$0 = S_v(\beta) <_{\text{lex}} S_w(\delta) <_{\mathtt{lex}} S_u(\alpha) = 2,$$

as required.

This ends the proof of the lemma, except for the fact that the sequences $s_u(\alpha)$ belong to $3^{<\Omega}$, but improvement to $2^{<\Omega}$ is easy. ∎

12. Decomposing thin OD sets in the Solovay model

Proof of Theorem 2. Let $\preccurlyeq$ be an OD PQO on ω^ω, $\approx$ be the associated equivalence relation, and $X^* \subseteq \omega^\omega$ be an OD $\preccurlyeq$-thin set. Assume towards a contradiction that the OD set $\mathbb{U}$ of all reals $x \in X^*$ which do not belong to any OD $\preccurlyeq$-chain is non-empty.

Let $\mathbb{F}$ consist of all LR order preserving (Section 2) OD maps $F : \langle\omega^\omega; \preccurlyeq\rangle \to \langle A; \leqslant_{\mathtt{lex}}\rangle$, where $A \subseteq 2^{<\Omega}$ is an antichain. If $R \subseteq \omega^\omega$ then let $\mathbb{F}_R$ consist of all maps $F \in \mathbb{F}$, *antichain-collapsing* on R in the sense that

$$\text{if } x, y \in R \text{ are } \preccurlyeq\text{-incomparable then } F(x) = F(y),$$

or equivalently $F(x) <_{\mathtt{lex}} F(y) \Rightarrow x \prec y$ for all $x, y \in R$. Any map $F \in \mathbb{F}_R$ is invariant with respect to the equivalence hull of the relation: $x \not\preccurlyeq y \wedge y \not\preccurlyeq x$.

Define an OD equivalence relation by $x \mathrel{\mathbb{E}_R} y$ iff $\forall F \in \mathbb{F}_R\, (F(x) = F(y))$.

If $R \subseteq R'$ then $\mathbb{F}_{R'} \subseteq \mathbb{F}_R$, and hence $x \mathrel{\mathbb{E}_R} y$ implies $x \mathrel{\mathbb{E}_{R'}} y$.

Lemma 20. *If $R \subseteq \omega^\omega$ is OD and $\mathbb{E}_R \subseteq S \subseteq \omega^\omega \times \omega^\omega$, and if S is OD, then there is a function $F \in \mathbb{F}_R$ such that $\forall x, y\ (F(x) = F(y) \Rightarrow S(x, y))$.*

Proof. Clearly $\mathtt{card}\,\mathbb{F}_R = \Theta$ and $\mathbb{F}_R$ admits an OD enumeration $\{F_\xi : \xi < \Theta\}$. If $x \in \omega^\omega$ then let $f(x) = F_0(x)^\wedge F_1(x)^\wedge \ldots {}^\wedge F_\xi(x)^\wedge\ \ldots$, the concatenation of all sequences $F_\xi(x)$. Then $f:\langle\omega^\omega;\preccurlyeq\rangle\to\langle W;\leqslant_{\mathtt{lex}}\rangle$ is an LR order preserving OD map, where $W=\operatorname{\mathtt{ran}} f=\{f(r):r\in\omega^\omega\}\subseteq 2^\Theta$, and $f(x)=f(y)\Rightarrow S(x,y)$ by construction. By Lemma 19 there is an OD isomorphism $g:\langle W;\leqslant_{\mathtt{lex}}\rangle\to\langle A;\leqslant_{\mathtt{lex}}\rangle$ onto an antichain $A\subseteq 2^{<\Omega}$. The superposition $F(x)=g(f(x))$ proves the lemma. ∎

Lemma 21. *If $R\subseteq\omega^\omega$ is OD then $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}}\ \mathbb{E}_R\ \dot{\boldsymbol{x}}_{\mathtt{ri}}$.*

Proof. Otherwise by Lemma 16 there is a function $F\in\mathbb{F}_R$, an ordinal $\xi<\Omega$, and a saturated condition $X\times Y$ in $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$ which $(\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P})$-forces $F(\dot{\boldsymbol{x}}_{\mathtt{le}})(\xi)=0\neq 1=F(\dot{\boldsymbol{x}}_{\mathtt{ri}})(\xi)$ (or forces $F(\dot{\boldsymbol{x}}_{\mathtt{le}})(\xi)=1\neq 0=F(\dot{\boldsymbol{x}}_{\mathtt{ri}})(\xi)$). Then $F(x)(\xi)=0\neq 1=F(y)(\xi)$ for any pair $\langle x,y\rangle\in X\times Y$, so that we have $F(x)\neq F(y)$ and hence $\neg(x\ \mathbb{E}_R\ y)$ whenever $\langle x,y\rangle\in X\times Y$. This contradicts the choice of $X\times Y$ in $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$. ∎

Lemma 22. *If $\emptyset\neq R\subseteq\mathbb{U}$ is OD then $R\times R$ $(\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}}\not\approx\dot{\boldsymbol{x}}_{\mathtt{ri}}$.*

