INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES MATHEMATICAE

518

NURULLA AZAMOV

Spectral flow inside essential spectrum

WARSZAWA 2016

Nurulla Azamov
School of Computer Science, Engineering and Mathematics
Flinders University, Tonsley
5042 Clovelly Park, SA, Australia
E-mail: nurulla.azamov@flinders.edu.au

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound in Poland by HermanDruK, Warszawa
Nakład 200 egz.

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

Available online at **http://journals.impan.pl**

DOI: 10.4064/dm740-7-2015

ISSN 0012-3862

Contents

Abstract

Spectral flow is a classical notion of functional analysis and differential geometry which was given different interpretations as Fredholm index, Witten index, and Maslov index. The classical theory treats spectral flow outside essential spectrum. Inside essential spectrum, the spectral shift function could be considered as a proper analogue of spectral flow, but unlike spectral flow, the spectral shift function is not an integer-valued function.

In this paper it is shown that the notion of spectral flow admits a natural extension for a.e. value of the spectral parameter inside essential spectrum too, and an appropriate theory is developed. The definition of spectral flow inside essential spectrum given in this paper applies to classical spectral flow and thus provides one more new alternative definition of it.

One of the results of this paper asserts that for trace class self-adjoint perturbations of self-adjoint operators the following four integer-valued functions are equal almost everywhere. The common value of these functions is spectral flow inside essential spectrum by definition.

1) Singular spectral shift function.
2) Singular part of the Pushnitski μ-invariant.
3) The so-called total resonance index.
4) The so-called total signature of resonance matrices.

Equality of the third and the fourth functions is proved under much weaker assumptions which cover Schrödinger operators. Some applications of this result are given.

Acknowledgements. I thank Thomas Daniels for a scrupulous and critical reading of this paper which greatly reduced the number of inaccuracies and typos. I also thank Prof. Peter Dodds and Prof. Jerzy Filar for their moral support. Finally, I would like to thank the referee for numerous helpful remarks and suggestions.

2010 *Mathematics Subject Classification*: Primary 47A55, 47A10, 47A70, 47A40; Secondary 35P99.

Key words and phrases: spectral flow, spectral shift function, singular spectral shift function, absolutely continuous spectral shift function, scattering matrix, Birman–Krein formula, Lippmann–Schwinger equation.

Received 6.2.2015; revised 22.7.2015.

Published online 2 August 2016.

1. Introduction

This paper develops the theory of spectral flow inside essential spectrum. In order to put the results into context, in this introduction a quick survey is given of relevant parts of the theory of spectral flow, the mathematical theory of scattering, and related notions, from the perspective of this paper. In fact, the introduction and the main body of the paper are quite independent; the reader may choose to omit reading this introduction (as long as he/she does not ask what is the point and origin of the results of the paper), or treat the introduction as an independent survey. This also explains the relatively large size of this introduction.

1.1. Spectral flow. In this subsection we briefly mention several papers on differential geometry and operator theory where the notion of spectral flow was introduced and studied. The subsection can safely be skipped, if necessary.

Spectral flow was introduced by M. Atiyah, V. Patodi and I. M. Singer [APS, APS_2], as the intersection number of eigenvalues of a continuous path D_u, $0 \le u \le 1$, of elliptic self-adjoint pseudo-differential operators on a compact manifold with the line $\lambda = -\varepsilon$, where ε is a small positive number. Atiyah, Patodi and Singer [APS_2] remarked that the spectral flow could in fact be defined for any continuous path of self-adjoint Fredholm operators. The essential spectrum of a self-adjoint Fredholm operator does not contain zero, and this allows one to define the spectral flow formally as the net number of eigenvalues crossing 0 in the positive direction, where it is assumed that if an eigenvalue crosses 0 in the negative direction then its contribution to the spectral flow is negative. I. M. Singer suggested in 1974 that it should be possible to express the spectral flow as an integral of a one-form defined in terms of the path of operators. Such an analytic formula was established by E. Getzler [Ge]:

$$\operatorname{sf}(\mathrm{D}, g^{-1}\mathrm{D}g) = \frac{1}{\sqrt{\pi}} \int_0^1 \operatorname{Tr}(\dot{\mathrm{D}}_u e^{-\mathrm{D}_u^2})\, du, \tag{1.1.1}$$

where D is a self-adjoint operator of an odd θ-summable Fredholm module (see [C] for definition) $(\mathcal{A}, \mathcal{H}, D)$ over a Banach $*$-algebra $\mathcal{A}$, g is a representative of an element $[g]$ of the odd K-theory group $K_1(\mathcal{A})$ (see e.g. [Bl, §8] or [Mu, Chapter 7] for definition), and $\mathrm{D}_u = (1-u)\mathrm{D} + ug^{-1}\mathrm{D}g$. For example by [Ge], in the case when $\mathcal{H} = L_2(\mathbb{T}, d\theta)$, $\mathcal{A} = C(\mathbb{T})$, $\mathrm{D} = \frac{1}{i}\frac{d}{d\theta}$, and $[g]$ is the class of the function $e^{in\theta}$, one has $\mathrm{D}_u = \mathrm{D} + nuI$, where I is the identity operator, so that $\sigma(\mathrm{D}_u) = \{k + nu \colon k \in \mathbb{Z}\}$. Thus, as u changes from 0 to 1, each real number including zero is crossed by n simple eigenvalues of D_u in the positive direction, and therefore $\operatorname{sf}(\mathrm{D}, g^{-1}\mathrm{D}g) = n$.

For a norm-continuous path $F\colon [a,b] \to \mathcal{B}(\mathcal{H})$ of self-adjoint Fredholm operators where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators, J. Phillips [Ph, Ph_2] gave an alternative definition of spectral flow:

$$\mathrm{sf}(\{F_t\}) = \sum_{i=1}^{n} \mathrm{ec}(P_{t_{i-1}}, P_{t_i}),$$

where $P_t = E^{F_t}_{[0,\infty)}$ is the spectral projection of F_t corresponding to the interval $[0,\infty)$ and $\mathrm{ec}(P,Q)$ is the essential co-dimension of a Fredholm pair of projections P, Q (see [ASS] for definition, see also [AS, K, Ka_3]), which is defined as the Fredholm index of the operator $PQ\colon Q\mathcal{H} \to P\mathcal{H}$. It was shown in [Ph, Ph_2] that this definition of spectral flow is correct, that it is independent of the choice of small enough partitions and that it is homotopically invariant. The spectral flow $\mathrm{sf}(F_0, F_1)$ for a pair of Fredholm operators F_0 and F_1 with compact difference is then defined by the above formula for the straight path $(1-t)F_0 + tF_1$ connecting F_0 and F_1, and for a pair of self-adjoint operators D_0, D_1 with compact resolvents and bounded difference the spectral flow is defined by

$$\mathrm{sf}(D_0, D_1) = \mathrm{sf}(\phi(D_0), \phi(D_1)),$$

where $\phi(x) = x(1+x^2)^{-1/2}$.

The analytic formula (1.1.1) was generalized by A. Carey and J. Phillips [CP, CP_2], who in particular proved the following formula [CP_2, Corollary 8.10] for the spectral flow for two θ-summable operators D_0 and D_1:

$$\begin{aligned}\mathrm{sf}(D_0, D_1) = \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{Tr}\left(\frac{dD_t}{dt}\, e^{-D_t^2} \right) dt + \eta_1(D_1) - \eta_1(D_0) \\ + \tfrac{1}{2}\,\mathrm{Tr}([\ker(D_1)]) - \tfrac{1}{2}\,\mathrm{Tr}([\ker(D_0)]),\end{aligned} \tag{1.1.2}$$

where $[\ker(D_j)]$ is the projection onto the kernel of D_j and where the real number

$$\eta_1(D) = \frac{1}{\sqrt{\pi}} \int_1^\infty \mathrm{Tr}(D\, e^{-tD^2})\, \frac{dt}{\sqrt{t}}$$

is the so-called η-invariant of D_j, a notion introduced for self-adjoint elliptic operators on compact manifolds by Atiyah, Patodi and Singer [APS_2]. A formula analogous to (1.1.2) was also established for p-summable operators. It was moreover shown in [CP_2] that the one-form on the affine space of θ-summable self-adjoint operators $\{D_0 + A\colon A$ is a bounded self-adjoint operator$\}$ given by

$$\alpha_D(A) = \frac{1}{\sqrt{\pi}}\,\mathrm{Tr}(A\, e^{-D^2})$$

is exact.

The nature of integral formulas for the spectral flow such as (1.1.1), (1.1.2) was clarified in [ACS], where it was proved [ACS, (35)] that for any two self-adjoint operators D_0 and D_1 with compact resolvent such that $D_1 - D_0$ is bounded we have

$$\mathrm{sf}(\lambda; D_0, D_1) = \xi_{D_1,D_0}(\lambda) + \tfrac{1}{2}\,\mathrm{Tr}([\ker(D_1 - \lambda)]) - \tfrac{1}{2}\,\mathrm{Tr}([\ker(D_0 - \lambda)]), \tag{1.1.3}$$

where $\xi_{D_1,D_0}(\lambda)$ is the so-called spectral shift function. The formula (1.1.3) is quite general in the sense that firstly it allows one to recover integral formulas for spectral flow by averaging over an appropriate probability distribution $\phi(\lambda)$, which in the case of the

formulas of Getzler (1.1.1) and Carey–Phillips (1.1.2) is the Gaussian, and secondly, unlike other integral formulas it does not impose any summability conditions on the operators D_0 and D_1.

Though in [ACS] the operators D_r were assumed to have compact resolvent, the same technique of proof shows that a connection between the spectral flow and the spectral shift function given by (1.1.3) holds for norm-continuous paths D_r of self-adjoint operators with trace class difference if λ does not belong to the common essential spectrum of the operators D_r (see also [Pu$_4$]).

1.2. Spectral shift function. The works on spectral flow discussed above were written by geometers, who were interested in it primarily as a topological invariant and in its connections with other topological invariants, such as Chern character (see e.g. [KN, We] for definition). See also, for instance, [BCPRSW, BF, BLP, CPRS, CPRS$_2$, CPRS$_3$, CM]. A notion closely related to spectral flow appeared in 1952 in the work of I. M. Lifshitz [L]. He introduced and developed a formalism for the *spectral shift function* $\xi(\lambda)$ of a pair of self-adjoint operators H_0 and H_1 with finite rank difference $V = H_1 - H_0$. The function $\xi(\lambda)$ in [L] was defined by

$$\xi(\lambda) = \operatorname{Tr}(E_\lambda^{H_1} - E_\lambda^{H_0}). \tag{1.2.1}$$

In particular, Lifshitz observed that the spectral shift function formally satisfies the following equality, called the *trace formula*:

$$\operatorname{Tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)\,d\lambda. \tag{1.2.2}$$

Lifshitz introduced the spectral shift function in connection with a problem of solid state physics, in which the initial operator H_0 is the Hamiltonian of a pure crystal and V is the perturbation generated by a point impurity, and his work had a formal character.

A mathematically rigorous theory of the spectral shift function was created one year later by M. G. Kreĭn [Kr]. He showed that for any pair of self-adjoint operators H_0 and H_1 with trace class difference $V = H_1 - H_0$ there exists a unique (up to a set of zero measure, of course) integrable function $\xi(\lambda)$ such that for all functions f from a class which includes $C_c^2(\mathbb{R})$, the trace formula (1.2.2) holds. Kreĭn also demonstrated by a counter-example that (1.2.1) cannot serve as a definition of the spectral shift function, since $E_\lambda^{H_1} - E_\lambda^{H_0}$ may fail to be trace class. Further, a description of the largest class of functions f for which (1.2.2) holds was given by V. V. Peller [Pel] in terms of Besov spaces (see also [Far]). There is a large literature on the spectral shift function: see e.g. [GM, GM$_2$, Pu, Pu$_3$, S].

M. Sh. Birman and M. Z. Solomyak [BS$_2$] showed that for any self-adjoint operator H_0 and any trace class self-adjoint operator V the spectral shift function $\xi_{H_1,H_0}(\lambda)$ satisfies the equality

$$\xi(\lambda) = \frac{d}{d\lambda}\int_0^1 \operatorname{Tr}(VE_\lambda^{H_r})\,dr \quad \text{a.e. } \lambda, \tag{1.2.3}$$

where

$$H_r = H_0 + rV, \quad r \in \mathbb{R},$$

and where E_λ^H is the spectral projection of H corresponding to the interval $(-\infty, \lambda]$. If we are to interpret the spectral shift function $\xi(\lambda)$ as a distribution $\xi(\phi)$, $\phi \in C_c^\infty(\mathbb{R})$, the Birman–Solomyak formula (1.2.3) can be rewritten as

$$\xi(\phi) = \int_0^1 \operatorname{Tr}(V\phi(H_r))\, dr \qquad \forall \phi \in C_c^\infty(\mathbb{R}). \tag{1.2.4}$$

The Birman–Solomyak formula (1.2.3) rewritten in the form (1.2.4) makes a clear connection between the integral formulas for spectral flow (1.1.1), (1.1.2), etc. and the spectral shift function: both are integrals of one-forms

$$\alpha_H^f(V) = \operatorname{Tr}(Vf(H)) \tag{1.2.5}$$

on a real affine space $H_0 + \mathcal{A}_0$ of self-adjoint operators, where $\mathcal{A}_0$ is a real vector space of self-adjoint operators. This connection was observed and used in [ACS] to derive a general integral formula for the spectral flow in the case of self-adjoint operators H with compact resolvent and $\mathcal{A}_0 = \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$. It was shown in [ACS] that the one-forms (1.2.5) are exact on the affine space $H + \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ for any compactly supported smooth function f, and therefore integrals over all piecewise smooth continuous paths connecting H_0 and $H_0 + V$ coincide and are equal to the right hand side of (1.2.4).

An analogue of this result was proved in [AzS] for so-called trace compatible perturbations, which include self-adjoint operators with compact resolvent and bounded perturbations, as well as arbitrary self-adjoint operators and trace class perturbations. An affine space $\mathcal{A} = H_0 + \mathcal{A}_0$ of self-adjoint operators is called *trace compatible* if for any $H \in \mathcal{A}$, any perturbation $V \in \mathcal{A}_0$, and any compactly supported continuous function ϕ we have $V\phi(H) \in \mathcal{L}_1(\mathcal{H})$, where $\mathcal{L}_1(\mathcal{H})$ is the class of operators with finite trace. This definition was motivated by the distribution version (1.2.4) of the Birman–Solomyak formula (1.2.3), since trace compatibility is the least requirement which one needs to impose on the operators $H_0 + rV$ to give a meaning to the integral in (1.2.4).

One of the important developments in the theory of the spectral shift function occurred when V. S. Buslaev and L. D. Faddeev [BuFa] observed a connection between the spectral shift function and the phase shift of the scattering matrix. This connection for trace class perturbations of self-adjoint operators was established by M. Sh. Birman and M. G. Kreĭn [BK]; namely, for self-adjoint operators H_0 and H_1 with trace class difference $V = H_1 - H_0$ they proved the formula

$$e^{-2\pi i \xi(\lambda)} = \det S(\lambda; H_1, H_0), \tag{1.2.6}$$

where $S(\lambda; H_1, H_0)$ is the scattering matrix for the pair (H_1, H_0) (see e.g. [Y]), whose definition is found in the next subsection, det is the Fredholm determinant (see e.g. [GK, Chapter 4], [S_2, Chapter 3] or [RS_4, §XIII.7]) and $\xi(\lambda)$ is the spectral shift function of (H_1, H_0).

1.3. Scattering theory. The *scattering operator* $\mathbf{S}(H_1, H_0)$ for a pair (H_1, H_0) of self-adjoint operators is defined by (see e.g. [BW, RS_3, Y])

$$\mathbf{S}(H_1, H_0) = W_+^*(H_1, H_0) W_-(H_1, H_0), \tag{1.3.1}$$

where the *Möller wave operators* $W_\pm(H_1, H_0)$ are defined, if they exist, as the strong operator limits

$$W_\pm(H_1, H_0) = \lim_{t \to \pm\infty} e^{itH_1} e^{-itH_0} P^{(a)}(H_0), \tag{1.3.2}$$

where $P^{(a)}(H_0)$ is the orthogonal projection onto the absolutely continuous subspace of H_0 (for definition, see e.g. [RS, Theorem VII.4 and the preceding definition]). The classical Kato–Rosenblum theorem ([Ka, R], see also [RS$_3$, Theorem XI.8], [Y, Theorem 6.2.3]) asserts that if $H_1 - H_0$ is trace class, then $W_\pm(H_1, H_0)$ exist and are therefore complete (by symmetry of the condition $H_1 - H_0 \in \mathcal{L}_1(\mathcal{H})$), which implies that (1.3.1) exists as well. Completeness of wave operators means that both $W_+(H_1, H_0)$ and $W_-(H_1, H_0)$ are partial isometries whose initial space is the absolutely continuous subspace $\mathcal{H}^{(a)}(H_0)$ with respect to H_0, and the final space is $\mathcal{H}^{(a)}(H_1)$.

One of the many versions of the spectral theorem asserts that, given a self-adjoint operator H_0, the absolutely continuous subspace $\mathcal{H}^{(a)}(H_0)$ admits a representation as a direct integral of Hilbert spaces

$$\mathcal{F}\colon \mathcal{H}^{(a)}(H_0) \to \int_{\hat\sigma_{H_0}}^{\oplus} \mathfrak{h}_\lambda \, \rho(d\lambda) \tag{1.3.3}$$

such that for any $f \in \mathcal{H}^{(a)}(H_0) \cap \operatorname{dom}(H_0)$ the equality

$$\mathcal{F}(H_0 f)(\lambda) = \lambda \mathcal{F}(f)(\lambda)$$

holds for a.e. $\lambda \in \hat\sigma_{H_0}$, where $\hat\sigma_{H_0}$ is a core of the absolutely continuous spectrum of H_0, $\{\mathfrak{h}_\lambda\colon \lambda \in \hat\sigma_{H_0}\}$ is a measurable family of Hilbert spaces, ρ is an absolutely continuous Borel measure with Borel support $\hat\sigma_{H_0}$, and $\mathcal{F}$ is a unitary operator; for definition of the direct integral of Hilbert spaces see e.g. [BW, BS].

By the Kato–Rosenblum theorem, the scattering operator $\mathbf{S}(H_1, H_0)$ is a partial isometry with initial and final space $\mathcal{H}^{(a)}(H_0)$; further, it commutes with H_0. These properties imply (see e.g. [BÈ, BY, BY$_2$, Y]) that in the spectral representation (1.3.3) the scattering operator (1.3.1) is represented by a direct integral

$$\mathbf{S}(H_1, H_0) = \int_{\hat\sigma_{H_0}}^{\oplus} S(\lambda; H_1, H_0) \, \rho(d\lambda), \tag{1.3.4}$$

where $\{S(\lambda; H_1, H_0)\colon \lambda \in \hat\sigma_{H_0}\}$ is a measurable family of unitary operators on fibre Hilbert spaces $\mathfrak{h}_\lambda$.

The spectral parameter λ has a physical meaning of energy E; the fibre Hilbert space $\mathfrak{h}_\lambda$ is often called an *energy shell*. Physicists call the unitary operator $S(\lambda; H_1, H_0)$ the *on-shell scattering operator*, while $\mathbf{S}(H_1, H_0)$ itself is called the *off-shell scattering operator* (see e.g. [T, §3-b]; see also [RS$_3$, Theorem XI.42 and the following discussion]). In physics there is a famous stationary formula due mainly to B. Lippmann and J. Schwinger [LSch] and Gell-Mann and Goldberger [GG] for the on-shell scattering operator (see e.g. [T], [RS$_3$, Theorem XI.42])

$$\begin{aligned}\langle \mathbf{p}' | \mathsf{S} | \mathbf{p} \rangle &= \delta_3(\mathbf{p}' - \mathbf{p}) - 2\pi i \delta(E_{p'} - E_p) \langle \mathbf{p}' | V(1 - G^0(E_p + i0)V)^{-1} | \mathbf{p} \rangle \\ &= \delta_3(\mathbf{p}' - \mathbf{p}) - 2\pi i \delta(E_{p'} - E_p) \langle \mathbf{p}' | (V + VG(E_p + i0)V) | \mathbf{p} \rangle, \end{aligned} \tag{1.3.5}$$

which follows from a combination of [T, (3.7), (8.11) and (8.22)]. This is a version of the stationary formula for one spinless particle, scattered by a potential V; there are stationary formulas for particles with a spin and for multi-particle systems as well (see e.g. [T]).

In rigorous mathematical notation the stationary formula (1.3.5) for a self-adjoint operator H_0 and its trace class perturbation $H_1 = H_0 + V$ should have been written as

$$S(\lambda; H_1, H_0) = 1 - 2\pi i \mathcal{F}_\lambda V (1 - R_{\lambda+i0}(H_0)V)^{-1} \mathcal{F}_\lambda^*, \tag{1.3.6}$$

where $\mathcal{F}_\lambda \colon \mathcal{H}^{(a)}(H_0) \to \mathfrak{h}_\lambda$ is a fibre of the unitary operator (1.3.3) ([1]). But, unfortunately, the expression on the right hand side of (1.3.6) does not make sense for two reasons: firstly, the limit of the resolvent $R_{\lambda+i0}(H_0) := (H_0 - \lambda - i0)^{-1}$ does not in general exist even in the weakest of all reasonable topologies (for a discussion of this question see e.g. [Y, §6.1]), and secondly, the operator $\mathcal{F}_\lambda$ is not well-defined for a particular value of λ.

A mathematically rigorous version of the stationary formula (1.3.5) for the scattering matrix was established by L. D. Faddeev [Fa] (see also [LF]) in the setting of the Friedrichs–Faddeev model [Fr, Fr_2, Fr_3, Y]. In that model the initial self-adjoint operator H_0 is multiplication by the independent variable x in the Hilbert space $L_2[a, b; \mathfrak{h}]$, $-\infty \le a < b \le \infty$, of square-integrable $\mathfrak{h}$-valued functions, where $\mathfrak{h}$ is a fixed Hilbert space, and the perturbation operator V is an integral operator

$$Vf(x) = \int_a^b v(x, y) f(y)\, dy$$

with sufficiently regular kernel $v \colon [a, b]^2 \to \mathcal{B}(\mathfrak{h})$. A detailed exposition of the stationary approach to scattering theory for the Friedrichs–Faddeev model can be found in [Y, Chapter 4].

Another important setting is short range potential scattering theory (see e.g. [Po, Po_2, I, Ka_4, Ag, Ku_2, Ku_3]; expositions of this theory and literature can be found in [Ag, Ku], see also [Y_2]). In potential scattering theory the initial operator H_0 is the Laplace operator

$$H_0 u = -\Delta u \tag{1.3.7}$$

on the Hilbert space $L_2(\mathbb{R}^n)$, where the domain of H_0 is the Sobolev space $\mathsf{H}_2(\mathbb{R}^n)$ (see e.g. [RS_2, IX.6] for definition); a short range perturbation V is multiplication by a measurable function $q \colon \mathbb{R}^n \to \mathbb{R}$ which satisfies an estimate $|q(x)| \le C(1 + |x|^2)^{-\rho/2}$, where $\rho > 1$ (in [Ag] short range potentials are defined by a weaker condition of integral type). The perturbed operator H is the Schrödinger operator

$$Hu(x) = -\Delta u(x) + q(x)u(x). \tag{1.3.8}$$

In this case the spectral structure of H_0 is completely transparent since it can be diagonalized by the Fourier transform $\mathcal{F}$, that is,

$$H_0 = \mathcal{F}^* M_{|\xi|^2} \mathcal{F}, \tag{1.3.9}$$

([1]) A sign mismatch in formulas (1.3.5) and (1.3.6) comes from the definitions of the resolvent $R_z(H) = (H - z)^{-1}$ and of the Green operator $G(z) = (z - H)^{-1}$, as in [T, §8-a].

where $M_{|\xi|^2}$ is multiplication by $|\xi|^2$. So, in this case $\mathcal{H}^{(a)}(H_0) = \mathcal{H}$, and in the decomposition (1.3.3) one can take a core of the absolutely continuous spectrum $\hat{\sigma}_{H_0}$ to be $(0,\infty)$, the measure $\rho(d\lambda)$ to be Lebesgue measure $d\lambda$ and the fibre Hilbert space $\mathfrak{h}_\lambda$ to be $L_2(\Sigma_{\sqrt{\lambda}})$, where $\Sigma_{\sqrt{\lambda}} = \{\xi \in \mathbb{R}^n_\xi \colon |\xi| = \sqrt{\lambda}\}$ is the sphere with surface measure inherited from $\mathbb{R}^n_\xi$. The scattering operator (1.3.1) for the pair (H, H_0) given by (1.3.8) and (1.3.7) exists and it admits the decomposition (1.3.4). Further, for all $\lambda > 0$ except possibly a discrete subset $e_+(H)$ of positive values of λ, the stationary formula for the scattering matrix holds in the form

$$\begin{aligned} S(\lambda) &= 1 - 2\pi i c(\lambda)\gamma_0(\lambda)\mathcal{F}V(1 + R_{\lambda+i0}(H_0)V)^{-1}\mathcal{F}^*\gamma_0^\diamondsuit(\lambda) \\ &= 1 - 2\pi i c(\lambda)\gamma_0(\lambda)\mathcal{F}(V - VR_{\lambda+i0}(H)V)\mathcal{F}^*\gamma_0^\diamondsuit(\lambda) \end{aligned} \tag{1.3.10}$$

(for details see [Ag, Ku]): Here $c(\lambda)$ is a constant which occurs as a result of the change from Cartesian coordinates to polar coordinates in the momentum space $\mathbb{R}^n_\xi$. For any $s \in \mathbb{R}$ let $L_{2,s}(\mathbb{R}^n)$ be the weighted Hilbert space of measurable functions $u \colon \mathbb{R}^n \to \mathbb{C}$ for which

$$\|u\|_{0,s} := \|(1 + |x|^2)^{s/2}u\|_{L_2(\mathbb{R}^n)} < \infty,$$

and let

$$\mathsf{H}_{m,s}(\mathbb{R}^n) = \{u \colon D^\alpha u \in L_{2,s}(\mathbb{R}^n),\ 0 \le |\alpha| \le m\}$$

be the weighted Sobolev space with norm

$$\|u\|_{m,s} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{0,s}\Big)^{1/2}.$$

A rigorous treatment of the stationary formula in potential scattering theory is based on the following theorems, whose proofs can be found in [Ag, Ku]. In general, a form of the Limiting Absorption Principle is of utmost importance for stationary scattering theory.

THEOREM 1.3.1. *If $q(x)$ is a short range potential, then there exists $\varepsilon' > 0$ such that for any $s \in \mathbb{R}$ and for all $\varepsilon \in (0, \varepsilon')$, multiplication by $q(x)$ is a compact operator from the Hilbert space $\mathsf{H}_{2,s}(\mathbb{R}^n)$ to the Hilbert space $L_{2,1+s+\varepsilon}(\mathbb{R}^n)$.*

THEOREM 1.3.2 (The Limiting Absorption Principle for $-\Delta$; see [Ag, Theorem 4.1], [Ku, §4.4]). *Let $H_0 = -\Delta$ with domain $\mathsf{H}_2(\mathbb{R}^n)$. For any $s > 1/2$ and any $\lambda > 0$ the resolvents $R_{\lambda\pm iy}(H_0)$ as operators from $L_{2,s}(\mathbb{R}^n)$ to $\mathsf{H}_{2,-s}(\mathbb{R}^n)$ converge in the uniform operator topology as $y \to 0$, so the bounded operators*

$$R_{\lambda\pm i0}(H_0) \in \mathcal{B}(L_{2,s}(\mathbb{R}^n), \mathsf{H}_{2,-s}(\mathbb{R}^n))$$

exist.

THEOREM 1.3.3 (see e.g. [Ag, Theorem 3.1], [RS$_4$, Theorem XIII.33]). *Let $H = -\Delta + V$ be a Schrödinger operator with domain $\mathsf{H}_2(\mathbb{R}^n)$, where V is a short range potential. The set $e_+(H)$ of positive eigenvalues of H is a discrete subset of $(0,\infty)$, all eigenvalues from $e_+(H)$ have finite multiplicity and the only possible limit points of $e_+(H)$ are 0 and ∞.*

THEOREM 1.3.4 (The Limiting Absorption Principle for $-\Delta + V$, see [Ag, Theorem 4.2], [Ku, §5.3]). *Let $H = -\Delta + V$ be a Schrödinger operator with short range potential V.*

For any $s > 1/2$ and any $\lambda > 0$ not in $e_+(H)$ the resolvents $R_{\lambda\pm iy}(H)$ as operators from $L_{2,s}(\mathbb{R}^n)$ to $\mathsf{H}_{2,-s}(\mathbb{R}^n)$ converge in the uniform operator topology as $y \to 0$, so the bounded operators

$$R_{\lambda\pm i0}(H) \in \mathcal{B}(L_{2,s}(\mathbb{R}^n), \mathsf{H}_{2,-s}(\mathbb{R}^n))$$

exist.

Further, for any $s \in \mathbb{R}$ the Fourier transform $\mathcal{F}$ is a unitary operator from $L_{2,s}(\mathbb{R}^n)$ onto $\mathsf{H}_s(\mathbb{R}^n)$. For any $s > 1/2$ the term $\gamma_0(\lambda)$ in (1.3.10) is a well-defined bounded operator from $\mathsf{H}_s(\mathbb{R}^n)$ to $L_2(\Sigma_{\sqrt{\lambda}})$ (the trace theorem, see e.g. [Ag, §2], [Ku, Theorem 4.2.1]); namely, the operator $\gamma_0(\lambda)$ is a continuous extension of the restriction operator

$$C_c^\infty(\mathbb{R}^n_\xi) \ni f \mapsto f|_{\Sigma_{\sqrt{\lambda}}} \in L_2(\Sigma_{\sqrt{\lambda}}).$$

Finally, the bounded operator $\gamma_0^\diamondsuit(\lambda)\colon L_2(\Sigma_{\sqrt{\lambda}}) \to \mathsf{H}_{-s}$ can be defined for any $s > 1/2$ by

$$\langle \gamma_0^\diamondsuit(\lambda) f, g\rangle_{-s,s} = \langle f, \gamma_0(\lambda) g\rangle_{L_2(\Sigma_{\sqrt{\lambda}})}, \tag{1.3.11}$$

where $f \in L_2(\Sigma_{\sqrt{\lambda}})$ and $g \in \mathsf{H}_s(\mathbb{R}^n)$ and $\langle\cdot,\cdot\rangle_{-s,s}$ is the natural pairing of Hilbert spaces $\mathsf{H}_{-s}(\mathbb{R}^n)$ and $\mathsf{H}_s(\mathbb{R}^n)$, defined by

$$\langle f, g\rangle_{-s,s} = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} g(\xi)\, d\xi.$$

So, the stationary formula (1.3.10) acquires a precise meaning if the factors on the right hand side are understood as acting between appropriate Hilbert spaces:

$$L_2(\Sigma_{\sqrt{\lambda}}) \xleftarrow{\gamma_0} \mathsf{H}_{\tilde{s}} \xleftarrow{\mathcal{F}} L_{2,\tilde{s}} \xleftarrow{V} \mathsf{H}_{2,-\tilde{s}} \xleftarrow{R_{\lambda+i0}(H)} L_{2,\tilde{s}} \xleftarrow{V} L_{2,-s} \xleftarrow{\mathcal{F}^*} \mathsf{H}_{-s} \xleftarrow{\gamma_0^\diamondsuit} L_2(\Sigma_{\sqrt{\lambda}}),$$

where

$$\tilde{s} = 1 - s + \varepsilon,$$

as long as s and ε are chosen so that $s, \tilde{s} > 1/2$ and

$$\tilde{s} = s - \varepsilon + \varepsilon'.$$

The last equality ensures compactness of the operator $V\colon \mathsf{H}_{2,-\tilde{s}} \to L_{2,\tilde{s}}$ according to Theorem 1.3.1. For instance, one can take $s = 1/2 + \varepsilon'/4$ and $\varepsilon = 3\varepsilon'/4$.

The set $e_+(H)$ of eigenvalues of H is related to the set of points λ for which the operator $1 + R_{\lambda+i0}(H_0)V$ is not invertible (see e.g. proof of [Ag, Theorem 4.2]), and the operator $HE^H_{(0,\infty)\setminus e_+(H)}$ is absolutely continuous [Ag, Theorem 6.1].

A mathematically rigorous version of the stationary formula (1.3.6) for arbitrary self-adjoint trace class perturbations of arbitrary self-adjoint operators was proved in [BÈ] (see also [Y]). To give (1.3.6) a rigorous meaning, one needs to introduce an artificial factorization of the perturbation operator V. Assuming that V is trace class, one can write V in the form G^*JG, where G is a Hilbert–Schmidt operator acting from the Hilbert space $\mathcal{H}$ to possibly another Hilbert space $\mathcal{K}$ and where J is a bounded operator on $\mathcal{K}$. Using the factorization $V = G^*JG$, the formal formula (1.3.6) can be rewritten as

$$S(\lambda; H_1, H_0) = 1 - 2\pi i\, (\mathcal{F}_\lambda G^*)\, J(1 - GR_{\lambda+i0}(H_0)G^*J)^{-1} G\, \mathcal{F}_\lambda^*, \qquad \text{a.e. } \lambda \in \mathbb{R},$$

or, introducing the notation

$$Z_0(\lambda; G) = \mathcal{F}_\lambda G^* \tag{1.3.12}$$

and

$$T_{\lambda+i0}(H_0) = GR_{\lambda+i0}(H_0)G^*,$$

as

$$S(\lambda; H_1, H_0) = 1 - 2\pi i\, Z_0(\lambda; G)J(1 - T_{\lambda+i0}(H_0)J)^{-1}Z_0^*(\lambda; G), \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (1.3.13)$$

In this formula the two hindrances mentioned above are overcome: the abstract Limiting Absorption Principle (proved in [BÈ, Br]; see Theorem 2.5.2 below) asserts that the limit $T_{\lambda+i0}(H_0)$ exists in Hilbert–Schmidt norm for a.e. λ, and the product $Z_0(\lambda; G) = \mathcal{F}_\lambda G^*$ also makes sense for a.e. λ as an operator from $\mathcal{K}$ to $\mathfrak{h}_\lambda$, and moreover this product is Hilbert–Schmidt.

Nevertheless, it should be noted that while $S(\lambda; H_1, H_0)$ is defined by the right hand side of (1.3.13) for almost every value of λ, still for no particular choices of $\lambda \in \mathbb{R}$ is the operator $S(\lambda; H_1, H_0)$ well-defined. The source of this uncertainty is in the factor $Z_0(\lambda; G)$, whose definition (1.3.12) involves the unitary operator $\mathcal{F}$ from (1.3.3). This uncertainty is not possible to eradicate, since in (1.3.3) the choice of a core $\hat{\sigma}$ of the absolutely continuous spectrum is arbitrary, partially due to the possible presence of pure point and singular continuous spectra, and the measure ρ can be replaced by any other measure of the same spectral type. This was not considered as a hindrance in abstract scattering theory, in which one works as a rule with two operators, initial H_0 and perturbed H_1. However, in [Az], in an attempt to find an operator version of the Birman–Kreĭn formula (1.2.6), the following formula was derived:

$$S(\lambda; H_1, H_0) = \operatorname{Texp}\left(-2\pi i \int_0^1 w_+(\lambda; H_0, H_r)Z_r(\lambda; G)JZ_r^*(\lambda; G)w_+(\lambda; H_r, H_0)\, dr\right), \quad (1.3.14)$$

where the subscript r on Z_r indicates that in (1.3.12) the unitary operator $\mathcal{F}$ is from the spectral representation of $H_r = H_0 + rV$, and where the so-called *wave matrix* (see e.g. [Y])

$$w_\pm(\lambda; H_1, H_0)\colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_1)$$

is taken from the direct integral representation of the wave operator $W_\pm(H_1, H_0)$:

$$W_\pm(H_1, H_0) = \int_{\hat{\sigma}_{H_0}}^{\oplus} w_\pm(\lambda; H_1, H_0)\, \rho(d\lambda), \quad (1.3.15)$$

analogous to the spectral representation (1.3.4) of the scattering operator $\mathbf{S}(H_1, H_0)$. (For a rigorous definition and basic properties of the *chronological exponential* $\operatorname{Texp}(\int_a^b A(s)\, ds)$ of a path of trace class operators $A(s)$ continuous in trace class norm which were used in the proof of (1.3.14) see [Az$_3$, Appendix A]; for a formal definition of Texp see e.g. [BoSh, Chapter 4].)

The proof of (1.3.14) relies on the validity of the stationary formula (1.3.13) for a continuous family $\{H_r\colon r \in [0,1]\}$ of operators, and more importantly, it requires the operators $w_+(\lambda; H_r, H_0)$ and $Z_r(\lambda; G)$ to be well-defined for a *continuous* set $[0,1]$ of values of r. For this reason, the proof of (1.3.14) only works under stringent conditions on H_0 and V which ensure the existence of $w_+(\lambda; H_r, H_0)$ and $Z_r(\lambda; G)$. As discussed

above, these conditions which were postulated in [Az] hold for a class of short-range Schrödinger operators.

Further, it was observed in [Az] that, provided $S(\lambda; H_1, H_0) - 1$ is trace class, the equality (1.3.14) implies the following modified Birman–Kreĭn formula:

$$e^{-2\pi i \xi^{(a)}(\lambda)} = \det S(\lambda; H_1, H_0), \quad \text{a.e. } \lambda \in \mathbb{R}, \tag{1.3.16}$$

where the function $\xi^{(a)}(\lambda) = \xi^{(a)}_{H_1,H_0}(\lambda)$, called in [Az] the *absolutely continuous spectral shift function*, can be defined as the density of the absolutely continuous measure $\xi^{(a)}(\phi)$, $\phi \in C_c(\mathbb{R})$, given by

$$\xi^{(a)}(\phi) = \int_0^1 \operatorname{Tr}(V\phi(H_r^{(a)}))\, dr, \quad \phi \in C_c(\mathbb{R}). \tag{1.3.17}$$

Here the self-adjoint operator $H_r^{(a)}$ is the absolutely continuous part of H_r. Analogously, one can define the *singular spectral shift function* $\xi^{(s)}(\lambda)$, the density of the absolutely continuous measure $\xi^{(s)}(\phi)$, $\phi \in C_c(\mathbb{R})$, defined by

$$\xi^{(s)}(\phi) = \int_0^1 \operatorname{Tr}(V\phi(H_r^{(s)}))\, dr, \quad \phi \in C_c(\mathbb{R}), \tag{1.3.18}$$

where the self-adjoint operator $H_r^{(s)}$ is the singular part of H_r. One can note that the definitions of $\xi^{(a)}$ and $\xi^{(s)}$ are modifications of the Birman–Solomyak formula (1.2.4) for the spectral shift function ξ, and these functions are related by

$$\xi = \xi^{(a)} + \xi^{(s)}, \tag{1.3.19}$$

which is an immediate consequence of (1.2.4), (1.3.17) and (1.3.18). In particular, absolute continuity of the measure $\xi^{(s)}$ follows from (1.3.19).

Now, the Birman–Kreĭn formula (1.2.6) combined with (1.3.16) implies $e^{-2\pi i \xi^{(s)}(\lambda)} = 1$ for a.e. λ, that is,

$$\xi^{(s)}(\lambda) \in \mathbb{Z} \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{1.3.20}$$

By Weyl's theorem on stability of the essential spectrum of a self-adjoint operator under relatively compact perturbations (see e.g. [Ka$_2$, §IV.5.6], [RS$_4$, §XIII.4]), the essential spectra of all operators $H_r = H_0 + rV$ are identical. Hence, it follows from (1.3.17) that $\xi^{(a)}$ vanishes outside the common essential spectrum of the operators H_r. Therefore, outside the essential spectrum the singular spectral shift function $\xi^{(s)}$ coincides with the spectral shift function; equivalently, it coincides with the spectral flow. But unlike the spectral flow, the singular spectral shift function is still defined inside the essential spectrum as an a.e. integer-valued function. On the basis of this observation, it was suggested in [Az] (see also [Az$_3$]) that the singular spectral shift function should be regarded as a natural extension of the spectral flow into the essential spectrum. This definition of spectral flow inside the essential spectrum has a significant drawback in the sense that the definition (1.3.18) is hard to work with, since it requires diagonalization of a continuous family of self-adjoint operators.

In [Az$_4$] a new equivalent definition of spectral flow inside essential spectrum, called *total resonance index*, was given. The total resonance index coincides with the singular spectral shift function $\xi^{(s)}(\lambda)$ for a.e. λ, but unlike $\xi^{(s)}(\lambda)$ the resonance index is quite

tangible and easy to work with. It is defined as the difference of two non-negative integers N_+ and N_-, and it makes sense outside the essential spectrum too, thus providing a new definition of spectral flow. In this paper we also show that the resonance index is equal to the signature of a finite-rank self-adjoint operator naturally associated with the data $(\lambda; H, V)$.

These considerations, however, are based on the formula (1.3.14). A rigorous justification and a proof of this formula, given in [Az$_3$] for trace class perturbations, required development of a new approach to stationary scattering theory. It turns out that (1.3.14) holds under much weaker conditions; the proof is based on an adjustment of this new approach, which is discussed in the next subsection.

1.4. Constructive approach to stationary scattering theory. In one of the basic settings of abstract mathematical scattering theory one studies an arbitrary initial self-adjoint operator H_0 and a relatively trace class perturbation $H_1 = H_0 + V$ of H_0. In this setting not only the proof of (1.3.14) given in [Az] does not work, but the formula itself does not make sense, since for any fixed value of the coupling constant r the ingredients of this formula such as $w_+(\lambda; H_0, H_r)$ and $Z_r(\lambda; G)$ are defined only for a.e. λ. Indeed, the right hand side of (1.3.14), which involves a continuous family of such operators, may be defined only for a set of values of λ which can potentially be as small as the empty set; more importantly, whatever this set is, one has no control over it. This is apparently a serious hindrance in any attempt to give a meaning to and to prove the formula (1.3.14). In fact, a proof of (1.3.14) for arbitrary self-adjoint trace class perturbations of arbitrary self-adjoint operators requires new definitions of basic notions and new proofs of basic theorems of abstract scattering theory. There are several reasons for this. Firstly, the definition of the operator $Z_r(\lambda; G)$ involves the operator $\mathcal{F}_\lambda$ from the spectral representation (1.3.3) for the operator H_r, and for this reason the set of values of the spectral parameter λ for which $Z_r(\lambda; G)$ is defined cannot be pinpointed: it is an arbitrary core of the spectrum of H_r. Secondly, in the classical approach to abstract scattering theory [BÈ, Y], the scattering matrix $S(\lambda; H_1, H_0)$ cannot be defined for a fixed single λ. This is analogous to the fact that while the notion of a measurable function makes perfect sense, its value at a given point does not. Thirdly, if one traces out a proof given in e.g. [BÈ, Y] of a formula involving the scattering matrix $S(\lambda; H_1, H_0)$, such as (1.3.13), then one finds that in numerous steps of the proof one throws away from an initial core of the absolutely continuous spectrum $\hat{\sigma}_{H_0}$ several finite and/or countable families of null sets. It is necessary to stress here that firstly an initial core of the absolutely continuous spectrum is chosen arbitrarily and it is not a constructive object, and secondly, the null sets that are thrown away depend on arbitrarily chosen objects, with no clear connections to the main objects of study, namely, the operators H_0 and V.

An approach to scattering theory which partly addresses this issue was given by Kato and Kuroda [KK]. They construct wave matrices $w_\pm(\lambda; H_1, H_0)$ for a set of full Lebesgue measure which depends on a fixed vector space $\mathfrak{X}$ in the Hilbert space. However, in [KK] only a fixed pair of self-adjoint operators (H_1, H_0) is studied, and it remains unclear how the theory presented could be applied to prove (1.3.14) and (1.3.20). On the other hand, numerous monographs and surveys on mathematical scattering theory, e.g.

[BW, RS_3, Y, BY_2], which appeared after the publication of [KK], do not discuss this problem.

An approach to scattering theory for trace class perturbations of arbitrary self-adjoint operators was developed in [Az_3] with primary aim to give a meaning to and to prove formula (1.3.14) for the scattering matrix $S(\lambda; H_1, H_0)$. Unlike the conventional approach of [BÈ, Y], in [Az_3] one first defines the wave matrices $w_\pm(\lambda; H_1, H_0)$ and the scattering matrix $S(\lambda; H_1, H_0)$ for all values of λ from an explicit set Λ of full Lebesgue measure, which is defined beforehand, while the wave operators $W_\pm(H_1, H_0)$ and the scattering operator $\mathbf{S}(H_1, H_0)$ thus become derivative objects which are *defined* by (1.3.15) and (1.3.4). Further, in constructing the theory, not a single number from Λ is removed, and all objects are explicitly constructed, in contrast to the conventional scattering theory. The main steps of this theory are described below. The proofs are given in [Az_3] in the case of a trace class perturbation V and will appear in [AzD] in the general case; see also [Az_6] for the general case.

I. The main data for constructing a scattering theory are a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$ and a self-adjoint perturbation operator V. The pair (H_0, V) is assumed to be compatible in a sense specified below. Besides these data, one needs an additional structure, namely a *rigging operator* F, which is a closed operator with trivial kernel and co-kernel which acts from $\mathcal{H}$ to some auxiliary Hilbert space $\mathcal{K}$, such that V admits a well-defined decomposition $V = F^*JF$ with a bounded self-adjoint operator J on $\mathcal{K}$. All objects of the scattering theory discussed below depend only on the data (H_0, V, F).

The pair (H_0, F) must be such that the operator

$$T_z(H_0) := FR_z(H_0)F^* = F(H_0 - z)^{-1}F^*,$$

called the *sandwiched resolvent*, is well-defined and compact for non-real z.

II. The next step is to define the set of values of λ for which the wave matrices $w_\pm(\lambda; H_1, H_0)$ are to be defined. We let $\Lambda(H_0, F)$ be the set of all real numbers λ such that the limits

$$\lim_{y\to 0} T_{\lambda\pm iy}(H_0)$$

exist in the uniform norm.

To ensure existence of the spectral shift functions (1.2.6) and (1.3.17) one has to impose an additional condition that the operator $\operatorname{Im} T_z(H_0)$ is trace class and that

$$\lim_{y\to 0^+} \operatorname{Im} T_{\lambda+iy}(H_0)$$

exists in trace class norm, but for the scattering theory this is not necessary and can be done at a later stage. It turns out that, unlike the situation with the functions ξ and $\xi^{(a)}$, to be able to define $\xi^{(s)}$ one probably does not need a trace class condition.

The set $\Lambda(H_0, F)$ is assumed to have full Lebesgue measure. In certain important cases this assumption holds. The corresponding theorems are called the Limiting Absorption Principle. Two of the main cases for which the principle holds are:

1. an arbitrary self-adjoint operator H_0 and a Hilbert–Schmidt rigging operator F (see e.g. [Y, Theorems 6.1.5 and 6.1.9]), and
2. a Schrödinger operator $H_0 = -\Delta + V_0$ and a rigging operator $F = \sqrt{|V|}$, where V_0 and V are short range potentials (Theorems 1.3.2 and 1.3.4).

The role of $\Lambda(H_0, F)$ in the constructive approach to stationary scattering theory is about the same as the role of $(0, \infty) \setminus e_+(H)$ from Theorem 1.3.3 in potential scattering theory. But while the structure of $e_+(H)$ is quite simple (see Theorem 1.3.3), the set $\mathbb{R} \setminus \Lambda(H_0, F)$ is more or less an arbitrary set of Lebesgue measure zero; for instance, the singular operator $HE^H_{\mathbb{R}\setminus\Lambda(H_0,F)}$ may contain, in the worst scenario, everywhere dense pure point and singular continuous spectra.

III. Since the wave operators $w_\pm(\lambda; H_1, H_0)$ act between the fibre Hilbert spaces $\mathfrak{h}_\lambda(H_0)$ and $\mathfrak{h}_\lambda(H_1)$, the next logical step is the construction of fibre Hilbert spaces of the spectral representation (1.3.3) and the direct integral on the right hand side of (1.3.3). The fibre Hilbert space $\mathfrak{h}_\lambda(H_0)$ is defined as a (closed) subspace of $\mathcal{K}$ by

$$\mathfrak{h}_\lambda(H_0) = \overline{\operatorname{im} \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}}, \tag{1.4.1}$$

the closure of the image of the compact non-negative operator $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}$. The family of Hilbert spaces

$$\{\mathfrak{h}_\lambda(H_0) \colon \lambda \in \Lambda(H_0, F)\}$$

is measurable, where as a measurability base one can take orthogonal projections of vectors from an orthonormal basis of $\mathcal{K}$ onto $\mathfrak{h}_\lambda(H_0) \subset \mathcal{K}$. Hence, one can define a direct integral of Hilbert spaces $\mathcal{H}(H_0)$ by the formula

$$\mathcal{H}(H_0) = \int^{\oplus}_{\Lambda(H_0,F)} \mathfrak{h}_\lambda(H_0)\, d\lambda. \tag{1.4.2}$$

The complement of the set $\Lambda(H_0, F)$ is a support of the singular spectrum of H_0 in the sense that the operator $H_0 E^{H_0}_{\Lambda(H_0,F)}$ is absolutely continuous. In other words, the singular spectrum of H_0 including all eigenvalues of H_0 is left out from $\Lambda(H_0, F)$. The dimensions of $\mathfrak{h}_\lambda(H_0)$ can be finite including zero. A core of the absolutely continuous spectrum of H_0 can be defined by

$$\hat{\sigma}_{H_0} = \{\lambda \in \Lambda(H_0, F) \colon \dim \mathfrak{h}_\lambda(H_0) > 0\}. \tag{1.4.3}$$

In particular, a measure ρ from the spectral representation (1.3.3) has the same spectral type as the restriction of Lebesgue measure $d\lambda$ to $\hat{\sigma}_{H_0}$. Therefore, if one wishes, in the direct integral (1.4.2) the set $\Lambda(H_0, F)$ can be replaced by the core (1.4.3), but it is more convenient to work with $\Lambda(H_0, F)$ itself.

IV. The next step is the construction of the unitary isomorphism $\mathcal{F}$ from the spectral representation (1.3.3) and its fibre $\mathcal{F}_\lambda$. To distinguish the non-constructive object $\mathcal{F}$ from its constructive counterpart to be defined, the latter is denoted by $\mathcal{E}$. By definition, for any vector ϕ from the dense linear manifold

$$F^*\mathcal{K} =: \mathcal{H}_+ \subset \mathcal{H}$$

the value of $\mathcal{E}_\lambda(H_0)$ at ϕ is defined by

$$\mathcal{E}_\lambda(H_0)\phi = \pi^{-1/2}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,\psi \in \mathfrak{h}_\lambda(H_0), \tag{1.4.4}$$

where ψ is the unique vector from $\mathcal{K}$ such that $\phi = F^*\psi$. Justification of these definitions is given by the following theorem.

THEOREM 1.4.1. *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with a rigging operator $F\colon \mathcal{H} \to \mathcal{K}$. The linear operator $\mathcal{E} = \mathcal{E}(H_0)$ which acts from the dense subspace $\mathcal{H}_+ = F^*\mathcal{K}$ of $\mathcal{H}$ to the direct integral Hilbert space* (1.4.2) *and which is defined by*

$$\mathcal{E}(F^*\psi)(\lambda) = \mathcal{E}_\lambda(H_0)(F^*\psi) = \pi^{-1/2}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,\psi$$

is a bounded operator whose continuous extension to $\mathcal{H}$ is a surjective isometric operator with initial subspace $\mathcal{H}^{(a)}(H_0)$. In particular, $\mathcal{E}$ is a natural isomorphism of the Hilbert spaces $\mathcal{H}^{(a)}(H_0)$ and (1.4.2) *provided there is a fixed rigging operator F in $\mathcal{H}$ compatible with H_0. Moreover, the restriction of H_0 to its absolutely continuous subspace $\mathcal{H}^{(a)}(H_0)$ in the representation of the latter by the direct integral* (1.4.2) *acts as follows: for any $f \in \mathcal{H}^{(a)}(H_0)$ and for a.e. $\lambda \in \Lambda(H_0, F)$,*

$$\mathcal{E}(H_0 f)(\lambda) = \lambda\mathcal{E}(f)(\lambda). \tag{1.4.5}$$

In other words, the operator $\mathcal{E}$ and the direct integral (1.4.2) *diagonalize the absolutely continuous part of the self-adjoint operator H_0.*

If a vector f belongs to the image of F^*, then (1.4.5) holds for all $\lambda \in \Lambda(H_0, F)$. Theorem 1.4.1 is in fact the spectral theorem for the absolutely continuous part of a self-adjoint operator. The importance of Theorem 1.4.1 comes from the fact that it gives an explicit diagonalization of the absolutely continuous part of an arbitrary self-adjoint operator. This is a difficult problem; for instance, in the case of potential scattering, while the free Hamiltonian $H_0 = -\Delta$ is easily diagonalized by the Fourier transform (see (1.3.9)), diagonalization of the Schrödinger operator $H = -\Delta + V$ requires, or in essence is equivalent to, calculation of the wave matrices (see e.g. [RS$_3$, (83)], [T, §10-a, (10.2)]) so that, in fact, often the wave operators are defined via eigenfunction expansion of the perturbed operator. Compared to this situation, in Theorem 1.4.1 the self-adjoint operator H_0 is *arbitrary*. This is a key circumstance, since once explicit eigenfunction expansions of an operator H_0 and of its perturbation $H = H_0 + V$ are found, one may try to define the wave matrices by a formula analogous to [RS$_3$, (83)] or [T, §10-a, (10.2)]. Having said this, the bulk of Theorem 1.4.1 is the Limiting Absorption Principle.

The operator $\mathcal{E}_\lambda(H_0)$ which acts from $\mathcal{H}$ to $\mathfrak{h}_\lambda(H_0)$ makes perfect sense for all values of λ from the full set $\Lambda(H_0, F)$. In this regard, it is different from $\mathcal{F}_\lambda$ of (1.3.3). The operator $\mathcal{E}_\lambda(H_0)$ will be called the *evaluation operator*. Theorem 1.4.1 implies that the operator (1.3.12) can be unambiguously defined for all $\lambda \in \Lambda(H_0, F)$ by

$$Z_0(\lambda; F) = \mathcal{E}_\lambda(H_0)F^*.$$

But, actually, this formula makes the operator $Z_0(\lambda; F)$ redundant, since the operator $\mathcal{E}_\lambda(H_0)$ on the right hand side is unambiguously defined for an explicit set $\Lambda(H_0, F)$ of full Lebesgue measure.

V. Once the fibre Hilbert spaces $\mathfrak{h}_\lambda(H_0)$ have been constructed, one can define wave matrices

$$w_\pm(\lambda; H_1, H_0)\colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_1), \tag{1.4.6}$$

for all real λ in $\Lambda(H_0, F) \cap \Lambda(H_1, F)$. Initially, $w_\pm(\lambda; H_1, H_0)$ is defined as a form on the dense subspace

$$\mathcal{E}_\lambda(H_1)F^*\mathcal{K} \times \mathcal{E}_\lambda(H_0)F^*\mathcal{K}$$

of the direct product $\mathfrak{h}_\lambda(H_1) \times \mathfrak{h}_\lambda(H_0)$ by [Az$_3$, Definition 5.2.1]: for any $F^*f, F^*g \in F^*\mathcal{K}$,

$$\langle \mathcal{E}_\lambda(H_1)F^*f, w_\pm(\lambda; H_1, H_0)\mathcal{E}_\lambda(H_0)F^*g\rangle = \left\langle f, [1 - T_{\lambda\mp i0}(H_1)J]\frac{1}{\pi}\operatorname{Im} T_{\lambda+i0}(H_0)g \right\rangle. \tag{1.4.7}$$

The idea to define the wave matrices by a formula similar to (1.4.7) was taken from [Y, Definition 2.7.2].

Theorem 1.4.2.

(1) *For any $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ the formula* (1.4.7) *correctly defines a bounded operator* (1.4.6)*. Moreover, this operator is unitary.*
(2) *For any three values, not necessarily distinct, r_1, r_2, r_3 of the coupling constant r such that*

$$\lambda \in \Lambda(H_{r_1}, F) \cap \Lambda(H_{r_2}, F) \cap \Lambda(H_{r_3}, F)$$

the following multiplicative property holds:

$$w_\pm(\lambda; H_{r_3}, H_{r_1}) = w_\pm(\lambda; H_{r_3}, H_{r_2})w_\pm(\lambda; H_{r_2}, H_{r_1}).$$

In particular, for any $\lambda \in \Lambda(H_0, F)$,

$$w_\pm(\lambda; H_0, H_0) = 1,$$

and for any $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$,

$$w_\pm^*(\lambda; H_1, H_0) = w_\pm(\lambda; H_0, H_1).$$

VI. Once the wave matrices $w_\pm(\lambda; H_1, H_0)$ are defined and their basic properties are proved, one can *define* the wave operators

$$W_\pm(H_1, H_0)\colon \mathcal{H}(H_0) \to \mathcal{H}(H_1) \tag{1.4.8}$$

by a formula similar to (1.3.15):

$$W_\pm(H_1, H_0) = \int_{\Lambda(H_0,F)\cap\Lambda(H_1,F)}^{\oplus} w_\pm(\lambda; H_1, H_0)\, d\lambda. \tag{1.4.9}$$

Here instead of the absolutely continuous subspaces $\mathcal{H}^{(a)}(H_0)$ and $\mathcal{H}^{(a)}(H_1)$, between which the wave operators act, one can use the Hilbert spaces $\mathcal{H}(H_0)$ and $\mathcal{H}(H_1)$ defined by (1.4.2), since according to Theorem 1.4.1 the Hilbert spaces $\mathcal{H}^{(a)}(H)$ and $\mathcal{H}(H)$ are naturally isomorphic via the unitary operator $\mathcal{E}(H)$. The following theorem demonstrates that (1.4.9) coincides with the classical definition (1.3.2).

Theorem 1.4.3. *The wave operators defined by* (1.4.9) *and* (1.4.7) *are equal to the right hand side of* (1.3.2).

Further, Theorems 1.4.1 and 1.4.2 immediately imply well-known properties of the wave operators [Az$_3$, Theorems 5.4.1, 5.4.2, Corollary 5.4.3]:

1. The wave operators (1.4.8) are unitary as operators from $\mathcal{H}(H_0)$ to $\mathcal{H}(H_1)$.
2. (Multiplicative property) For any real numbers r_1, r_2, r_3, not necessarily distinct,
$$W_\pm(H_{r_3}, H_{r_1}) = W_\pm(H_{r_3}, H_{r_2})W_\pm(H_{r_2}, H_{r_1}).$$
3. $W_\pm^*(H_1, H_0) = W_\pm(H_0, H_1)$.
4. $W_\pm(H_0, H_0)$ is the identity operator on $\mathcal{H}(H_0)$.
5. $H_1 W_\pm(H_1, H_0) = W_\pm(H_1, H_0)H_0$ (intertwining property).
6. For any bounded measurable function h on $\mathbb{R}$,
$$h(H_1)W_\pm(H_1, H_0) = W_\pm(H_1, H_0)h(H_0).$$
7. The absolutely continuous parts of H_0 and H_1 are unitarily equivalent (Kato–Rosenblum theorem).

VII. The scattering matrix $S(\lambda; H_1, H_0)$ is defined as an operator $\mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_0)$ for all $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ by (see [Az$_3$, Definition 7.1.1]):
$$S(\lambda; H_1, H_0) = w_+^*(\lambda; H_1, H_0)w_-(\lambda; H_1, H_0). \tag{1.4.10}$$
Note that in the conventional approach this is a theorem (see e.g. [Y]) which is proved for a.e. λ from an unspecified set of full measure. Many well-known properties of $S(\lambda; H_1, H_0)$ such as unitarity follow immediately from this definition and Theorem 1.4.2 [Az$_3$, Theorem 7.1.2]. The scattering operator $\mathbf{S}(H_1, H_0)$ is defined by
$$\mathbf{S}(H_1, H_0) = \int_{\Lambda(H_0,F)\cap\Lambda(H_1,F)}^{\oplus} S(\lambda; H_1, H_0)\, d\lambda. \tag{1.4.11}$$
Equalities (1.4.9) and (1.4.11) imply the classical definition (1.3.1) of $\mathbf{S}(H_1, H_0)$.

VIII. Now we return to (1.3.14). Before proceeding, one needs to give a meaning to the right hand side of the formula. This raises the following question: if $H_r = H_0 + rV$ and if $\lambda \in \Lambda(H_0, F)$, for which values of r does one have
$$\lambda \in \Lambda(H_r, F)? \tag{1.4.12}$$
This question is important, since the wave matrices $w_\pm(\lambda; H_r, H_0)$ and the scattering matrix $S(\lambda; H_r, H_0)$ are defined for those values of r for which (1.4.12) holds. The following well-known theorem answers this question; for a proof see e.g. [Az$_3$, Theorem 4.1.11].

THEOREM 1.4.4. *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with a rigging operator $F\colon \mathcal{H} \to \mathcal{K}$, let $V = F^*JF$, where J is a bounded operator on $\mathcal{K}$, and let $H_r = H_0 + rV$. If $\lambda \in \Lambda(H_0, F)$ (so in particular $T_{\lambda+i0}(H_0)$ exists and is compact), then for any $r \in \mathbb{R}$ the number λ belongs to $\Lambda(H_r, F)$ if and only if one (and hence each) of the following four operators is invertible:*
$$1 + rJT_{\lambda\pm i0}(H_0), \qquad 1 + rT_{\lambda\pm i0}(H_0)J.$$
In particular, the set of r for which $\lambda \notin \Lambda(H_r, F)$ is a discrete subset of the real line.

The set $\{r \in \mathbb{R}\colon \lambda \notin \Lambda(H_r, F)\}$ is of importance; its elements will be called the *resonance points* of the triple $(\lambda; H_0, V)$, the set itself will be called the *resonance set*

and will be denoted by $R(\lambda; H_0, V)$. This set depends on the rigging operator F too, but this dependence will not be indicated in the notation. One of the reasons λ may fail to belong to $\Lambda(H_r, F)$ is that λ may be an eigenvalue of H_r [Az$_3$, Proposition 4.1.10].

Theorem 1.4.4 states that the perturbed operator $H_r = H_0 + rV$ has a *coupling constant regularity property*. This was already observed by N. Aronszajn [Ar] (see also [AD]) in the study of boundary value perturbations of singular Sturm–Liouville equations. Later, coupling constant regularity for general rank-one perturbations was used by B. Simon and T. Wolff [SW], Simon [S$_2$, Chapters 12,13] and others (e.g. [RMS, RJMS, Gor]; see [S$_2$] for more references) in a study of the singular continuous spectrum and Anderson localization for random Hamiltonians.

A corollary of Theorem 1.4.4 is that the operators

$$\mathcal{E}_\lambda(H_r), \quad w_\pm(\lambda; H_r, H_0) \quad \text{and} \quad S(\lambda; H_r, H_0)$$

are defined for all r *outside the discrete resonance set* $R(\lambda; H_0, V)$.

Now we are in a position to present the stationary formula for the scattering matrix.

THEOREM 1.4.5. *Let $\lambda \in \Lambda(H_0, F)$. For all $r \notin R(\lambda; H_0, V)$ the scattering matrix $S(\lambda; H_r, H_0)$, defined by* (1.4.10) *as an operator on the fibre Hilbert space* (1.4.1)*, satisfies*

$$S(\lambda; H_r, H_0) = 1 - 2i\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, rJ(1 + rT_{\lambda+i0}(H_0)J)^{-1}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}. \quad (1.4.13)$$

The right hand side of (1.4.13), known as a *modified scattering matrix*, is defined on the whole auxiliary Hilbert space $\mathcal{K}$, and it is not difficult to check by a direct calculation that it is a unitary operator on the whole Hilbert space. The equality (1.4.13) shows that its right hand side can be interpreted as a proper scattering matrix, given that the fibre Hilbert space is defined by (1.4.1). Recalling the definition of the evaluation operator (1.4.4), the equality (1.4.13) can be rewritten in more familiar terms as follows:

$$S(\lambda; H_r, H_0) = 1 - 2\pi i \mathcal{E}_\lambda(H_0) F^* rJ(1 + rT_{\lambda+i0}(H_0)J)^{-1} F\mathcal{E}_\lambda^*(H_0). \quad (1.4.14)$$

REMARK 1.4.6. The expression $\mathcal{E}_\lambda^*(H_0)$ itself does not make sense since $\mathcal{E}_\lambda(H_0)$ as an operator $\mathcal{H} \to \mathfrak{h}_\lambda(H_0)$ with domain $F^*\mathcal{K}$ as defined by (1.4.4) is not in general closable, but the product $F\mathcal{E}_\lambda^*(H_0)$ is a well-defined compact operator from the Hilbert space $\mathfrak{h}_\lambda(H_0)$ to the Hilbert space $\mathcal{K}$ for every $\lambda \in \Lambda(H_0, F)$; for details see [Az$_3$, §§2.6, 2.15, 5.1].

The formula (1.4.14) coincides with (1.3.13), but, unlike the latter, in (1.4.14) the full set $\Lambda(H_0, F) \cap \Lambda(H_r, F)$ of values of λ (energy) for which it makes sense is explicitly given. Finally, (1.4.14) can be written as (see [Az$_3$, (7.6)])

$$S(\lambda; H_r, H_0) = 1 - 2\pi i \mathcal{E}_\lambda(H_0)\, rV(1 + rR_{\lambda+i0}(H_0)V)^{-1}\mathcal{E}_\lambda^\diamondsuit(H_0),$$

provided the operators $\mathcal{E}_\lambda(H_0)$, V and $R_{\lambda+i0}(H_0)$ are interpreted as acting between the appropriate pairs of Hilbert spaces $\mathcal{H}_-$, $\mathcal{H}_+$ and $\mathfrak{h}_\lambda$ (see [Az$_3$, §§5.1, 2.15] for details):

$$\mathfrak{h}_\lambda \xleftarrow{\mathcal{E}_\lambda(H_0)} \mathcal{H}_+ \xleftarrow{V} \mathcal{H}_- \xleftarrow{R_{\lambda+i0}(H_0)} \mathcal{H}_+ \xleftarrow{V} \mathcal{H}_- \xleftarrow{\mathcal{E}_\lambda^\diamondsuit(H_0)} \mathfrak{h}_\lambda.$$

Here $\mathcal{E}_\lambda^\diamondsuit(H_0)$ is a *modified adjoint* (see [Az$_3$, §2.6.1]), defined by

$$\langle \mathcal{E}_\lambda^\diamondsuit(H_0) f, g\rangle_{-1,1} = \langle f, \mathcal{E}_\lambda(H_0) g\rangle_{\mathfrak{h}_\lambda}, \qquad f \in \mathfrak{h}_\lambda,\ g \in \mathcal{H}_+.$$

This definition is an abstract version of (1.3.11); for definition of the Hilbert spaces $\mathcal{H}_\pm$ see also p. 47.

Theorem 1.4.5 allows us to overcome a hindrance on the way to a proof of (1.3.14).

THEOREM 1.4.7 ([Az_3, Theorem 7.3.4]). *For all λ from the set $\Lambda(H_0, F) \cap \Lambda(H_1, F)$ of full Lebesgue measure,*

$$S(\lambda; H_1, H_0) = \operatorname{Texp}\left(-2\pi i \int_0^1 w_+(\lambda; H_0, H_r)\mathcal{E}_\lambda(H_r)F^* JF\mathcal{E}_\lambda^*(H_r)w_+(\lambda; H_r, H_0)\, dr\right). \quad (1.4.15)$$

This yields the following theorem.

THEOREM 1.4.8 ([Az_3, Corollary 8.2.5]). *Let H_0 be a self-adjoint operator, let V be a trace class self-adjoint operator and let $H_r = H_0 + rV$. For a.e. $\lambda \in \mathbb{R}$,*

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi^{(a)}(\lambda)},$$

where $\xi^{(a)}(\lambda)$ is the absolutely continuous spectral shift function defined as the density of the absolutely continuous measure (1.3.17).

This formula, combined with Birman–Kreĭn's formula (1.2.6), implies that the singular spectral shift function $\xi^{(s)}(\lambda)$ of the pair (H_0, H_1) defined as the density of the measure (1.3.18) is a.e. integer-valued (see (1.3.20)). In fact, in [Az_3] another proof of (1.3.20) was given, so that the Birman–Kreĭn formula becomes its corollary. This proof is relevant to the content of the present paper; for this reason its main idea is outlined in the next paragraphs.

Let $U(r)$, $r \in [a, b]$, be a path of unitary operators such that $U(a) = 1$, $U(r) - 1$ is trace class for all $r \in [a, b]$, and the function $r \mapsto U(r) - 1$ is continuous in trace class norm. These conditions imply that the spectrum of $U(r)$ consists of isolated eigenvalues on the unit circle with 1 as the only point in the essential spectrum of $U(r)$. As r decreases from b to a, the eigenvalues of $U(r)$ converge continuously to 1. So, given a point $e^{i\theta}$, $\theta \in [0, 2\pi)$, on the unit circle, one can calculate the spectral flow through the point $e^{i\theta}$, which, following [Pu_2], is called the *μ-invariant* of the path $U(r)$.

The scattering matrix $S(\lambda; H_1, H_0)$ for any given λ from the full set $\Lambda(H_0, F) \cap \Lambda(H_1, F)$ is a unitary matrix from the class $1 + \mathcal{L}_1(\mathfrak{h}_\lambda(H_0))$ (that is, $S(\lambda; H_1, H_0) - 1$ is a trace class operator on $\mathfrak{h}_\lambda(H_0)$). There exist two natural paths which continuously connect the scattering matrix $S(\lambda; H_1, H_0)$ to the identity operator on $\mathfrak{h}_\lambda(H_0)$. In the first path one changes the imaginary part of the spectral parameter $y = \operatorname{Im} z$ in the stationary formula (1.4.14) or (1.4.13) for the scattering matrix from ∞ to 0:

$$[0, \infty] \ni y \mapsto S(\lambda + iy; H_1, H_0) = 1 - 2\pi i \mathcal{E}_{\lambda+iy}(H_0)F^* J(1 + T_{\lambda+iy}(H_0)J)^{-1} F\mathcal{E}_{\lambda+iy}^*(H_0), \quad (1.4.16)$$

where, in accordance with (1.4.4),

$$\mathcal{E}_{\lambda+iy}(H_0)F^* = \pi^{-1/2}\sqrt{\operatorname{Im} T_{\lambda+iy}(H_0)}.$$

One can show that this path is continuous in trace class norm. In order to get the second

way of connecting $S(\lambda; H_1, H_0)$ to the identity operator the following theorem (initially observed in [Az$_2$]) is used.

PROPOSITION 1.4.9 ([Az$_3$, Proposition 7.2.5]). *The scattering matrix $S(\lambda; H_r, H_0)$ as a meromorphic function of r admits analytic continuation to the real axis.*

REMARK 1.4.10. In [Az$_3$] this proposition in fact precedes Theorem 1.4.7 and is used in its proof. Indeed, though the integrand of the chronological exponential in (1.4.15) is defined for all r outside the discrete resonance set $R(\lambda; H_0, V)$, to define the chronological exponential itself one needs the integrand to be continuous in trace class norm.

Proposition 1.4.9 provides the second way of continuously connecting the scattering matrix $S(\lambda; H_1, H_0)$ to the identity operator via the continuous mapping

$$[0,1] \ni r \mapsto S(\lambda; H_r, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}_\lambda(H_0)). \tag{1.4.17}$$

The μ-invariant of the path (1.4.16) was introduced in [Pu$_2$] where it was denoted by $\mu(\theta, \lambda; H_1, H_0)$, $\theta \in [0, 2\pi)$. The μ-invariant of the path (1.4.17) was introduced in [Az$_2$, Az$_3$] where it was denoted by $\mu^{(a)}(\theta, \lambda; H_1, H_0)$. The relations of these μ-invariants to the spectral shift functions ξ, $\xi^{(a)}$ and $\xi^{(s)}$ are given by the following theorems.

THEOREM 1.4.11 ([Pu$_2$]). *For a.e. $\lambda \in \mathbb{R}$,*

$$\xi_{H_1,H_0}(\lambda) = -\frac{1}{2\pi} \int_0^{2\pi} \mu(\theta, \lambda; H_1, H_0)\, d\theta.$$

THEOREM 1.4.12 ([Az$_3$, Theorem 9.2.2]). *For a.e. $\lambda \in \mathbb{R}$,*

$$\xi^{(a)}_{H_1,H_0}(\lambda) = -\frac{1}{2\pi} \int_0^{2\pi} \mu^{(a)}(\theta, \lambda; H_1, H_0)\, d\theta.$$

THEOREM 1.4.13 ([Az$_3$, Theorem 9.7.3]). *The difference*

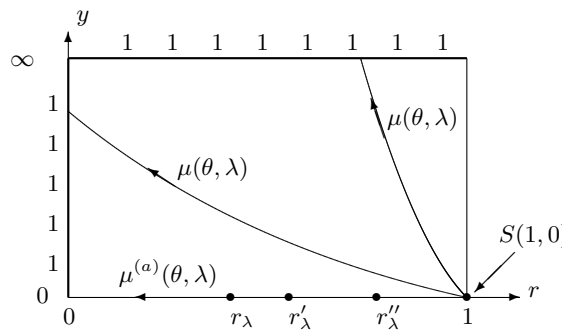
$$\mu^{(s)}(\theta, \lambda; H_1, H_0) := \mu(\theta, \lambda; H_1, H_0) - \mu^{(a)}(\theta, \lambda; H_1, H_0)$$

does not depend on the angle θ, and for a.e. $\lambda \in \mathbb{R}$ it is equal to minus the density, $-\xi^{(s)}(\lambda; H_1, H_0)$, of the singular spectral shift measure $\xi^{(s)}(\phi)$ as defined by (1.3.18). *In particular, $\xi^{(s)}(\lambda; H_1, H_0)$ is almost everywhere integer-valued.*

Theorem 1.4.11 of A. Pushnitski was given a new proof in [Az$_3$, Theorem 9.6.1]. Theorems 1.4.13 and 1.4.8 give a new proof of the Birman–Kreĭn formula (1.2.6).

The last assertion of Theorem 1.4.13 justifies calling the function $\xi^{(s)}(\lambda)$ the *spectral flow inside the essential spectrum*, since $\xi^{(s)}(\lambda)$ coincides with the spectral flow outside the essential spectrum and it is a.e. integer-valued inside the essential spectrum as well.

The following diagram demonstrates the relationship between the μ- and $\mu^{(a)}$-invariants. In this diagram for a fixed real λ we consider the scattering matrix $S(\lambda+iy; H_r, H_0)$ as a function of (r, y), where r is the coupling constant and y is the imaginary part of the spectral parameter.



The three points $r_\lambda, r'_\lambda, r''_\lambda$ represent resonance points from $[0,1]$. A comment on the figure: $S(r,y) := S(\lambda+iy; H_r, H_0)$ is continuous in the rectangle except at the resonance points. On the left $r=0$ and upper $y=\infty$ rims of this rectangle, $S(\lambda+iy; H_r, H_0) = 1$. Next, $\mu(\theta,\lambda)$ is the spectral flow of eigenvalues of $S(r,y)$ through $e^{i\theta}$ corresponding to any path which connects $(1,0)$ to the left or the upper rim as long as it avoids the resonance points. Finally, $\mu^{(a)}(\theta,\lambda)$ is the spectral flow of eigenvalues of $S(\lambda; H_r, H_0)$ as r goes from 1 to 0.

The operators $S(\lambda; H_r, H_0)$ and $S(\lambda+i0; H_r, H_0)$ are identical outside the resonance points in $[0,r]$. The group $\mathcal{U}_1$ of unitary operators of the form "1 + trace class" has a non-trivial homotopical structure, and the difference between $S(\lambda; H_r, H_0)$ and $S(\lambda+i0; H_r, H_0)$ is revealed in the way one connects them to the base point 1 of $\mathcal{U}_1$.

The functions $\xi(\lambda)$, $\xi^{(a)}(\lambda)$ and $\xi^{(s)}(\lambda)$ are integrable, and in general one cannot talk about their value at a given point λ. But Theorems 1.4.11–1.4.13 allow one to define values of these functions explicitly on the set

$$\Lambda(H_0, F) \cap \Lambda(H_1, F) \tag{1.4.18}$$

of full measure, because the right hand sides of the equalities in these theorems are well-defined for all λ from (1.4.18). This is an important point, since if the perturbed operator H_1 is replaced by $H_r = H_0 + rV$ with an arbitrary real number r, then for every fixed value of λ from $\Lambda(H_0, F)$ the expressions

$$\xi(\lambda; H_r, H_0), \quad \xi^{(a)}(\lambda; H_r, H_0) \quad \text{and} \quad \xi^{(s)}(\lambda; H_r, H_0)$$

can be considered as functions of the coupling constant r. Their behaviour is explained by the following theorem.