Proof. Otherwise there is a condition $X\times Y$ in $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$ with $X,Y\subseteq R$ which forces $\dot{\boldsymbol{x}}_{\mathtt{le}}\approx\dot{\boldsymbol{x}}_{\mathtt{ri}}$. Let $W=\{\langle x,x'\rangle\in X\times X: x\ \mathbb{E}_R\ x'\wedge x'\not\approx x\}$. We claim that $W=\emptyset$. Indeed, otherwise the forcing $\mathbb{P}(W)=\{P\subseteq W:\emptyset\neq P\in\mathrm{OD}\}\neq\emptyset$ is just a 2-dimensional version of $\mathbb{P}$ with the same basic properties. In particular $\mathbb{P}(W)$ adds pairs $\langle x_{\mathtt{le}},x_{\mathtt{ri}}\rangle\in W$ with $x_{\mathtt{le}}\ \mathbb{E}_R\ x_{\mathtt{ri}}$ and $x_{\mathtt{le}}\not\approx x_{\mathtt{ri}}$. If $P\in\mathbb{P}(W)$ then obviously $[\operatorname{\mathtt{dom}}P]_{\mathbb{E}_R}=[\operatorname{\mathtt{ran}}P]_{\mathbb{E}_R}$.

Consider the forcing notion $\mathscr{P}=\mathbb{P}(W)\times_{\mathbb{E}_R}\mathbb{P}(Y)$ of all pairs $P\times Y'$, where $P\in\mathbb{P}(W)$, $Y'\in\mathbb{P}$, $Y'\subseteq Y$, and $[\operatorname{\mathtt{dom}}P]_{\mathbb{E}_R}\cap[Y']_{\mathbb{E}_R}\neq\emptyset$. For instance, $W\times Y\in\mathscr{P}$. Then $\mathscr{P}$ adds a pair $\langle x_{\mathtt{le}},x_{\mathtt{ri}}\rangle\in W$ and a real $x\in Y$ such that the pairs $\langle x_{\mathtt{le}},x\rangle$ and $\langle x_{\mathtt{ri}},x\rangle$ belong to $X\times Y$ and are $(\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P})$-generic, hence $x_{\mathtt{le}}\approx x\approx x_{\mathtt{ri}}$ by the choice of $X\times Y$. On the other hand, $x_{\mathtt{le}}\not\approx x_{\mathtt{ri}}$ as $\langle x_{\mathtt{le}},x_{\mathtt{ri}}\rangle\in W$, which is a contradiction.

Thus $W=\emptyset$. Then X is a $\preccurlyeq$-chain: if $x,y\in X$ are $\preccurlyeq$-incomparable then by definition $x\ \mathbb{E}_R\ y$, hence $x\approx y$, a contradiction. Thus X is an OD $\preccurlyeq$-chain with $\emptyset\neq X\subseteq\mathbb{U}$, contrary to the definition of $\mathbb{U}$. ∎

Lemma 23. *Let $R\subseteq\mathbb{U}$ be a non-empty OD set. Then $R\times R$ does **not** $(\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}},\dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable.*

Proof. Suppose to the contrary that $R\times R$ forces the comparability. Then by Corollary 17 and Lemma 22 there is a condition $X\times Y$ in $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$ with $X,Y\subseteq R$ which $(\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}}\prec\dot{\boldsymbol{x}}_{\mathtt{ri}}$. (For if it forces $\dot{\boldsymbol{x}}_{\mathtt{ri}}\prec\dot{\boldsymbol{x}}_{\mathtt{le}}$ then replace $X\times Y$ by $Y\times X$.)

Claim 24. *If $x\in X$, $y\in Y$, and $x\ \mathbb{E}_R\ y$ then $x\prec y$.*

Proof. Otherwise the OD set $W=\{\langle x,y\rangle\in X\times Y: x\ \mathbb{E}_R\ y\wedge x\not\prec y\}$ is non-empty. Let $X'=\operatorname{\mathtt{dom}}W$. As $X'\subseteq R$, there is a condition $A\times B$ in $\mathbb{P}\times_{\mathbb{E}_R}\mathbb{P}$ with $A\cup B\subseteq X'$ which forces $\dot{\boldsymbol{x}}_{\mathtt{le}}\prec\dot{\boldsymbol{x}}_{\mathtt{ri}}$. (See the choice of $X\times Y$.)

Let $Z = \{\langle x,y\rangle \in W : x \in A\}$ and consider the forcing notion $\mathscr{P} = \mathbb{P}(Z) \times_{\mathbb{E}_R} \mathbb{P}(B)$ of all non-empty OD sets $P \times B'$, where $P \subseteq Z$, $B' \subseteq B$, and $[B']_{\mathbb{E}_R} \cap [\mathtt{dom}\, P]_{\mathbb{E}_R} \neq \emptyset$ (equivalently, $[B']_{\mathbb{E}_R} \cap [\mathtt{ran}\, P]_{\mathbb{E}_R} \neq \emptyset$). For instance, $Z \times B \in \mathscr{P}$. It adds a pair $\langle x_{\mathtt{le}}, x_{\mathtt{ri}}\rangle \in Z$ and a separate real $x \in B$ such that both pairs $\langle x_{\mathtt{le}}, x\rangle$ and $\langle x_{\mathtt{ri}}, x\rangle$ are $(\mathbb{P} \times_{\mathbb{E}_R} \mathbb{P})$-generic. It follows that $\mathscr{P}$ forces both $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}$ (as this pair belongs to $A \times B$) and $\dot{\boldsymbol{x}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (this one belongs to $X \times Y$), hence it forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. Yet $\mathscr{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (as this pair belongs to $Z \subseteq W$), a contradiction. ■

COROLLARY 25. *The OD set $C = \{x' : \exists x \in X\ (x \mathbb{E}_R x' \wedge x' \preccurlyeq x)\}$ is downwards $\preccurlyeq$-closed in each $\mathbb{E}_R$-class, $X \subseteq C$, and $Y \cap C = \emptyset$.* ■

CLAIM 26. *If $x \in C \cap R$, $y \in R \smallsetminus C$, and $y \mathbb{E}_R x$, then $x \prec y$.*

Proof. Otherwise the set $H_0 = \{y \in R \smallsetminus C : \exists x \in C \cap R\ (x \mathbb{E}_R y \wedge x \not\prec y)\}$ in OD is non-$\emptyset$. As above (the choice of X, Y), there is a condition $H \times H'$ in $\mathbb{P} \times_{\mathbb{E}_R} \mathbb{P}$ with $H \cup H' \subseteq H_0$ which forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. Similarly to Claim 24, if $\langle y, y'\rangle \in H \times H'$ and $y \mathbb{E}_R y'$ then $y \prec y'$. Then the set

$$C_1 = \{x \in C \cap R : \exists y' \in H'\ (x \mathbb{E}_R y' \wedge x \not\prec y')\}$$

satisfies $[C_1]_{\mathbb{E}_R} = [H]_{\mathbb{E}_R} = [H']_{\mathbb{E}_R}$ by construction, hence $C_1 \times H$ belongs to $\mathbb{P} \times_{\mathbb{E}_R} \mathbb{P}$. Let $\langle x_1, y\rangle \in C_1 \times H$ be any $(\mathbb{P} \times_{\mathbb{E}_R} \mathbb{P})$-generic pair. Then $x_1 \mathbb{E}_R y$ and $x_1 \prec y$ or $y \prec x_1$ by Lemmas 21 and 22. But $y \prec x_1$ fails by Claim 24. Thus in fact $x_1 \prec y$. Hence if $y' \in H'$ and $x_1 \mathbb{E}_R y'$ then $x_1 \prec y \prec y'$, which contradicts $x_1 \in C_1$. ■

CLAIM 27. $[X]_{\mathbb{E}_R} \cap [Y]_{\mathbb{E}_R} = \emptyset$.

Proof. By Corollary 25, Claim 26, and Lemma 20, there is a function $F \in \mathbb{F}_R$ such that if $x \in C$, $y \notin C$, and $F(x) = F(y)$ then $y \not\preccurlyeq x$, and if in addition $x, y \in R$ then even $x \prec y$. We will prove that the *derived function*

$$G(x) = \begin{cases} F(x)^\wedge 0 & \text{whenever } x \in C, \\ F(x)^\wedge 1 & \text{whenever } x \in \omega^\omega \smallsetminus C, \end{cases}$$

belongs to $\mathbb{F}_R$. Let $x \preccurlyeq y$. Then $F(x) \leqslant_{\mathtt{lex}} F(y)$ as $F \in \mathbb{F}_R$, and moreover $G(x) \leqslant_{\mathtt{lex}} G(y)$, since if $F(x) = F(y)$ then $y \in C \wedge x \notin C$ is impossible by the choice of F, x, y. Now suppose that $x, y \in R$ and $G(x) <_{\mathtt{lex}} G(y)$. Then either $F(x) <_{\mathtt{lex}} F(y)$—then immediately $x \prec y$ since $F \in \mathbb{F}_R$, or $F(x) = F(y)$—then $x \in C$ and $y \notin C$ by the definition of G, hence $x \prec y$ by the choice of F. Thus $G \in \mathbb{F}_R$. Now if $x \in X$, $y \in Y$ then $x \in C$ and $y \notin C$, hence $G(y) \neq G(x)$, and $x \mathbb{E}_R y$ fails as $G \in \mathbb{F}_R$. ■

But $[X]_{\mathbb{E}_R} \cap [Y]_{\mathbb{E}_R} \neq \emptyset$ as $X \times Y$ belongs to $\mathbb{P} \times_{\mathbb{E}_R} \mathbb{P}$. ■ Lemma 23

Lemmas 23 and 18 imply Theorem 2. ■ Theorem 2

13. The theorem on analytic thin subsets. Here we begin the proof of Theorem 3. The theorem is essentially established in [5, Theorems 3.1 and 5.1]. Literally, only the case of Δ^1_1 subsets X^* is considered in [5], but the case of Σ^1_1 sets X^* can be obtained by a rather straightforward modification of the arguments in [5]. See also [9] as regards the additional requirement in claim (II) of the theorem, which is also presented in [5] implicitly. Nevertheless, following a referee's advice, we decided to incorporate a sketch of the proof of Theorem 3, just to make the text self-contained and more reader-friendly, and to make the proof of Theorem 1(A) complete.

The proof presented here will largely follow the arguments in [5], but of necessity we modify them here and there in order to distinguish some key ingredients of the proof, in some cases mixed in [5] in the general flow of arguments. In particular, we give more attention to the details of coding of Δ^1_1 maps, presented in [5] with extreme brevity.