THEOREM 1.4.14 ([Az$_3$, Proposition 8.2.3, Theorem 9.7.6, Corollary 9.7.7]). *For every λ from the set $\Lambda(H_0, F)$ of full Lebesgue measure the following assertions hold:*

(1) *The function $r \mapsto \xi^{(a)}(\lambda; H_r, H_0)$ is analytic in a neighbourhood of $\mathbb{R}$.*
(2) *The function $r \mapsto \xi^{(s)}(\lambda; H_r, H_0)$ is a locally constant integer-valued function with a discrete set of discontinuity points which coincides with the resonance set $R(\lambda; H_0, V)$ (see Theorem 1.4.4 and the paragraph after it for the definition).*
(3) *As a consequence, $r \mapsto \xi(\lambda; H_r, H_0)$ is a piecewise continuous locally analytic function and its discontinuity points are resonance points of $(\lambda; H_0, V)$.*

1.5. Resonance index. Theorem 1.4.14 implies, in particular, that if for $\lambda \in \Lambda(H_0, F)$ there are no resonance points in an interval $[a,b]$, then $\xi^{(a)}(\lambda; H_b, H_a) = \xi(\lambda; H_b, H_a)$. It

also suggests that the integer jump of the singular spectral shift function $\xi^{(s)}(\lambda; H_1, H_0)$ at a resonance point $r_\lambda \in [0,1]$ should depend only on the triple $(\lambda; H_{r_\lambda}, V)$. Indeed, to a triple $(\lambda; H_{r_\lambda}, V)$ one can assign an integer number, which in this paper is called the *resonance index* and is denoted by

$$\mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V).$$

This number is defined as follows. Firstly, by Theorem 1.4.4 a real number r_λ is a resonance point of $(\lambda; H_0, V)$ if and only if

$$\sigma_\lambda = -r_\lambda^{-1}$$

is an eigenvalue of the compact operator $T_{\lambda+i0}(H_0)J$. Further, r_λ is a pole of the meromorphic factor

$$(1 + rT_{\lambda+i0}(H_0)J)^{-1}$$

which is part of the stationary formula (1.4.14) for the scattering matrix $S(\lambda; H_r, H_0)$. Still, according to Proposition 1.4.9, $S(\lambda; H_r, H_0)$ does not have a singularity at $r = r_\lambda$. This is due to the fact that this singularity belongs to the singular subspace of H_0, which is eliminated by the factors $\mathcal{E}_\lambda(H_0)F^*$ and $F\mathcal{E}_\lambda^*(H_0)$ of the stationary formula. In order to reveal this hidden singularity, one has to shift the spectral parameter $\lambda + i0$ slightly off the real axis. Since σ_λ is an isolated eigenvalue of the compact operator $T_{\lambda+i0}(H_0)J$, it is stable but it may split into several eigenvalues

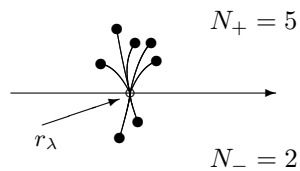
$$\sigma^1_{\lambda+iy}, \dots, \sigma^N_{\lambda+iy}, \tag{1.5.1}$$

where N is the multiplicity of σ_λ, which are therefore eigenvalues of the compact operator $T_{\lambda+iy}(H_0)J$ from the group of σ_λ. It is well-known and not difficult to show that none of the shifted eigenvalues (1.5.1) is a real number. Therefore, the following definition makes sense: the resonance index $\mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ is the difference

$$N_+ - N_-, \tag{1.5.2}$$

where N_+ (respectively, N_-) is the number of shifted eigenvalues of the group of σ_λ in the upper (respectively, lower) complex half-plane. This definition is correct in the sense that it does not depend on the choice of the initial operator H_0, as the following lemma with a simple proof asserts.

Lemma 1.5.1. *Let $\lambda \in \Lambda(H_0, F)$. Let a real number s be such that λ also belongs to the full set $\Lambda(H_s, F)$. Further, let r_λ be a real resonance point of $(\lambda; H_0, V)$ (that is, $\lambda \notin \Lambda(H_{r_\lambda}, F)$). Then the real number $\sigma_\lambda(s) = (s - r_\lambda)^{-1}$ is an eigenvalue of $T_{\lambda+i0}(H_s)J$ of the same algebraic multiplicity N as that of the eigenvalue $\sigma_\lambda(0) = (0 - r_\lambda)^{-1}$ of $T_{\lambda+i0}(H_0)J$, and if λ is shifted off the real axis to $\lambda + iy$ with a small and positive y, then the number of split eigenvalues from the group of $(s - r_\lambda)^{-1}$ in the upper complex half-plane is equal to N_+.*



Introduction of this notion is justified by the following theorem; see [Az$_4$, Theorem 3.8]. Since [Az$_4$] is not published, an outline of the proof is given in Section 6.

THEOREM 1.5.2. *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with a Hilbert–Schmidt rigging operator $F\colon \mathcal{H} \to \mathcal{K}$. Let V be a trace class self-adjoint operator which admits a decomposition $V = F^*JF$ with a bounded operator J on $\mathcal{K}$, and let $a < b$. Then for every real number λ from the set $\Lambda(H_a, F) \cap \Lambda(H_b, F)$ of full Lebesgue measure we have*

$$\xi^{(s)}(\lambda; H_b, H_a) = \sum_{r_\lambda} \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V), \tag{1.5.3}$$

where the sum is taken over all resonance points r_λ of $(\lambda; H_0, V)$ in $[a, b]$.

In other words, as r changes from a to b, the locally constant function

$$[a, b] \ni r \mapsto \xi^{(s)}(\lambda; H_r, H_a)$$

jumps at every resonance point $r_\lambda \in [a, b]$ by $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$. Theorem 1.5.2 gives a computable and tangible representation for values of the function $\xi^{(s)}(\cdot; H_b, H_a)$, which is initially defined as the density of the singular spectral shift measure (1.3.18), and as such seems to be difficult to handle (indeed, (1.3.18) requires in particular calculation of the singular parts of a continuous family of self-adjoint operators). In particular, this theorem allows us to prove the following [Az$_4$, Theorem 4.3]:

THEOREM 1.5.3. *There exist a self-adjoint operator H_0 and a rank-one self-adjoint operator V such that the pair (H_0, V) is irreducible and the restriction of the singular spectral shift function $\xi^{(s)}(\cdot; H_0 + V, H_0)$ to the absolutely continuous spectrum $\sigma_{\mathrm{a.c.}}(H_0)$ is a non-zero element of $L_1(\sigma_{\mathrm{a.c.}}(H_0), dx)$.*

The construction of such a pair may be uninteresting, but at least this theorem shows that the decomposition (1.3.19) is non-trivial.

The expression on the right hand side of (1.5.3) will be called the *total resonance index* for the pair (H_a, H_b). For λ outside the essential spectrum of H_0, the singular spectral shift function coincides with the spectral flow, and therefore it follows from (1.5.3) that the total resonance index provides a new definition of the spectral flow. Moreover, the notion of the resonance index which was discovered in the course of the study of the singular spectral shift function makes sense even in finite dimensions. The resonance index represents a new approach to calculation of spectral flow, which is in essence the "flow of eigenvalues". Indeed, in order to find out how many eigenvalues of a path of self-adjoint operators $\{H_0 + rV\colon 0 \le r \le 1\}$ crossed in the positive direction a fixed point λ outside of the spectra of the initial H_0 and the final H_1, one can either try to keep track of each eigenvalue and count how many times and in which direction it crossed λ, or instead one can try to detect "times" (coupling constants) r_λ for which the event "λ is an eigenvalue of H_{r_λ}" occurs and then to decide where the eigenvalue has come from and where it is going to.

The first approach requires continuous enumeration of eigenvalues (which for general continuous paths is not a trivial problem even in finite dimensions, see [Ka$_2$, §II.5.2]), but inside the essential spectrum this approach does not work since eigenvalues embedded into

the essential spectrum are extremely unstable (for some striking examples see e.g. [S_2, §12.5]). In the second approach a detector of eigenvalues needs to be told how to decide in which direction a detected eigenvalue is moving. The answer is to tell the counter: calculate the resonance index of the triple $(\lambda; H_{r_\lambda}, V)$, that is, choose any real s such that λ is not an eigenvalue of H_s and find those eigenvalues of the operator

$$R_{\lambda+iy}(H_s)V \tag{1.5.4}$$

with a very small $y > 0$ which are close to $(s - r_\lambda)^{-1}$. Then the difference $N_+ - N_-$ of the eigenvalues in $\mathbb{C}_+$ and $\mathbb{C}_-$ will give the net number of eigenvalues crossing λ in the positive direction at "time" $r = r_\lambda$. Remarkably, this algorithm works equally well for eigenvalues embedded into the essential spectrum, so even if an eigenvalue appears suddenly from the continuous spectrum and then dissolves in it immediately afterwards, one is still able to determine which direction it appeared from and in which direction it dissolved. The difference is that the condition "λ is an eigenvalue of H_r" should be replaced by $\lambda \notin \Lambda(H_r, F)$, or equivalently, $r \in R(\lambda; H_0, V)$. As a consequence, to define the spectral flow inside the essential spectrum one has to consider singular points instead of eigenvalues, as a non-trivial spectral flow inside the essential spectrum may be a result of moving the singular continuous spectrum.

Finally, we discuss the origin of the terminology "resonance points", "resonance index" etc. used in this paper. This paragraph of the introduction has a formal character as it frequently refers to physical concepts and phenomena; its partial aim is to explain/justify usage of the word "resonance", though this formal and remote connection with quantum scattering may be found interesting. The justification of this terminology can be even more necessary since the word "resonance" has several meanings, and this word is used in this paper since it is associated with a quantum scattering phenomenon. A resonance in quantum scattering is associated with a sharp variation of the scattering cross-section as a function of energy (see e.g. [Bö, §XVIII.6]). The value of the energy λ_0 of a projectile at which this sharp variation occurs is called the *resonance energy*.

Physicists associate resonances with other phenomena (see e.g. [RS_4, §XII.6], [T, Chapter 13] or [Bö, §XVIII.6, more specifically, see e.g. the last sentence on p. 431 of that section and (6.1)]):

1. poles of the scattering matrix as a function of energy which are close to the real axis,
2. a rapid increase of a scattering phase $\theta_j(\lambda)$ $(= 2\delta_l(E)$ in physical notation) by 2π as the energy λ, of a projectile crosses a resonance value λ_0,
3. existence of a quasi-stationary (or meta-stable) state with energy λ_0, and finally
4. a time delay for the interval of time between the moments of entering and leaving the interaction region around the target by the projectile compared to the same time interval for a non-interacting projectile.

These phenomena are non-trivially related to one another and to the fact that at a resonance energy the projectile can be captured by the target into a nearly bound meta-stable "target-projectile" state (see e.g. introduction to [T, Chapter 13]). These phenomena except the time delay will have mathematical analogues in our setting if one fixes the value of energy λ and considers the coupling constant r as a variable:

1° a resonance point r_λ is a real pole of the factor $(1+rT_{\lambda+i0}(H_0)J)^{-1}$ from the stationary formula (1.4.13) for the scattering matrix,

2° Theorem 1.4.13 and (1.5.3) express the fact that as the coupling constant r crosses a resonant value r_λ, at least one of the scattering phases jumps by a multiple of 2π,

3° by Theorem 1.4.4 the value r of the coupling constant at a given energy λ is resonant if and only if the equation

$$(1 + rJT_{\lambda+i0}(H_0))\psi = 0 \tag{1.5.5}$$

has a non-trivial solution ψ, which can be interpreted as a quasi-stationary state.

Further, unlike the physical resonances, in this paper an idealized situation is considered in the sense that (1) the pole of the scattering matrix is not near the real axis, but exactly on it, (2) the scattering phase does not *change rapidly* by 2π at a resonance point, but *jumps* by a multiple of 2π, and finally (3) while a physical quasi-stationary state is nevertheless a scattering state in the sense that sooner or later the projectile leaves the target and can be observed, the quasi-stationary state represented by a solution of (1.5.5) is not a scattering state, in the sense that it does not belong to the fibre Hilbert space $\mathfrak{h}_\lambda(H_r)$. The latter may be attributed to the possibility that in this idealized situation, i.e. a pole exactly on the real axis, the projectile gets captured by the target and never leaves it; see e.g. Pearson's example in [RS_3, §XI.4, p. 70], which shows that this scenario is mathematically possible. This is also in accordance with a physical fact that time delay is proportional to the inverse width $1/\Gamma$ of the resonance bump (= imaginary part of the resonance pole), which (the width Γ) is zero (see e.g. [T, (13.10)], [Bö, §XVIII.6, p. 432]).

1.6. Main results. We assume the following:

1. $\mathcal{H}$ and $\mathcal{K}$ are separable complex Hilbert spaces, and $F\colon \mathcal{H} \to \mathcal{K}$ is a closed operator with zero kernel and co-kernel. The operator F will be called the *rigging operator*.
2. H_0 is a self-adjoint operator on $\mathcal{H}$ with dense domain $\mathcal{D}$ such that $\mathcal{D} \subset \mathrm{dom}(F)$ and the operator
$$T_z(H_0) := FR_z(H_0)F^*$$
is compact for some non-real z, where $R_z(H_0) = (H_0 - z)^{-1}$ is the resolvent of H_0. If this operator is compact for some non-real z then it is compact for any z from the complement of the essential spectrum of H_0.
3. $\mathcal{A}_0$ is a real vector space of self-adjoint operators on $\mathcal{H}$ such that any operator V from $\mathcal{A}_0$ satisfies these assumptions:
   (a) V admits a factorization $V = F^*JF$, where J is a bounded operator on $\mathcal{K}$, such that
   $$JF\mathcal{D} \subset \mathrm{dom}(F^*);$$
   (b) assumption (a) implies that the domain of V contains $\mathcal{D}$; we assume that V is essentially self-adjoint on $\mathcal{D}$;

(c) V is a relatively compact perturbation of H_0, that is, the product $VR_z(H_0)$ is compact.

4. $\mathcal{A}$ is the real affine space $H_0 + \mathcal{A}_0$. The previous assumptions imply that all operators from the affine space $\mathcal{A}$ have common domain $\mathcal{D}$ and, according to the second resolvent identity, for any operator H from $\mathcal{A}$ the sandwiched resolvent

$$T_z(H) = FR_z(H)F^*$$

is compact, where z is a non-real number. Hence, all operators from $\mathcal{A}$ satisfy the same assumptions, and there is no distinguished element of $\mathcal{A}$.
5. Since perturbations $V \in \mathcal{A}_0$ are relatively compact, by Weyl's theorem, all operators from $\mathcal{A}$ have common essential spectrum, which we denote by $\sigma_{\mathrm{ess}}(\mathcal{A})$ or simply σ_{ess}.
6. Let $H_0 \in \mathcal{A}$. The set $\Lambda(H_0, F)$ of real λ such that the norm limit

$$T_{\lambda+i0}(H_0) := \lim_{y\to 0^+} T_{\lambda+iy}(H_0)$$

exists, and therefore is compact, has full Lebesgue measure. This is the main assumption, called the Limiting Absorption Principle. We denote by $\Lambda(\mathcal{A}, F)$ the union of all sets $\Lambda(H, F)$, $H \in \mathcal{A}$. Numbers from the full set $\Lambda(\mathcal{A}, F)$ will be called *essentially regular* for $\mathcal{A}$.

An operator $H \in \mathcal{A}$ will be called λ-*regular* or *regular at* λ if $\lambda \in \Lambda(H, F)$; otherwise, H will be called λ-*resonant* or *resonant at* λ.

Mainly we shall work with one operator H_0 from the real affine space $\mathcal{A}$, one perturbation operator V from the real vector space $\mathcal{A}_0$ and one value $\lambda \in \Lambda(H_0, F)$. By H_r, where r is a real number, we denote the operator $H_0 + rV$. The set of all real r such that $\lambda \notin \Lambda(H_r, F)$ is denoted by $R(\lambda; H_0, V)$ and called the *resonance set*. It is not difficult to show that $R(\lambda; H_0, V)$ is a discrete subset of $\mathbb{R}$, that is, it has no accumulation points. A real number r will be called *resonant* (respectively, *regular*) at λ if H_r is resonant (respectively, regular) at λ. The set $R(\lambda; H_0, V)$ also depends on F, but this dependence will not be indicated.

If the rigging operator F is bounded, then one can take $\mathcal{A}_0$ to be the vector space $F^*\mathcal{B}_{\mathrm{sa}}(\mathcal{K})F = \{F^*JF\colon J \in \mathcal{B}_{\mathrm{sa}}(\mathcal{K})\}$, where $\mathcal{B}_{\mathrm{sa}}(\mathcal{K})$ is the algebra of all bounded self-adjoint operators on $\mathcal{K}$. But to allow some flexibility, we shall not impose any other conditions on the real vector space $\mathcal{A}_0$ of self-adjoint perturbations.

Let Π be the disjoint union of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}$ and of two copies $\Lambda(\mathcal{A}, F) + i0$ and $\Lambda(\mathcal{A}, F) - i0$ of $\Lambda(\mathcal{A}, F)$:

$$\Pi = (\mathbb{C} \setminus \sigma_{\mathrm{ess}}) \cup (\Lambda(\mathcal{A}, F) + i0) \cup (\Lambda(\mathcal{A}, F) - i0).$$

If $\lambda \notin \sigma_{\mathrm{ess}}$, then $\lambda = \lambda + i0 = \lambda - i0$, but otherwise $\lambda + i0 \neq \lambda - i0$. Thus, the operator $T_z(H_r)$ as a function of z is defined on Π except at those $z = \lambda \pm i0$ for which r is a resonance point.

For $z \in \Pi$ let

$$A_z(s) = T_z(H_s)J, \qquad B_z(s) = JT_z(H_s).$$

Given $z \in \Pi$, a number $r_z \in \mathbb{C}$ is called a *resonance point* corresponding to z if r_z is a

pole of the meromorphic function $s \mapsto A_z(s)$. We define the vector spaces

$$\Upsilon_z^k(r_z) = \{u \in \mathcal{K}\colon (1 + (r_z - s)T_z(H_s)J)^k u = 0\}, \qquad \Upsilon_z(r_z) = \bigcup_{k=1,2,\dots} \Upsilon_z^k(r_z),$$

$$\Psi_z^k(r_z) = \{\psi \in \mathcal{K}\colon (1 + (r_z - s)JT_z(H_s))^k \psi = 0\}, \qquad \Psi_z(r_z) = \bigcup_{k=1,2,\dots} \Psi_z^k(r_z),$$

and idempotents

$$P_z(r_z) = \frac{1}{2\pi i} \oint_{C(\sigma_z(s))} (\sigma - A_z(s))^{-1}\, d\sigma, \qquad Q_z(r_z) = \frac{1}{2\pi i} \oint_{C(\sigma_z(s))} (\sigma - B_z(s))^{-1}\, d\sigma,$$

where $C(\sigma_z(s))$ is a small circle enclosing the eigenvalue $\sigma_z(s) = (s - r_z)^{-1}$ of $A_z(s)$, such that there are no other eigenvalues inside or on the circle. These vector spaces and idempotents do not depend on the choice of $s \in \mathbb{R}$, as long as, for $z \in \partial\Pi$, the operator $A_z(s)$ exists (Propositions 3.1.2 and 3.2.1). Many properties of $\Upsilon_z^k(r_z)$ and $P_z(r_z)$ are similar to those of $\Psi_z^k(r_z)$ and $Q_z(r_z)$; for this reason only properties of the former are given. The idempotent $P_z(r_z)$ satisfies

$$P_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} A_z(s)\, ds$$

(see (3.2.6)), and for any two different resonance points r_z^1 and r_z^2 (see (3.2.8)),

$$P_z(r_z^1) P_z(r_z^2) = 0.$$

With every resonance point r_z the following three non-negative integers are associated, called respectively the *geometric multiplicity*, *algebraic multiplicity* and *order* of r_z:

$$m = \dim \Upsilon_z^1(r_z), \qquad N = \dim \Upsilon_z(r_z), \qquad d = \min\{k \in \mathbb{N}\colon \Upsilon_z^k(r_z) = \Upsilon_z(r_z)\}.$$

A number r_z is resonant for z if and only if $\bar{r}_z$ is resonant for $\bar{z}$, in which case the numbers m, N, d are the same for r_z and $\bar{r}_z$.

The nilpotent operators $\mathbf{A}_z(r_z)$ and $\mathbf{B}_z(r_z)$ are defined by

$$\mathbf{A}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) A_z(s)\, ds, \qquad \mathbf{B}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) B_z(s)\, ds,$$

where $C(r_z)$ is a small contour which encloses the resonance point r_z and no other resonance points.

Section 3 also contains an exposition of other properties of the idempotents $P_z(r_z)$ and $Q_z(r_z)$ and the nilpotent operators $\mathbf{A}_z(r_z)$ and $\mathbf{B}_z(r_z)$ which are used repeatedly throughout this paper, such as

$$(P_z(r_z))^* = Q_{\bar{z}}(\bar{r}_z), \qquad (\mathbf{A}_z(r_z))^* = \mathbf{B}_{\bar{z}}(\bar{r}_z),$$
$$JP_z(r_z) = Q_z(r_z)J, \qquad J\mathbf{A}_z(r_z) = \mathbf{B}_z(r_z)J.$$

Further, for a fixed $z \in \Pi$, $A_z(s)$ is a meromorphic function of s whose poles are exactly the resonance points corresponding to z. The Laurent expansion of $A_z(s)$ at a pole r_z is

$$A_z(s) = \tilde{A}_{z,r_z}(s) + \frac{1}{s - r_z} P_z(r_z) + \frac{1}{(s - r_z)^2} \mathbf{A}_z(r_z) + \cdots + \frac{1}{(s - r_z)^d} \mathbf{A}_z^{d-1}(r_z),$$

where $\tilde{A}_{z,r_z}(s)$ is the holomorphic part.

In Section 4 we study the relationship between eigenvectors of H_{r_λ} corresponding to an eigenvalue λ and resonance vectors of order 1.

THEOREM 1.6.1 (Theorem 4.1.1). *Let λ be an essentially regular point, let $H_0 \in \mathcal{A}$ be a λ-regular operator, let $V \in \mathcal{A}_0$, let r_λ be a real resonance point of $(\lambda; H_0, V)$ and let r be a regular point of $(\lambda; H_0, V)$. If λ is an eigenvalue of $H_{r_\lambda} = H_0 + r_\lambda V$ with eigenvector $\chi \in \mathcal{D} = \mathrm{dom}(H_0)$, then $u = F\chi$ is a resonance vector of order 1, that is,*

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)u = 0.$$

COROLLARY 1.6.2 (Corollary 4.1.2). *If λ is an essentially regular point, then the geometric multiplicity of λ as an eigenvalue of the self-adjoint operator $H_{r_\lambda} = H_0 + r_\lambda V$ does not exceed the dimension of the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$, that is,*

$$\dim \mathcal{V}_\lambda \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda),$$

where $\mathcal{V}_\lambda$ is the eigenspace of H_{r_λ} corresponding to λ.

THEOREM 1.6.3 (Theorem 4.2.1). *If λ is an eigenvalue of infinite multiplicity for at least one self-adjoint operator H from the affine space $\mathcal{A} = H_0 + \mathcal{A}_0$, then λ is not an essentially regular point of the pair $(\mathcal{A}, F)$, that is, $\lambda \notin \Lambda(\mathcal{A}, F)$.*

Now we return to the discussion of spectral flow inside essential spectrum. Since inside the essential spectrum a non-trivial spectral flow can be generated in absence of any eigenvalues, the notion of multiplicity of an eigenvalue needs to be properly generalized. To this end, the following is used

THEOREM 1.6.4 (Theorem 4.3.2). *Let λ be a real number which does not belong to the essential spectrum and let r_λ be a real resonance point of the triple $(\lambda; H_0, V)$ (that is, λ is an eigenvalue of H_{r_λ}). Let s be any non-resonant point of $(\lambda; H_0, V)$. The rigging operator F is a linear isomorphism of the vector space $\mathcal{V}_\lambda$ of eigenvectors of H_{r_λ} corresponding to the eigenvalue λ and the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$ of eigenvectors of the operator $T_{\lambda+i0}(H_s)J$ corresponding to the eigenvalue $(s - r_\lambda)^{-1}$.*

Theorems 1.6.1 and 1.6.4 give a rationale to call the integer number $\dim \Upsilon^1_{\lambda+i0}(r_\lambda)$ the *multiplicity of the singular spectrum* of the self-adjoint operator H_{r_λ} at λ. That this is a reasonable definition is further confirmed by the U-turn Theorem 1.6.14.

THEOREM 1.6.5 (Theorem 4.4.1). *If H_{r_λ} is resonant at an essentially regular point λ, then the vector space*

$$\Upsilon^1_{\lambda+i0}(r_\lambda) = \Upsilon^1_\lambda(H_{r_\lambda}, V)$$

does not depend on the regularizing operator V.

In Section 5 we introduce a class $\mathcal{R}$ of finite-rank operators which do not have non-zero real eigenvalues. The so-called *R-index* for operators A of class $\mathcal{R}$ is defined as

$$\mathcal{R}(A) = N_+ - N_-,$$

where N_+ and N_- are the numbers of eigenvalues of A in the upper ($\mathbb{C}_+$) and lower ($\mathbb{C}_-$) half-plane respectively. Some elementary properties of the R-index and a new proof of Kreĭn's theorem [Kr]

$$\mathcal{R}(R_z(H)V) = \mathrm{sign}(V),$$

where H is a self-adjoint operator and V is a finite rank self-adjoint operator, are given.

Further, in Section 5 the *resonance index* of the triple $(\lambda; H_{r_\lambda}, V)$ is introduced, which can be defined by

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \mathcal{R}(A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)) \quad \text{for all small enough } y.$$

Given a finite set $\Gamma = \{r_z^1, \dots, r_z^M\}$ of resonance points corresponding to $z \in \Pi$, we denote by $P_z(\Gamma)$ and $Q_z(\Gamma)$ the idempotents

$$P_z(\Gamma) = \sum_{r_z \in \Gamma} P_z(r_z) \quad \text{and} \quad Q_z(\Gamma) = \sum_{r_z \in \Gamma} Q_z(r_z).$$

We denote by $\bar{\Gamma}$ the set $\{\bar{r}_z^1, \dots, \bar{r}_z^M\}$.

The following theorem is one of the main technical results of this paper.

THEOREM 1.6.6 (Theorem 7.1.4). *If $\Gamma = \{r_z^1, \dots, r_z^M\}$ is a finite set of resonance points with positive imaginary part corresponding to a non-real number z, then the operator*

$$\operatorname{Im} z\, Q_{\bar{z}}(\bar{\Gamma}) J P_z(\Gamma)$$

is non-negative and its rank is equal to the rank of $P_z(\Gamma)$.

THEOREM 1.6.7 (Theorem 7.2.1). *If $\Gamma = \{r_z^1, \dots, r_z^M\}$ is a finite set of resonance points corresponding to a non-real number z, then the signature of the finite-rank self-adjoint operator*

$$Q_{\bar{z}}(\bar{\Gamma}) J P_z(\Gamma)$$

is equal to the R-index of the operator

$$\operatorname{Im} z\, A_z(s) P_z(\Gamma).$$

Theorems 1.6.6 and 1.6.7 are non-trivial even when $\dim \mathcal{H} < \infty$, that is, for matrices.

In Section 8 we prove the following

PROPOSITION 1.6.8. *Let λ be an essentially regular point, let $\{H_0 + rV : r \in \mathbb{R}\}$ be a line regular at λ, let r_λ be a real resonance point of the path $\{H_0 + rV : r \in \mathbb{R}\}$ at λ and let k be a positive integer. If $u_{\lambda\pm i0}(r_\lambda) \in \Upsilon_{\lambda\pm i0}(r_\lambda)$ is a resonance vector of order $k \geq 1$ at $\lambda \pm i0$, then for all non-resonant values of s,*

$$\langle J u_{\lambda\pm i0}(r_\lambda), \operatorname{Im} T_{\lambda\pm i0}(H_s)\, J u_{\lambda\pm i0}(r_\lambda)\rangle = \frac{c_{\pm 2}}{(s-r_\lambda)^2} + \dots + \frac{c_{\pm k}}{(s-r_\lambda)^k}, \tag{1.6.1}$$

where, in case $k \geq 2$, for $j = 2, \dots, k$,

$$c_{\pm j} = \operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}_{\lambda\pm i0}^{j-1}(r_\lambda) u_{\lambda\pm i0}(r_\lambda)\rangle = -\operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}_{\lambda\mp i0}^{j-1}(r_\lambda) u_{\lambda\pm i0}(r_\lambda)\rangle.$$

In particular, if $u_{\lambda\pm i0}(r_\lambda) \in \Upsilon_{\lambda\pm i0}(r_\lambda)$ is a resonance vector of order 1, then

$$\langle J u_{\lambda\pm i0}(r_\lambda), \operatorname{Im} T_{\lambda\pm i0}(H_s)\, J u_{\lambda\pm i0}(r_\lambda)\rangle = 0.$$

Further, in Section 8 we introduce and study the so-called *vectors of type I*. These are vectors which satisfy any of the following equivalent conditions.

THEOREM 1.6.9. *Let r_λ be a real resonance point of the line $\gamma = \{H_r : r \in \mathbb{R}\}$, corresponding to a real number $\lambda \in \Lambda(\gamma, F)$. Let $u \in \mathcal{K}$. The following assertions are equivalent:*

(1) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and for all non-resonant real numbers s,*
$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Ju = 0.$$
(2) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and for all non-resonant real numbers s,*
$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Ju = 0.$$
(3) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and for all non-resonant real numbers s,*
$$A_{\lambda+i0}(s)u = A_{\lambda-i0}(s)u.$$
(4) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and for all non-resonant real numbers s,*
$$A_{\lambda+i0}(s)u = A_{\lambda-i0}(s)u.$$
(5) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *or* $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and for all $j = 0, 1, \dots, d-1$, where d is the order of r_λ,*
$$\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = \mathbf{A}^j_{\lambda-i0}(r_\lambda)u.$$
(6) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and there exists a non-resonant real number r such that for all $j = 0, 1, 2, \dots,$*
$$(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = 0.$$
(7) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and there exists a non-resonant real number r such that for all $j = 0, 1, 2, \dots,$*
$$(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda-i0}(r_\lambda)u = 0.$$
(8) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and all the coefficients c_{+j} in* (1.6.1) *are zero.*
(9) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and all the coefficients c_{-j} in* (1.6.1) *are zero.*

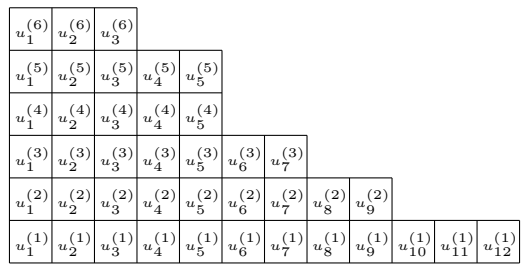
The set $\Upsilon^{\mathrm{I}}_{\lambda+i0}(r_\lambda)$ of vectors which satisfy any of these equivalent conditions is a vector subspace of $\Upsilon_{\lambda+i0}(r_\lambda) \cap \Upsilon_{\lambda-i0}(r_\lambda)$, and the vector space $\Upsilon^{\mathrm{I}}_{\lambda+i0}(r_\lambda)$ is invariant with respect to both $\mathbf{A}^j_{\lambda+i0}(r_\lambda)$ and $\mathbf{A}^j_{\lambda-i0}(r_\lambda)$.

For the nilpotent operator $\mathbf{A}_z(r_z)$ on $\Upsilon_z(r_z)$ there exists a Jordan basis $(u^{(j)}_\nu)$, $\nu = 1, \dots, m$, $j = 1, \dots, d_\nu$, where we assume that $d_1 \geq \cdots \geq d_m$, that is, a basis of $\Upsilon_z(r_z)$ such that $\mathbf{A}_z(r_z)u^{(j)}_\nu = u^{(j-1)}_\nu$; where $u^{(0)}_\nu = 0$. Every Jordan basis $(u^{(j)}_\nu)$ induces a decomposition
$$\Upsilon_z(r_z) = \Upsilon^{[1]}_z(r_z) \dotplus \cdots \dotplus \Upsilon^{[m]}_z(r_z),$$
where $\Upsilon^{[\nu]}_z(r_z)$ is the linear span of $u^{(1)}_\nu, \dots, u^{(d_\nu)}_\nu$ and where $\dotplus$ denotes direct sum of linear spaces. We call it a *Jordan decomposition* of $\Upsilon_z(r_z)$.

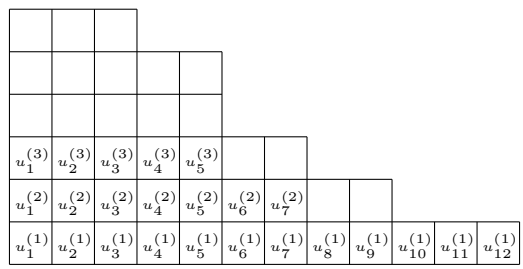
Proposition 1.6.8 and Theorem 1.6.9 are used to prove the following theorem which in its turn is essentially used in Section 10.

THEOREM 1.6.10. *If a resonance vector $u^{(k)} \in \Upsilon_{\lambda\pm i0}(r_\lambda)$ has order k then the vectors*
$$u^{(1)}, \dots, u^{(\lceil k/2 \rceil)}$$
are of type I, where $\lceil k/2 \rceil$ is the smallest integer not less than $k/2$, and $u^{(j)} = \mathbf{A}^{k-j}_{\lambda\pm i0}(r_\lambda)u^{(k)}$.

For example, assume that the geometric multiplicity m is 12 and the order d is 6. If a Jordan basis $(u_\nu^{(j)})$ of $\Upsilon_{\lambda+i0}(r_\lambda)$ is represented by the Young diagram



then according to Theorem 1.6.10 all vectors shown in the Young diagram



are of type I.

In Section 9 we prove that the resonance index is equal to the signature of the resonance matrix.

THEOREM 1.6.11 (Theorem 9.1.1). *The idempotents $P_{\lambda\pm i0}(r_\lambda)$ are linear isomorphisms of the vector spaces $\Upsilon_{\lambda\mp i0}(r_\lambda)$ and $\Upsilon_{\lambda\pm i0}(r_\lambda)$.*

Theorem 1.6.11 is used in the proof of the following theorem, which is one of the main results of this paper.

THEOREM 1.6.12 (Theorem 9.2.1). *For any real resonance point r_λ,*

$$\operatorname{sign}(Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)) = \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V).$$

In Section 10 we prove Theorem 1.6.14 which is one of the main results of this paper. It is a corollary to Theorem 1.6.12 and the following

THEOREM 1.6.13 (Theorem 10.1.5). *If r_λ is a real resonance point corresponding to $z = \lambda \pm i0$, then*

$$|\operatorname{sign}(Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda))| \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

THEOREM 1.6.14 (Theorem 10.1.6). *For all real resonance points r_λ,*

$$|\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)| \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

Theorem 1.6.14 has the following meaning: the increment of the spectral flow inside essential spectrum which occurs at a real resonance point r_λ cannot be greater than the multiplicity of the singular spectrum of H_{r_λ} at λ.

The numbers $N_\pm$ from the definition of the resonance index give more information about the behaviour of points of the singular spectrum than the difference $N_+ - N_-$. Striking a shop-keeper's doorbell, a customer may open the door and leave without entering the shop. In this case the doorbell rings but the number of customers in the shop remains the same (that is, the increment of the spectral flow is zero). In other words, a ring of the doorbell condition $r \in R(\lambda; H_0, V)$ does not necessarily mean that an "eigenvalue" crossed λ, e.g., if λ is outside the essential spectrum, an eigenvalue can make a U-turn at λ. Theorems 1.5.2 and 1.6.14 imply that if there is an eigenvalue $\lambda_j(r)$ of a path H_r making a U-turn at λ when $r = r_\lambda$, then $N_+, N_- > 0$ so that the contributions of that eigenvalue to N_+ and N_- cancel each other. In particular, if the eigenvalue $\lambda_j(r_\lambda) = \lambda$ of H_{r_λ} making a U-turn is non-degenerate, then $N_+ = N_-$ so that $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ is zero. On p. 106 of this paper eight diagrams are given which correspond to eight qualitatively different eigenvalue behaviours in case $N_+ = 5$ and $N_- = 2$.

The main result of Section 11 is Theorem 11.2.7. Its proof relies on certain algebraic relations between the operators $P_{\lambda\pm i0}(r_\lambda)$ and $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$ which are proved in that section.

A real resonance point r_λ will be said to have *property C* if the vector spaces $\Upsilon_{\lambda\pm i0}(r_\lambda)$ admit Jordan decompositions

$$\Upsilon_{\lambda+i0}(r_\lambda) = \Upsilon^{[1]}_{\lambda+i0}(r_\lambda) \dotplus \cdots \dotplus \Upsilon^{[m]}_{\lambda+i0}(r_\lambda), \qquad \Upsilon_{\lambda-i0}(r_\lambda) = \Upsilon^{[1]}_{\lambda-i0}(r_\lambda) \dotplus \cdots \dotplus \Upsilon^{[m]}_{\lambda-i0}(r_\lambda)$$

such that for all $\nu = 1, \dots, m$,

$$P_{\lambda+i0}(r_\lambda)\Upsilon^{[\nu]}_{\lambda-i0}(r_\lambda) = \Upsilon^{[\nu]}_{\lambda+i0}(r_\lambda) \quad \text{and} \quad P_{\lambda-i0}(r_\lambda)\Upsilon^{[\nu]}_{\lambda+i0}(r_\lambda) = \Upsilon^{[\nu]}_{\lambda-i0}(r_\lambda).$$

THEOREM 1.6.15 (Theorems 11.2.7 and 11.2.8). *For any $z = \lambda \pm i0 \in \partial\Pi$, for any real resonance point $r_\lambda \in \mathbb{R}$ with property C corresponding to z and for any $j = 1, 2, \dots$:*

(1) *the restriction of the idempotent operator $P_{\lambda\pm i0}(r_\lambda)$ to $\Upsilon^j_{\lambda\mp i0}(r_\lambda)$ is a linear isomorphism of $\Upsilon^j_{\lambda\mp i0}(r_\lambda)$ and $\Upsilon^j_{\lambda\pm i0}(r_\lambda)$,*
(2) *the idempotent $Q_{\lambda\pm i0}(r_\lambda)$ is a linear isomorphism of $\Psi^j_{\lambda\mp i0}(r_\lambda)$ and $\Psi^j_{\lambda\pm i0}(r_\lambda)$ for all $j = 1, 2, \dots$.*

In other words, for points r_λ with property C, for all $j = 1, 2, \dots$ we have commutative diagrams of linear isomorphisms:

$$\begin{array}{ccc}
\Psi^j_{\lambda+i0}(r_\lambda) & \xleftarrow{\ J\ } & \Upsilon^j_{\lambda+i0}(r_\lambda) \\
{\scriptstyle Q_{\lambda+i0}(r_\lambda)}\uparrow & & \uparrow{\scriptstyle P_{\lambda+i0}(r_\lambda)} \\
\Psi^j_{\lambda-i0}(r_\lambda) & \xleftarrow{\ J\ } & \Upsilon^j_{\lambda-i0}(r_\lambda)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Psi^j_{\lambda+i0}(r_\lambda) & \xleftarrow{\ J\ } & \Upsilon^j_{\lambda+i0}(r_\lambda) \\
{\scriptstyle Q_{\lambda-i0}(r_\lambda)}\downarrow & & \downarrow{\scriptstyle P_{\lambda-i0}(r_\lambda)} \\
\Psi^j_{\lambda-i0}(r_\lambda) & \xleftarrow{\ J\ } & \Upsilon^j_{\lambda-i0}(r_\lambda)
\end{array}$$

Real resonance points for which the conclusion of this theorem holds are called points with *property U*. Thus, property C implies property U. Plainly, every point of geometric multiplicity 1 has property C and therefore property U too. We conjecture that every real resonance point has properties C and U.