On the other hand, we skip the proof of the non-thinness lemma (Lemma 34), with a reference to both [5] and our Lemma 18 (with a very similar proof). We also replace reflection arguments in [5] with more transparent constructions, beginning with the following:

Lemma 28 (Kreisel selection). *Let D be the set of all Δ^1_1 points in ω^ω. If $P \subseteq \omega^\omega \times D$ is a Π^1_1 set, and $X \subseteq \mathtt{dom}\, P$ is Σ^1_1, then there is a Δ^1_1 set $Y \subseteq \mathtt{dom}\, P$ and a Δ^1_1 function $F : Y \to D$ such that $X \subseteq Y$ and $F \subseteq P$.*

Proof. The set $X_0 = \mathtt{dom}\, P$ is Π^1_1 since Π^1_1 is closed under $\exists y \in \Delta^1_1$. Therefore by separation there is a Δ^1_1 set Y such that $X \subseteq Y \subseteq X_0$. By Π^1_1 uniformization, there is a Π^1_1 set $F \subseteq P$ such that $\mathtt{dom}\, F = Y$ and Y is a function. To show that F is in fact Δ^1_1, note that $F(x) = y$ iff $x \in Y$ and $\forall y' \in D\ (y \neq y' \Rightarrow \langle x, n' \rangle \notin F)$, which leads to a Σ^1_1 definition. ∎

14. Ingredient 1: coding Δ^1_1 functions. Let $\preccurlyeq$ be a Δ^1_1 PQO on ω^ω and $\approx$ be the associated equivalence relation.

Recall that ω_1^{CK} is the least non-recursive (= the least non-Δ^1_1) ordinal. If $\alpha < \omega_1^{\mathrm{CK}}$ then let $\mathbf{F}(\alpha)$ be the set of all *LR order preserving* (Section 2) Δ^1_1 maps $F : \langle \omega^\omega; \preccurlyeq \rangle \to \langle 2^\alpha; \leqslant_{\mathtt{lex}} \rangle$. Let $\mathbf{F} = \bigcup_{\alpha < \omega_1^{\mathrm{CK}}} \mathbf{F}(\alpha)$.

If $R \subseteq \omega^\omega$ then let $\mathbf{F}_R$ consist of all maps $F \in \mathbf{F}$ *antichain-collapsing on R* in the sense of Section 12. We define

$$x\ \mathbf{E}_{\mathbf{F}}\ y \quad \text{iff} \quad \forall F \in \mathbf{F}\ (F(x) = F(y)),$$

$$x\ \mathbf{E}_R\ y \quad \text{iff} \quad \forall F \in \mathbf{F}_R\ (F(x) = F(y)); \quad \mathbf{E}_{\mathbf{F}} = \mathbf{E}_\emptyset.$$

Lemma 29. *Let $R \subseteq \omega^\omega$ be a Σ^1_1 set. Then both $\mathbf{E}_{\mathbf{F}}$ and $\mathbf{E}_R$ are Σ^1_1 equivalence relations, and $\approx\ \subseteq \mathbf{E}_{\mathbf{F}} \subseteq \mathbf{E}_R$.*

If $\mathbf{E}_{\mathbf{F}} \subseteq S \subseteq \omega^\omega \times \omega^\omega$ and S is Π^1_1 then there is $F \in \mathbf{F}$ such that $\forall x, y\ (F(x) = F(y) \Rightarrow S(x, y))$. The same is true for $\mathbf{E}_R$, with $F \in \mathbf{F}_R$.

Proof. This looks similar to Lemma 20, yet the proof is quite different. We focus on $\mathbf{E_F}$; the case of $\mathbf{E}_R$ does not differ much. We begin with a coding of functions in $\mathbf{F}$, based on a standard coding system for Δ^1_1 sets.

(I) (see, e.g., [11, 2.8.1].) There is a Π^1_1 *set of codes* $\mathbf{Code} \subseteq \omega$, and for any $k \in \mathbf{Code}$ a Δ^1_1 set $B_k \subseteq \omega^\omega \times \omega^\omega$, and two Π^1_1 sets $W, W' \subseteq \omega \times \omega^\omega \times \omega^\omega$ such that, first, $\{B_k : k \in \mathbf{Code}\}$ is exactly the family of all Δ^1_1 sets $B \subseteq \omega^\omega \times \omega^\omega$, and second, if $k \in \mathbf{Code}$ and $x, y \in \omega^\omega$ then

$$\langle x, y\rangle \in B_k \;\Leftrightarrow\; W(k, x, y) \;\Leftrightarrow\; \neg W'(k, x, y).$$

(II) We define $\mathbf{CF} = \{k \in \mathbf{Code} : B_k \text{ is a total map } \omega^\omega \to \omega^\omega\}$, the set of codes of all Δ^1_1 functions $F : \omega^\omega \to \omega^\omega$; this is still a Π^1_1 set because the key condition $\mathtt{dom}\, B_k = \omega^\omega$ can be expressed by $\forall x\ \exists y \in \Delta^1_1(x)\ W(k, x, y)$, where the quantifier $\exists y \in \Delta^1_1(x)$ is known to preserve the type Π^1_1.

(III) If $\varepsilon \in \omega^\omega$ then let $\leq_\varepsilon = \{\langle i, j\rangle : \varepsilon(2^i \cdot 3^j) = 0\}$. Let $\mathbf{WO}$ consist of all ε such that $\leq_\varepsilon$ is a (non-strict) wellordering of the set $\mathtt{dom}(\leq_\varepsilon)$. If $\varepsilon \in \mathbf{WO}$ then we let $|\varepsilon| = \mathtt{otp}(\varepsilon) < \omega_1$ be the order type of $\leq_\varepsilon$, let $\beta_\varepsilon : \mathtt{dom}(\leq_\varepsilon) \to |\varepsilon|$ be the order-preserving bijection, and let $H_\varepsilon : \omega^\omega \to (\omega^\omega)^{|\varepsilon|}$ be the induced homeomorphism. If in addition $k \in \mathbf{CF}$ then let F^ε_k be the Δ^1_1 map $\omega^\omega \to (\omega^\omega)^{|\varepsilon|}$ defined by $F^\varepsilon_k(x) = H_\varepsilon(B_k(x))$ for all $x \in \omega^\omega$.

(V) Let $\mathbf{\Pi}$ be the set of all pairs $\langle\varepsilon, k\rangle$ such that $\varepsilon \in \mathbf{WO}$ is Δ^1_1, $k \in \mathbf{CF}$, and $F^\varepsilon_k \in \mathbf{F}$. If $R \subseteq \omega^\omega$ is Σ^1_1 then let $\mathbf{\Pi}(R) = \{\langle\varepsilon, k\rangle \in \mathbf{\Pi} : F^\varepsilon_k \in \mathbf{F}_R\}$.

Claim 30 (routine). *$\mathbf{\Pi} \subseteq \omega^\omega \times \omega$ is a countable Π^1_1 set of Δ^1_1 elements, and $\mathbf{F} = \{F^\varepsilon_k : \langle\varepsilon, k\rangle \in \mathbf{\Pi}\}$. If $R \subseteq \omega^\omega$ is a Σ^1_1 set then $\mathbf{\Pi}(R) \subseteq \omega^\omega \times \omega$ is a countable Π^1_1 set of Δ^1_1 elements, and $\mathbf{F}_R = \{F^\varepsilon_k : \langle\varepsilon, k\rangle \in \mathbf{\Pi}(R)\}$.* ■

It follows that $\mathbf{E_F}$ is Σ^1_1, since $x\ \mathbf{E_F}\ y$ is equivalent to either of the formulas $\forall\langle\varepsilon, k\rangle \in \mathbf{\Pi}\ (F^\varepsilon_k(x) = F^\varepsilon_k(y))$ and $\forall\langle\varepsilon, k\rangle \in \mathbf{\Pi}\ (B_k(x) = B_k(y))$.

We will prove the claim of Lemma 29 related to S. We rewrite the assumption as $\forall x, y\ (\neg S(x, y) \Rightarrow \neg(x\ \mathbf{E_F}\ y))$, or equivalently by Claim 30 as

$$\forall x, y\ \big(\neg S(x, y) \;\Rightarrow\; \exists\langle\varepsilon, k\rangle \in \Delta^1_1\ \underbrace{(\langle\varepsilon, k\rangle \in \mathbf{\Pi} \wedge F^\varepsilon_k(x) \neq F^\varepsilon_k(y))}_{P(x,y;\,\varepsilon,k)}\big)\,.$$

The relation P is Π^1_1 by means of (I) and Claim 30. Lemma 28 yields a Δ^1_1 set $W \subseteq \omega^\omega \times \omega^\omega$ satisfying $\neg S(x, y) \Rightarrow W(x, y)$, and a Δ^1_1 map $\Phi(x, y) = \langle\varepsilon(x, y), k(x, y)\rangle : W \to \mathbf{\Pi}$ with $F^{\varepsilon(x,y)}_{k(x,y)}(x) \neq F^{\varepsilon(x,y)}_{k(x,y)}(y)$ for all $\langle x, y\rangle \in W$ —then, in particular, for all x, y with $\neg S(x, y)$.

Then $Z = \{\Phi(x, y) : \langle x, y\rangle \in W\}$ is a Σ^1_1 subset of the Π^1_1 set $\mathbf{\Pi}$. By separation, there is a Δ^1_1 set D such that $Z \subseteq D \subseteq \mathbf{\Pi}$. As a countable Δ^1_1 set, it admits a Δ^1_1 enumeration $D = \{\langle\varepsilon_n, k_n\rangle : n \in \mathbb{N}\}$, and by construction $\forall n\ (F^{\varepsilon_n}_{k_n}(x) = F^{\varepsilon_n}_{k_n}(y))$ implies $S(x, y)$. Then the map $F(x) = F^{\varepsilon_0}_{k_0}(x)^\wedge F^{\varepsilon_1}_{k_1}(x)^\wedge$ belongs to $\mathbf{F}$ and satisfies $F(x) = F(y) \Rightarrow S(x, y)$. ■Lemma 29

15. Ingredient 2: invariant separation. Under the assumptions of Theorem 3, let E be a Σ^1_1 equivalence relation. A set $X \subseteq \omega^\omega$ is *downwards*, resp., *upwards* $\preccurlyeq$*-closed in each* E*-class* iff $x \in X \Rightarrow y \in X$ whenever $x \mathrel{\mathsf{E}} y$ and $y \preccurlyeq x$, resp. $x \preccurlyeq y$.