In Section 12 we consider some questions of independence from the choice of the rigging operator F.

THEOREM 1.6.16 (Theorem 12.1.2). *The resonance index* $\operatorname{ind}_{\mathrm{res}}(\lambda; H, V)$ *does not depend on the choice of the rigging operator* F *as long as* λ *is essentially regular for the pair* $(\mathcal{A}, F)$*, where* $\mathcal{A} = \{H + rV : r \in \mathbb{R}\}$ *and* V *is a regularizing direction for an operator* H *which is resonant at* λ*.*

In Section 13 we study the class of so-called *real resonance points of type I*. By definition, a real resonance point r_λ is of type I if all resonance vectors satisfy at least one and therefore all assertions of Theorem 1.6.9.

THEOREM 1.6.17 (Theorem 13.1.9). *Let* λ *be an essentially regular point for* $(\mathcal{A}, F)$*. Let* $H_0 \in \mathcal{A}$ *be regular at* λ *and let* $V \in \mathcal{A}_0$*. Let* $r_\lambda \in \mathbb{R}$ *be a real resonance point of the path* $\{H_0 + rV : r \in \mathbb{R}\}$*. The following assertions are all equivalent to* r_λ *being of type I:*

- ($\mathrm{i}_\pm$) *For any regular point* r, $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, JP_{\lambda\pm i0}(r_\lambda) = 0$.
- ($\mathrm{i}^*_\pm$) *There exists a regular point* r *such that* $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, JP_{\lambda\pm i0}(r_\lambda) = 0$.
- ($\mathrm{ii}_\pm$) *For any regular point* r, $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, Q_{\lambda\pm i0}(r_\lambda) = 0$.
- ($\mathrm{ii}^*_\pm$) *There exists a regular point* r *such that* $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, Q_{\lambda\pm i0}(r_\lambda) = 0$.
- ($\mathrm{iii}_\pm$) *The meromorphic function*

$$w_\pm(s) := \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,[1 + sJT_{\lambda\pm i0}(H_0)]^{-1}$$

  is holomorphic at $s = r_\lambda$*.*
- ($\mathrm{iii}'_\pm$) *The meromorphic function*

$$w_\pm(s)J = \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, J[1 + sT_{\lambda\pm i0}(H_0)J]^{-1}$$

  is holomorphic at $s = r_\lambda$*.*
- ($\mathrm{iv}_\pm$) *The meromorphic function*

$$w^\dagger_\pm(s) = [1 + sT_{\lambda\mp i0}(H_0)J]^{-1}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}$$

  is holomorphic at $s = r_\lambda$*.*
- ($\mathrm{v}_\pm$) *The residue of the function* $w_\pm(s)$ *at* $s = r_\lambda$ *is zero.*
- ($\mathrm{vi}_\pm$) *For all* $\pm$*-resonance vectors the real numbers* c_{-j} *from Proposition* 8.1.1 *are all zero.*
- (vii) *The function* $s \mapsto \operatorname{Im} T_{\lambda+i0}(H_s)$ *is holomorphic at* $s = r_\lambda$*.*
- (viii) *The function* $s \mapsto J \operatorname{Im} T_{\lambda+i0}(H_s)J$ *is holomorphic at* $s = r_\lambda$*.*

Moreover, the assertions obtained from ($\mathrm{i}_\pm$)–($\mathrm{ii}_\pm$) *and* ($\mathrm{i}^*_\pm$)–($\mathrm{ii}^*_\pm$) *by removing the square root are also equivalent to those above.*

The following theorem shows that being of type I is a generic property of real resonance points.

THEOREM 1.6.18 (Theorems 13.2.1–13.2.3). *Let* λ *be an essentially regular point, let* $H_0 \in \mathcal{A}$ *and let* $V \in \mathcal{A}_0$ *be a regularizing direction at* λ*. Let* r_λ *be a real resonance point of* $(\lambda; H_0, V)$*. If at least one of the following three conditions holds:*

(1) *λ does not belong to the (necessarily common) essential spectrum of operators from $\mathcal{A}$,*
(2) *the order of r_λ is equal to 1,*
(3) *the operator V is non-negative or non-positive,*

then r_λ is a point of type I.

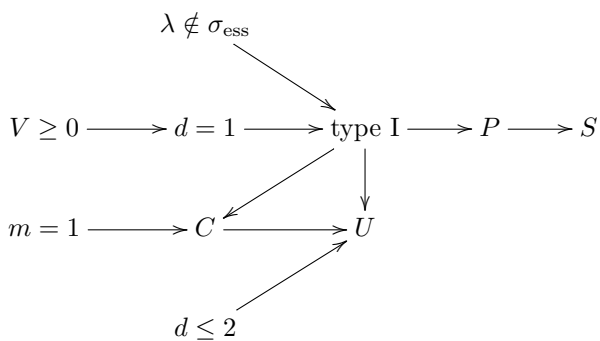
For every real resonance point r_λ of type I the idempotents $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$ coincide. We say that a real resonance point r_λ has *property S* if the kernels of $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$ coincide.

PROPOSITION 1.6.19 (Proposition 13.3.1). *Let λ be an essentially regular point and let r_λ be a real resonance point of $(\lambda; H_0, V)$. The following assertions are equivalent:*

(i) *r_λ has property S.*
(ii) $P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) = P_{\lambda+i0}(r_\lambda)$ *and* $P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$.
(iii) $\operatorname{im} Q_{\lambda+i0}(r_\lambda) = \operatorname{im} Q_{\lambda-i0}(r_\lambda)$.
(iv) $Q_{\lambda+i0}(r_\lambda)Q_{\lambda-i0}(r_\lambda) = Q_{\lambda-i0}(r_\lambda)$ *and* $Q_{\lambda-i0}(r_\lambda)Q_{\lambda+i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)$.
(v) $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = JP_{\lambda+i0}(r_\lambda)$.
(vi) $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = JP_{\lambda-i0}(r_\lambda)$.
(vii) $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = Q_{\lambda-i0}(r_\lambda)J$.
(viii) $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)J$.
(ix) $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$.

PROPOSITION 1.6.20 (Proposition 13.3.2). *Every real resonance point of type I has property S. There are real resonance points which do not have property S, and there are points with property S which are not of type I.*

Let us say that a real resonance point r_λ has *property P* if $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$. The relations between real resonance points with different properties are given in the following diagram, where arrows stand for implications:



In Section 14 we study the behaviour of a non-degenerate eigenvalue embedded into the essential spectrum under a regularizing perturbation V. In particular, we construct real resonance points which do not have property S, and real resonance points with property S, but without property P.

Assume that λ is an eigenvalue of a self-adjoint operator H_{r_λ} with eigenvector χ. Then the Hilbert space $\mathcal{H}$ on which H_{r_λ} acts can be represented as

$$\mathcal{H} = \hat{\mathcal{H}} \oplus \mathbb{C},$$

so that the operator H_{r_λ} takes the form

$$H_{r_\lambda} = \begin{pmatrix} \hat{H}_{r_\lambda} & 0 \\ 0 & \lambda \end{pmatrix},$$

where $\hat{H}_{r_\lambda}$ is the restriction of H_{r_λ} to $\hat{\mathcal{H}}$. Let

$$V = \begin{pmatrix} \hat{V} & \hat{v} \\ \langle \hat{v}, \cdot \rangle & \alpha \end{pmatrix}$$

be the representation of the operator V. We assume that the rigging operator $F\colon \mathcal{H} \to \mathcal{K}$ has a representation

$$F = \begin{pmatrix} \hat{F} & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case $V = F^* J F$, where J has a representation

$$J = \begin{pmatrix} \hat{J} & \hat{\psi} \\ \langle \hat{\psi}, \cdot \rangle & \alpha \end{pmatrix}$$

such that $\hat{V} = \hat{F}^* \hat{J} \hat{F}$ and $\hat{v} = \hat{F}^* \hat{\psi}$. The vector $\hat{\psi}$ is related to the eigenvector χ by $\hat{\psi} = JF\chi - \alpha F\chi$. Finally, we assume that λ is a regular point of $(\hat{H}_{r_\lambda}, \hat{F})$:

$$\lambda \in \Lambda(\hat{H}_{r_\lambda}, \hat{F}). \tag{1.6.2}$$

Let

$$\hat{u}_z(s) = \hat{F} R_z(\hat{H}_s) \hat{F}^* \hat{\psi},$$

where $T_z(\hat{H}_s) = \hat{F} R_z(\hat{H}_s) \hat{F}^*$, and let $\hat{A}_z(s) = T_z(\hat{H}_s)\hat{J}$. The operator $A_{\lambda+i0}(r_\lambda)$ does not exist since λ is an eigenvalue of H_{r_λ} and therefore $\lambda \notin \Lambda(H_{r_\lambda}, F)$, but the sliced operator $\hat{A}_{\lambda+i0}(r_\lambda)$ and the vector $\hat{u}_{\lambda+i0}(r_\lambda)$ exist due to (1.6.2).

The following lemma and theorem describe the properties of the real resonance point r_λ.

Lemma 1.6.21. *The order of the real resonance point r_λ is greater than 1 if and only if $\alpha = 0$. In that case the vector space $\Upsilon^2_{\lambda+i0}(r_\lambda)$ is two-dimensional and is generated by the vectors*

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix},$$

which have orders 1 and 2 respectively.

Theorem 1.6.22 (Theorem 14.2.11). *Let d be an integer greater than 1. The order of the real resonance point r_λ is equal to d if and only if the vectors*

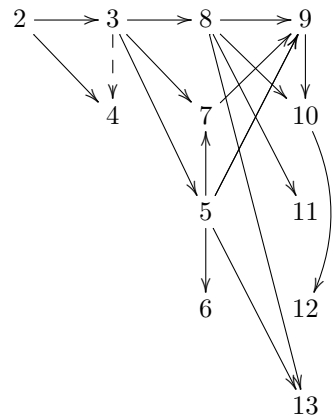
$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_{\lambda+i0}(r_\lambda),\ \dots,\ \hat{A}^{d-3}_{\lambda+i0}(r_\lambda)\hat{u}_{\lambda+i0}(r_\lambda)$$

are orthogonal to $\hat{\psi}$ but $\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_{\lambda+i0}(r_\lambda)$ is not. In that case for all $j = 1, \dots, d$ the vector space $\Upsilon^j_{\lambda+i0}(r_\lambda)$ is j-dimensional and is generated by the vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix},\ \begin{pmatrix} \hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix},\ \dots,\ \begin{pmatrix} \hat{A}^{j-2}_{\lambda+i0}(r_\lambda)\hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix},$$

which have orders $1, \dots, j$ respectively.

The following diagram shows the interdependence of Sections 2–14. A dashed arrow means that the dependence is of notational and terminological character. In particular, Section 14 is almost independent of the other sections, but motivation for it comes from the previous ones. The core of this paper are Sections 6, 7, 9 and 10. Having said this, ideologically all sections are interconnected in the sense that they represent different aspects of the same subject given in the title of this paper.



In Section 15 some open problems are stated. Finally, for the reader's convenience there is a detailed index.

1.7. Future work

1.7.1. Singular spectral shift function for relatively trace class perturbations. So far the property (1.3.20) of the singular spectral shift function has been proved for trace class perturbations. There is a paper in preparation [AzD] in which this result will be proved for relatively trace class perturbations. A special case of this result is

THEOREM 1.7.1. *Let $H_0 = -\Delta + V_0(x)$ be a Schrödinger operator acting on $L_2(\mathbb{R}^\nu)$, where $\nu = 1$, 2 or 3 and where $V_0(x)$ is a bounded measurable real-valued function. Let V be the operator of multiplication by a real-valued measurable function $V(x)$ such that $|V(x)| \le \mathrm{const}(1+|x|^2)^{-\nu/2-\varepsilon}$ for some $\varepsilon > 0$, and let $H_r = H_0 + rV$. Let*

$$\xi^{(s)}(\phi) = \int_0^1 \mathrm{Tr}(V\phi(H_r^{(s)}))\,dr, \qquad \phi \in C_c(\mathbb{R}),$$

where $H_r^{(s)}$ is the singular part of H_r. Then $\xi^{(s)}$ is an absolutely continuous measure whose density $\xi^{(s)}(\lambda)$ (denoted by the same symbol) is integer-valued for a.e. λ.

The bulk of the proof of this theorem is a modification to relatively trace class perturbations of the approach to scattering theory given in [Az$_3$] and discussed in this introduction. This modification was given in [Az$_6$] with the aim to prove Theorem 1.7.1. For reasons mentioned in this introduction classical approaches to scattering theory do not allow one to prove this theorem.

There is a work in progress aimed at proving an analogue of Theorem 1.7.1 for n-dimensional Schrödinger operators.

1.7.2. Resonance index and singular μ-invariant. For trace class perturbations the singular spectral shift function admits three other equivalent descriptions, as the singular μ-invariant, the total resonance index and the total signature of the resonance matrix. These three definitions do not require the perturbation to be trace class or to be relatively trace class: all we need to assume is that the perturbation is relatively compact and that the Limiting Absorption Principle holds. In this paper it is shown that the resonance index and the signature of resonance matrix are equal under these two conditions. In [Az$_5$] it is proved that the singular μ-invariant is equal to the total resonance index under the same conditions.

1.7.3. Resonance index outside the essential spectrum. The theory of the resonance index given in this paper applies to the spectral flow outside the essential spectrum. Outside the essential spectrum one can consider many questions which do not make sense inside the essential spectrum. There is work in progress where these questions will be studied.

2. Preliminaries

2.1. Operators on a Hilbert space. Details concerning the material of this section can be found in [GK, Ka$_2$, RS, S$_2$]. A partial aim of these preliminaries is to fix notation and terminology.

Throughout this paper, $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers. The calligraphic letters $\mathcal{H}$ and $\mathcal{K}$ will denote complex separable Hilbert spaces, which can be finite- or infinite-dimensional. The scalar product $\langle \cdot, \cdot \rangle$ is assumed to be linear with respect to the second argument and anti-linear with respect to the first. If it is necessary to distinguish the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the former will be called the *main Hilbert space*, and the latter the *auxiliary Hilbert space*; note, however, that most of the operators considered in this paper act on $\mathcal{K}$ rather than on $\mathcal{H}$. The letter H with possible subscripts will denote a self-adjoint operator on $\mathcal{H}$. The letter F will always denote a fixed densely defined closed operator from $\mathcal{H}$ to $\mathcal{K}$ which has trivial kernel and co-kernel. The letter Λ with arguments will always denote a measurable subset of $\mathbb{R}$ of full Lebesgue measure. Throughout this paper the word "operator" means a linear operator.

The letter V will be used to denote a self-adjoint operator on $\mathcal{H}$ with some conditions imposed on it. We shall consider a perturbation $H_r = H_0 + rV$ of a self-adjoint operator H_0 by a real multiple of V; the multiple itself, called a *coupling constant*, will be denoted by the letter s or r (with possible subscripts).

A subset A of a metric space X is *discrete* if the intersection of A with any compact subset of X is finite.

If L_1 and L_2 are closed subspaces of a Hilbert space such that $L_1 \cap L_2 = \{0\}$, then we denote by $L_1 \dotplus L_2$ the direct sum of L_1 and L_2. If in addition L_1 and L_2 are orthogonal then their direct sum is denoted by $L_1 \oplus L_2$.

The *kernel* of an operator A is denoted by $\ker(A)$, and $\operatorname{im}(A)$ will denote the *range* or the *image* of A. The *resolvent set* ρ_T of a densely defined closed operator T on a Hilbert space $\mathcal{H}$ consists of all complex numbers z such that $T - z$ is a bijection of $\operatorname{dom}(T)$ onto $\mathcal{H}$; for such z the bounded inverse

$$R_z(T) = (T - z)^{-1},$$

called the *resolvent* of T, exists. The *spectrum* σ_T or $\sigma(T)$ of a densely defined closable operator T on a Hilbert space is the complement of the resolvent set. For two bounded operators S and T one has (see e.g. [BR, Proposition 2.2.3])

$$\sigma_{ST} \cup \{0\} = \sigma_{TS} \cup \{0\}. \tag{2.1.1}$$

Let T be a closed operator on a Hilbert space $\mathcal{K}$ and let $z \in \mathbb{C}$. Non-zero vectors u from $\mathcal{K}$ such that $(T-z)^k u = 0$ for some $k = 1, 2, \dots$ are called *root vectors* of T corresponding to the eigenvalue z. A point z of the spectrum of T is called an *isolated eigenvalue* of finite algebraic multiplicity if z is an isolated point of $\sigma(T)$ and if the *algebraic multiplicity* $\mu_T(z)$ of z defined by

$$\mu_T(z) := \dim\{u \in \mathcal{K} \colon \exists k \in \mathbb{Z}_+ \ (T-z)^k u = 0\}$$

is finite. The set of all isolated eigenvalues of finite algebraic multiplicity of an operator T is denoted by $\sigma_d(T)$. If T is compact, the function μ_T of z will be called the *eigenvalue counting measure* of T. If S and T are bounded operators such that ST and TS are both compact, then the following stronger version of (2.1.1) holds:

$$\mu_{ST}|_{\mathbb{C}\setminus\{0\}} = \mu_{TS}|_{\mathbb{C}\setminus\{0\}}. \tag{2.1.2}$$

Further, for any compact operator T,

$$\mu_{T^*} = \bar{\mu}_T, \tag{2.1.3}$$

where $\bar{\mu}_T(z) = \mu_T(\bar{z})$.

A closed operator T is said to be *Fredholm* if the range of T is a closed subspace of finite co-dimension and the kernel of T is finite-dimensional (see [Ka$_2$, IV.5.1]). A bounded operator T is Fredholm if and only if there exists a bounded operator S such that the operators $ST-1$ and $TS-1$ are compact; such an operator S is called a *parametrix* of T. In other words, bounded Fredholm operators are invertible up to compact operators. By definition, the *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ of a closed operator T consists of all complex numbers z such that the operator $T-z$ is not Fredholm; in this regard note that in [Ka$_2$, §IV.5] the essential spectrum is defined as the set of all complex numbers z such that the operator $T-z$ is not semi-Fredholm. There are also other definitions, but they all coincide for self-adjoint operators. Since in this paper we shall be concerned with the essential spectrum of self-adjoint operators and of their relatively compact perturbations, this definition suffices. The essential spectrum of a self-adjoint operator H admits another characterization: the essential spectrum of H is the spectrum of H from which all isolated eigenvalues of finite multiplicity are removed. In general,

$$\sigma_{\mathrm{ess}}(T) \subset \sigma(T) \setminus \sigma_d(T),$$

but this inclusion may be strict [Ka$_2$].

Let H and V be two self-adjoint operators on a Hilbert space $\mathcal{H}$. Then V is said to be *relatively compact* with respect to H if $R_z(H)V$ is a bounded operator on $\mathrm{dom}(V) \subset \mathcal{H}$ for some $z \in \rho_H$ such that its continuous extension to $\mathcal{H}$ is a compact operator. In this case the operator $R_z(H)V$ is bounded with compact extension for any $z \in \rho_H$. Weyl's theorem asserts that the essential spectrum of a self-adjoint operator is stable under relatively compact perturbations (see e.g. [Ka$_2$, §IV.5.6], [RS$_4$, §XIII.4]).

The spectrum of a closed operator T on a Hilbert space is upper semicontinuous: for any neighbourhood O of the spectrum of T there exists $\delta > 0$ such that for all bounded S with $\|S\| < \delta$ the spectrum of $S+T$ is a subset of O.

In general, the spectrum is not continuous in the sense that for a bounded operator T there may exist $z \in \sigma(T)$ and a neighbourhood O of z such that for any $\delta > 0$ there exists a bounded operator S with $\|S\| < \delta$ such that $\sigma(T+S) \cap O = \emptyset$.

For brevity, the identity operator on a Hilbert space is denoted by 1; in particular, the scalar operator of multiplication by a number c will be denoted by c instead of cI. An *idempotent operator* is a bounded operator P such that $P^2 = P$. If A and B are bounded operators such that $z \notin \sigma_{AB} \cup \{0\}$, then

$$(z - AB)^{-1}A = A(z - BA)^{-1}. \tag{2.1.4}$$

The condition $z \notin \sigma_{AB} \cup \{0\}$ implies that $z \notin \sigma_{BA}$, so that the right hand side of the above equality makes sense. Hence, the equality itself follows from the obvious equality $A(z - BA) = (z - AB)A$.

The *real* $\operatorname{Re} A$ and *imaginary* $\operatorname{Im} A$ parts of a bounded operator A on a Hilbert space are defined by

$$\operatorname{Re} A = \frac{A + A^*}{2} \quad \text{and} \quad \operatorname{Im} A = \frac{A - A^*}{2i}.$$

The *Rank* of an operator A is the dimension of the image of A. The *signature* $\operatorname{sign}(A)$ of a finite-rank self-adjoint operator A is the integer defined as follows:

$$\operatorname{sign}(A) = \operatorname{rank} A_+ - \operatorname{rank} A_-, \tag{2.1.5}$$

where A_+ (respectively, A_-) is the positive (respectively, negative) part of A. In this regard note that, given a self-adjoint operator A, the word "signature" is also used for the operator $f(A)$, where $f(x)$ is the sign-function, but in this paper this notion will not be used and therefore there is no danger of confusion.

LEMMA 2.1.1. *If A is an operator of rank $N < \infty$, then there exists $\varepsilon > 0$ such that for any operator B of norm less than ε the inequality* $\operatorname{rank}(A + B) \le N$ *implies the equality* $\operatorname{rank}(A + B) = N$.

In other words, small enough perturbations of finite rank operators which do not increase the rank preserve the rank. This lemma is a direct consequence of the upper semicontinuity of spectrum.

LEMMA 2.1.2. *Let M be a finite-rank self-adjoint operator on a Hilbert space $\mathcal{K}$. If $\mathcal{L}$ is a vector subspace of $\mathcal{K}$ such that for any non-zero $f \in \mathcal{L}$ the scalar product $\langle f, Mf\rangle$ is positive, then*

$$\dim \mathcal{L} \le \operatorname{rank} M_+.$$

Proof. Let $\mathcal{M}_+$ be the vector space spanned by the eigenvectors of M corresponding to positive eigenvalues and assume contrary to the claim that $\dim \mathcal{L} > \dim \mathcal{M}_+$. Then the intersection $\mathcal{M}_+^\perp \cap \mathcal{L}$ is a vector subspace of dimension at least 1. If f is a non-zero vector from $\mathcal{M}_+^\perp \cap \mathcal{L}$, then $\langle f, Mf\rangle > 0$ since $f \in \mathcal{L}$ and $\langle f, Mf\rangle \le 0$ since $f \in \mathcal{M}_+^\perp$. ∎

LEMMA 2.1.3. *Let M be a self-adjoint finite rank operator on a Hilbert space $\mathcal{K}$ and let $F\colon \mathcal{H} \to \mathcal{K}$ be a closed operator with zero kernel and co-kernel. If* $\operatorname{im}(M) \subset \operatorname{dom}(F^*)$ *and if* $\operatorname{im}(M) = \operatorname{im}(MF)$, *then the product F^*MF is a well-defined finite-rank self-adjoint operator such that*

$$\operatorname{rank}(M) = \operatorname{rank}(F^*MF) \quad \textit{and} \quad \operatorname{sign}(M) = \operatorname{sign}(F^*MF).$$

Proof. Let $\mathcal{M}_+$ (respectively, $\mathcal{M}_-$) be the vector space spanned by the eigenvectors of M corresponding to positive (respectively, negative) eigenvalues of M, and let $\mathcal{M} =$

$\mathcal{M}_+ \oplus \mathcal{M}_-$. Since $\operatorname{im}(M) \subset \operatorname{dom}(F^*)$, the product F^*MF is well-defined. Since F^* has zero kernel and $\operatorname{im}(M) = \operatorname{im}(MF)$, the ranks of the operators F^*MF and M are the same. Further, the equality $\operatorname{im}(M) = \operatorname{im}(MF)$ implies that there exist vector spaces $\mathcal{L}_\pm$ such that $\mathcal{L}_\pm = F^{-1}\mathcal{M}_\pm$ and $\dim \mathcal{M}_\pm = \dim \mathcal{L}_\pm$. For any non-zero vector $f = F^{-1}g \in \mathcal{L}_+$, where $g \in \mathcal{M}_+$, we have

$$\langle f, F^*MFf\rangle = \langle Ff, MFf\rangle = \langle g, Mg\rangle > 0.$$

It follows from this and Lemma 2.1.2 that $\dim \mathcal{L}_+ = \dim \mathcal{M}_+$ is not greater than the rank of the positive part of F^*MF. Similarly, one shows that $\dim \mathcal{M}_-$ is not greater than the rank of the negative part of F^*MF. Combining this with the equality $\operatorname{rank}(M) = \operatorname{rank}(F^*MF)$ implies that $\operatorname{sign}(M) = \operatorname{sign}(F^*MF)$. ∎

If T is a compact operator on a Hilbert space, then the *s-numbers* $s_1(T), s_2(T), \dots$ of T are the eigenvalues of the compact operator $|T| := \sqrt{T^*T}$ listed in non-increasing order, with each eigenvalue repeated according to its multiplicity. Let $p \in [1, \infty]$. The notation $\mathcal{L}_p(\mathcal{H})$ means the class of all compact operators T acting on $\mathcal{H}$ such that the p-norm $\|T\|_p$ of T, defined by

$$\|T\|_p^p := \sum_{n=1}^{\infty} s_n(T)^p \quad \text{if } p < \infty, \qquad \|T\|_\infty := s_1(T) = \|T\| \quad \text{if } p = \infty,$$

is finite. The linear space $\mathcal{L}_p(\mathcal{H})$ with this norm is an invariant operator ideal [GK], called the *pth Schatten ideal*. This means, in particular, that if $T \in \mathcal{L}_p(\mathcal{H})$ and if A, B are bounded operators, then $ATB \in \mathcal{L}_p(\mathcal{H})$ and $\|ATB\|_p \le \|A\|\,\|T\|_p\|B\|$. The ideal $\mathcal{L}_\infty$ consists of all compact operators on $\mathcal{H}$. Operators from the first Schatten ideal $\mathcal{L}_1(\mathcal{H})$ are called *trace class operators*, and those from the second Schatten ideal $\mathcal{L}_2(\mathcal{H})$ are called *Hilbert–Schmidt operators*. The *trace class norm* $\|T\|_1$ of a trace class operator T is equal to $\operatorname{Tr}(|T|)$, and the *Hilbert–Schmidt norm* $\|T\|_2$ of a Hilbert–Schmidt operator T is equal to $\sqrt{\operatorname{Tr}(|T|^2)}$. The *trace* $\operatorname{Tr}\colon \mathcal{L}_1(\mathcal{H}) \to \mathbb{C}$ is a linear continuous functional, defined for trace class operators by $\operatorname{Tr}(T) = \sum_{n=1}^{\infty}\langle \phi_n, T\phi_n\rangle$, where $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. If A, B are bounded operators such that AB and BA are trace class, then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$

Further, for any trace class operator T we have $\operatorname{Tr}(T^*) = \overline{\operatorname{Tr}(T)}$. For any trace class operator T the sequence $\{\lambda_j(T)\}_{j=1}^{\infty}$ of eigenvalues of T is summable; the Lidskiĭ theorem asserts that

$$\operatorname{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j(T). \tag{2.1.6}$$

Lemma 2.1.4. *Let $p \ge 1$. If $A, A_1, A_2, \dots$ is a sequence of finite-rank operators on a Hilbert space such that the sequence of ranks of A_n is bounded, then A_n converges to A as $n \to \infty$ in the uniform norm if and only if A_n converges to A in p-norm as $n \to \infty$.*

Proof. If N is the largest of the ranks of the operators $A, A_1, A_2, \dots$, then

$$\|A\| \le \|A\|_p \le N\|A\|. \ ∎$$

2.2. Analytic operator-valued functions. For definition and a detailed study of vector-valued holomorphic functions see e.g. [HPh, Ka$_2$, RS]. Let $T(\kappa)$ be a single-valued holomorphic function with values in bounded operators; assume that $T(\kappa)$ is defined in some domain G of the complex plane off a discrete set of singular points. In a deleted neighbourhood $0 < |\kappa - \kappa_0| < \delta$ of a singular point $\kappa_0 \in G$ the function $T(\kappa)$ admits a Laurent expansion at κ_0,

$$T(\kappa) = \tilde{T}(\kappa) + \sum_{j=1}^{\infty} (\kappa - \kappa_0)^{-j} T_j, \tag{2.2.1}$$

where $\tilde{T}(\kappa)$ is a function of κ holomorphic in the neighbourhood of κ_0 (including κ_0) and $T_1, T_2, \dots$ are some bounded operators. A function T defined on G is said to be *meromorphic* in G if it is holomorphic everywhere on G except possibly on a discrete subset of singular points, such that at each singular point κ_0 the sum in its Laurent expansion (2.2.1) is finite.

THEOREM 2.2.1 (Analytic Fredholm alternative). *Let G be an open connected subset of $\mathbb{C}$. Let $T\colon G \to \mathcal{L}_\infty(\mathcal{H})$ be a holomorphic family of compact operators in G. If the operators $1 + T(\kappa)$ are all invertible at some point $\kappa_1 \in G$, then they are invertible at all points of G except on the discrete set*

$$\mathcal{N} := \{\kappa \in G\colon -1 \in \sigma(T(\kappa))\}.$$

Further, the operator-function $F(\kappa) := (1 + T(\kappa))^{-1}$ is meromorphic in G and the set of its poles is $\mathcal{N}$. Moreover, in the Laurent expansion of $F(\kappa)$ in a neighbourhood of any point $\kappa_0 \in \mathcal{N}$ the coefficients of negative powers of $\kappa - \kappa_0$ are finite-rank operators.

For a proof of this theorem see e.g. [RS, Theorem VI.14], [Y, Theorem 1.8.2].

2.3. Divided differences. If $f(s)$ is a function of one variable, then the *divided difference of f of first order* is the function

$$f^{[1]}(s_1, s_2) = \frac{f(s_2) - f(s_1)}{s_2 - s_1}.$$

The *divided difference of f of order k* is the function $f^{[k]}(s_1, \dots, s_{k+1})$ of $k+1$ variables $s_1, \dots, s_{k+1}$ which is defined inductively by

$$f^{[k]}(s_1, \dots, s_{k+1}) = \frac{f^{[k-1]}(s_2, \dots, s_{k+1}) - f^{[k-1]}(s_1, \dots, s_k)}{s_{k+1} - s_1}.$$

We shall use two facts about divided differences; the proofs can be found in e.g. [Ba]:

LEMMA 2.3.1. *The divided difference of order $k-1$ of f is equal to*

$$f^{[k-1]}(s_1, \dots, s_k) = \sum_{j=1}^{k} f(s_j) \prod_{i=1,\, i\neq j}^{k} \frac{1}{s_j - s_i}.$$

LEMMA 2.3.2. *The divided difference of order $k-1$ of f is zero if and only if f is a polynomial of degree $\le k-2$.*

2.4. Rigged Hilbert spaces. A *rigging* of a Hilbert space $\mathcal{H}$ is a triple $(X, \mathcal{H}, X^*)$ where X is a normed or more generally a locally convex space, and X^* is its dual such that X is continuously embedded into $\mathcal{H}$ and $\mathcal{H}$ is continuously embedded into X^* and the embeddings have dense ranges (see e.g. [BeSh]). The rigging normed space X is often introduced as the range of a certain operator acting on $\mathcal{H}$. In this case it is possible to consider the operator itself as the rigging. In this paper we follow this view-point. Further, the normed space X can itself be a Hilbert or pre-Hilbert space. In this case elements $f, g, \dots$ of X can be considered as elements of both X^* via the Riesz–Fisher theorem and of $\mathcal{H}$ via the natural embedding $X \hookrightarrow \mathcal{H}$, and then it is assumed that $\langle f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{(X,X^*)}$, where $\langle f, g\rangle_{(X,X^*)}$ is the value of $g \in X^*$ at $f \in X$. The number $\langle f, g\rangle_{(X,X^*)}$ is often denoted by $\langle f, g\rangle_{1,-1}$.

A *rigging operator* F on a Hilbert space $\mathcal{H}$ is a closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ with trivial kernel and co-kernel. Endowing a Hilbert space $\mathcal{H}$ with a rigging operator F generates a triple of Hilbert spaces

$$\mathcal{H}_+,\ \mathcal{H},\ \mathcal{H}_-, \tag{2.4.1}$$

where $\mathcal{H}_+$ is the completion of $\operatorname{im}(|F|)$ endowed with the scalar product

$$\langle f, g\rangle_{\mathcal{H}_+} = \langle |F|^{-1} f, |F|^{-1} g\rangle_{\mathcal{H}}$$

and $\mathcal{H}_-$ is the completion of $\operatorname{dom}(|F|)$ endowed with the scalar product

$$\langle f, g\rangle_{\mathcal{H}_-} = \langle |F| f, |F| g\rangle_{\mathcal{H}}.$$

Similarly, the operator F^* considered as a rigging operator in $\mathcal{K}$ generates a triple of Hilbert spaces

$$\mathcal{K}_+,\ \mathcal{K},\ \mathcal{K}_-.$$

The mapping $|F|$ extends to an isomorphism of $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha-1}$, $\alpha = 0, 1$. Similarly, $|F^*|$ extends to an isomorphism of $\mathcal{K}_\alpha$ and $\mathcal{K}_{\alpha-1}$, $\alpha = 0, 1$. The (extension of the) rigging operator F itself can be considered as an isomorphism

$$F\colon \mathcal{H} \simeq \mathcal{K}_+ \quad \text{or} \quad F\colon \mathcal{H}_- \simeq \mathcal{K}.$$

Similarly, F^{-1} can be treated as an isomorphism $\mathcal{K}_+ \simeq \mathcal{H}$ or $\mathcal{K} \simeq \mathcal{H}_-$.

2.5. Limiting Absorption Principle. Let $\mathcal{H}$ and $\mathcal{K}$ be two complex separable Hilbert spaces and let

$$F\colon \mathcal{H} \to \mathcal{K} \tag{2.5.1}$$

be a fixed rigging operator in $\mathcal{H}$. Let $\mathcal{A}_0$ be a real normed space of self-adjoint operators V of the form

$$V = F^* J F, \tag{2.5.2}$$

where J is an element of a real subspace of the algebra of bounded self-adjoint operators on $\mathcal{K}$. The norm on $\mathcal{A}_0$ is defined by $\|V\|_F = \|J\|$. Let H be a self-adjoint operator on $\mathcal{H}$. The affine space of self-adjoint operators of the form $H + V$, where $V \in \mathcal{A}_0$, will be denoted by $\mathcal{A}$, that is,

$$\mathcal{A} = H + \mathcal{A}_0. \tag{2.5.3}$$

Here we have firstly introduced the rigging operator F, and then using it we have introduced the affine space $\mathcal{A}$. In fact, the operator F is of auxiliary nature while the affine space $\mathcal{A}$ comes directly from the formulation of a problem. Hence, in practice, given an affine space $\mathcal{A}$ one has to find a rigging operator F which makes the pair $(\mathcal{A}, F)$ compatible in the sense that all the conditions imposed on this pair are satisfied.

We frequently use the notation

$$T_z(H) := FR_z(H)F^*. \tag{2.5.4}$$

The operator $T_z(H)$ is often called a *sandwiched resolvent*.

We assume that all operators V from the real vector space $\mathcal{A}_0$ are relatively compact perturbations of some operator H from the real affine space $\mathcal{A}$, that is, $\mathrm{dom}(H) \subset \mathrm{dom}(V)$ and that the operator $R_z(H)V$ is bounded and its continuous extension is compact:

$$\text{the operator } R_z(H)V \text{ is compact.} \tag{2.5.5}$$

Since all perturbation operators $V = H_1 - H_0$, where H_1, H_0 is any pair of operators from $\mathcal{A}$, are supposed to be relatively compact with respect to H_0, this implies that H_0 and $H_1 = H_0 + V$ have the same domain. That is, the domains of all operators H from the affine space $\mathcal{A}$ coincide; we denote this common domain by $\mathcal{D}$:

$$\text{for any } H \in \mathcal{A}, \quad \mathrm{dom}(H) = \mathcal{D}. \tag{2.5.6}$$

Further, since all perturbations $V \in \mathcal{A}_0$ of operators H from $\mathcal{A}$ are relatively compact, Weyl's theorem implies that all operators H from $\mathcal{A}$ have a common essential spectrum:

$$\forall H_0, H_1 \in \mathcal{A} \sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ess}}(H_1).$$

This common essential spectrum is denoted by σ_{ess}. The subset σ_{ess} of $\mathbb{R}$ depends only on $\mathcal{A}$. This allows us to talk about the essential spectrum of the affine space $\mathcal{A}$.