Lemma 31. *Let E be a Σ^1_1 equivalence relation containing $\approx$, and let X, Y be Σ^1_1 sets such that $y \not\preccurlyeq x$ whenever $x \in X \wedge y \in Y \wedge x \mathrel{\mathsf{E}} y$. Then there is a Δ^1_1 set Z that is downwards $\preccurlyeq$-closed in each E-class and satisfies $X \subseteq Z$ and $Y \cap Z = \emptyset$.*

Proof. Let $Y' = \{y' : \exists y \in Y\,(y \preccurlyeq y')\}$; still $Y' \cap X = \emptyset$ and Y' is Σ^1_1. Using separation, define an increasing sequence of sets

$$X = X_0 \subseteq A_0 \subseteq X_1 \subseteq A_1 \subseteq \cdots \subseteq X_n \subseteq A_n \subseteq \cdots \subseteq \omega^\omega \smallsetminus Y'$$

so that $A_n \in \Delta^1_1$ and $X_{n+1} = \{x' \in \omega^\omega : \exists x \in A_n\ (x' \mathrel{\mathsf{E}} x \wedge x' \preccurlyeq x)\}$ for all n. If $A_n \cap Y' = \emptyset$ then $X_{n+1} \cap Y' = \emptyset$ as well since Y' is upwards closed, which justifies the inductive construction. Furthermore, a proper execution of the construction yields the final set $Z = \bigcup_n A_n = \bigcup_n X_n$ in Δ^1_1. (We refer to the proof of an "invariant" effective separation theorem 5.1 in [4] or a similar construction in [11, Lemma 10.4.2].) Note that by definition $X \subseteq Z$, but $Z \cap Y = \emptyset$, and Z is downwards $\preccurlyeq$-closed in each E-class. ∎

Coming back to the relations introduced in Section 14, we prove

Corollary 32.

(i) *If $X, Y \subseteq \omega^\omega$ are Σ^1_1 sets, and $y \not\preccurlyeq x$ whenever $x \in X$, $y \in Y$ and $x \mathbf{E_F}\, y$, then $[X]_{\mathbf{E_F}} \cap [Y]_{\mathbf{E_F}} = \emptyset$.*

(ii) *If $X, Y \subseteq R \subseteq \omega^\omega$ are Σ^1_1 sets, and $x \prec y$ whenever $x \in X$, $y \in Y$ and $x \mathbf{E}_R\, y$, then $[X]_{\mathbf{E}_R} \cap [Y]_{\mathbf{E}_R} = \emptyset$.*

Proof. Otherwise by Lemma 31 there is a Δ^1_1 set C such that $X \subseteq C$, $Y \cap C = \emptyset$, and C is downwards $\preccurlyeq$-closed in each $\mathbf{E_F}$-class. By Lemma 29, there is a map $F \in \mathbf{F}$, resp. $F \in \mathbf{F}_R$ in case (ii), such that if $F(x) = F(y)$ and $x \preccurlyeq y$ then $y \in C \Rightarrow x \in C$. Then the derived function

$$G(x) = \begin{cases} F(x)^\wedge 0 & \text{whenever } x \in C, \\ F(x)^\wedge 1 & \text{whenever } x \in \omega^\omega \smallsetminus C, \end{cases}$$

belongs to $\mathbf{F}$, resp. $\mathbf{F}_R$ in case (ii) (see the proof of Claim 27).

Now if $x \in X$ and $y \in Y$, then $x \in Z$ and $y \notin Z$, hence $G(x) \neq G(y)$ and $\neg(x \mathbf{E_F}\, y)$, resp. $\neg(x \mathbf{E}_R\, y)$ in case (ii), which is a contradiction. ∎

16. Ingredient 3: the Gandy–Harrington forcing. The Gandy–Harrington forcing notion $\mathbf{P}$ is the set of all Σ^1_1 sets $\emptyset \neq X \subseteq \omega^\omega$ ordered so that smaller sets are stronger conditions. It is known that $\mathbf{P}$ adds a point of ω^ω, whose name will be $\dot{\boldsymbol{x}}$.

If E is a Σ^1_1 equivalence relation on ω^ω then a related forcing $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ defined in [5] consists of all sets of the form $X \times Y$, where $X, Y \in \mathbf{P}$ and $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \emptyset$. A condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is *saturated* iff $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$. Similarly to Lemma 8, if $X \times Y$ is a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ then there is a stronger saturated subcondition $X' \times Y' \subseteq X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$.

The forcings $\mathbf{P}$ and $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ will be used below as forcing notions over the ground set universe $\mathbf{V}$. Given a Σ^1_1 (or Π^1_1) set X in the ground universe $\mathbf{V}$, *we denote by the same letter X the extended set* (i.e., defined by the same formula) *in any generic extension of* $\mathbf{V}$. This does not lead to ambiguities by Shoenfield absoluteness (see [14, 2.4] for more details).

LEMMA 33 ([11, A.5.4]). *If $X \in \mathbf{P}$ then X $\mathbf{P}$-forces that $\dot{\boldsymbol{x}} \in X$. Therefore if $\Phi(x)$ is a Π^1_2 formula and $\Phi(x)$ holds for all $x \in X$ then X $\mathbf{P}$-forces $\Phi(\dot{\boldsymbol{x}})$. The same is true for forcing notions of the form $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$, where E is a Σ^1_1 equivalence relation.* ■

The next two important results are similar to (and are direct prototypes of) our Lemma 18, resp. Lemma 21, hence we skip the proofs.