The operator F is not assumed to be bounded; therefore, one needs to clarify the meaning of the operators (2.5.2) and (2.5.4). The domain of any perturbation operator V contains $\mathcal{D}$:

$$\mathcal{D} \subset \mathrm{dom}(V). \tag{2.5.7}$$

Additionally we assume that

$$\mathcal{D} \subset \mathrm{dom}(F), \tag{2.5.8}$$

and that any operator J from (2.5.2) satisfies

$$JF\mathcal{D} \subset \mathrm{dom}(F^*). \tag{2.5.9}$$

Since by (2.5.6) for any $H \in \mathcal{A}$ the range of the resolvent $R_z(H)$ is equal to $\mathcal{D}$, and on this subspace the operator F is defined by the assumption (2.5.8), the sandwiched resolvent (2.5.4) is defined at least on the dense domain of F^*. It will always be assumed that the operator (2.5.4) is bounded on $\mathrm{dom}(F^*)$ and that its continuous extension to $\mathcal{K}$ is compact:

$$T_z(H) \text{ is compact.} \tag{2.5.10}$$

This also implies that, for any bounded subset Δ of $\mathbb{R}$,

$$FE^H_\Delta \text{ is compact.} \tag{2.5.11}$$

Indeed, by (2.5.10), the operator $\operatorname{Im} T_z(H) = (F\sqrt{\operatorname{Im} R_z(H)})(F\sqrt{\operatorname{Im} R_z(H)})^*$ is compact, and hence so is $F\sqrt{\operatorname{Im} R_z(H)}$. This implies (2.5.11). Using this one can show that for a bounded rigging operator F the condition (2.5.10) implies (2.5.5).

LEMMA 2.5.1. *If $FR_z(H)F^*$ is compact for some $z \in \rho(H)$, then $FR_w(H)F^*$ is compact for any other $w \in \rho(H)$. Further, the function $\mathbb{C} \setminus \mathbb{R} \ni z \mapsto T_z(H)$ is holomorphic.*

Proof. Without loss of generality we can assume that $y = \operatorname{Im} z > 0$.

If $FR_z(H)F^*$ is compact then so is

$$FR_{\bar z}(H)F^* = (FR_z(H)F^*)^*,$$

and therefore the operator

$$\begin{aligned}(F\sqrt{\operatorname{Im} R_z(H)})(F\sqrt{\operatorname{Im} R_z(H)})^* &= F(\operatorname{Im} R_z(H))F^* \\ &= (FR_z(H)F^* - FR_{\bar z}(H)F^*)/(2i)\end{aligned}$$

is also compact. It follows that $F\sqrt{\operatorname{Im} R_z(H)}$ is compact. Since the function

$$\mathbb{R} \ni x \mapsto R_z(x)/\sqrt{\operatorname{Im} R_z(x)},$$

where $R_z(x) = (x - z)^{-1}$, is bounded (by $y^{-1/2}$ as can be easily checked), the operator $R_z(H)/\sqrt{\operatorname{Im} R_z(H)}$ is also bounded. It follows that $FR_z(H)$ is compact. Since the function

$$\mathbb{R} \ni x \mapsto R_w(x)/R_z(x)$$

is bounded, it follows that $FR_w(H)$ is compact. Hence, so is $FR_z(H)R_w(H)F^*$. Since

$$FR_z(H)R_w(H)F^* = (z - w)F(R_z(H) - R_w(H))F^*$$

and since $FR_z(H)F^*$ is also compact, it follows that $FR_w(H)F^*$ is compact too. The second assertion follows from the equivalence of weak and strong analyticity. ∎

Given an operator H from the affine space $\mathcal{A}$, the notation

$$\Lambda(H, F) \tag{2.5.12}$$

will stand for the set of all real numbers λ for which the limit

$$T_{\lambda+i0}(H) := \lim_{y\to 0^+} T_{\lambda+iy}(H) \ \text{ exists in the uniform topology.} \tag{2.5.13}$$

Since $(T_z(H))^* = T_{\bar z}(H)$ and since the operation of taking adjoint is continuous in the uniform topology, it follows that the norm limit $T_{\lambda+i0}(H)$ exists if and only if the norm limit $T_{\lambda-i0}(H)$ exists. Thus, if $\lambda \in \Lambda(H, F)$ then also

$$\operatorname{Im} T_{\lambda+i0}(H) := \lim_{y\to 0^+} \operatorname{Im} T_{\lambda+iy}(H) \ \text{ exists in the norm topology.} \tag{2.5.14}$$

In fact, one often requires a stronger form of convergence for the imaginary part $\operatorname{Im} T_{\lambda+i0}(H)$, and this additional condition is imposed when needed.

L. A. P. ASSUMPTION. Throughout this paper we assume that the pair $(\mathcal{H}, F)$ and the affine space $\mathcal{A}$ satisfy the Limiting Absorption Principle: *For any self-adjoint operator $H \in \mathcal{A}$ on the Hilbert space $\mathcal{H}$ with rigging operator F the set* (2.5.12) *has full Lebesgue measure.*

Though $\Lambda(H, F)$ has full Lebesgue measure in certain cases of interest, for the development of the theory of spectral flow inside essential spectrum this is not quite necessary. As long as $\Lambda(H, F)$ contains at least one point, one may study spectral flow through that point.

As mentioned in the introduction, the L. A. P. Assumption holds for Schrödinger operators with short range potentials. Another setting in which the L. A. P. Assumption holds is given by the following theorem.

THEOREM 2.5.2 ([BÈ, Br], [Y, Theorem 6.1.9]). *If H_0 is a self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and if F is a Hilbert–Schmidt operator from $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, then for a.e. $\lambda \in \mathbb{R}$ the operator-valued function $FR_{\lambda+iy}(H_0)F^*$ has a limit in Hilbert–Schmidt norm as $y \to 0$.*

The Limiting Absorption Principle plays an important role in the stationary approach to scattering theory (see [BÈ, Br, KK, Y]). Proving the Limiting Absorption Principle in the cases of interest is a difficult problem. But for this paper it is a postulate and of utmost importance.

2.6. λ-resonant and λ-regular operators. Given a self-adjoint operator H and a perturbation $V = F^*JF$ one is usually interested in points λ for which the Limiting Absorption Principle (2.5.13) holds. In contrast, in this work we are mainly interested in points λ for which the Limiting Absorption Principle fails. However, there can be points λ for which (2.5.13) fails for any operator $H \in \mathcal{A}$. This indicates that λ is a very singular value of the spectral parameter. We exclude such points from our study; for the present work we are interested in those points λ for which (2.5.13) fails for some but not all operators from $\mathcal{A}$. We introduce the appropriate notation and terminology.

Let

$$\Lambda(\mathcal{A}, F) := \bigcup_{H \in \mathcal{A}} \Lambda(H, F) \subset \mathbb{R}. \tag{2.6.1}$$

Since the sets $\Lambda(H, F)$, for $H \in \mathcal{A}$, have full Lebesgue measure, so does $\Lambda(\mathcal{A}, F)$. Any real number in $\Lambda(\mathcal{A}, F)$ will be called an *essentially regular point* [Az_3, §4.2]. Points which are not essentially regular exist; for example, an eigenvalue of infinite multiplicity cannot be essentially regular (Theorem 4.2.1). But a real number may fail to be essentially regular even if it is not an eigenvalue. This may happen inside the essential spectrum only, since outside the essential spectrum all points are essentially regular. This indicates the nature of non-essentially regular points as those of infinite singularity.

The notation

$$\Pi_+ = \Pi_+(\mathcal{A}, F), \quad \text{respectively} \quad \Pi_- = \Pi_-(\mathcal{A}, F),$$

will be used for the union of the open upper complex half-plane, respectively of the open lower complex half-plane, and the set $\Lambda(\mathcal{A}, F)$. The letter Π will denote the disjoint union of Π_+ and Π_-. Thus, the boundary $\partial\Pi$ of Π is the disjoint union of two copies $\partial\Pi_+ = \Lambda(\mathcal{A}, F)$ and $\partial\Pi_- = \Lambda(\mathcal{A}, F)$ of the same set. The conjugation $z \mapsto \bar{z}$ swaps Π_+ and Π_-. Elements of $\partial\Pi_\pm$ are written as $\lambda \pm i0$, where $\lambda \in \mathbb{R}$. Elements of Π will usually be denoted by z; the real part of z is denoted as a rule by λ, and the imaginary part of z

by y. Thus, $y \in (-\infty, 0-] \cup [0+, \infty)$. The real number λ will be fixed throughout most of this paper.

Let λ be an essentially regular point of the pair $(\mathcal{A}, F)$ and let $H \in \mathcal{A}$. We say that the operator

$$H \text{ is } \textit{resonant at } \lambda, \text{ or } \lambda\textit{-resonant}, \text{ if } \lambda \notin \Lambda(H, F). \tag{2.6.2}$$

Thus, H is resonant at λ if and only if the limit (2.5.13) does not exist. Otherwise,

$$H \text{ is } \textit{regular at } \lambda, \text{ or } \lambda\textit{-regular}, \text{ if } \lambda \in \Lambda(H, F).$$

The set of all λ-resonant operators in $\mathcal{A}$ will be denoted by

$$R(\lambda; \mathcal{A}, F), \tag{2.6.3}$$

and called the *resonance set* at λ. The following theorem is well-known; what may be new is the way we interpret it.

THEOREM 2.6.1. *Let λ be an essentially regular point of the pair $(\mathcal{A}, F)$, let $H_0 \in \mathcal{A}$ be an operator regular at λ and let $V = F^*JF \in \mathcal{A}_0$. The following five assertions are equivalent:*

(i) *The operator $H_0 + V$ is resonant at λ.*
(ii$_\pm$) *The operator $1 + JT_{\lambda \pm i0}(H_0)$ is not invertible.*
(iii$_\pm$) *The operator $1 + T_{\lambda \pm i0}(H_0)J$ is not invertible.*

Proof. The equivalence of (i) and (ii$_+$) can easily be derived from the equality

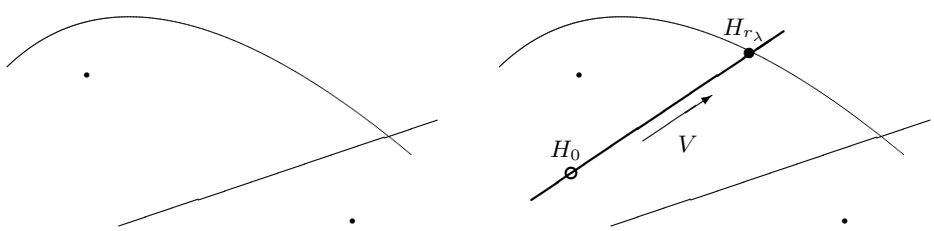
$$T_{\lambda+iy}(H_0 + V) = [1 + T_{\lambda+iy}(H_0)J]^{-1} T_{\lambda+iy}(H_0),$$

which in its turn follows from the second resolvent identity (see (2.7.2) below). Equivalence of (i) and the other items is proved similarly. ∎

This theorem has the following simple but important corollary. The proof follows verbatim that of [Az$_3$, Theorem 4.2.5].

THEOREM 2.6.2. *For every essentially regular point $\lambda \in \mathbb{R}$, the resonance set $R(\lambda; \mathcal{A}, F)$ is a closed nowhere dense subset of $\mathcal{A}$. Moreover, the intersection of any real-analytic path in $\mathcal{A}$ with $R(\lambda; \mathcal{A}, F)$ is either a discrete set or coincides with the path itself.*

The left figure below shows what a more or less typical two-dimensional section of the resonance set $R(\lambda; \mathcal{A}, F)$, which has two resonance lines and two resonance points, may look like.



Let

$$\gamma = \{H_r \colon r \in \mathbb{R}\}$$

be a straight line or a path of operators in the affine space $\mathcal{A}$, where $H_r = H_0 + rV$. If $\lambda \in \mathbb{R}$ is an essentially regular point of $(\mathcal{A}, F)$, then according to Theorem 2.6.2 there are two possible scenarios: all points of γ except a discrete subset are regular at λ, or all points of γ are resonant at λ. In the first case we say that γ is *regular at* λ. A real number r will be said to be a *resonance point* of γ at λ if H_r is resonant at λ. A regular line γ may only have a discrete set of resonance points. We shall mainly be concerned with only one of them which will be denoted by r_λ. The right figure above shows a λ-regular operator $H_0 \in \mathcal{A}$ and a direction $V \in \mathcal{A}_0$; the line γ intersects the resonance set $R(\lambda; \mathcal{A}, F)$ at the point H_{r_λ}. If an operator $H \in \mathcal{A}$ is resonant at λ, then a perturbation $V \in \mathcal{A}_0$ will be called a *regularizing direction* for H at λ if the straight line γ which passes through H in the direction of V is regular at λ. In the picture the operator H_{r_λ} is resonant at λ, and V is a regularizing direction for H_{r_λ} at λ; in fact, in the case of the figure every direction parallel to the two-dimensional section of $\mathcal{A}$ shown in the figure is regularizing for H_{r_λ} at λ.

PROPOSITION 2.6.3. *If $\lambda \in \Lambda(\mathcal{A}, F)$ is an eigenvalue of $H \in \mathcal{A}$, then H is resonant at λ.*

The proof is the same as that of [Az$_3$, Proposition 4.1.10]. Proposition 2.6.3 shows one source of real resonance points, but a point r can be resonant even if λ is not an eigenvalue of H_r.

COROLLARY 2.6.4. *Suppose $\lambda \in \Lambda(\mathcal{A}, F) \setminus \sigma_{\mathrm{ess}}(\mathcal{A})$. An operator $H \in \mathcal{A}$ is resonant at λ if and only if λ is an eigenvalue of H.*

Proof. The "if" implication follows from Proposition 2.6.3. We prove the "only if" part. Assume the contrary: λ is not an eigenvalue of H. Since also $\lambda \notin \sigma_{\mathrm{ess}}(H)$, it follows that λ belongs to the resolvent set of H. In this case the norm limit $R_{\lambda+i0}(H)$ exists even without sandwiching by F and F^*, and therefore H is regular at λ. ∎

2.7. The operators $A_z(s)$ and $B_z(s)$. Let $z \in \Pi$. We shall frequently use the notation

$$A_z(s) = T_z(H_s)J. \tag{2.7.1}$$

The sandwiched version of the second resolvent identity

$$T_z(H_r) - T_z(H_s) = (s - r)T_z(H_r)JT_z(H_s) \tag{2.7.2}$$

implies

$$A_z(r) - A_z(s) = (s - r)A_z(r)A_z(s). \tag{2.7.3}$$

From this equality one can infer that $1 + (s - r)A_z(r)$ must be invertible. Hence,

$$A_z(s) = (1 + (s - r)A_z(r))^{-1}A_z(r), \tag{2.7.4}$$

which also implies that

$$A_z(s)A_z(r) = A_z(r)A_z(s). \tag{2.7.5}$$

Since $A_z(r)$ is compact, by the analytic Fredholm alternative (Theorem 2.2.1), the equality (2.7.4) gives a meromorphic continuation of the function $A_z(s)$ of s to the whole complex plane $\mathbb{C}$. The equality (2.7.5) also holds for this meromorphic continuation. Moreover, Theorem 2.2.1 and (2.7.4) imply

LEMMA 2.7.1. *The function $(z,s) \mapsto A_z(s)$ is a meromorphic function of two complex variables z and s in $\Pi^\circ \times \mathbb{C} \subset \mathbb{C}^2$, where Π° is the interior of Π.*

The equality (2.7.3) implies that

$$\frac{d^n}{ds^n} A_z(s) = (-1)^n n! A_z^{n+1}(s). \tag{2.7.6}$$

We also use the notation

$$B_z(s) = JT_z(H_s). \tag{2.7.7}$$

One can check that the following analogue of (2.7.4) holds:

$$B_z(s) = (1 + (s-r)B_z(r))^{-1} B_z(r), \tag{2.7.8}$$

which implies

$$B_z(s)B_z(r) = B_z(r)B_z(s). \tag{2.7.9}$$

Using (2.7.4) and (2.7.8) one can check that

$$(A_z(s))^* = B_{\bar z}(\bar s). \tag{2.7.10}$$

We shall also use the following well-known equality (see e.g. [KK, p. 144], [RS_3, (99)], [Az_3, (4.8)]):

$$\begin{aligned} \operatorname{Im} T_z(H_s) &= (1 + (s-r)T_{\bar z}(H_r)J)^{-1} \operatorname{Im} T_z(H_r)(1 + (s-r)JT_z(H_r))^{-1} \\ &= (1 + (s-r)A_{\bar z}(r))^{-1} \operatorname{Im} T_z(H_r)(1 + (s-r)B_z(r))^{-1}. \end{aligned} \tag{2.7.11}$$

It holds for all real numbers s and r if z does not belong to $\partial\Pi$; otherwise, if $z = \lambda \pm i0$ and if the line $\{H_r = H_0 + rV \colon r \in \mathbb{R}\}$ is regular at λ, then (2.7.11) holds for all real numbers s and r outside $R(\lambda; H_0, V)$. In particular, the right hand side of (2.7.11) provides a meromorphic continuation of the left hand side as a function of s to the whole complex plane.

LEMMA 2.7.2. *As $y \to 0$, the holomorphic function $T_{\lambda+iy}(H_s)$ of s converges to $T_{\lambda+i0}(H_s)$ uniformly on any compact subset of Π which does not contain resonance points corresponding to $\lambda + i0$.*

In what follows, the spectra of $A_z(s)$ and $B_z(s)$ will play an important role. Since $A_z(s)$ and $B_z(s)$ are compact operators, their spectra consist of isolated eigenvalues of finite multiplicity and zero. By (2.1.2), the eigenvalue counting measures of $A_z(s)$ and $B_z(s)$ coincide, and therefore it suffices to consider the spectrum of $A_z(s)$. Eigenvalues of $A_z(s)$ will be denoted by $\sigma_z = \sigma_z(s)$. As will be seen later (Proposition 3.1.2), eigenvalues (with their multiplicities) of $A_z(s)$ for different s are connected by a simple relation: $\sigma_z(s) = (s - r_z)^{-1}$, where r_z is a complex number independent of s.

Occasionally we also consider the operators

$$\underline{A}_z(s) = R_z(H_s)V \quad \text{and} \quad \underline{B}_z(s) = VR_z(H_s). \tag{2.7.12}$$

Their spectral properties are identical to those of $A_z(s)$ and $B_z(s)$. By the Limiting Absorption Principle $A_z(s)$ and $B_z(s)$ have well-defined limits $A_{\lambda\pm i0}(s)$ and $B_{\lambda\pm i0}(s)$ as $z = \lambda + iy$ approaches $\lambda \pm i0$, unlike $\underline{A}_z(s)$ and $\underline{B}_z(s)$; since eventually the limit $z = \lambda + iy \to \lambda \pm i0$ will be taken, this is the main reason to work with the former pair of operators rather than the latter. But as long as z stays outside the real axis or outside

the common essential spectrum of the operators H_s, practically all other properties of these two pairs of operators are almost identical, and as a consequence they will be stated only for $A_z(s)$ and $B_z(s)$. Nearly all objects, such as $P_z(r_z)$, $\mathbf{A}_z(r_z)$, etc., to be introduced later, which are naturally associated with $A_z(s)$ and $B_z(s)$ have their analogues for $\underline{A}_z(s)$ and $\underline{B}_z(s)$; these analogues will be distinguished by underlining, e.g. $\underline{P}_z(r_z)$, $\underline{\mathbf{A}}_z(r_z)$, etc.

The following lemma is well-known.

LEMMA 2.7.3. *Let s be a real number. If z is a non-real number, then the compact operators $\underline{A}_z(s)$, $\underline{B}_z(s)$, $A_z(s)$ and $B_z(s)$ do not have real eigenvalues except possibly zero. Moreover, if the operator V is non-negative (respectively, non-positive), then all eigenvalues of $\underline{A}_z(s)$, $\underline{B}_z(s)$, $A_z(s)$ and $B_z(s)$ belong to the open complex half-plane $\mathbb{C}_\pm$ which z belongs to (respectively, does not belong to).*

However, if z belongs to $\partial\Pi$, then $A_z(s)$ and $B_z(s)$ may have non-zero real eigenvalues. In fact, it is these real eigenvalues of $A_z(s)$ which are of top interest for the present paper, and with a bit of exaggeration it can be said that this paper is mainly devoted to investigation of these real eigenvalues.

3. Analytic properties of $A_z(s)$

3.1. The vector spaces $\Upsilon_z(r_z)$ and $\Psi_z(r_z)$. Throughout this paper we assume that H_0 is a self-adjoint operator from the affine space (2.5.3) and that V is a self-adjoint operator from the real vector space $\mathcal{A}_0$ with factorization (2.5.2). Let λ be a fixed real number. We assume that the line

$$\gamma := \{H_0 + rV : r \in \mathbb{R}\}$$

is *regular* at λ; by definition this means that there exists a non-resonant value of the coupling constant r, that is, for some value of r the inclusion $H_r \in R(\lambda; \mathcal{A}, F)$ fails (equivalently, the inclusion $\lambda \in \Lambda(H_r, F)$ holds). In this case the set $R(\lambda; H_0, V)$ of resonance points r_λ is a discrete subset of $\mathbb{R}$, by Theorem 2.6.1.

Let $z \in \Pi$, let $r_z \in \mathbb{C}$ and let k be a positive integer. Let s be any number for which the operator $A_z(s)$ is defined. Then

$$(1 + (r_z - s)A_z(s))^k u = 0 \tag{3.1.1}$$

will be called the *resonance equation of order k* for the pair (z, r_z). The resonance equation of order 1 is nothing but the Lippmann–Schwinger equation.

Definition 3.1.1. A complex number r_z will be called a *resonance point* corresponding to $z \in \Pi$ if the resonance equation (3.1.1) of order $k = 1$ has a non-zero solution.

In other words, r_z is a resonance point if and only if the number

$$\sigma_z(s) := (s - r_z)^{-1} \tag{3.1.2}$$

is a non-zero eigenvalue of the compact operator $A_z(s)$. Real resonance points r_λ were defined in the paragraph following Theorem 2.6.2, and these definitions are consistent with each other. It will be shown below (Proposition 3.1.2) that the definition of resonance point does not depend on s. Hence, r_z depends only on z, H_0, V and, in case $z \in \partial\Pi$, also on F. If $z \notin \partial\Pi$, then this definition does not depend on the rigging operator F, since in this case both operators $A_z(s) = FR_z(H_s)F^*J$ and $R_z(H_s)F^*JF = R_z(H_s)V$ make sense and they have the same non-zero eigenvalues by (2.1.1).

According to the correspondence (3.1.2) between resonance points r_z and eigenvalues $\sigma_z(s)$ of $A_z(s)$, the set of resonance points corresponding to a given $z \in \Pi$ is a discrete subset of $\mathbb{C}$. Also, (2.7.4) shows that the resonance points corresponding to z are exactly the poles of the meromorphic function $A_z(s)$. For this reason, resonance points may sometimes be called *poles*.

Solutions of the resonance equation (3.1.1) of order k will usually be denoted by u, u_z or $u_z(r_z)$ and will be called *resonance vectors of order* $\le k$. The order k of a

resonance vector u is the smallest positive integer such that u is a solution of the resonance equation (3.1.1) of order k. The order of a resonance vector will be denoted by $\mathrm{d}(u)$. If necessary, we write $\mathrm{d}_z(u)$ instead of $\mathrm{d}(u)$; also, instead of $\mathrm{d}_{\lambda\pm i0}(u)$ we often write $\mathrm{d}_\pm(u)$.

The finite-dimensional vector space of all resonance vectors of order $\le k$ will be denoted by $\Upsilon_z^k(r_z)$. To be precise, one should indicate dependence of this vector space on the operators H_0, V by writing, say, $\Upsilon_z^k(r_z; H_0, V)$, but since throughout this paper the operators H_0 and V are fixed, the simpler notation will be used. The same remark applies to many other objects to be introduced later. A vector $u = u_z(r_z)$ will be called a *resonance vector of order* k if u is a resonance vector of order $\le k$ but not a resonance vector of order $\le k-1$. It was proved in [Az$_4$] that the set of solutions of (3.1.1) does not depend on s. We give here the proof for completeness and for the readers' convenience.

PROPOSITION 3.1.2. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The vector space $\Upsilon_z^k(r_z)$ of solutions of* (3.1.1) *does not depend on $s \in \mathbb{R}$.*

Proof. We use induction on k. Let u be a solution of (3.1.1) with $k = 1$ for the value of $s = r$, so that $A_z(r)u = (r - r_z)^{-1}u$. It follows from this and (2.7.4) that

$$A_z(s)u = (1 + (s-r)A_z(r))^{-1}A_z(r)u = \left[1 + (s-r)\cdot\frac{1}{r-r_z}\right]^{-1}\frac{1}{r-r_z}u = \frac{1}{s-r_z}u.$$

Hence, if u is a solution of (3.1.1) with $k = 1$ for one value of s, then u is a solution of (3.1.1) with $k = 1$ for any other regular value of s too. Now assume that the assertion is true for $k = n$ and let u be a solution of (3.1.1) with $k = n + 1$ for $s = r$. Then

$$(1 + (r_z - r)A_z(r))(1 + (r_z - r)A_z(r))^n u = 0.$$

It follows from this and induction base, applied to the vector $(1 + (r_z - r)A_z(r))^n u$, that

$$(1 + (r_z - s)A_z(s))(1 + (r_z - r)A_z(r))^n u_z = 0.$$

Since, by (2.7.5), the operators $A_z(s)$ and $A_z(r)$ commute, it follows that

$$(1 + (r_z - r)A_z(r))^n(1 + (r_z - s)A_z(s))u = 0.$$

By the induction assumption, applied to the vector $(1 + (r_z - s)A_z(s))u$, this implies that

$$(1 + (r_z - s)A_z(s))^n(1 + (r_z - s)A_z(s))u = 0. \ \blacksquare$$

The sequence

$$\Upsilon_z^1(r_z) \subset \cdots \subset \Upsilon_z^k(r_z) \subset \cdots \subset \mathcal{K}$$

stabilizes. The union of the vector spaces $\Upsilon_z^1(r_z), \Upsilon_z^2(r_z), \ldots$ will be denoted by $\Upsilon_z(r_z)$.

A resonance point r_z will be said to have *order* d if there are resonance vectors of order d, but there are no resonance vectors of order $d + 1$. In other words, the order d of a resonance point r_z is the integer

$$d = \min\{k \in \mathbb{N}\colon \Upsilon_z^k(r_z) = \Upsilon_z^{k+1}(r_z)\} = \min\{k \in \mathbb{N}\colon \Upsilon_z^k(r_z) = \Upsilon_z(r_z)\}. \tag{3.1.3}$$

Apart from the order d, with every resonance point r_z another two positive integers are naturally associated: the geometric multiplicity m defined by

$$m = \dim \Upsilon_z^1(r_z) \tag{3.1.4}$$

and the algebraic multiplicity N defined by

$$N = \dim \Upsilon_z(r_z). \tag{3.1.5}$$

Obviously, $d + m - 1 \le N$. Throughout this paper the letters d, m and N will be used only with these meanings, unless specifically stated otherwise.

The equation

$$(1 + (r_z - s)B_z(s))^k \psi = 0 \tag{3.1.6}$$

will be called the *co-resonance equation of order k*. Its solutions will be denoted by ψ, ψ_z or $\psi_z(r_z)$, and will be called *co-resonance vectors* of order $\le k$. The finite-dimensional vector space of all co-resonance vectors of order $\le k$ will be denoted by $\Psi_z^k(r_z)$. A co-resonance vector ψ has order k if it has order $\le k$ but not $\le k-1$. The sequence

$$\Psi_z^1(r_z) \subset \cdots \subset \Psi_z^k(r_z) \subset \cdots \subset \mathcal{K}$$

stabilizes; its union will be denoted by $\Psi_z(r_z)$. Similarly to Proposition 3.1.2, one can prove the following

PROPOSITION 3.1.3. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The vector space $\Psi_z^k(r_z)$ of solutions of* (3.1.6) *does not depend on $s \in \mathbb{R}$.*

This proposition also follows from Proposition 3.1.2 and Lemma 3.1.4.

LEMMA 3.1.4. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The dimensions of the four vector spaces*

$$\Upsilon_z^j(r_z),\ \Upsilon_{\bar z}^j(\bar r_z),\ \Psi_z^j(r_z) \text{ and } \Psi_{\bar z}^j(\bar r_z)$$

coincide for all $j = 1, 2, \dots$. Moreover, for all $j = 1, 2, \dots$ and all non-resonant real numbers s the mappings

$$J\colon \Upsilon_z^j(r_z) \to \Psi_z^j(r_z) \quad \text{and} \quad T_z(H_s)\colon \Psi_z^j(r_z) \to \Upsilon_z^j(r_z)$$

are linear isomorphisms.

In particular, the dimensions of the four vector spaces

$$\Upsilon_z(r_z),\ \Upsilon_{\bar z}(\bar r_z),\ \Psi_z(r_z) \text{ and } \Psi_{\bar z}(\bar r_z)$$

coincide and J is a linear isomorphism of $\Upsilon_z(r_z)$ and $\Psi_z(r_z)$.

Proof. Let j be a positive integer and s a real number. The resonance equation (3.1.1) implies that if $u \in \Upsilon_z^j(r_z)$, then $Ju \in \Psi_z^j(r_z)$. Also, if $Ju = 0$, where u is a solution of (3.1.1), then it follows from this equation, after expanding brackets, that $u = 0$. Hence, J is an injective linear operator from $\Upsilon_z^j(r_z)$ into $\Psi_z^j(r_z)$.

Similarly, the co-resonance equation (3.1.6) implies that if $\psi \in \Psi_z^j(r_z)$, then $T_z(H_s)\psi \in \Upsilon_z^j(r_z)$; further, if $T_z(H_s)\psi = 0$, where ψ is a solution of (3.1.6), then it follows from this equation that $\psi = 0$. Hence, $T_z(H_s)$ is an injective linear operator from $\Psi_z^j(r_z)$ into $\Upsilon_z^j(r_z)$.

Thus, the vector spaces $\Upsilon_z^j(r_z)$ and $\Psi_z^j(r_z)$ are linearly isomorphic and the mappings $J\colon \Upsilon_z^j(r_z) \to \Psi_z^j(r_z)$ and $T_z(H_s)\colon \Psi_z^j(r_z) \to \Upsilon_z^j(r_z)$ are linear isomorphisms.

Further, let $S = [1 + (s - r_z)A_z(s)]^j$; then

$$\dim \Upsilon_z^j(r_z) = \dim \ker S = \dim \ker S^* = \dim \Psi_{\bar z}^j(\bar r_z),$$

where the first and the third equalities directly follow from the definitions of $\Upsilon_z^j(r_z)$ and $\Psi_{\bar z}^j(\bar r_z)$, and the second equality follows from the fact that the Fredholm index of S is zero, since the operator $S-1$ is compact. Consequently, the dimensions of $\Upsilon_z^j(r_z)$, $\Upsilon_{\bar z}^j(\bar r_z)$, $\Psi_z^j(r_z)$ and $\Psi_{\bar z}^j(\bar r_z)$ are the same. ∎

COROLLARY 3.1.5. *If r_z is a resonance point of algebraic multiplicity N, order d and geometric multiplicity m, corresponding to z, then $\bar r_z$ is a resonance point of algebraic multiplicity N, order d and geometric multiplicity m, corresponding to $\bar z$.*

COROLLARY 3.1.6. *The vector spaces $\Upsilon_z(r_z)$ and $\Upsilon_z^k(r_z)$, $k=1,2,\dots$, are invariant under the operator $A_z(s)=T_z(H_s)J$ for any non-resonant $s\in\mathbb{R}$.*

LEMMA 3.1.7. *For any non-real z and any resonance point r_z corresponding to z,*

$$\Upsilon_z(r_z)\subset F\mathcal{D} \quad \textit{and} \quad \Psi_z(r_z)\subset JF\mathcal{D},$$

where $\mathcal{D}=\mathrm{dom}(H_0)$. In particular, $\Psi_z(r_z)\subset\mathrm{dom}(F^*)$.

Proof. An element u of $\Upsilon_z(r_z)$ is a solution of the resonance equation (3.1.1). Hence, any such vector belongs to the range of $A_z(s)$. This range is a subset of the range of $FR_z(H_s)$, which is $F\mathcal{D}$. This proves the first equality. The second equality follows similarly from the co-resonance equation (3.1.6). Finally, the last inclusion follows from the assumption (2.5.9). ∎

3.2. The idempotents $P_z(r_z)$ and $Q_z(r_z)$. For a given element z of Π with a corresponding resonance point $r_z\in\mathbb{C}$ an idempotent operator $P_z(r_z)$, which acts on the Hilbert space $\mathcal{K}$ and has range $\Upsilon_z(r_z)$, will be defined by

$$P_z(r_z)=\frac{1}{2\pi i}\oint_{C(\sigma_z(s))}(\sigma-A_z(s))^{-1}\,d\sigma, \tag{3.2.1}$$

where $C(\sigma_z(s))$ is a small circle enclosing the eigenvalue (3.1.2) of $A_z(s)$, so that there are no other eigenvalues of this operator on or inside the circle. The contour integral in (3.2.1) and in all the following formulas is taken in the uniform operator topology.

Apart from the operator $P_z(r_z)$ we shall sometimes need its modification

$$\underline{P}_z(r_z)=\frac{1}{2\pi i}\oint_{C(\sigma_z(s))}(\sigma-\underline{A}_z(s))^{-1}\,d\sigma, \tag{3.2.2}$$

where $\underline{A}_z(s)=R_z(H_s)V$. As long as the variable z is non-real, the properties of $P_z(r_z)$ and $\underline{P}_z(r_z)$ are quite similar; for this reason they are given only for $P_z(r_z)$. An essential difference between $P_z(r_z)$ and $\underline{P}_z(r_z)$ is that the former operator has the limit $P_{\lambda\pm i0}(r_z)$ as z approaches its real part λ from above or below, while the latter may not have such a limit. In fact, this is the main reason for considering $P_z(r_z)$ instead of $\underline{P}_z(r_z)$. The same remark applies to other "underlined" versions of operators to be introduced later.

The following assertion was proved in [Az$_4$]; its proof is given below for completeness.

PROPOSITION 3.2.1. *The idempotent operator $P_z(r_z)$ defined by* (3.2.1) *does not depend on s.*

Proof. Let P_1 and P_2 be two idempotents $P_z(r_z)$ defined for two different values s_1 and s_2 of s. Since by Proposition 3.1.2 these idempotents have the same range $\Upsilon_z(r_z)$, we have

$P_1P_2 = P_2$ and $P_2P_1 = P_1$. Since, by (2.7.5), the operators $A_z(s_1)$ and $A_z(s_2)$ commute, it follows from (3.2.1) that so do P_1 and P_2. Therefore,

$$P_1 = P_2P_1 = P_1P_2 = P_2. \ \blacksquare$$

This result also follows from Proposition 3.2.3.

We define an idempotent operator $Q_z(r_z)$, which acts on the Hilbert space $\mathcal{K}$ and has range $\Psi_z(r_z)$, by

$$Q_z(r_z) = \frac{1}{2\pi i} \oint_{C(\sigma_z(s))} (\sigma - B_z(s))^{-1}\, d\sigma, \tag{3.2.3}$$

where the contour $C(\sigma_z(s))$ is the same as in (3.2.1). The "underlined" version of $Q_z(r_z)$ is defined by

$$\underline{Q}_z(r_z) = \frac{1}{2\pi i} \oint_{C(\sigma_z(s))} (\sigma - \underline{B}_z(s))^{-1}\, d\sigma. \tag{3.2.4}$$

The proof of the following proposition is similar to that of Proposition 3.2.1.

PROPOSITION 3.2.2. *The idempotent operator $Q_z(r_z)$ defined by* (3.2.3) *does not depend on s.*

The following equality follows from the definitions (3.2.1) and (3.2.3) of the idempotents $P_z(r_z)$ and $Q_z(r_z)$, norm continuity of taking the adjoint $T \mapsto T^*$, and (2.7.10):

$$(P_z(r_z))^* = Q_{\bar{z}}(\bar{r}_z). \tag{3.2.5}$$

PROPOSITION 3.2.3. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The idempotent $P_z(r_z)$ is equal to the residue of the function $A_z(s)$ of s corresponding to the pole r_z:*

$$P_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} A_z(s)\, ds, \tag{3.2.6}$$

where $C(r_z)$ is a small circle enclosing r_z in counter-clockwise direction.

Proof. Let r be a complex number which lies outside of the circle $C(r_z)$. Then (2.7.4) implies

$$\begin{aligned}\oint_{C(r_z)} A_z(s)\, ds &= \oint_{C(r_z)} (1 + (s-r)A_z(r))^{-1} A_z(r)\, ds \\ &= \oint_{C(r_z)} \frac{1}{s-r}\big(1 - (1 + (s-r)A_z(r))^{-1}\big)\, ds.\end{aligned}$$

Since r lies outside of $C(r_z)$, the integral of $1/(s-r)$ vanishes. Hence,

$$\oint_{C(r_z)} A_z(s)\, ds = \oint_{C(r_z)} \frac{1}{r-s}(1 + (s-r)A_z(r))^{-1}\, ds.$$

We make the change of variables $\sigma = 1/(r-s)$. When s goes around r_z in counter-clockwise direction, so does σ around $\sigma_z(r) := 1/(r - r_z)$. Hence, from the last displayed equality we obtain

$$\begin{aligned}\oint_{C(r_z)} A_z(s)\, ds &= \oint_{C(\sigma_z(r))} \sigma(1 - \sigma^{-1}A_z(r))^{-1}\sigma^{-2}\, d\sigma \\ &= \oint_{C(\sigma_z(r))} (\sigma - A_z(r))^{-1}\, d\sigma = 2\pi i P_z(r_z),\end{aligned}$$

where $C(\sigma_z(r))$ is the image of the contour $C(r_z)$ under the mapping $s \mapsto \sigma = 1/(r-s)$. $\blacksquare$

Similarly one proves:

PROPOSITION 3.2.4. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The idempotent $Q_z(r_z)$ is equal to the residue of the function $B_z(s)$ of s corresponding to the pole r_z:*

$$Q_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} B_z(s)\, ds, \tag{3.2.7}$$

where $C(r_z)$ is a small circle enclosing r_z in counter-clockwise direction.

The next proposition directly follows from the definition (3.2.1) and standard properties of Riesz idempotents (see [Ka$_2$]), but nonetheless we give another proof of it.

PROPOSITION 3.2.5. *If for a given $z \in \Pi$ the operator $A_z(s)$ has two different poles r_z^1 and r_z^2, then the corresponding idempotents $P_z(r_z^1)$ and $P_z(r_z^2)$ satisfy*

$$P_z(r_z^1)P_z(r_z^2) = 0. \tag{3.2.8}$$

Proof. Proposition 3.2.3 and (2.7.3) imply that

$$\begin{aligned} P_z(r_z^1)P_z(r_z^2) &= \frac{1}{(2\pi i)^2} \oint_{C_t(r_z^1)} \oint_{C_s(r_z^2)} A_z(t)A_z(s)\, dt\, ds \\ &= \frac{1}{(2\pi i)^2} \oint_{C_t(r_z^1)} \oint_{C_s(r_z^2)} \frac{A_z(t) - A_z(s)}{s-t}\, dt\, ds, \end{aligned} \tag{3.2.9}$$

where the contours $C_t(r_z^1)$ and $C_s(r_z^2)$, enclosing (only) the points r_z^1 and r_z^2 respectively, can be chosen so that they do not intersect and (therefore) do not enclose one another. Under this choice of the contours, for any $t \in C_t(r_z^1)$ the function $A_z(t)/(s-t)$ of s is holomorphic on and inside $C_s(r_z^2)$, and therefore its integral vanishes. For an analogous reason, the integral of $A_z(s)/(s-t)$ vanishes too. ∎

Similarly, one shows that

$$Q_z(r_z^1)Q_z(r_z^2) = 0.$$

Now we record some relations between $P_z(r_z)$, $Q_z(r_z)$, $T_z(H_s)$ and J which will be used later. Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. The equality (2.1.4) combined with the definitions (3.2.1) and (3.2.3) of the idempotents $P_z(r_z)$ and $Q_z(r_z)$ implies

$$JP_z(r_z) = Q_z(r_z)J, \tag{3.2.10}$$

$$P_z(r_z)T_z(H_s) = T_z(H_s)Q_z(r_z). \tag{3.2.11}$$

From Lemma 3.1.4 and (3.2.10) we deduce

$$JP_z(r_z) = Q_z(r_z)JP_z(r_z) = Q_z(r_z)J. \tag{3.2.12}$$

The equality

$$A_z(s)P_z(r_z) = P_z(r_z)A_z(s) \tag{3.2.13}$$

is a direct consequence of (3.2.1).

3.3. The nilpotent operators $\mathbf{A}_z(r_z)$ and $\mathbf{B}_z(r_z)$. Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. We introduce a compact operator $\mathbf{A}_z(r_z)$ on the auxiliary Hilbert space $\mathcal{K}$ by

$$\mathbf{A}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) A_z(s)\, ds, \tag{3.3.1}$$

where $C(r_z)$ is a small circle which contains only one resonance point r_z and which is counter-clockwise oriented around r_z. Quite often the dependence of $\mathbf{A}_z(r_z)$ on r_z will not be indicated, especially in proofs. Also, instead of $\mathbf{A}_z(r_z)^j$ we shall write $\mathbf{A}_z^j(r_z)$.

Similarly, one introduces the operator

$$\mathbf{B}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) B_z(s)\, ds. \tag{3.3.2}$$

Apart from $\mathbf{A}_z(r_z)$ and $\mathbf{B}_z(r_z)$ we may sometimes need their "underlined" versions

$$\underline{\mathbf{A}}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) \underline{A}_z(s)\, ds, \tag{3.3.3}$$

$$\underline{\mathbf{B}}_z(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z) \underline{B}_z(s)\, ds. \tag{3.3.4}$$

But since many properties of $\underline{\mathbf{A}}_z(r_z)$ and $\mathbf{A}_z(r_z)$, etc., are similar, they are given only for $\mathbf{A}_z(r_z)$, etc.

PROPOSITION 3.3.1. *Let $z \in \Pi$ and let $r_z \in \mathbb{C}$ be a resonance point corresponding to z. For any positive integer j,*

$$\mathbf{A}_z^j(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z)^j A_z(s)\, ds. \tag{3.3.5}$$

Proof. Let $\mathbf{A}_z^{(j)}$ be the right hand side above. We have $\mathbf{A}_z^{(1)} = \mathbf{A}_z(r_z)$. The claim will be proved if it is shown that $\mathbf{A}_z^{(m)}\mathbf{A}_z^{(k)} = \mathbf{A}_z^{(m+k)}$ for any positive integers m and k. We have

$$\begin{aligned}\mathbf{A}_z^{(m)}\mathbf{A}_z^{(k)} &= \frac{1}{(2\pi i)^2} \oint_{C_s(r_z)} (s - r_z)^m A_z(s) \left(\oint_{C_t(r_z)} (t - r_z)^k A_z(t)\, dt \right) ds \\ &= \frac{1}{(2\pi i)^2} \oint_{C_s(r_z)} \oint_{C_t(r_z)} (s - r_z)^m (t - r_z)^k \frac{A_z(s) - A_z(t)}{t - s}\, dt\, ds.\end{aligned}$$

In this expression it can be assumed that the contour $C_s(r_z)$ lies strictly inside $C_t(r_z)$. Under this choice of contours the second summand of the integrand which contains $A_z(t)$ is holomorphic inside $C_s(r_z)$ with respect to s, and therefore its integral over $C_s(r_z)$ vanishes:

$$\begin{aligned}\mathbf{A}_z^{(m)}\mathbf{A}_z^{(k)} &= \frac{1}{(2\pi i)^2} \oint_{C_s(r_z)} \oint_{C_t(r_z)} (s - r_z)^m (t - r_z)^k \frac{A_z(s)}{t - s}\, dt\, ds \\ &= \frac{1}{(2\pi i)^2} \oint_{C_s(r_z)} (s - r_z)^m A_z(s) \left(\oint_{C_t(r_z)} \frac{(t - r_z)^k}{t - s}\, dt \right) ds\end{aligned}$$

$$= \frac{1}{2\pi i} \oint_{C_s(r_z)} (s - r_z)^m A_z(s) \cdot (s - r_z)^k \, ds$$
$$= \frac{1}{2\pi i} \oint_{C_s(r_z)} (s - r_z)^{m+k} A_z(s) \, ds = \mathbf{A}_z^{(m+k)},$$

where in the third equality the Cauchy integral formula is used. ∎

Proposition 3.2.3 and (3.3.5) allow us to write a bit informally

$$P_z(r_z) = \mathbf{A}_z^0(r_z). \tag{3.3.6}$$

With this convention, (3.3.5) holds for $j = 0$ too, according to (3.2.6).

Relation (2.7.5), combined with (3.2.6) and (3.3.1), implies that

$$\mathbf{A}_z(r_z)P_z(r_z) = P_z(r_z)\mathbf{A}_z(r_z) = \mathbf{A}_z(r_z). \tag{3.3.7}$$

This also follows from the general theory of operator-valued holomorphic functions [Ka_2].

If r_z^1 and r_z^2 are two different resonance points corresponding to z, then

$$\mathbf{A}_z(r_z^1)\mathbf{A}_z(r_z^2) = 0. \tag{3.3.8}$$

Indeed, $\mathbf{A}_z(r_z^1)\mathbf{A}_z(r_z^2) = \mathbf{A}_z(r_z^1)P_z(r_z^1)P_z(r_z^2)\mathbf{A}_z(r_z^2) = 0$, where the first equality follows from (3.3.7) and the second from (3.2.8).

The equalities

$$Q_z(r_z^1)Q_z(r_z^2) = 0, \tag{3.3.9}$$
$$\mathbf{B}_z(r_z^1)\mathbf{B}_z(r_z^2) = 0 \tag{3.3.10}$$

can be proved by the same argument; they also follow from (3.2.8) and (3.3.8), by using (3.2.5) and (3.3.11). It follows from (3.3.1) and (3.3.2) that for any $z \in \Pi$ and any resonance point r_z corresponding to z,

$$\mathbf{A}_z^*(r_z) = \mathbf{B}_{\bar{z}}(\bar{r}_z), \tag{3.3.11}$$

and since $JA_z(s) = B_z(s)J$,

$$J\mathbf{A}_z(r_z) = \mathbf{B}_z(r_z)J. \tag{3.3.12}$$

Similarly to (3.3.5)–(3.3.7) we have

$$\mathbf{B}_z^j(r_z) = \frac{1}{2\pi i} \oint_{C(r_z)} (s - r_z)^j B_z(s) \, ds, \tag{3.3.13}$$
$$Q_z(r_z) = \mathbf{B}_z^0(r_z), \tag{3.3.14}$$
$$\mathbf{B}_z Q_z(r_z) = Q_z(r_z)\mathbf{B}_z(r_z) = \mathbf{B}_z(r_z). \tag{3.3.15}$$

Recall that r_z is a pole of the meromorphic function $A_z(s)$ of s. Proposition 3.3.1 implies that the Laurent series of $A_z(s)$ in a neighbourhood of r_z is, for some positive integer d,

$$A_z(s) = \tilde{A}_{z,r_z}(s) + \frac{1}{s - r_z} P_z(r_z) + \frac{1}{(s - r_z)^2} \mathbf{A}_z(r_z) + \cdots + \frac{1}{(s - r_z)^d} \mathbf{A}_z^{d-1}(r_z), \tag{3.3.16}$$

where $\tilde{A}_{z,r_z}(s)$ is the holomorphic part of the Laurent series. It will be shown later that d is equal to the order of r_z. This Laurent series is an analogue of (2.2.1); the difference is that (3.3.16) is a Laurent series of a function of the coupling constant, while (2.2.1)

is a Laurent series of a function of the spectral parameter (energy). The finiteness of the Laurent series follows from the fact that $(s-r_z)^{-1}$ is an isolated eigenvalue of finite multiplicity of the compact operator $A_z(s)$. It follows from (3.3.16) that if r_z^1 and r_z^2 are two resonance points, then

$$A_z(s) = \tilde{A}_{z,r_z^1,r_z^2}(s) + \frac{1}{s-r_z^1}P_z(r_z^1) + \frac{1}{(s-r_z^1)^2}\mathbf{A}_z(r_z^1) + \cdots + \frac{1}{(s-r_z^1)^{d_1}}\mathbf{A}_z^{d_1-1}(r_z^1) + \frac{1}{s-r_z^2}P_z(r_z^2) + \frac{1}{(s-r_z^2)^2}\mathbf{A}_z(r_z^2) + \cdots + \frac{1}{(s-r_z^2)^{d_2}}\mathbf{A}_z^{d_2-1}(r_z^2), \tag{3.3.17}$$

where d_ν is the order of r_z^ν and where the meromorphic function $\tilde{A}_{z,r_z^1,r_z^2}(s)$ is holomorphic at r_z^1 and r_z^2. Similarly, the expansion (3.3.17) can be written for any finite set of resonance points $r_z^1, r_z^2, \ldots$. If the perturbation operator V has finite rank, then the set of resonance points r_z is finite and the Laurent expansion, similar to (3.3.17) but written for the set of all resonance points, gives the Mittag–Leffler representation of the meromorphic function $A_z(s)$. Whether this is true for infinite-rank V is unknown to me. The equalities (3.3.16) and (3.3.7) imply that

$$\tilde{A}_{z,r_z}(s)P_z(r_z) = P_z(r_z)\tilde{A}_{z,r_z}(s).$$

In fact, it will be shown later that this product is zero.

LEMMA 3.3.2. *Let $z \in \Pi$ and let r_z be a resonance point corresponding to z. For any non-negative k and any non-resonance point r,*

$$\oint_{C(\sigma_z(r))} (\sigma - \sigma_z(r))^k (\sigma - A_z(r))^{-1}\, d\sigma = \frac{1}{(r-r_z)^k}\oint_{C(r_z)} \left(\frac{s-r_z}{r-r_z} + \left(\frac{s-r_z}{r-r_z}\right)^2 + \cdots\right)^k A_z(s)\, ds, \tag{3.3.18}$$

where $C(\sigma_z(r))$ is an anti-clockwise oriented contour which encloses the pole

$$\sigma_z(r) = (r-r_z)^{-1},$$

and where $C(r_z)$ is an anti-clockwise oriented small enough contour which encloses only the pole r_z and such that the above series converges for all $s \in C(r_z)$.

Proof. This is a calculation similar to the one from the proof of Proposition 3.2.3, but it is given here for the sake of completeness.

The contour $C(r_z)$ can be chosen as a small enough circle with centre at r_z such that the number r lies outside of it. In this case the geometric series on the right hand side of (3.3.18) converges. We denote the right hand side by (E), and compute:

$$(E) = \frac{1}{(r-r_z)^k}\oint_{C(r_z)} \left(\frac{s-r_z}{r-r_z}\cdot\left(1-\frac{s-r_z}{r-r_z}\right)^{-1}\right)^k A_z(s)\, ds = \frac{1}{(r-r_z)^k}\oint_{C(r_z)} \left(\frac{s-r_z}{r-s}\right)^k A_z(s)\, ds.$$

Now, following the proof of Proposition 3.2.3, we obtain

$$(E) = \frac{1}{(r-r_z)^k}\oint_{C(r_z)} \left(\frac{s-r_z}{r-s}\right)^k \frac{1}{s-r}\big(1-(1+(s-r)A_z(r))^{-1}\big)\, ds.$$

Since r lies outside of the contour $C(r_z)$, it follows that

$$(E) = \frac{1}{(r-r_z)^k} \oint_{C(r_z)} \left(\frac{s-r_z}{r-s}\right)^k \frac{1}{r-s}(1+(s-r)A_z(r))^{-1}\,ds.$$

Let $\sigma = 1/(r-s)$. When s goes around r_z in counter-clockwise direction, so does σ around $\sigma_z(r) = 1/(r-r_z)$. Noting that

$$\frac{1}{(r-r_z)^k}\left(\frac{s-r_z}{r-s}\right)^k = (\sigma-\sigma_z(r))^k,$$
$$\frac{1}{r-s}(1+(s-r)A_z(r))^{-1}ds = (\sigma - A_z(r))^{-1}d\sigma$$

completes the proof. ∎

PROPOSITION 3.3.3. *Let $z \in \Pi$ and let $r_z \in \mathbb{C}$ be a resonance point corresponding to z. The terms with negative powers in the Laurent expansion of the function $(\sigma - A_z(r))^{-1}$ of σ at $\sigma = \sigma_z(r) = (r-r_z)^{-1}$ are linear combinations of powers of $\mathbf{A}_z(r_z)$. In particular, taking $k = 1$ in* (3.3.18) *gives the coefficient of $(\sigma-\sigma_z)^{-2}$:*

$$\frac{1}{2\pi i}\oint_{C(\sigma_z(r))} (\sigma-\sigma_z(r))(\sigma-A_z(r))^{-1}\,d\sigma = \sigma_z^2(r)\mathbf{A}_z(r_z) + \sigma_z^3(r)\mathbf{A}_z^2(r_z) + \cdots.$$

Taking $k = d-1$ in (3.3.18)*, where d is the order of r_z, gives the coefficient of $(\sigma-\sigma_z)^{-d}$:*

$$\oint_{C(\sigma_z(r))} (\sigma-\sigma_z(r))^{d-1}(\sigma-A_z(r))^{-1}\,d\sigma = \sigma_z^{2d-2}(r)\mathbf{A}_z^{d-1}(r_z). \tag{3.3.19}$$

For other values of k the coefficient of $(\sigma-\sigma_z)^{-k-1}$ in (3.3.18) *has the form*

$$\sigma_z^{2k}(r)\mathbf{A}_z^k(r_z) + \cdots, \tag{3.3.20}$$

where the dots denote terms containing $\mathbf{A}_z^j(r_z)$ with $j > k$.

Proof. This immediately follows from (3.3.16) and (3.3.18). ∎

One can prove an assertion similar to Proposition 3.3.3 for the operator $B_z(s)$.

PROPOSITION 3.3.4. *The terms with negative powers in the Laurent expansion of the function $(\sigma - B_z(r))^{-1}$ of σ at $\sigma = \sigma_z(r)$ are linear combinations of powers of $\mathbf{B}_z(r_z)$.*

Similarly to (3.3.16) we have

$$B_z(s) = \tilde{B}_{z,r_z}(s) + \frac{1}{s-r_z}Q_z(r_z) + \frac{1}{(s-r_z)^2}\mathbf{B}_z(r_z) + \cdots + \frac{1}{(s-r_z)^d}\mathbf{B}_z^{d-1}(r_z), \tag{3.3.21}$$

where $\tilde{B}_{z,r_z}(s)$ is the holomorphic part of the Laurent series.

Formulas (3.3.16), (3.3.21), (3.2.10) and (3.3.12) imply that the holomorphic parts $\tilde{A}_{z,r_z}(s)$ and $\tilde{B}_{z,r_z}(s)$ satisfy

$$J\tilde{A}_{z,r_z}(s) = \tilde{B}_{z,r_z}(s)J.$$

3.4. Resonance vectors of order k. Using a polarization type argument and (2.7.3) allows one to rewrite the left hand side of the resonance equation (3.1.1) of order k as an expression linearly dependent on $A_z(s_j)$. This is done in the following proposition, which will prove useful.

PROPOSITION 3.4.1. *If $z \in \Pi$, if r_z is a resonance point corresponding to z and if distinct numbers $s_1, \dots, s_k$ are non-resonant, then*

$$\prod_{j=1}^{k}[1+(r_z-s_j)A_z(s_j)] = \sum_{j=1}^{k}(s_j-r_z)^{k-1}(1+(r_z-s_j)A_z(s_j)) \prod_{i=1,\, i\neq j}^{k}(s_j-s_i)^{-1}. \quad (3.4.1)$$

Proof. For $k = 1$ this equality is trivial. In the case of $k = 2$, the second resolvent identity (2.7.3) implies

$$\begin{aligned}
&[1+(r_z-s)A_z(s)][1+(r_z-r)A_z(r)] \\
&\quad = 1+(r_z-s)A_z(s)+(r_z-r)A_z(r)+\frac{(r_z-s)(r_z-r)}{s-r}(A_z(r)-A_z(s)) \\
&\quad = \frac{s-r_z}{s-r}(1+(r_z-s)A_z(s))+\frac{r-r_z}{r-s}(1+(r_z-r)A_z(r)), \qquad (3.4.2)
\end{aligned}$$

and this gives (3.4.1) for $k = 2$. Assuming that (3.4.1) holds for $k-1$ instead of k, we have

$$\begin{aligned}
(E) &:= \prod_{j=1}^{k}[1+(r_z-s_j)A_z(s_j)] = (1+(r_z-s_k)A_z(s_k)) \prod_{j=1}^{k-1}[1+(r_z-s_j)A_z(s_j)] \\
&= (1+(r_z-s_k)A_z(s_k)) \sum_{j=1}^{k-1}(s_j-r_z)^{k-2}(1+(r_z-s_j)A_z(s_j)) \prod_{i=1,\, i\neq j}^{k-1}(s_j-s_i)^{-1}.
\end{aligned}$$

Applying (3.4.2) to the product $(1+(r_z-s_k)A_z(s_k))(1+(r_z-s_j)A_z(s_j))$ gives

$$\begin{aligned}
(E) &= \sum_{j=1}^{k-1}(s_j-r_z)^{k-2}\left[\frac{s_k-r_z}{s_k-s_j}(1+(r_z-s_k)A_z(s_k))+\frac{s_j-r_z}{s_j-s_k}(1+(r_z-s_j)A_z(s_j))\right] \\
&\qquad \times \prod_{i=1,\, i\neq j}^{k-1}(s_j-s_i)^{-1} \\
&= \sum_{j=1}^{k-1}(s_j-r_z)^{k-1}(1+(r_z-s_j)A_z(s_j)) \prod_{i=1,\, i\neq j}^{k}(s_j-s_i)^{-1} \\
&\quad -(s_k-r_z)(1+(r_z-s_k)A_z(s_k)) \sum_{j=1}^{k-1}(s_j-r_z)^{k-2} \prod_{i=1,\, i\neq j}^{k}(s_j-s_i)^{-1}.
\end{aligned}$$

Thus the proof will be complete if it is shown that

$$\sum_{j=1}^{k}(s_j-r_z)^{k-2} \prod_{i=1,\, i\neq j}^{k}(s_j-s_i)^{-1} = 0. \quad (3.4.3)$$

By Lemma 2.3.1, the left hand side is the divided difference of order $k-1$ of $f(s) = (s-r_z)^{k-2}$. Hence, (3.4.3) follows from Lemma 2.3.2. ∎

Propositions 3.4.1 and 3.1.2 imply the following assertion.

THEOREM 3.4.2. *The resonance equation* (3.1.1) *of order k is equivalent to any of the following two equations:*

$$\prod_{j=1}^{k}(1+(r_z-s_j)A_z(s_j))u=0 \tag{3.4.4}$$

or

$$\sum_{j=1}^{k}(s_j-r_z)^{k-1}(u+(r_z-s_j)A_z(s_j)u)\prod_{i=1,\,i\neq j}^{k}(s_j-s_i)^{-1}=0, \tag{3.4.5}$$

where $s_1,\dots,s_k$ is any set of k non-resonance points.

Proof. The commutativity property (2.7.5) of $A_z(s)$ and Proposition 3.1.2 imply that the resonance equation (3.1.1) is equivalent to (3.4.4). Proposition 3.4.1 implies that (3.4.4) is equivalent to (3.4.5). ∎

THEOREM 3.4.3. *If $u^{(k)}$ is a resonance vector of order k, then*

$$A_z(s)u^{(k)}=\sum_{j=0}^{k-1}\frac{u^{(k-j)}}{(s-r_z)^{j+1}}, \tag{3.4.6}$$

where $u^{(k-j)}$ is a resonance vector of order $k-j$. Moreover,

$$u^{(k-j)}=\mathbf{A}_z^j(r_z)u^{(k)}, \tag{3.4.7}$$

and thus the operator $\mathbf{A}_z^j(r_z)$ lowers the order of a resonance vector $u\in\Upsilon_z(r_z)$ by j, where $j=1,2,\dots$.

In particular, the operator $\mathbf{A}_z(r_z)$ is nilpotent: $\mathbf{A}_z^d(r_z)=0$, where d is the order of r_z, and the geometric multiplicity m of r_z is equal to $m=\dim\ker\mathbf{A}_z(r_z)$.

Proof of Theorem 3.4.3. We use induction on k. For $k=1$ the equality (3.4.6) is equivalent to the resonance equation (3.1.1) of order $k=1$. Assume that the assertion holds for $k=n-1$ and let $u=u^{(n)}$ be a vector of order n. Since u satisfies the resonance equation of order n, it follows from Theorem 3.4.2 that u satisfies (3.4.5). Hence, taking in (3.4.5) (with $k=n$) $s=s_n$ we deduce that

$$A_z(s)u=\sum_{j=0}^{n-1}\frac{u^{(n-j)}}{(s-r_z)^{j+1}}, \tag{3.4.8}$$

where $u^{(n-j)}$, $j=0,1,\dots,n-1$, are some vectors; we have to show that $u^{(n-j)}$ has order $n-j$ for all $j=0,1,\dots,n-1$. Applying to both sides of (3.4.8) the operator $1+(r_z-r)A_z(r)$ and using the commutativity (2.7.5) of $A_z(s)$ and $A_z(r)$ we obtain

$$A_z(s)[1+(r_z-r)A_z(r)]u=[1+(r_z-r)A_z(r)]A_z(s)u=\sum_{j=0}^{n-1}\frac{\phi^{(n-j-1)}}{(s-r_z)^{j+1}}, \tag{3.4.9}$$

where

$$\phi^{(n-j-1)}=[1+(r_z-r)A_z(r)]u^{(n-j)},\quad j=0,1,\dots,n-1, \tag{3.4.10}$$

and where $\phi^{(0)}=0$. Since u is a resonance vector of order n, $[1+(r_z-r)A_z(r)]u$ is a resonance vector of order $n-1$. Hence, by induction assumption, it follows from (3.4.9)

that $\phi^{(n-j-1)}$ has order $n-j-1$. Since $1+(r_z-r)A_z(r)$ decreases the order of a resonance vector by 1, (3.4.10) implies that $u^{(n-j)}$ is a vector of order $n-j$. The proof of the first part of the theorem is complete.

The equality (3.4.7) follows from (3.3.5) and (3.4.6). ∎

The equality (3.4.6) can be rewritten as

$$A_z(s)P_z(r_z) = \sum_{j=0}^{d-1}(s-r_z)^{-j-1}\mathbf{A}_z^j(r_z), \tag{3.4.11}$$

where d is the order of the resonance point r_z.

COROLLARY 3.4.4. *The holomorphic part $\tilde{A}_{z,r_z}(s)$ of the meromorphic function $A_z(s)$ in a neighbourhood of r_z satisfies*

$$\tilde{A}_{z,r_z}(s)P_z(r_z) = P_z(r_z)\tilde{A}_{z,r_z}(s) = 0. \tag{3.4.12}$$

Proof. This follows from (3.3.16), (3.3.7) and (3.4.11). ∎

PROPOSITION 3.4.5. *If r_z is a resonance point of order d and if r and $s+r$ are regular points such that $|s| < |r-r_z|$, then*

$$[1+sA_z(r)]^{-1}P_z(r_z) = \sum_{j=0}^{d-1}(r-r_z)^{-j}R_j\left(\frac{s}{r_z-r}\right)\mathbf{A}_z^j(r_z), \tag{3.4.13}$$

where $R_j(w)$, $j=0,1,2,\dots$, are some holomorphic functions given by power series centred at $w=0$ with radius of convergence equal to 1.

Proof. The numbers r and $s+r$ are to be regular points for (3.4.13) to hold, since otherwise the operator $A_z(r)$ does not exist or $1+sA_z(r)$ is not invertible.

It follows from (2.7.6) and (3.4.11) that

$$\begin{aligned} A_z^{n+1}(r)P_z(r_z) &= \frac{(-1)^n}{n!}\frac{d^n}{dr^n}\sum_{j=0}^{d-1}\frac{1}{(r-r_z)^{j+1}}\mathbf{A}_z^j(r_z) \\ &= \frac{1}{n!}\sum_{j=0}^{d-1}\frac{(j+1)(j+2)\dots(j+n)}{(r-r_z)^{j+n+1}}\mathbf{A}_z^j(r_z) = \sum_{j=0}^{d-1}\frac{C_{n+j}^n}{(r-r_z)^{j+n+1}}\mathbf{A}_z^j(r_z), \end{aligned}$$

where C_{n+j}^n is the binomial coefficient. Using this, for small enough s we have

$$\begin{aligned} [1+sA_z(r)]^{-1}P_z(r_z) &= \sum_{n=0}^{\infty}(-s)^nA_z^n(r)P_z(r_z) = \sum_{n=0}^{\infty}(-s)^n\sum_{j=0}^{d-1}\frac{C_{n+j-1}^{n-1}}{(r-r_z)^{j+n}}\mathbf{A}_z^j(r_z) \\ &= \sum_{j=0}^{d-1}(r-r_z)^{-j}\sum_{n=0}^{\infty}\left(\frac{-s}{r-r_z}\right)^n C_{n+j-1}^{n-1}\mathbf{A}_z^j(r_z). \end{aligned}$$

The functions $R_j(w) = \sum_{n=0}^{\infty}C_{n+j-1}^{n-1}w^n$, $j=1,2,\dots$, are holomorphic with radius of convergence 1. It follows that (3.4.13) holds for all small enough s, and therefore by analytic continuation it holds for all s such that $1+sA_z(r)$ is invertible and $|s| < |r-r_z|$. The last displayed equality also shows that if $|s| < |r-r_z|$, then the function

$[1 + sA_z(r)]^{-1}P_z(r_z)$ admits analytic continuation to non-regular points s which belong to the disk $|s| < |r - r_z|$. ∎

Recall that the "underlined" versions $\underline{P}_z(r_z)$ and $\underline{\mathbf{A}}_z(r_z)$ of $P_z(r_z)$ and $\mathbf{A}_z(r_z)$ are defined by (3.2.2) and (3.3.3). In the following proposition we use the "underlined" operators, since for "non-underlined" operators it does not make sense.

PROPOSITION 3.4.6. *For any resonance point r_z corresponding to a non-real number z,*

$$(H_{r_z} - z)\underline{P}_z(r_z) = -V\underline{\mathbf{A}}_z(r_z). \tag{3.4.14}$$

Proof. From (2.7.4) we have

$$(1 + (s - r)\underline{A}_z(r))\underline{A}_z(s) = \underline{A}_z(r).$$

Substituting for $\underline{A}_z(s)$ its Laurent expansion (3.3.16), we find a Laurent expansion of the left hand side as a function of s. Since the right hand side is constant, all coefficients except one in this Laurent expansion are zero. In particular, calculating the coefficient of $(s - r_z)^{-1}$ we find that

$$(1 + (r_z - r)\underline{A}_z(r))\underline{P}_z(r_z) = -\underline{A}_z(r)\underline{\mathbf{A}}_z(r_z).$$

Multiplying both sides by $H_r - z$ gives (3.4.14). ∎

The equality (3.4.14) is plainly equivalent to the following proposition.

COROLLARY 3.4.7. *Let z be a non-real number and let r_z be a resonance point corresponding to z. If $u_z^{(k)} = F\chi_z^{(k)}$ is a vector of order k, then*

$$\begin{aligned}(H_{r_z} - z)\chi_z^{(1)} &= 0,\\ (H_{r_z} - z)\chi_z^{(2)} &= -V\chi_z^{(1)},\\ &\cdots\\ (H_{r_z} - z)\chi_z^{(k)} &= -V\chi_z^{(k-1)},\end{aligned}$$

where the vectors $u_z^{(j)} = F\chi_z^{(j)}$ satisfy (3.4.7).

3.5. The holomorphic part of $A_z(s)$. In this subsection we study the holomorphic part $\tilde{A}_{z,r_z}(s)$ of the Laurent expansion (3.3.16) of $A_z(s)$ at a resonance point $s = r_z$.

PROPOSITION 3.5.1. *If $z \in \Pi$ and if r_z is a resonance point corresponding to z, then for any non-resonant value of s we have*

$$\tilde{A}_{z,r_z}(r) = \tilde{A}_{z,r_z}(s)(1 + (s - r)\tilde{A}_{z,r_z}(r)) \tag{3.5.1}$$

as equality of two holomorphic functions of r.

Proof. Using (2.7.4) and the Laurent expansion (3.3.16) of $A_z(s)$ we have

$$\begin{aligned}A_z(r) &= A_z(s)(1 + (s - r)A_z(r))\\ &= A_z(s)\Big(1 + (s - r)\tilde{A}_{z,r_z}(r) + (s - r)\sum_{j=1}^{d}(r - r_z)^{-j}\mathbf{A}_z^{j-1}\Big).\end{aligned}$$

Here we consider both sides as meromorphic functions of r, so s is a fixed number. One can see that the holomorphic part of $(s - r)\sum_{j=1}^{d}(r - r_z)^{-j}\mathbf{A}_z^{j-1}$ at $r = r_z$ is $-P_z(r_z)$.

Hence, comparing holomorphic parts at $r = r_z$ of the last equality, we find that

$$\tilde{A}_{z,r_z}(r) = A_z(s)\big(1 + (s-r)\tilde{A}_{z,r_z}(r) - P_z(r_z)\big). \tag{3.5.2}$$

Formulas (3.4.12) and (3.3.7) combined with (3.3.16) imply $A_z(s)(1-P_z(r_z)) = \tilde{A}_{z,r_z}(s)$ and $A_z(s)\tilde{A}_{z,r_z}(r) = \tilde{A}_{z,r_z}(s)\tilde{A}_{z,r_z}(r)$. Combining these equalities with (3.5.2) gives (3.5.1). ∎

Another way to prove this proposition is to observe that, since $P_z(r_z)$ and $A_z(s)$ commute, the kernel of $P_z(r_z)$ reduces $A_z(s)$ and by (3.3.16) the reduction is $\tilde{A}_{z,r_z}(s)$. Hence, the claim follows from (2.7.4) and (3.4.12). From this observation it also follows that the kernel and range of $\tilde{A}_{z,r_z}(r)$ do not depend on r. Using (3.5.1) and a standard Fredholm alternative argument one can show that $1+(s-r)\tilde{A}_{z,r_z}(r)$ is invertible, so that

$$\tilde{A}_{z,r_z}(s) = (1 + (s-r)\tilde{A}_{z,r_z}(r))^{-1}\tilde{A}_{z,r_z}(r). \tag{3.5.3}$$

Similar equalities also hold for $\tilde{A}_{z,r_z^1,r_z^2}$, etc.

Since r_z is a pole of $A_z(s)$ the expression $A_z(r_z)$ does not make sense, but the value $\tilde{A}_{z,r_z}(r_z)$ of the holomorphic part $\tilde{A}_{z,r_z}(s)$ at $s = r_z$ is defined. In particular,

$$\tilde{A}_{z,r_z}(s) = (1 + (s-r_z)\tilde{A}_{z,r_z}(r_z))^{-1}\tilde{A}_{z,r_z}(r_z). \tag{3.5.4}$$

The equality (3.5.4) allows us to find the Taylor series of $\tilde{A}_{z,r_z}(s)$ at $s = r_z$:

$$\tilde{A}_{z,r_z}(s) = \tilde{A}_{z,r_z}(r_z) - \tilde{A}^2_{z,r_z}(r_z)(s-r_z) + \tilde{A}^3_{z,r_z}(r_z)(s-r_z)^2 - \cdots .$$

It is possible that $\tilde{A}_{z,r_z}(r_z) = 0$, but this is very unlikely, since it would imply that $\tilde{A}_{z,r_z}(s) = 0$ for all s and therefore according to (3.3.16) that r_z is the only resonance point corresponding to z.

Similar properties hold for the holomorphic part $\tilde{B}_z(s)$ of $B_z(s)$. One can also see that

$$(\tilde{A}_{z,r_z}(s))^* = \tilde{B}_{\bar{z},\bar{r}_z}(\bar{s}) \quad \text{and} \quad J\tilde{A}_{z,r_z}(s) = \tilde{B}_{z,r_z}(s)J.$$

4. Geometric meaning of $\Upsilon^1_{\lambda+i0}(r_\lambda)$

This and subsequent sections are independent of each other.

In scattering theory one may distinguish three types of states: scattering states, bound states and states with some kind of erratic behaviour, which we shall call erratic states (see e.g. [T, RS_3]). Bound states describe localized particles, while scattering states describe particles which are free at $t \to \pm\infty$. Erratic states include trapped states, which describe particles which are free at $t \to -\infty$ or respectively at $t \to \infty$ but localized at $t \to \infty$, or respectively at $t \to -\infty$, and as such, trapped states describe processes of capture or respectively decay. For this reason we may sometimes refer to erratic states as trapped states.

Bound states are eigenvectors of the full Hamiltonian $H = H_0 + V$, so they are attributed to point spectrum; the vector space of scattering states of a fixed energy λ can be seen as a fibre Hilbert space $\mathfrak{h}_\lambda$ (on-shell Hilbert space) and thus they can be attributed to absolutely continuous spectrum; finally, erratic states should be attributed to singular continuous spectrum. While the states ψ of all three types are eigenvectors of the full Hamiltonian in the sense that they satisfy the eigenvector equation $H\psi = \lambda\psi$, only bound states belong to the Hilbert space. Scattering and erratic states are usually called generalized eigenvectors. For the Schrödinger operator $-\Delta+V$, the scattering and erratic states are given by functions which do not belong to $L_2(\mathbb{R}^\nu)$. In an abstract setting one may consider a rigged Hilbert space (2.4.1) to describe generalized eigenvectors. That is, proper eigenvectors are elements of $\mathcal{H}$ while generalized eigenvectors are elements of $\mathcal{H}_-$. Since the rigging operator F provides natural isomorphisms of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}_+$ on the one hand, and of Hilbert spaces $\mathcal{H}_-$ and $\mathcal{K}$ on the other hand, one may also treat proper eigenvectors as elements of $\mathcal{K}_+$ and generalized eigenvectors as elements of $\mathcal{K}$.

Let H_0 be a self-adjoint operator from the affine space (2.5.3) which is regular at an essentially regular point λ, and let $V \in \mathcal{A}_0$ be a perturbation. At the discrete set of real resonance points r_λ of the triple $(\lambda; H_0, V)$ the operator $H_0 + r_\lambda V$ ceases to be regular at λ. A natural question is why this can happen. By Proposition 2.6.3, one reason is that λ can be an eigenvalue of $H_0 + r_\lambda V$. For λ outside the essential spectrum this is the only reason. But if λ belongs to the essential spectrum then the operator $H_0 + r_\lambda V$ may still fail to be regular at λ even if λ is not an eigenvalue. Intuitively, if H_r is regular at λ then all generalized eigenvectors are scattering states which form the fibre Hilbert space $\mathfrak{h}_\lambda$. Therefore it is natural to expect that if λ is not an eigenvalue of H_r but nevertheless H_r is not regular at λ, then the operator $H_0 + r_\lambda V$ should have trapped eigenvectors, that is, generalized eigenvectors which are neither proper eigenvectors nor the elements of

the Hilbert space $\mathfrak{h}_\lambda$ of scattering states. Results of this section formally confirm this assertion. Namely, it is shown that the vector space

$$\Upsilon^1_{\lambda+i0}(r_\lambda)$$

of solutions of the equation

$$u + (r_\lambda - r)T_{\lambda+i0}(H_r)Ju = 0$$

can be considered as a proper replacement of the vector space of proper eigenvectors in the sense that the latter space is naturally linearly isomorphic to a subspace of $\Upsilon^1_{\lambda+i0}(r_\lambda)$. The linear isomorphism is natural in the sense that it is given by the rigging operator F. Thus, the dimension of $\Upsilon^1_{\lambda+i0}(r_\lambda)$ is the sum of the dimension of the vector space of proper eigenvectors and the dimension of the factor space of trapped vectors defined up to an eigenvector.

The eigenvalue equation for the perturbed operator $H_r = H_0 + rV$,

$$(H_0 + rV)\chi = \lambda\chi,$$

can be rewritten formally as the homogeneous Lippmann–Schwinger equation ([LSch], see also e.g. [RS$_3$, (81)], [T])

$$\chi + r(H_0 - \lambda)^{-1}V\chi = 0. \tag{4.0.1}$$

If λ lies outside the essential spectrum, then the Lippmann–Schwinger equation makes perfect sense and is equivalent to the eigenvalue equation, but if λ belongs to the essential spectrum, then the Lippmann–Schwinger equation should be rewritten to make sense. One way of doing so is to factorize the perturbation V as F^*JF, where F is an operator acting from the main Hilbert space to an auxiliary Hilbert space $\mathcal{K}$, and to rewrite the Lippmann–Schwinger equation as an equation for a vector $u = F\chi$ in $\mathcal{K}$ as follows (see e.g. [Y, Lemma 4.7.8]):

$$u + rF(H_0 - \lambda - i0)^{-1}F^*Ju = 0. \tag{4.0.2}$$

This can be done as long as the limiting absorption principle holds, that is, as long as the limit operator $F(H_0 - \lambda - i0)^{-1}F^*$ acting on $\mathcal{K}$ exists. The vector χ may afterwards be recovered by $\chi = F^{-1}u$, but this vector may or may not belong to $\mathcal{H}$. The number λ for which (4.0.2) has a non-zero solution is an eigenvalue of $H_0 + V$ if and only if the vector $\chi = F^{-1}u$ exists and belongs to $\mathcal{H}$, that is, iff u belongs to the range of F. But even if u does not belong to that range, the number λ is still to be considered as a singular point of the spectrum of H_r due to the presence of trapped states.