LEMMA 34 ([5, 2.9]). *Suppose that $\preccurlyeq$ is a Δ^1_1 PQO on ω^ω, and, for any $R \in \mathbf{P}$, E_R is a Σ^1_1 equivalence relation on ω^ω such that if $R \subseteq R'$ then $x \mathrel{\mathsf{E}_R} y$ implies $x \mathrel{\mathsf{E}_{R'}} y$. Assume that $X^* \in \mathbf{P}$, and if $R \in \mathbf{P}$ and $R \subseteq X^*$ then $R \times R$ does **not** $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable. Then X^* is not $\preccurlyeq$-thin.* ■

LEMMA 35 ([5, 2.7]). *If $R \subseteq \omega^\omega$ is Σ^1_1 then the forcing $\mathbf{P} \times_{\mathbf{E}_R} \mathbf{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \mathrel{\mathbf{E}_R} \dot{\boldsymbol{x}}_{\mathtt{ri}}$. In particular, as $\mathbf{E}_\emptyset = \mathbf{E}_{\mathbf{F}}$, $\mathbf{P} \times_{\mathbf{E}} \mathbf{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \mathrel{\mathbf{E}} \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* ■

17. Bounding thin analytic partial orderings

Proof of Theorem 3(I). So let $\preccurlyeq$ be a Δ^1_1 PQO on ω^ω, $\approx$ be the associated equivalence, and $X^* \subseteq \omega^\omega$ be a Σ^1_1 $\preccurlyeq$-thin set. Then $\approx$ is a subrelation of the Σ^1_1 equivalence relation $\mathbf{E} = \mathbf{E}_{\mathbf{F}}$ by Lemma 29. We claim that *the relations $\approx$ and $\mathbf{E}$ coincide on X^**, in particular $x \mathrel{\mathbf{E}} y \Leftrightarrow x \approx y$ for $x, y \in X^*$. Then by Lemma 29, there is a function $F \in \mathbf{F}$ such that $F(x) = F(y)$ implies $x \approx y$ for all $x, y \in X^*$. This yields (I) of Theorem 3.

Assume towards a contradiction that $\approx$ is a *proper* subrelation of $\mathbf{E}$ on X^*; then $\mathbf{V} = \{x \in X^* : \exists y \in X^* \ (x \not\approx y \wedge x \mathrel{\mathbf{E}} y)\}$, a Σ^1_1 set, is non-empty.

LEMMA 36 ([5, Section 3, Claims 2 and 3]). *The condition $\mathbf{V} \times \mathbf{V}$ $(\mathbf{P} \times_{\mathbf{E}} \mathbf{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}$ and $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-incomparable.*

Proof. Suppose to the contrary that a subcondition $X \times Y$ either forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We will get a contradiction in both cases. Note that $X, Y \subseteq \mathbf{V}$ are non-empty Σ^1_1 sets and $[X]_{\mathbf{E}} \cap [Y]_{\mathbf{E}} \neq \emptyset$.

Case A: $X \times Y$ *forces* $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$. Similarly to the proof of Lemma 22, the Σ^1_1 set $W = \{\langle x, x'\rangle \in X \times X : x \mathbf{E} x' \wedge x' \not\approx x\}$ is empty, so that $\mathbf{E}$ and $\approx$ coincide on X. Then, as $X \subseteq \mathbf{V}$, at least one of the Σ^1_1 sets

$$B = \{x' : \exists x \in X \ (x \mathbf{E} x' \wedge x' \not\preccurlyeq x)\}, \quad B' = \{x' : \exists x \in X \ (x \mathbf{E} x' \wedge x \not\preccurlyeq x')\}$$

is non-empty, let say $B \neq \emptyset$. Consider the Σ^1_1 set

$$A = \{x' : \exists x \in X \, (x \mathbf{E} x' \wedge x' \preccurlyeq x)\};$$

then $X \subseteq A$. We have $A \cap B = \emptyset$, A is downwards closed while B is upwards closed in each $\mathbf{E}$-class, therefore $y \not\preccurlyeq x$ whenever $x \in A$, $y \in B$, and $x \mathbf{E} y$. Then $[A]_{\mathbf{E}} \cap [B]_{\mathbf{E}} = \emptyset$ by Corollary 32(i). Yet by definition $[X]_{\mathbf{E}} \cap [B]_{\mathbf{E}} \neq \emptyset$ and $X \subseteq A$, which is a contradiction.

Case B: $X \times Y$ *forces* $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We claim that the Σ^1_1 set $W = \{\langle x, y\rangle \in X \times Y : x \mathbf{E} y \wedge y \preccurlyeq x\}$ is empty. Indeed, suppose that $W \neq \emptyset$. Let $\mathbf{P}(W)$ contain all non-empty Σ^1_1 sets $P \subseteq W$; if $P \in \mathbf{P}(W)$ then $[\mathtt{dom}\, P]_{\mathbf{E}} = [\mathtt{ran}\, P]_{\mathbf{E}}$. Let $\mathbf{P}(W) \times_{\mathbf{E}} \mathbf{P}(W)$ contain all products $P \times Q$, where $P, Q \in \mathbf{P}(W)$ and $[\mathtt{dom}\, P]_{\mathbf{E}} \cap [\mathtt{dom}\, Q]_{\mathbf{E}} \neq \emptyset$; then $W \times W \in \mathbf{P}(W) \times_{\mathbf{E}} \mathbf{P}(W)$.