As a final remark we note that though a factorization F^*JF of the perturbation V seems to be an unnatural nuisance, which is however necessary for technical reasons, in the current setting there is a fixed rigging operator F and the perturbation V admits a factorization F^*JF by the very definition.

4.1. Eigenvectors and regular resonance vectors of order 1. In this section we shall use two well-known properties of a self-adjoint operator H: for any real number λ,

$$\frac{(H-\lambda)^2}{(H-\lambda)^2 + y^2} \to 1 \quad \text{strongly as } y \to 0, \tag{4.1.1}$$

and if a real number λ is not an eigenvalue of H then

$$\frac{y(H-\lambda)}{(H-\lambda)^2+y^2} \to 0 \quad \text{weakly as } \ y \to 0. \tag{4.1.2}$$

THEOREM 4.1.1. *Let λ be an essentially regular point, let $H_0 \in \mathcal{A}$ be a λ-regular operator, let $V \in \mathcal{A}_0$, let r_λ be a real resonance point of $(\lambda; H_0, V)$ and let r be a regular point of $(\lambda; H_0, V)$. If λ is an eigenvalue of*

$$H_{r_\lambda} = H_0 + r_\lambda V$$

with an eigenvector $\chi \in \mathcal{D} = \operatorname{dom}(H_{r_\lambda})$, then $u = F\chi$ is a resonance vector of order 1, that is,

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)u = 0. \tag{4.1.3}$$

Proof. Firstly we note that by (2.5.8) the vector $F\chi$ is well-defined, since the domain of F contains the common domain of operators $H \in \mathcal{A}$. The eigenvalue equation $H_{r_\lambda}\chi = \lambda\chi$ implies the equality

$$(H_r - \lambda)\chi = (r - r_\lambda)V\chi. \tag{4.1.4}$$

Here both sides are well-defined since H_r and H_{r_λ} have common domain $\mathcal{D}$ by (2.5.6), and by (2.5.7) the domain of V contains $\mathcal{D}$. Hence, for any z with $\operatorname{Im} z \neq 0$ we have

$$FR_z(H_r)(H_r - \lambda)\chi = (r - r_\lambda)FR_z(H_r)V\chi.$$

Since $V\chi = F^*JF\chi$ and $\lambda \in \Lambda(H_r, F)$, by the Limiting Absorption Principle Assumption (see (2.5.12) and (2.5.13)) the limit of the right hand side of the above equality exists in the uniform operator topology as $z = \lambda \pm iy \to \lambda \pm i0$ and therefore so does the limit of the left hand side:

$$FR_{\lambda\pm i0}(H_r)(H_r - \lambda)\chi = (r - r_\lambda)FR_{\lambda\pm i0}(H_r)V\chi.$$

Adding these equalities gives

$$F \operatorname{Re} R_{\lambda+i0}(H_r)(H_r - \lambda)\chi = (r - r_\lambda)F \operatorname{Re} R_{\lambda+i0}(H_r)V\chi.$$

Since, by (4.1.1), $\operatorname{Re} R_{\lambda+iy}(H_r)(H_r - \lambda) \to 1$ in the strong operator topology as $y \to 0$, it follows that

$$F\chi = (r - r_\lambda)F \operatorname{Re} R_{\lambda+i0}(H_r)V\chi. \tag{4.1.5}$$

Since r is a regular point of the path $\{H_s \mid s \in \mathbb{R}\}$, by Proposition 2.6.3, λ is not an eigenvalue of H_r. It follows from this and (4.1.2) that

$$\operatorname{Im} R_{\lambda+iy}(H_r)(H_r - \lambda) \to 0$$

in the weak operator topology as $y \to 0$. Since $FE_\Delta(H_r)$ is compact by (2.5.11), we get

$$F \operatorname{Im} R_{\lambda+i0}(H_r)(H_r - \lambda)\chi = 0.$$

Combining this with (4.1.4) gives

$$0 = (r - r_\lambda)F \operatorname{Im} R_{\lambda+i0}(H_r)V\chi.$$

Multiplying this equality by i and adding it to (4.1.5), one gets

$$F\chi = (r - r_\lambda)FR_{\lambda+i0}(H_r)V\chi.$$

Since $V = F^*JF$, this can be rewritten as

$$(1 + (r_\lambda - r)FR_{\lambda+i0}(H_r)F^*J)F\chi = 0.$$

This is (4.1.3) with $u = F\chi$. Hence, $u = F\chi$ is a resonance vector of order 1. ∎

A resonance vector u will be called *regular* if $u \in \mathcal{K}_+$. Since the rigging operator F has trivial kernel, Theorem 4.1.1 implies that to linearly independent eigenvectors $\chi_1, \dots, \chi_N$ of H_0 there correspond linearly independent regular resonance vectors

$$u_1 = F\chi_1, \dots, u_N = F\chi_N \in \Upsilon^1_{\lambda+i0}(r_\lambda).$$

Hence:

COROLLARY 4.1.2. *If λ is an essentially regular point, then the geometric multiplicity of λ as an eigenvalue of the self-adjoint operator $H_{r_\lambda} = H_0 + r_\lambda V$ does not exceed the dimension of the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$, that is,*

$$\dim \mathcal{V}_\lambda \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda),$$

where $\mathcal{V}_\lambda$ is the eigenspace of H_{r_λ} corresponding to the eigenvalue λ.

4.2. Example of an essentially singular point. Corollary 4.1.2 allows us to present an example of a point λ which is not essentially regular.

THEOREM 4.2.1. *If λ is an eigenvalue of infinite multiplicity for at least one self-adjoint operator H from the affine space $\mathcal{A} = H_0 + \mathcal{A}_0$, then λ is not an essentially regular point of the pair $(\mathcal{A}, F)$, that is, $\lambda \notin \Lambda(\mathcal{A}, F)$.*

Proof. Without loss of generality it can be assumed that $H = H_0$. Assume to the contrary that for some $V \in \mathcal{A}_0$ and some non-zero $r \in \mathbb{R}$ the number λ belongs to $\Lambda(H_r, F)$, where $H_r = H_0 + rV$. Since λ is an eigenvalue of infinite multiplicity of H_0 and $\mathcal{V}_\lambda$ is the corresponding infinite-dimensional subspace of eigenvectors, by Theorem 4.1.1 for the non-resonant point r the linear subspace $F(\mathcal{V}_\lambda)$ consists of eigenvectors of a compact operator $A_{\lambda+i0}(r) = T_{\lambda+i0}(H_r)J$ corresponding to the eigenvalue $1/r$. Since F has trivial kernel, $F(\mathcal{V}_\lambda)$ is also infinite-dimensional. This contradicts the compactness of $T_{\lambda+i0}(H_r)J$. ∎

4.3. The case of $\lambda \notin \sigma_{\mathrm{ess}}$. So far in this section no conditions were imposed on λ except that of essential regularity. If, however, λ lies outside the essential spectrum, then one can prove a more refined version of Theorem 4.1.1.

LEMMA 4.3.1. *Let λ be an essentially regular point, let H_{r_λ} be resonant at λ and let V be a regularizing direction. If λ is an isolated eigenvalue of H_{r_λ}, then all resonance vectors of first order are regular vectors, that is, all vectors $u \in \Upsilon^1_{\lambda+i0}(r_\lambda)$ are of the form $u = F\chi$ for some vector $\chi \in \mathcal{H}$.*

Proof. Assume that u is a resonance vector of order 1:

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)u = 0. \tag{4.3.1}$$

Since V is a regularizing direction, by Corollary 2.6.4 for some r the number λ belongs to the resolvent set of H_r, so $R_{\lambda+i0}(H_r) = (H_r - \lambda)^{-1}$ exists as a bounded operator in $\mathcal{H}$.

Hence,

$$\chi := (r - r_\lambda)R_{\lambda+i0}(H_r)F^*Ju \in \mathcal{H}_-$$

is a well-defined element of $\mathcal{H}$, where F^*Ju is well-defined by (2.5.9). It follows from this and (4.3.1) that $u = F\chi$. Hence, $u = F\chi$ belongs to $\mathcal{K}_+ \supset F\mathcal{H}$. ∎

THEOREM 4.3.2. *If λ does not belong to the essential spectrum σ_{ess}, then the rigging operator F is a linear isomorphism of the eigenspace $\mathcal{V}_\lambda$ of H_{r_λ} and the vector space $\Upsilon^1_\lambda(r_\lambda)$. In particular,*

$$\dim \mathcal{V}_\lambda = \dim \Upsilon^1_\lambda(r_\lambda).$$

Proof. Since F has trivial kernel, it follows from Theorem 4.1.1 that F is an injective linear mapping from $\mathcal{V}_\lambda$ to $\Upsilon^1_\lambda(r_\lambda)$. To show that F maps $\mathcal{V}_\lambda$ onto $\Upsilon^1_\lambda(r_\lambda)$, let $u \in \Upsilon^1_\lambda(r_\lambda)$, so that u satisfies (4.1.3). By Lemma 4.3.1, there exists $\chi \in \mathcal{H}$ such that $u = F\chi$. Since V is a regularizing direction and λ is an isolated eigenvalue, the resolvent $(H_r - \lambda)^{-1}$ exists (as a bounded operator) for any non-resonant r. Hence, the equation (4.1.3), which is satisfied by the vector u by definition, can be written as

$$u + (r_\lambda - r)FR_{\lambda+i0}(H_r)F^*Ju = 0,$$

where $R_{\lambda+i0}(H_r) = (H_r - \lambda)^{-1}$ is a bounded operator. Replacing u by $F\chi$ gives

$$F\chi + (r_\lambda - r)FR_{\lambda+i0}(H_r)F^*JF\chi = 0.$$

Since F has trivial kernel, it follows that

$$\chi + (r_\lambda - r)R_{\lambda+i0}(H_r)F^*JF\chi = 0.$$

Applying $H_r - \lambda$ to both sides gives

$$(H_r - \lambda)\chi + (r_\lambda - r)V\chi = 0.$$

Thus, $H_{r_\lambda}\chi = \lambda\chi$, that is, $u = F\chi$ is the image of an eigenvector $\chi \in \mathcal{V}_\lambda$. ∎

The statement of Theorem 4.3.2 is not final in the sense that the condition $\lambda \notin \sigma_{\text{ess}}$ in fact might be redundant. In this regard, see Conjecture 7 in Section 15.

4.4. Multiplicity of singular spectrum. Theorem 4.3.2 implies that if $\lambda \notin \sigma_{\text{ess}}$, then $\Upsilon^1_\lambda(r_\lambda) = \Upsilon^1_\lambda(H_{r_\lambda}, V)$ does not depend on V. This raises a natural question: is this true in general? It turns out that the answer is positive. This is a simple but interesting fact, since it allows one to introduce the multiplicity of singular spectrum at an essentially regular point λ as the dimension of $\Upsilon^1_\lambda(H_{r_\lambda}, V)$.

THEOREM 4.4.1. *If H_{r_λ} is resonant at an essentially regular point λ, then the vector space*

$$\Upsilon^1_{\lambda+i0}(r_\lambda) = \Upsilon^1_\lambda(H_{r_\lambda}, V)$$

does not depend on the regularizing operator $V \in \mathcal{A}_0$.

Proof. To simplify formulas, without loss of generality we assume that $r_\lambda = 0$.

Let $V = F^*JF$ and $V' = F^*J'F$ be two regularizing operators. We have to show that if $u \in \mathcal{K}$ satisfies

$$[1 - FR_{\lambda+i0}(H_0 + V)F^*J]u = 0, \tag{4.4.1}$$

then also

$$[1 - FR_{\lambda+i0}(H_0 + V')F^*J']u = 0. \tag{4.4.2}$$

For $y > 0$ we have

$$\begin{aligned}
&FR_{\lambda+iy}(H_0 + V)F^*Ju - FR_{\lambda+iy}(H_0 + V')F^*J'u \\
&\quad = F[R_{\lambda+iy}(H_0 + V) - R_{\lambda+iy}(H_0 + V')]F^*Ju - FR_{\lambda+iy}(H_0 + V')F^*[J' - J]u \\
&\quad = F[R_{\lambda+iy}(H_0 + V')(V' - V)R_{\lambda+iy}(H_0 + V)]F^*Ju - FR_{\lambda+iy}(H_0 + V')F^*[J' - J]u \\
&\quad = FR_{\lambda+iy}(H_0 + V')F^*(J' - J)[FR_{\lambda+iy}(H_0 + V)F^*Ju - u].
\end{aligned}$$

Since u satisfies (4.4.1), the expression in the last square brackets vanishes as $y \to 0^+$. Since $FR_{\lambda+iy}(H_0 + V')F^*$ converges in norm as $y \to 0^+$, it follows that

$$FR_{\lambda+i0}(H_0 + V)F^*Ju - FR_{\lambda+i0}(H_0 + V')F^*J'u = 0.$$

Adding this equality to (4.4.1) we obtain (4.4.2). ∎

Theorem 4.4.1 allows us to consider the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$ as an analogue of the vector space of eigenvectors when a point λ of the singular spectrum belongs to the essential spectrum.

Later, in Section 10, we will show that $\dim \Upsilon^1_{\lambda+i0}(r_\lambda)$ does not depend on the choice of the rigging operator F either.

5. Resonance index

5.1. R-index

DEFINITION 5.1.1. Let $\mathcal{K}$ be a Hilbert space. The class $\mathcal{R} = \mathcal{R}(\mathcal{K})$ consists of all finite-rank operators $A\colon \mathcal{K} \to \mathcal{K}$ which satisfy the following conditions:

(1) The spectrum of A does not contain real numbers apart from zero: $\sigma_A \cap \mathbb{R} = \{0\}$.
(2) For any $f \in \mathcal{K}$ the equality $A^2 f = 0$ implies $Af = 0$.

By definition, the *R-index* of an operator $A \in \mathcal{R}$ is the integer $\mathcal{R}(A) = N_+ - N_-$, where N_+ and N_- are the numbers of eigenvalues of A (counted with multiplicities) in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively.

The second condition in the definition of $\mathcal{R}$ means that zero is an eigenvalue of order 1 for any operator A from $\mathcal{R}$.

If N is a positive integer, then $\mathcal{R}_N$ will denote the subset of $\mathcal{R}$ which consists of operators of rank N. The union $\bigcup_{n \le N} \mathcal{R}_n$ will be denoted by $\mathcal{R}_{\le N}$.

A list of some elementary properties of the R-index is given in the following lemma.

LEMMA 5.1.2. *Let A and B be bounded operators and let N be a positive integer.*

(i) *If $AB, BA \in \mathcal{R}$, then $\mathcal{R}(AB) = \mathcal{R}(BA)$.*
(ii) *If A belongs to $\mathcal{R}$ and if S is a bounded invertible operator, then $S^{-1}AS$ also belongs to $\mathcal{R}$ and $\mathcal{R}(S^{-1}AS) = \mathcal{R}(A)$.*
(iii) *If $A \in \mathcal{R}$, then also $A^* \in \mathcal{R}$ and $\mathcal{R}(A^*) = -\mathcal{R}(A)$.*
(iv) *If $A \in \mathcal{R}_N$, then there exists a neighbourhood of A in $\mathcal{R}_{\le N}$ which is a subset of $\mathcal{R}_N$ and such that $\mathcal{R}(B) = \mathcal{R}(A)$ for all B from the neighbourhood. That is, the R-index is a locally constant function on $\mathcal{R}_{\le N}$.*
(v) *If $A \in \mathcal{R}_N$ and if k is a non-negative integer, then there exists a neighbourhood of A in $\mathcal{R}_{\le N+k}$ such that $|\mathcal{R}(B) - \mathcal{R}(A)| \le k$ for all B from the neighbourhood.*
(vi) *If A and B belong to $\mathcal{R}$ and if $AB = BA = 0$, then $A + B$ also belongs to $\mathcal{R}$ and $\mathcal{R}(A + B) = \mathcal{R}(A) + \mathcal{R}(B)$.*

Proof. (i) This equality follows from (2.1.2), which asserts that the eigenvalue counting measures of AB and BA coincide outside of zero.

(ii) It is easy to check that if $A \in \mathcal{R}$ and S is a bounded invertible operator, then $S^{-1}AS \in \mathcal{R}$. Hence, the equality $\mathcal{R}(S^{-1}AS) = \mathcal{R}(A)$ follows from (i) applied to AS and S^{-1}.

(iii) If A satisfies the first condition of the definition of $\mathcal{R}$, then so does A^* by (2.1.3). Since A and A^* are finite-rank, we may assume that $\mathcal{K}$ is finite-dimensional. In this case

the second condition for A^* also follows from (2.1.3). The equality in (iii) follows from (2.1.3).

(iv) Small enough perturbations of A cannot decrease the rank of A. Hence, a small enough neighbourhood O of A in $\mathcal{R}_{\leq N}$ is a subset of $\mathcal{R}_N$. The half-plane $\mathbb{C}_+$ or $\mathbb{C}_-$ to which an eigenvalue belongs is stable under small enough perturbations. For any operator from O no other non-zero eigenvalues can emerge from zero, since this would increase the rank of A. Thus, any operator B from a small enough neighbourhood has the same R-index as that of A.

(v) Small enough perturbations of A do not change the half-plane $\mathbb{C}_\pm$ which the non-zero eigenvalues of A belong to. Hence, if B belongs to a small enough neighbourhood O of A in $\mathcal{R}_{\leq N+k}$ then, since $\operatorname{rank}(B) \leq N + k$, no more than k non-zero eigenvalues can emerge from zero as A is perturbed to B. Therefore, the R-indices of A and B may differ by no more than k.

(vi) Let v be a root vector of order k corresponding to a non-zero eigenvalue σ of A, that is, $(A-\sigma)^k v = 0$ and $(A-\sigma)^{k-1} v \neq 0$. The equality $BA = 0$ implies that $0 = B(A-\sigma)^k v = \sigma^k Bv$, or $Bv = 0$. Therefore, since A and B commute, $(B+A-\sigma)^k v = (A-\sigma)^k v = 0$ and $(B+A-\sigma)^{k-1} v = (A-\sigma)^{k-1} v \neq 0$. It follows that a non-zero number σ is an eigenvalue for A if and only if it is also an eigenvalue of the same algebraic multiplicity for $A+B$. The same assertion holds for B instead of A. Hence, the eigenvalue counting measure of $A+B$ is the sum of the eigenvalue counting measures of A and B, which implies that $A+B$ satisfies the first condition of Definition 5.1.1.

If $(A+B)^2 f = 0$, then $A^2 f + B^2 f = 0$; this implies $A^3 f = 0$. Therefore, $A^2 f = 0$ and hence $Af = 0$. Similarly, $Bf = 0$. Hence, $A+B$ satisfies the second condition too.

The equality $\mathcal{R}(A+B) = \mathcal{R}(A) + \mathcal{R}(B)$ follows. ∎

LEMMA 5.1.3. *Let $A\colon \mathcal{K} \to \mathcal{H}$ be a finite-rank operator, and $F\colon \mathcal{H} \to \mathcal{K}$ be a closed operator with zero kernel and co-kernel such that:*

(1) *AF is a bounded finite-rank operator and* $\operatorname{rank}(AF) = \operatorname{rank}(A)$,
(2) *FA is a well-defined bounded finite-rank operator and* $\operatorname{rank}(FA) = \operatorname{rank}(A)$,
(3) *both AF and FA belong to the class $\mathcal{R}$.*

Then

$$\mathcal{R}(AF) = \mathcal{R}(FA).$$

Proof. Since F has zero kernel and co-kernel, in the polar decomposition $F = U|F|$ the partial isometry U is a unitary operator. Hence, without loss of generality, we can assume that F is a self-adjoint operator with zero kernel.

Let $F_n = FE^F_{[-n,n]}$. Since AF is compact and $E^F_{[-n,n]} \to 1$ strongly as $n \to \infty$, the product AF_n converges to AF in norm. Hence, since the rank of AF_n is not larger than $\operatorname{rank}(A) = \operatorname{rank}(AF)$, for large enough n we have, using part (iv) of Lemma 5.1.2,

$$\mathcal{R}(AF_n) = \mathcal{R}(AF).$$

Similarly, for large enough n we have $\mathcal{R}(F_n A) = \mathcal{R}(FA)$. Hence, using part (i) of Lemma 5.1.2, for large enough n we get

$$\mathcal{R}(FA) = \mathcal{R}(F_n A) = \mathcal{R}(AF_n) = \mathcal{R}(AF). \quad ∎$$

It is easy to check that

$$\text{if } \operatorname{Im} z > 0 \text{ then } \operatorname{Im} T_z(H) > 0.$$

LEMMA 5.1.4. *Let z be a non-real number and let s be a real number. The eigenvalue counting measures of the operators $R_z(H_s)V$ and $T_z(H_s)J$ coincide. Moreover, for any resonance point r_z corresponding to z,*

$$F\underline{P}_z(r_z) = P_z(r_z)F \quad \text{and} \quad \underline{Q}_z(r_z)F^* = F^*Q_z(r_z). \tag{5.1.1}$$

Proof. If F is bounded then the first assertion follows directly from (2.1.2), while the second follows from the definition of $P_z(r_z)$, $\underline{P}_z(r_z)$, $Q_z(r_z)$, $\underline{Q}_z(r_z)$ and (2.1.4).

In general, it is not difficult to see that if u is a solution of

$$(1 + (r_z - s)A_z(s))^k u = 0,$$

then for some unique χ we have $u = F\chi$ where χ is a solution of

$$(1 + (r_z - s)\underline{A}_z(s))^k \chi = 0;$$

and vice versa, if a vector χ is a solution of this last equation then $u = F\chi$ is a solution of the previous one. It follows that the eigenvalue counting measures of $A_{\lambda+iy}(s)$ and $\underline{A}_{\lambda+iy}(s)$ coincide and that $F\underline{P}_z(r_z) = P_z(r_z)F$.

The equality $\underline{Q}_z(r_z)F^* = F^*Q_z(r_z)$ can be proved similarly using the co-resonance equation. ∎

For bounded F this follows from (2.1.2); in general this can be seen from Lemma 3.1.4.

LEMMA 5.1.5. *If H is a self-adjoint operator and V is a finite-rank self-adjoint operator, then for any non-real number z the operators $R_z(H)V$ and $T_z(H)J$ belong to $\mathcal{R}$.*

Proof. We prove this for $R_z(H)V$ only, since the proof for $T_z(H)J$ is similar. The operator $R_z(H)V$ is finite-rank and it satisfies the first condition of Definition 5.1.1 according to Lemma 2.7.3. Let $f \in \mathcal{H}$ be such that $(R_z(H)V)^2 f = 0$. Since $R_z(H)$ has zero kernel, this implies $VR_z(H)Vf = 0$ and $\langle Vf, R_z(H)Vf\rangle = 0$. Therefore $\langle Vf, R_{\bar{z}}(H)Vf\rangle = 0$ and thus $\langle Vf, \operatorname{Im} R_z(H)Vf\rangle = 0$. The operator $\operatorname{Im} R_z(H)$ is strictly positive if $\operatorname{Im} z > 0$ or is strictly negative if $\operatorname{Im} z < 0$. Hence, $\langle Vf, \operatorname{Im} R_z(H)Vf\rangle = 0$ implies $Vf = 0$. ∎

The following theorem is proved in [Kr]. We give here a new proof which is based on properties of the R-index and which has topological character.

THEOREM 5.1.6 ([Kr]). *If H is a self-adjoint operator and V is a finite-rank self-adjoint operator, then for any $y = \operatorname{Im} z > 0$ the operator $R_z(H)V$ has exactly* $\operatorname{rank}(V_\pm)$ *eigenvalues in $\mathbb{C}_\pm$, where V_+ is the positive part of V and V_- is the negative part of V. In particular,*

$$\mathcal{R}(R_{\lambda\pm iy}(H)V) = \pm \operatorname{sign}(V).$$

Proof. By Lemma 5.1.5 the operator $R_z(H)V$ belongs to $\mathcal{R}$.

(A) Assume first that either V or $-V$ is non-negative. Let N be the rank of V. Since $R_{\lambda+iy}(H)$ has trivial kernel, the dimension of the image of $R_{\lambda+iy}(H)V$ is also N. Hence, $R_{\lambda+iy}(H)V$ has N non-zero eigenvalues (counting multiplicities). That all these non-zero eigenvalues belong either to $\mathbb{C}_+$ in the case of $V \geq 0$ or to $\mathbb{C}_-$ in the case of $V \leq 0$ follows from Lemma 2.7.3.

(B) If a finite-rank self-adjoint operator V has at least one positive eigenvalue, then one of them can be continuously deformed so that it crosses 0 from $\mathbb{R}_+$ to $\mathbb{R}_-$. For instance, if

$$V = \sum_{j=1}^{N} \alpha_j \langle v_j, \cdot \rangle v_j$$

is the Schmidt representation of V and $\alpha_1 > 0$, then the path of operators

$$V_t = (1-2t)\alpha_1 \langle v_1, \cdot \rangle v_1 + \sum_{j=2}^{N} \alpha_j \langle v_j, \cdot \rangle v_j, \quad 0 \le t \le 1,$$

deforms the positive eigenvalue α_1 to $-\alpha_1$. By definition, the R-index of $R_z(H)V_t$ is constant before and after the eigenvalue being deformed reaches zero. According to Lemma 5.1.2(v), when the eigenvalue of V being deformed crosses 0 to the other half-line, the R-index of $R_z(H)V$ can change by no more than 2. According to (A), if V is non-negative, then the R-index of $R_z(H)V$ is N. When all eigenvalues of V become negative one by one as the operator V is deformed to a non-positive operator $-V$, the R-index of $R_z(H)V$ has to become $-N$. From this one can infer that every time one positive eigenvalue of V crosses 0 from $\mathbb{R}_+$ to $\mathbb{R}_-$, the R-index of $R_z(H)V$ has to change by -2. ∎

5.2. The idempotents $P_z(r_\lambda)$ and $Q_z(r_\lambda)$. Given a set

$$\Gamma = \{r_z^1, \dots, r_z^n\}$$

of resonance points corresponding to $z \in \Pi$, let

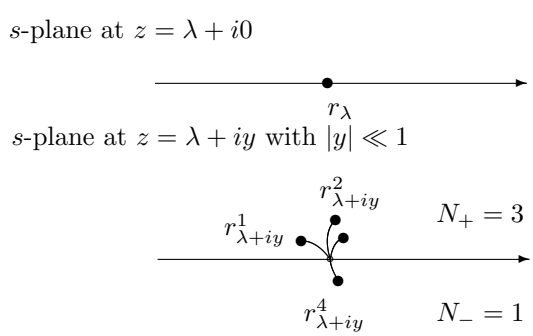
$$P_z(\Gamma) = P_z(r_z^1) + \cdots + P_z(r_z^n).$$

It follows from (3.2.8) that $P_z(\Gamma)$ is an idempotent. It will be called the *idempotent of* Γ. Similarly, one defines $Q_z(\Gamma)$. The range of $P_z(\Gamma)$ (respectively, $Q_z(\Gamma)$) will be denoted by $\Upsilon_z(\Gamma)$ (respectively, $\Psi_z(\Gamma)$).

We are mainly interested in the case when the number $z = \lambda \pm iy$ belongs to $\partial\Pi$ and the corresponding resonance point $r_z = r_\lambda$ is real. If the point $z = \lambda + i0$ is slightly shifted off the real axis, then the pole $s = r_\lambda$ of the meromorphic function $A_z(s)$ in general splits into several poles

$$r_z^1, \dots, r_z^N, \tag{5.2.1}$$

as schematically shown in the figure below.



In this kind of figures the word "s-plane" means that the plane of the figure is the domain of values of the variable s. The poles (5.2.1) will be said to belong to the *group of* r_λ; the number of these poles (counted with multiplicities) will be denoted by $N = N_+ + N_-$, where $N_\pm$ is the number of poles in $\mathbb{C}_\pm$; for numbers z outside of $\partial\Pi$ the poles r_z^ν, $\nu = 1, \dots, N$, cannot be real, according to Lemma 2.7.3. We denote by $P_z(r_\lambda) = P_{\lambda+iy}(r_\lambda)$ the idempotent of the group of resonance points (5.2.1):

$$P_z(r_\lambda) = P_z(r_z^1) + \cdots + P_z(r_z^N). \tag{5.2.2}$$

Similarly, $Q_z(r_\lambda)$ will denote the sum of the idempotents $Q_z(r_z^\nu)$, $\nu = 1, \dots, N$:

$$Q_z(r_\lambda) = Q_z(r_z^1) + \cdots + Q_z(r_z^N). \tag{5.2.3}$$

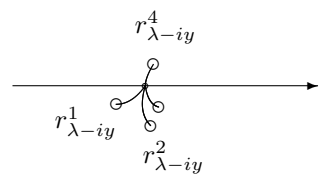
The range of $P_z(r_\lambda)$ will be denoted by $\Upsilon_z(r_\lambda)$, and the range of $Q_z(r_\lambda)$ will be denoted by $\Psi_z(r_\lambda)$.

We denote by $P^\uparrow_{\lambda+iy}(r_\lambda)$ the sum of the idempotents $P_{\lambda+iy}(r^\nu_{\lambda+iy})$ for which the poles $r^\nu_{\lambda+iy}$ belong to $\mathbb{C}_+$, and $P^\downarrow_{\lambda+iy}(r_\lambda)$ will denote the sum of the idempotents $P_{\lambda+iy}(r^\nu_{\lambda+iy})$ for which $r^\nu_{\lambda+iy} \in \mathbb{C}_-$. Similarly, one defines $Q^\uparrow_{\lambda+iy}(r_\lambda)$ and $Q^\downarrow_{\lambda+iy}(r_\lambda)$.

We remark that a priori the idempotents $P_{\lambda+iy}(r_\lambda)$, $P^\uparrow_{\lambda+iy}(r_\lambda)$, etc. are defined for small enough values of y, depending on how far away the point r_z as a function of z can be continued analytically (a possible hindrance is that it can get absorbed by ∞; see in this regard Subsection 15.7).

Similarly, one defines the operators $P_{\bar z}(r_\lambda)$, $Q_{\bar z}(r_\lambda)$ as idempotents of the group of resonance points of r_λ as $z = \lambda - i0$ is shifted to $z = \lambda - iy$.

In the following figures, resonance points will be depicted by dark circles and anti-resonance points by light circles (see Subsection 5.3 for definition of anti-resonance points). The next figure shows poles of the group of r_λ for the idempotents $P_{\lambda-iy}(r_\lambda)$ and $Q_{\lambda-iy}(r_\lambda)$.



The following result is [Az$_4$, Theorem 3.3]. Here we give a different proof.

PROPOSITION 5.2.1. *For any $z = \lambda \pm i0 \in \partial\Pi$ and any real resonance point r_λ corresponding to $\lambda \pm i0$, we have, for all small enough $y > 0$,*

$$\frac{1}{\pi} \oint_{C(r_\lambda)} \operatorname{Im} T_{\lambda+iy}(H_s) J \, ds = P_{\lambda+iy}(r_\lambda) - P_{\lambda-iy}(r_\lambda),$$

where $C(r_\lambda)$ is a contour which encloses all poles $r^1_{\lambda+iy}, \dots, r^N_{\lambda+iy}$ of the group of r_λ and their conjugates $\bar r^1_{\lambda+iy}, \dots, \bar r^N_{\lambda+iy}$.

Proof. Since

$$\frac{1}{\pi} \operatorname{Im} T_{\lambda+iy}(H_s) J = \frac{1}{2\pi i}(A_{\lambda+iy}(s) - A_{\lambda-iy}(s)),$$

the equality to be proved follows from the Laurent expansion (3.3.16) of $A_z(s)$. ∎

LEMMA 5.2.2. *For any real resonance point r_λ,*

$$P_{\lambda\pm i0}(r_\lambda) = \lim_{y\to 0^+} P_{\lambda\pm iy}(r_\lambda) \quad \textit{and} \quad Q_{\lambda\pm i0}(r_\lambda) = \lim_{y\to 0^+} Q_{\lambda\pm iy}(r_\lambda),$$

where the limits are taken in the trace class norm.

Proof. It follows from the definition of $P_z(r_\lambda)$ that $P_{\lambda\pm iy}(r_\lambda)$ converges to $P_{\lambda\pm i0}(r_\lambda)$ in uniform norm. By a well-known stability property of isolated eigenvalues, for small enough y the rank of $P_{\lambda\pm iy}(r_\lambda)$ is constant and is equal to the rank N of $P_{\lambda\pm i0}(r_\lambda)$. It follows that only the first N singular values of $P_{\lambda\pm iy}(r_\lambda)$ can be non-zero. Hence, the only first $2N$ s-numbers of the compact operator $P_{\lambda\pm iy}(r_\lambda) - P_{\lambda\pm i0}(r_\lambda)$ can be non-zero. This implies the estimate

$$\begin{aligned}\|P_{\lambda\pm iy}(r_\lambda) - P_{\lambda\pm i0}(r_\lambda)\|_1 &\le \sum_{j=1}^{2N} s_j(P_{\lambda\pm iy}(r_\lambda) - P_{\lambda\pm i0}(r_\lambda)) \\ &\le 2N s_1(P_{\lambda\pm iy}(r_\lambda) - P_{\lambda\pm i0}(r_\lambda)) = 2N\|P_{\lambda\pm iy}(r_\lambda) - P_{\lambda\pm i0}(r_\lambda)\|,\end{aligned}$$

which shows that the trace class norm on the left hand side also converges to zero as $y \to 0$. ∎

Similarly to the definition of $P_z(r_\lambda)$ one can introduce nilpotent operators

$$\mathbf{A}_z(r_\lambda) = \mathbf{A}_z(r_z^1) + \cdots + \mathbf{A}_z(r_z^N), \tag{5.2.4}$$

$$\mathbf{B}_z(r_\lambda) = \mathbf{B}_z(r_z^1) + \cdots + \mathbf{B}_z(r_z^N), \tag{5.2.5}$$

where $r_z^1, \dots, r_z^N$ are resonance points of the group of r_λ (see (5.2.1)). It follows from (3.3.8) and (3.3.10) that $\mathbf{A}_z(r_\lambda)$ and $\mathbf{B}_z(r_\lambda)$ are indeed nilpotent.

LEMMA 5.2.3. *We have*

$$\mathbf{A}_{\lambda\pm i0}(r_\lambda) = \lim_{y\to 0^+} \mathbf{A}_{\lambda\pm iy}(r_\lambda) \quad \textit{and} \quad \mathbf{B}_{\lambda\pm i0}(r_\lambda) = \lim_{y\to 0^+} \mathbf{B}_{\lambda\pm iy}(r_\lambda),$$

where the limits are in the trace class norm.

Proof. Since $A_z(s)$ converges to $A_{\lambda+i0}(s)$ in the uniform norm, it follows from (5.2.4) and (3.3.1) that so do the limits in question. Hence, the claim is a consequence of (3.3.7), Lemma 5.2.2 and the joint continuity of $\mathcal{L}_\infty \times \mathcal{L}_1 \ni (A, B) \mapsto AB \in \mathcal{L}_1$. ∎

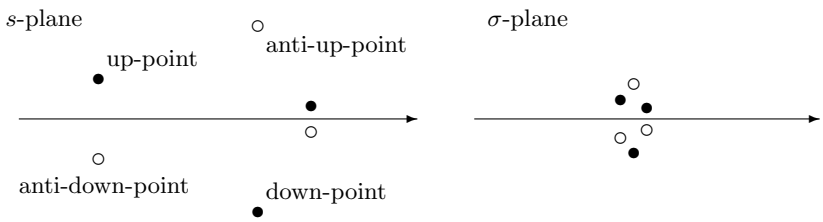
5.3. Resonance index. Let $z \in \Pi$ and let H_0 and V be as usual. A resonance point r_z (see Definition 3.1.1) corresponding to z will be said to be an *up-point* (respectively, *down-point*) if $\operatorname{Im} r_z > 0$ (respectively, $\operatorname{Im} r_z < 0$). Further, if r_z is an up-point corresponding to z, then $\bar{r}_z$ will be called an *anti-down-point* corresponding to z; similarly, if r_z is a down-point of z, then $\bar{r}_z$ will be called an *anti-up-point* of z. Anti-up-points and anti-down-points of z will be called *anti-resonance points* of z. By Corollary 3.1.5, for any $z \in \Pi$, resonance points of $\bar{z}$ are anti-resonance points of z and vice versa. In figures, resonance points are denoted by dark circles and anti-resonance points by light circles.

If $z = \lambda + i0 \in \partial\Pi$ is an essentially regular point and if r_λ is a corresponding real resonance point, then the *resonance index* of a triple $(\lambda; H_{r_\lambda}, V)$ will be defined as the difference of the number N_+ of up-points and the number N_- of down-points which

belong to the group of r_λ, corresponding to $z = \lambda + iy$ with small enough $y > 0$. The resonance index of $(\lambda; H_{r_\lambda}, V)$ will be denoted by

$$\operatorname{ind}_{\rm res}(\lambda; H_{r_\lambda}, V). \tag{5.3.1}$$

Given a real number s, the resonance index can also be defined as the difference of the number of eigenvalues $\sigma^\nu_{\lambda+iy}(s)$ in $\mathbb{C}_+$ and in $\mathbb{C}_-$ of the operator $A_{\lambda+iy}(s)$ which are obtained from the resonance points of the group of r_λ for $z = \lambda + iy$ after the transformation $\sigma_z(s) = (s - r_z)^{-1}$, since this transformation maps the upper half-plane to the upper half-plane for any real s. This is demonstrated by the following figure, where the label "s-plane", respectively "σ-plane", means that the plane of the figure represents the range of values of the variable s, respectively σ. Thus, to calculate the resonance index $N_+ - N_-$ one can use either of these two figures.