Let $\langle x, y; x', y'\rangle$ be a $\mathbf{P}(W) \times_{\mathbf{E}} \mathbf{P}(W)$-generic quadruple in $W \times W$, so that both $\langle x, y\rangle \in W$ and $\langle x', y'\rangle \in W$ are $\mathbf{P}(W)$-generic pairs in W, and both $y \preccurlyeq x$ and $y' \preccurlyeq x'$ hold by the definition of W. On the other hand, an easy argument shows that both criss-cross pairs $\langle x, y'\rangle \in X \times Y$ and $\langle x', y\rangle \in X \times Y$ are $\mathbf{P} \times_{\mathbf{E}} \mathbf{P}$-generic, hence $x \prec y'$ and $x' \prec y$ by the choice of $X \times Y$. Altogether $y \preccurlyeq x \prec y' \preccurlyeq x' \prec y$, which is a contradiction.

Thus $W = \emptyset$. Then the Σ^1_1 sets

$$X_0 = \{x' : \exists x \in X \ (x \mathbf{E} x' \wedge x' \preccurlyeq x)\}, \quad Y_0 = \{y' : \exists y \in Y \ (y \mathbf{E} y' \wedge y \preccurlyeq y')\}$$

are disjoint and $\preccurlyeq$-closed, resp. downwards and upwards, in each $\mathbf{E}$-class, hence $[X_0]_{\mathbf{E}} \cap [Y_0]_{\mathbf{E}} = \emptyset$ by Corollary 32(i). However $[X]_{\mathbf{E}} \cap [Y]_{\mathbf{E}} \neq \emptyset$, which is a contradiction as $X \subseteq X_0$ and $Y \subseteq Y_0$. ∎

By Lemmas 36 and 34 ($\mathsf{E}_R = \mathbf{E}$ for all R), the set $\mathbf{V}$, and hence the bigger set X^* as well, are not $\preccurlyeq$-thin, contrary to our assumptions.

18. Decomposing thin analytic partial orderings

Proof of Theorem 3(II). Let $\mathbf{U}$ be the set of all reals $x \in X^*$ such that there is no Δ^1_1 $\preccurlyeq$-chain C containing x. Note that $\mathbf{U}$ is Σ^1_1. Indeed, $x \in \mathbf{U}$ iff $x \in X^*$ and $\forall C \in \Delta^1_1$, if $x \in C$ then C is not a chain, i.e., $\exists y, z \in C \ (z \not\preccurlyeq y \wedge y \not\preccurlyeq z)$. And $\forall C \in \Delta^1_1$ preserves Σ^1_1.

We assume to the contrary that $\mathbf{U} \neq \emptyset$.

Here we will use equivalence relations of the form $\mathbf{E}_R$ (Section 14).

Lemma 37 ([5, Section 5, Claims 1–3]). *Let* $\emptyset \neq R \subseteq \mathbf{U}$ *be* Σ^1_1. *Then* $R \times R$ *does* ***not*** $(\mathbf{P} \times_{\mathbf{E}_R} \mathbf{P})$*-force that* $\dot{\boldsymbol{x}}_{\mathtt{le}}$ *and* $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ *are* $\preccurlyeq$*-comparable.*

Proof. Suppose to the contrary that $R \times R$ forces the comparability. Then there is a subcondition $X \times Y$ which forces either $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$; $X, Y \subseteq R$ are non-empty Σ^1_1 sets and $[X]_{\mathbf{E}_R} \cap [Y]_{\mathbf{E}_R} \neq \emptyset$.

Case A: *$X \times Y$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* Similarly to the proof of Lemma 22, the Σ^1_1 set $W = \{\langle x, x' \rangle \in X \times X : x \,\mathbf{E}_R\, x' \wedge x' \not\approx x\}$ is empty, and hence X is a $\preccurlyeq$-chain by Lemma 35. To cover X by a Δ^1_1 chain, make use of a typical trick. Let C be the Π^1_1 set of all reals $\preccurlyeq$-comparable with each $x \in X$; then $X \subseteq C$. By separation there is a Δ^1_1 set D with $X \subseteq D \subseteq C$. Let C' be the Π^1_1 set of all reals in D that are $\preccurlyeq$-comparable with each $d \in D$; then $X \subseteq C' \subseteq D$. Take any Δ^1_1 set B with $X \subseteq B \subseteq C'$. By construction B is a Δ^1_1 $\preccurlyeq$-chain with $\emptyset \neq X \subseteq B$, contrary to the definition of $\mathbf{U}$.

Case B: *$X \times Y$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* Similarly to the proof of Claim 24, the Σ^1_1 set $W = \{\langle x, y \rangle \in X \times Y : x \,\mathbf{E}_R\, y \wedge x \not\prec y\}$ is empty, so that if $x \in X$, $y \in Y$ and $x \,\mathbf{E}_R\, y$ then $x \prec y$. Thus $[X]_{\mathbf{E}_R} \cap [Y]_{\mathbf{E}_R} = \emptyset$ by Corollary 32(ii), which contradicts the choice of $X \times Y$ in $\mathbf{P} \times_{\mathbf{E}_R} \mathbf{P}$. ■

Lemmas 37 and 34 imply claim (II) of Theorem 3. ■$_{\text{Theorem 3}}$

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