LEMMA 5.3.1. *For any real resonance point r_λ, for any real number s and for all small enough $y > 0$,*

$$\operatorname{ind}_{\rm res}(\lambda; H_{r_\lambda}, V) = \mathcal{R}(A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)). \tag{5.3.2}$$

Proof. Let

$$\sigma^\nu_{\lambda+iy}(s) = (s - r^\nu_{\lambda+iy})^{-1}$$

be an eigenvalue of $A_{\lambda+iy}(s)$ corresponding to a resonance point $r^\nu_{\lambda+iy}$ of the group of r_λ for $z = \lambda + iy$. Further, let

$$u^1_{\lambda+iy,+}, \dots, u^{N_+}_{\lambda+iy,+} \quad \text{and} \quad u^1_{\lambda+iy,-}, \dots, u^{N_-}_{\lambda+iy,-}$$

be linearly independent root vectors of $A_{\lambda+iy}(s) = T_{\lambda+iy}(H_s)J$, such that the eigenvalue $\sigma^\nu_{\lambda+iy,\pm}(s)$ corresponding to $u^\nu_{\lambda+iy,\pm}$ lies in $\mathbb{C}_\pm$. Since r_z and $\sigma_z(s) = (s - r_z)^{-1}$ belong to the same half-plane, by definition of the resonance index it follows that

$$\operatorname{ind}_{\rm res}(\lambda; H_{r_\lambda}, V) = N_+ - N_-. \tag{5.3.3}$$

On the other hand, using (3.2.8), we have $P_{\lambda+iy}(r_\lambda)u^\nu_{\lambda+iy,\pm} = u^\nu_{\lambda+iy,\pm}$, and therefore for eigenvectors $u^\nu_{\lambda+iy,\pm}$,

$$\sigma^\nu_{\lambda+iy,\pm}(s)u^\nu_{\lambda+iy,\pm} = A_{\lambda+iy}(s)u^\nu_{\lambda+iy,\pm} = A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)u^\nu_{\lambda+iy,\pm}.$$

It follows that $A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)$ has $N_\pm$ eigenvalues in $\mathbb{C}_\pm$. This implies that

$$\mathcal{R}(A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)) = N_+ - N_-.$$

Combining this with (5.3.3) completes the proof. ∎

Since resonance points r_z corresponding to z are anti-resonance points corresponding to $\bar z$, the same argument shows that if $y > 0$, then

$$\operatorname{ind}_{\rm res}(\lambda; H_{r_\lambda}, V) = -\mathcal{R}(A_{\lambda-iy}(s)P_{\lambda-iy}(r_\lambda)). \tag{5.3.4}$$

Further, Lemma 5.3.1 combined with (3.2.10) and Lemma 5.1.2(i) imply that, for $y > 0$,

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \Re(B_{\lambda+iy}(s)Q_{\lambda+iy}(r_\lambda)) = -\Re(B_{\lambda-iy}(s)Q_{\lambda-iy}(r_\lambda)).$$

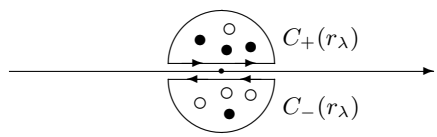
The definition of the resonance index can also be written in the form

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \operatorname{Tr}(P^{\uparrow}_{\lambda+iy}(r_\lambda) - P^{\downarrow}_{\lambda+iy}(r_\lambda)) = \operatorname{rank}(P^{\uparrow}_{\lambda+iy}(r_\lambda)) - \operatorname{rank}(P^{\downarrow}_{\lambda+iy}(r_\lambda)).$$

From Lemma 3.1.4 one can infer that $\operatorname{Tr}(P^{\downarrow}_{\lambda+iy}(r_\lambda)) = \operatorname{Tr}(P^{\uparrow}_{\lambda-iy}(r_\lambda))$; hence, also

$$\begin{aligned}\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) &= \operatorname{Tr}(P^{\uparrow}_{\lambda+iy}(r_\lambda) - P^{\uparrow}_{\lambda-iy}(r_\lambda)) \\ &= \operatorname{rank}(P^{\uparrow}_{\lambda+iy}(r_\lambda)) - \operatorname{rank}(P^{\uparrow}_{\lambda-iy}(r_\lambda)). \end{aligned} \tag{5.3.5}$$

According to Corollary 3.1.5, up-points of z are anti-up-points of $\bar{z}$, and down-points of z are anti-down-points of $\bar{z}$. Let $C_+(r_\lambda)$ be a contour which encloses in anticlockwise direction only up-points and anti-up-points of the group of r_λ, and similarly, let $C_-(r_\lambda)$ be a contour which encloses in anticlockwise direction only down-points and anti-down-points of the group of r_λ, as shown in the figure below.



Proposition 5.3.2 ([Az$_4$]). *If $C_+(r_\lambda)$ and $C_-(r_\lambda)$ are contours as defined above, then for small enough $y > 0$,*

$$\begin{aligned}\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) &= \frac{1}{\pi} \operatorname{Tr}\biggl(\oint_{C_+(r_\lambda)} \operatorname{Im} T_{\lambda+iy}(H_s) J \, ds \biggr) \\ &= -\frac{1}{\pi} \operatorname{Tr}\biggl(\oint_{C_-(r_\lambda)} \operatorname{Im} T_{\lambda+iy}(H_s) J \, ds \biggr). \end{aligned} \tag{5.3.6}$$

Proof. By Proposition 3.2.3 we have

$$\begin{aligned}\frac{1}{\pi} \oint_{C_+(r_\lambda)} \operatorname{Im} T_{\lambda+iy}(H_s) J \, ds &= \frac{1}{2\pi i} \oint_{C_+(r_\lambda)} (A_{\lambda+iy}(H_s) - A_{\lambda-iy}(H_s)) \, ds \\ &= P^{\uparrow}_{\lambda+iy}(r_\lambda) - P^{\uparrow}_{\lambda-iy}(r_\lambda). \end{aligned}$$

This equality shows that the integral over $C_+(r_\lambda)$ is trace class. After taking traces of both sides, the first equality of (5.3.6) follows from (5.3.5). The second equality is proved similarly. ∎

6. Total resonance index as the singular spectral shift function

In this section we give a sketch of the proof of Theorem 1.5.2, given in my unpublished paper [Az$_4$]. This section is not used in the remaining part of this paper and it may be safely skipped. On the other hand, the results of this subsection provide one of the main motivations for this work.

Theorem 1.5.2 holds under a weaker relatively trace class assumption which makes it applicable to Schrödinger operators $H_0u(x) = -\Delta u(x) + V_0(x)u(x)$ with bounded potentials $V_0(x)$ and quickly falling bounded perturbations $V(x)$ in dimensions 1, 2 and 3. The proof of this more general result relies on an appropriate modification of the constructive approach to the stationary scattering theory discussed in the introduction (see [Az$_6$]). This modification is lengthy and therefore the proof has not been included here. It will appear in [AzD].

6.1. A lemma. In this and only in this section we assume that the perturbation operator V is of trace class. This is achieved by assuming that the rigging operator F is Hilbert–Schmidt.

Let

$$F_z(s) = \frac{1}{\pi}\operatorname{Tr}(\operatorname{Im} R_z(H_s)V) = \frac{1}{2\pi i}\operatorname{Tr}(A_z(s) - A_{\bar z}(s)).$$

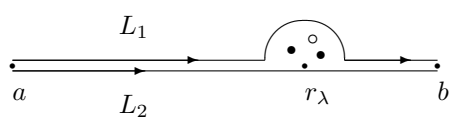
The operator $\operatorname{Im} R_z(H_s)V$ is equal to $\frac{1}{2i}(\underline{A}_z(s) - \underline{A}_{\bar z}(s))$ but the cyclic property $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ of the trace allows us to replace the underlined operators by the non-underlined counterparts.

LEMMA 6.1.1. *Let λ be any number from the set $\Lambda(H_0, F)$ of full Lebesgue measure. Assume that the interval $[a, b]$ of the real axis contains only one resonance point r_λ of the triple $(\lambda; H_0, V)$. Then for all small enough $y > 0$,*

$$\int_{L_2} F_{\lambda+iy}(s)\,ds = \int_{L_1} F_{\lambda+iy}(s)\,ds + \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V), \tag{6.1.1}$$

where L_1 and L_2 are the contours of integration from a to b shown below; namely, L_2 goes straight from a to b, while L_1 circumvents the resonance and anti-resonance points of the group of r_λ from above.

(s-*plane*) ($z = \lambda + iy$, $0 < y \ll 1$)



(6.1.2)

Proof. By Cauchy's theorem, for all small enough $y > 0$ we have

$$\int_{L_2} F_{\lambda+iy}(s)\,ds = \int_{L_1} F_{\lambda+iy}(s)\,ds + \oint_{C_+(r_\lambda)} F_{\lambda+iy}(s)\,ds,$$

where the half-circle $C_+(r_\lambda)$ encloses all the resonance and anti-resonance points of the group of r_λ which are in $\mathbb{C}_+$ (and anly these). Hence, (6.1.1) follows from Proposition 5.3.2. ∎

PROPOSITION 6.1.2. *For a.e.* $\lambda \in \mathbb{R}$,

$$\lim_{y\to 0+} \int_0^1 F_{\lambda+iy}(s)\,ds = \xi(\lambda; H_1, H_0),$$

where $\xi(\lambda; H_1, H_0)$ *is the spectral shift function of the pair* (H_1, H_0).

This proposition is in essence the Birman–Solomyak formula (1.2.3) for the spectral shift function. The difference is that (1.2.3) uses the derivative of the distributive function of the spectral shift measure, while in the formula above it is replaced by $1/\pi$ times the imaginary part of the limit of the Cauchy transform of the distributive function. By a well-known theorem of complex analysis, these two functions are equal a.e. Details of the proof can be found in e.g. [Az$_3$, §§9.5, 9.6].

6.2. Absolutely continuous part of the spectral shift function. Now we discuss the absolutely continuous part of the spectral shift function $\xi^{(a)}(\lambda; H_1, H_0)$. By definition, $\xi^{(a)}(\lambda; H_1, H_0)$ is the density of the measure defined by

$$\Delta \mapsto \int_0^1 \operatorname{Tr}(V E_\Delta^{H_s} P^{(a)}(H_s))\,ds,$$

where $P^{(a)}(H_s)$ is the (orthogonal) projection onto the absolutely continuous subspace of the self-adjoint operator H_s.

It was shown in [Az$_3$] that for a.e. λ the number $\xi^{(a)}(\lambda; H_s, H_0)$ is equal to

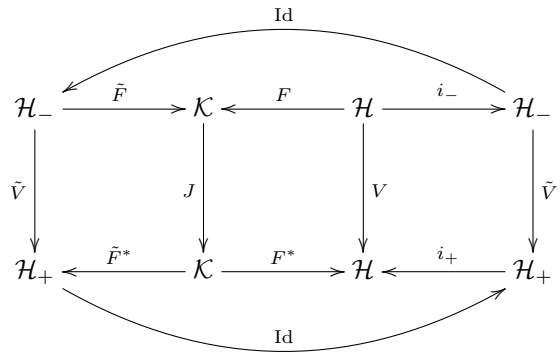
$$\int_0^s \operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}(\mathcal{E}_\lambda(H_r) V \mathcal{E}_\lambda^\diamondsuit(H_r))\,dr, \tag{6.2.1}$$

where $\mathcal{E}_\lambda(H_r)\colon \mathcal{H}_+ \to \mathfrak{h}_\lambda(H_r)$ is the *evaluation operator* defined by (1.4.4). Since $\mathcal{E}_\lambda(H_r)$ was introduced in a recent and lengthy paper, the meaning of this formula may need some explanations. Here $\mathcal{H}_+ = F^*\mathcal{H}$ is the rigging Hilbert space and $\mathfrak{h}_\lambda(H_r)$ is the subspace of the auxiliary Hilbert space $\mathcal{K}$ defined by

$$\mathfrak{h}_\lambda(H_r) = \overline{\operatorname{im} \operatorname{Im} T_{\lambda+i0}(H_r)}.$$

It was shown in [Az$_3$] that $\mathfrak{h}_\lambda(H_r)$ can be treated as the fibre Hilbert space. The operator $\mathcal{E}_\lambda^\diamondsuit(H_r)$ acts from the Hilbert space $\mathfrak{h}_\lambda(H_r)$ to the Hilbert space $\mathcal{H}_-$ which comes from the rigging operator F; the definition of $\mathcal{E}_\lambda^\diamondsuit(H_r)$ will follow shortly. The fact that the trace class perturbation $V\colon \mathcal{H} \to \mathcal{H}$ admits the factorization $V = F^*JF$ with Hilbert–Schmidt F and bounded J allows us to treat V as a bounded operator from $\mathcal{H}_-$ to $\mathcal{H}_+$, since F can be treated as a unitary isomorphism $\mathcal{H}_- \xrightarrow{\sim} \mathcal{K}$ and F^* can be treated as a unitary isomorphism $\mathcal{K} \xrightarrow{\sim} \mathcal{H}_+$. These unitary isomorphisms can be denoted by the same F and F^*, but we will be pedantic for a moment and denote them by $\tilde{F}\colon \mathcal{H}_- \xrightarrow{\sim} \mathcal{K}$

and $\tilde{F}^*\colon \mathcal{K} \xrightarrow{\sim} \mathcal{H}_+$. Now, the equality $V = F^*JF$ can be understood in several ways as shown in the following commutative diagram:



Here $i_\pm$ are the Hilbert–Schmidt inclusion operators. In (6.2.1) the symbol V denotes the bounded operator $\tilde{V}\colon \mathcal{H}_- \to \mathcal{H}_+$. The operator $\mathcal{E}_\lambda^\diamondsuit(H_r)$ acts from $\mathfrak{h}_\lambda(H_r)$ to $\mathcal{H}_-$ according to

$$\langle \mathcal{E}_\lambda^\diamondsuit(H_r)g, f\rangle_{-1,1} = \langle g, \mathcal{E}_\lambda(H_r)f\rangle, \quad g \in \mathfrak{h}_\lambda(H_r),\ f \in \mathcal{H}_+.$$

This definition of $\mathcal{E}_\lambda^\diamondsuit$ is equivalent to

$$\mathcal{E}_\lambda^\diamondsuit = \tilde{F}^{-1}(\tilde{F}^*)^{-1}\mathcal{E}_\lambda^*,$$

where $\mathcal{E}_\lambda^*\colon \mathfrak{h}_\lambda(H_r) \to \mathcal{H}_+$ is the usual adjoint and $\tilde{F}$ and $\tilde{F}^*$ are unitary isomorphisms shown in the diagram. The product $\mathcal{E}_\lambda(H_r)V\mathcal{E}_\lambda^\diamondsuit(H_r)$ is of trace class since $\mathcal{E}_\lambda(H_r)$ and $\mathcal{E}_\lambda^\diamondsuit(H_r)$ are Hilbert–Schmidt and $V\colon \mathcal{H}_- \to \mathcal{H}_+$ is bounded.

Note that for any fixed point λ from the set $\Lambda(H_0, F)$ the operator $\mathcal{E}_\lambda(H_r)$ is defined for all non-resonant values of r, according to the definition of this operator:

$$\mathcal{E}_\lambda(H_r)\tilde{F}^*\psi = \frac{1}{\pi}\sqrt{\operatorname{Im} T_{\lambda+i0}\,(H_r)}\,\psi, \quad \psi \in \mathcal{H}.$$

To avoid ambiguity, we write $\tilde{V}$ instead of V, when we treat V as an operator : $\mathcal{H}_- \to \mathcal{H}_+$. Note that, as the left square of the diagram above clearly shows, $\tilde{V}$ is unitarily equivalent to J.

The following proposition is proved in [Az$_3$, Corollary 7.3.5]. We give a sketch of that proof.

Proposition 6.2.1. *For any $\lambda \in \Lambda(H_0, F)$, the operator-valued function of $r \in \mathbb{R}$ defined by*

$$r \mapsto \operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}\big(\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)\big)$$

is analytic and admits holomorphic continuation to some neighbourhood of $\mathbb{R}$.

Proof. For any $\lambda \in \Lambda(H_0, F)$ and any real non-resonant r the following equality holds:

$$w_+^*(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)w_+(\lambda; H_r, H_0) = \left(\frac{d}{dr}S(\lambda; H_r, H_0)\right)S^*(\lambda; H_r, H_0), \qquad (6.2.2)$$

where

$$S(\lambda; H_r, H_0)\colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_0)$$

is the scattering matrix and

$$w_+(\lambda; H_r, H_0)\colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_r)$$

is the wave matrix. According to [Az$_3$, §5 and §7], the right hand side is defined for all non-resonant values of r. According to [Az$_3$, Proposition 7.2.5], the scattering matrix $S(\lambda; H_r, H_0)$ is an analytic function of r in the whole real axis $\mathbb{R}$, and therefore so is the right hand side of (6.2.2). It follows that the trace of the left hand side is also analytic. Since $w_+(\lambda; H_r, H_0)$ is unitary, this trace is equal to

$$\operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}\big(\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)\big) \quad \blacksquare$$

This proposition should not be surprising in the light of the general coupling constant regularity phenomenon observed first by Aronszajn back in 1957.

THEOREM 6.2.2 ([Az$_3$, Lemma 8.2.1, Theorem 8.1.3]). *For a.e. λ the absolutely continuous spectral shift function $\xi^{(a)}(\lambda; H_1, H_0)$ is equal to*

$$\int_0^1 \operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}\big(\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)\big)\,dr. \tag{6.2.3}$$

6.3. Singular spectral shift function and resonance index. Now we return to the equality (6.1.1). It is not difficult to see that as $y \to 0^+$, the limit

$$\int_{L_1} F_{\lambda+i0}(s)\,ds = \int_{L_1} \operatorname{Tr}_{\mathcal{K}}\left(\frac{1}{\pi}\operatorname{Im} T_{\lambda+i0}(H_r)J\right) ds \tag{6.3.1}$$

of the second integral over the contour L_1 shown in figure (6.1.2) exists, where L_1 indicates that all resonance points in the interval $[0,1]$ are circumvented in the upper half-plane.

LEMMA 6.3.1. *For all λ from the set $\Lambda(H_0, F)$ of full Lebesgue measure the integrals* (6.3.1) *and* (6.2.3) *are equal.*

Proof. For the purpose of this proof, we shall denote by $L_1(s)$ the part of the contour L_1, shown in (6.1.2), which projects onto $[0,s]$, where $s \in [0,1]$. In particular, for small enough s the contour $L_1(s)$ coincides with $[0,s]$.

By the definition (1.4.4) of $\mathcal{E}_\lambda(H_r)$, for all non-resonant values of r the integrand in (6.2.3) is equal to

$$\operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}\big(\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)\big) = \operatorname{Tr}_{\mathcal{H}_-}\big(\mathcal{E}_\lambda^\diamondsuit(H_r)\mathcal{E}_\lambda(H_r)\tilde{V}\big) = \operatorname{Tr}_{\mathcal{K}}\left(\frac{1}{\pi}\operatorname{Im} T_{\lambda+i0}(H_r)J\right).$$

Hence, for small enough s the integrals

$$\int_{L_1(s)} F_{\lambda+i0}(r)\,dr \tag{6.3.2}$$

and

$$\int_0^s \operatorname{Tr}_{\mathfrak{h}_\lambda(H_r)}\big(\mathcal{E}_\lambda(H_r)\tilde{V}\mathcal{E}_\lambda^\diamondsuit(H_r)\big)\,dr \tag{6.3.3}$$

are equal, since their integrands and contours of integration are equal.

We have to show that the integrals are equal for large values of s too, in particular for $s = 1$.

The integral (6.3.3) is holomorphic in some neighbourhood of $[0, 1]$, since so is its integrand according to Proposition 6.2.1. If we show that the first integral is also holomorphic in some neighbourhood of $[0, 1]$, the proof will be complete by the uniqueness theorem for holomorphic functions.

The integrand in (6.3.2) has singularities at resonance points in $[0, 1]$, but the integral itself is a single-valued holomorphic function in a neighbourhood of $[0, 1]$ with possibly the resonance points removed. Indeed, the integral (6.3.2) does not change its value if the contour $L_1(s)$ is changed so that the resonance points are circumvented not from above but from below: the result of analytic continuation of the integral from small values of s to the value $s = 1$ will be the same since, according to the second equality of Proposition 5.3.2, the integral over the circle which encloses a resonance point on $[0, 1]$ is zero.

Hence, both (6.3.2) and (6.3.3) are holomorphic single-valued functions in some neighbourhood of $[0, 1]$ minus a finite set of resonance points, and both integrals coincide for small values of s. Since (6.3.3) is holomorphic in a neighbourhood of $[0, 1]$, it follows that so is the function (6.3.2) and these two functions coincide. ∎

Combining Lemma 6.1.1, Proposition 6.1.2, Theorem 6.2.2 and Lemma 6.3.1, we conclude that after taking the limit as $y \to 0^+$ the equality (6.1.1) with $a = 0$ and $b = 1$ turns into

$$\xi(\lambda; H_1, H_0) = \xi^{(a)}(\lambda; H_1, H_0) + \sum_{r_\lambda} \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V),$$

where the sum is taken over all resonance points from $[0, 1]$.

Since $\xi^{(s)}(\lambda; H_1, H_0) = \xi(\lambda; H_1, H_0) - \xi^{(a)}(\lambda; H_1, H_0)$, this gives

THEOREM 6.3.2. *For a.e. λ,*

$$\xi^{(s)}(\lambda; H_1, H_0) = \sum_{r_\lambda \in [0,1]} \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V).$$

7. Signature of the resonance matrix

In this section we prove Theorem 7.2.1, which is one of the main technical results of this paper. This theorem allows us to express the signature of the finite-rank self-adjoint operator

$$Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda),$$

which we call the resonance matrix, in terms of the R-index of the operator

$$A_{\lambda+iy}(r_\lambda)P_{\lambda+iy}(r_\lambda),$$

where y is any small enough positive number.

Assume that we are given a finite set

$$\Gamma = \{r_z^1, \dots, r_z^M\}$$

of resonance points corresponding to a fixed number $z \in \Pi$. We denote by $\bar{\Gamma}$ the set $\{\bar{r}_z^1, \dots, \bar{r}_z^M\}$. The finite-rank self-adjoint operator

$$Q_{\bar{z}}(\bar{\Gamma})JP_z(\Gamma) \tag{7.0.1}$$

will be called the *resonance matrix* of Γ. The main difficulty in the proof of Theorem 7.2.1 is to show (Theorem 7.1.4) that if z and the numbers in Γ all belong to the open upper half-plane $\mathbb{C}_+$, then (7.0.1) is non-negative and its rank is equal to the sum of the algebraic multiplicities of the resonance points from Γ.

To start with, we shall give a proof of this assertion in the trivial special case where Γ consists of only one resonance point $r_z \in \mathbb{C}_+$ of algebraic multiplicity one and where $y = \operatorname{Im} z > 0$. In this case the self-adjoint operator (7.0.1) has rank one and its positivity is equivalent to the positivity of the real number

$$\langle \chi_z, V\chi_z \rangle,$$

where χ_z is a non-zero resonance vector from $\underset{\sim}{\Upsilon}_z(r_z)$. Since χ_z has order one, it is an eigenvector of the operator H_{r_z} corresponding to the eigenvalue z:

$$H_{r_z}\chi_z = z\chi_z.$$

We take the scalar product of both sides with χ_z:

$$\langle \chi_z, H_{r_z}\chi_z \rangle = z\langle \chi_z, \chi_z \rangle.$$

Now we take conjugates of both sides. This gives

$$\langle \chi_z, H_{\bar{r}_z}\chi_z \rangle = \bar{z}\langle \chi_z, \chi_z \rangle.$$

Finally, we subtract this equality from the previous one to get

$$(r_z - \bar{r}_z)\langle \chi_z, V\chi_z\rangle = 2iy\langle \chi_z, \chi_z\rangle.$$

Since $2iy/(r_z - \bar{r}_z)$ is positive, the claim follows.

In the general case, where Γ may consist of any finite number of resonance points of arbitrary geometric and algebraic multiplicities, the proof becomes far more tedious.

7.1. Positivity of the resonance matrix for a set of up-points. Recall that a symmetric matrix $\alpha \in \mathbb{C}^{n\times n}$ is *positive definite* if for any non-zero $x \in \mathbb{C}^n$ we have $\langle x, \alpha x\rangle > 0$. In particular, the rank of a positive definite matrix is equal to the dimension of the vector space on which it acts.

LEMMA 7.1.1. *Let $y > 0$. Let M and $d_1, \dots, d_M$ be positive integers. Assume that we are given M sets of vectors*

$$\chi_\mu^{(1)}, \dots, \chi_\mu^{(d_\mu)}, \qquad \mu = 1, \dots, M,$$

from a pre-Hilbert space, such that all

$$D := d_1 + \cdots + d_M$$

vectors $\chi_\mu^{(j)}$ are linearly independent. Let β be the positive definite $D \times D$ matrix

$$\beta_{\mu\nu}^{kj} = \langle \chi_\mu^{(k)}, \chi_\nu^{(j)}\rangle, \tag{7.1.1}$$

and define another $D \times D$ matrix γ by the recurrent formula

$$\gamma_{\mu\nu}^{kj} = 2y\beta_{\mu\nu}^{kj} - i(\gamma_{\mu\nu}^{k-1,j} - \gamma_{\mu\nu}^{k,j-1}), \tag{7.1.2}$$

where it is assumed that $\gamma_{\mu\nu}^{kj} = 0$ if at least one of the indices k or j is equal to 0. Then γ is positive definite.

Proof. Plainly, the matrix γ is symmetric.

In the proof we will use induction on the positive integer

$$d = \max\{d_1, \dots, d_M\}$$

which will be called *order*. If $d = 1$ then the second term in (7.1.2) is zero, and so in this case the claim follows from the positive definiteness of (7.1.1). Now assuming that the claim holds for orders $< d$ we show that it holds for order d.

Rows of a $D \times D$ matrix will be enumerated by pairs (μ, k) so that $(\mu, k) < (\nu, j)$ if and only if $\mu < \nu$, or both $\mu = \nu$ and $k < j$. We can look at a $D \times D$ matrix γ as composed of $M \times M$ cells, so that (μ, ν) indicates a cell and (k, j) indicates an element of the cell. The second index k in the pair (μ, k), denoting a row/column, will be called the *order* of the row/column. The following figure shows the structure of a $D \times D$ matrix γ:

$$\begin{bmatrix}
\begin{bmatrix} \gamma_{11}^{11} & \dots & \gamma_{11}^{1d_1} \\ \dots & \dots & \dots \\ \gamma_{11}^{d_1 1} & \dots & \gamma_{11}^{d_1 d_1} \end{bmatrix} & & & \begin{bmatrix} \gamma_{1M}^{11} & \dots & \gamma_{1M}^{1d_M} \\ \dots & \dots & \dots \\ \gamma_{1M}^{d_1 1} & \dots & \gamma_{1M}^{d_1 d_M} \end{bmatrix} \\
\dots & & \dots & \dots \\
& \begin{bmatrix} \gamma_{\mu\nu}^{11} & \dots & \gamma_{\mu\nu}^{1j} & \dots & \gamma_{\mu\nu}^{1d_\nu} \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_{\mu\nu}^{k1} & \dots & \gamma_{\mu\nu}^{kj} & \dots & \gamma_{\mu\nu}^{kd_\nu} \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_{\mu\nu}^{d_\mu 1} & \dots & \gamma_{\mu\nu}^{d_\mu j} & \dots & \gamma_{\mu\nu}^{d_\mu d_\nu} \end{bmatrix} & \\
\dots & & \dots & \dots \\
\begin{bmatrix} \gamma_{M1}^{11} & \dots & \gamma_{M1}^{1d_1} \\ \dots & \dots & \dots \\ \gamma_{M1}^{d_M 1} & \dots & \gamma_{M1}^{d_M d_1} \end{bmatrix} & & & \begin{bmatrix} \gamma_{MM}^{11} & \dots & \gamma_{MM}^{1d_M} \\ \dots & \dots & \dots \\ \gamma_{MM}^{d_M 1} & \dots & \gamma_{MM}^{d_M d_M} \end{bmatrix}
\end{bmatrix}.$$

We apply to the matrix γ the following elementary row and column operations: if a row (μ, k) has order $k \geq 2$, then we add to this row the previous row $(\mu, k-1)$ multiplied by i, and if a column (ν, j) is of order $j \geq 2$, then we add to this column the previous column $(\nu, j-1)$ multiplied by $-i$. We still have to specify in which order to execute these row and column operations. The rule is this: we start with rows of largest orders d_μ and finish with rows of order 2; the same rule applies to column operations. If two rows have the same order, then the corresponding row operations are interchangeable and so in this case we do not need to specify the order of execution of these operations. Also, a row operation and a column operation are always interchangeable. The following formula explains what happens to a 2×2 submatrix of γ after a row operation and a column operation (here for convenience the indices k and j are replaced by integers $3, 3$):

$$\begin{pmatrix} \gamma_{\mu\nu}^{22} & \gamma_{\mu\nu}^{23} \\ \gamma_{\mu\nu}^{32} & \gamma_{\mu\nu}^{33} \end{pmatrix} \to \begin{pmatrix} \gamma_{\mu\nu}^{22} & \gamma_{\mu\nu}^{23} \\ \gamma_{\mu\nu}^{32} + i\gamma_{\mu\nu}^{22} & \gamma_{\mu\nu}^{33} + i\gamma_{\mu\nu}^{23} \end{pmatrix} \to \begin{pmatrix} \gamma_{\mu\nu}^{22} & \gamma_{\mu\nu}^{23} - i\gamma_{\mu\nu}^{22} \\ \gamma_{\mu\nu}^{32} + i\gamma_{\mu\nu}^{22} & \gamma_{\mu\nu}^{33} + i\gamma_{\mu\nu}^{23} - i\gamma_{\mu\nu}^{32} + \gamma_{\mu\nu}^{22} \end{pmatrix}.$$

After performing other row and column operations this 2×2 block of γ takes the form

$$\begin{pmatrix} \gamma_{\mu\nu}^{22} + i\gamma_{\mu\nu}^{12} - i\gamma_{\mu\nu}^{21} + \gamma_{\mu\nu}^{11} & \gamma_{\mu\nu}^{23} + i\gamma_{\mu\nu}^{13} - i\gamma_{\mu\nu}^{22} + \gamma_{\mu\nu}^{12} \\ \gamma_{\mu\nu}^{32} + i\gamma_{\mu\nu}^{22} - i\gamma_{\mu\nu}^{31} + \gamma_{\mu\nu}^{21} & \gamma_{\mu\nu}^{33} + i\gamma_{\mu\nu}^{23} - i\gamma_{\mu\nu}^{32} + \gamma_{\mu\nu}^{22} \end{pmatrix}.$$

Now (7.1.2) implies that after these row and column operations have been performed in the specified order, the matrix γ will take the form

$$2y\beta + \tilde{\gamma},$$

where $\tilde{\gamma}$ is obtained from γ by the rule

$$\tilde{\gamma}_{\mu\nu}^{kj} = \begin{cases} \gamma_{\mu\nu}^{k-1,j-1} & \text{if } k, j \geq 2, \\ 0 & \text{if otherwise.} \end{cases}$$

This definition shows that after removing zero rows and columns the matrix $\tilde{\gamma}$ can be deemed as having been obtained by the same formula (7.1.2) but using the system of sets of vectors

$$\chi_\mu^{(1)}, \dots, \chi_\mu^{(d_\mu - 1)}, \quad \mu = 1, \dots, M.$$

The order of this system is $d-1$ and therefore by induction assumption the (original with zero rows and columns) matrix $\tilde{\gamma}$ is non-negative. Hence, $2y\beta+\tilde{\gamma}$ is positive definite, since so is β. Finally, since γ can be represented as $C(2y\beta+\tilde{\gamma})C^*$, where C is the matrix corresponding to the row operations, it follows that γ itself is also positive definite. ∎

LEMMA 7.1.2. *Let M and $d_1,\dots,d_M$ be positive integers and let $D=d_1+\dots+d_M$. Let $r_1,\dots,r_M$ be complex numbers with positive imaginary parts. Let*

$$\gamma=(\gamma_{\mu\nu}^{kj})$$

be a block-matrix of size $D\times D$, where $\mu,\nu=1,\dots,M$, $k=1,\dots,d_\mu$, $j=1,\dots,d_\nu$. If γ is positive definite, then so is

$$\left(\frac{i}{r_\nu-\bar{r}_\mu}\gamma_{\mu\nu}^{kj}\right).$$

Proof. For any positive number $p>0$ the $D\times D$ matrix with elements

$$e^{ip(r_\nu-\bar{r}_\mu)}\gamma_{\mu\nu}^{kj}$$

is positive definite. Hence, so is the matrix with elements

$$\int_0^\infty e^{ip(r_\nu-\bar{r}_\mu)}\gamma_{\mu\nu}^{kj}\,dp=\frac{i\gamma_{\mu\nu}^{kj}}{r_\nu-\bar{r}_\mu}.$$

Since $p>0$ and $\operatorname{Im} r_\mu>0$ for all $\mu=1,\dots,M$, the integral above converges absolutely. ∎

LEMMA 7.1.3. *Let $y>0$. Let M and $d_1,\dots,d_M$ be positive integers. Assume that we are given sets of vectors*

$$\chi_\mu^{(1)},\dots,\chi_\mu^{(d_\mu)},\qquad \mu=1,\dots,M,$$

from a pre-Hilbert space, such that all

$$D:=d_1+\dots+d_M$$

vectors $\chi_\mu^{(j)}$ are linearly independent. Assume further that we are given complex numbers $r_1,\dots,r_M$ with positive imaginary parts. Let β be the positive definite $D\times D$ matrix

$$\beta_{\mu\nu}^{kj}=\langle\chi_\mu^{(k)},\chi_\nu^{(j)}\rangle,$$

and define another $D\times D$ matrix α by the recurrent formula

$$\alpha_{\mu\nu}^{kj}=\frac{2iy}{r_\nu-\bar{r}_\mu}\beta_{\mu\nu}^{kj}+\frac{1}{r_\nu-\bar{r}_\mu}(\alpha_{\mu\nu}^{k-1,j}-\alpha_{\mu\nu}^{k,j-1}),\tag{7.1.3}$$

where it is assumed that $\alpha_{\mu\nu}^{kj}=0$ if at least one of the indices k or j is equal to 0. Then α is positive definite.

Proof. This follows immediately from Lemmas 7.1.1 and 7.1.2. ∎

As can be seen from the proof, if y is negative then the matrix α is negative definite.

THEOREM 7.1.4. *If $\Gamma=\{r_z^1,\dots,r_z^M\}$ is a finite set of resonance up-points corresponding to a non-real number z, then the operator*

$$\operatorname{Im} z\,Q_{\bar{z}}(\bar{\Gamma})JP_z(\Gamma)$$

is non-negative and its rank is equal to the rank of $P_z(\Gamma)$.

Proof. Without loss of generality we assume that $y = \operatorname{Im} z > 0$.

By Lemma 5.1.4, we have

$$F^* Q_{\bar z}(\bar\Gamma) J P_z(\Gamma) F = \underline{Q}_{\bar z}(\bar\Gamma) V \underline{P}_z(\Gamma).$$

Lemma 3.1.7 implies that the range of the finite-rank self-adjoint operator $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is a subset of both $\operatorname{dom}(F^*)$ and the range of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma) F$. Hence, by Lemma 2.1.3, the operators

$$Q_{\bar z}(\bar\Gamma) J P_z(\Gamma) \quad \text{and} \quad \underline{Q}_{\bar z}(\bar\Gamma) V \underline{P}_z(\Gamma)$$

have equal ranks and signatures. So, it is sufficient to prove the claim for the latter operator.

(A) For notational convenience we assume that the same point r_z^μ may appear in the list $r_z^1, \dots, r_z^M$ more than once. More exactly, each point r_z^μ appears in the list m^μ times, where m^μ is the geometric multiplicity of r_z^μ. In what follows we often write r_μ instead of r_z^μ. For each point $r_z^\mu \in \Gamma$ let

$$\chi_\mu^{(j)}, \quad j = 1, \dots, d_\mu,$$

be a basis of $\Upsilon_z(r_z^\mu)$ such that $\underline{\mathbf{A}}_z(r_z^\mu)\chi_\mu^{(j)} = \chi_\mu^{(j-1)}$. We can assume the existence of such a basis since, as mentioned above, the resonance points r_z^μ appear in the list according to their geometric multiplicities. Let

$$\alpha_{\mu\nu}^{kj} = \langle \chi_\mu^{(k)}, V\chi_\nu^{(j)} \rangle \quad \text{and} \quad \beta_{\mu\nu}^{kj} = \langle \chi_\mu^{(k)}, \chi_\nu^{(j)} \rangle.$$

Then

$$\alpha_{\mu\nu}^{kj} = \frac{2iy}{r_\nu - \bar r_\mu} \beta_{\mu\nu}^{kj} + \frac{1}{r_\nu - \bar r_\mu} (\alpha_{\mu\nu}^{k-1,j} - \alpha_{\mu\nu}^{k,j-1}). \tag{7.1.4}$$

Indeed, by Corollary 3.4.7,

$$(H_{r_\nu} - z)\chi_\nu^{(j)} = -V\chi_\nu^{(j-1)}.$$

It follows that

$$\langle \chi_\mu^{(k)}, (H_{r_\nu} - z)\chi_\nu^{(j)} \rangle = -\langle \chi_\mu^{(k)}, V\chi_\nu^{(j-1)} \rangle.$$

In this equality we swap the pairs of indices (μ, k) and (ν, j) and then take conjugates of both sides of the resulting equality:

$$\langle \chi_\mu^{(k)}, (H_{\bar r_\mu} - \bar z)\chi_\nu^{(j)} \rangle = -\langle \chi_\mu^{(k-1)}, V\chi_\nu^{(j)} \rangle.$$

Subtracting from this equality the previous one gives

$$\langle \chi_\mu^{(k)}, (-r_\nu V + \bar r_\mu V + z - \bar z)\chi_\nu^{(j)} \rangle = -\langle \chi_\mu^{(k-1)}, V\chi_\nu^{(j)} \rangle + \langle \chi_\mu^{(k)}, V\chi_\nu^{(j-1)} \rangle.$$

This can be written as

$$(r_\nu - \bar r_\mu)\langle \chi_\mu^{(k)}, V\chi_\nu^{(j)} \rangle = (z - \bar z)\langle \chi_\mu^{(k)}, \chi_\nu^{(j)} \rangle + \langle \chi_\mu^{(k-1)}, V\chi_\nu^{(j)} \rangle - \langle \chi_\mu^{(k)}, V\chi_\nu^{(j-1)} \rangle,$$

which is equivalent to (7.1.4).

(B) Since the vectors

$$\chi_\mu^{(j)}, \quad j = 1, \dots, d_\mu,\ \mu = 1, \dots, M,$$

form a basis of the range of $\underline{P}_z(\Gamma)$, to prove the theorem it is enough to prove the positive definiteness of $(\alpha_{\mu\nu}^{kj})$. But this follows from Lemma 7.1.3 and (7.1.4). ∎

An analogue of Theorem 7.1.4 holds also for sets of resonance down-points. Namely, if Γ is a finite set of resonance down-points, then the operator $\operatorname{Im} z\, Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is non-positive and its rank is equal to the rank of $P_z(\Gamma)$.

7.2. Signature of the resonance matrix and R-index. The following theorem is the main result of this section.

THEOREM 7.2.1. *If $\Gamma = \{r_z^1, \dots, r_z^M\}$ is a finite set of resonance points corresponding to a non-real number z, then the signature of the finite-rank self-adjoint operator $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is equal to the R-index of $\operatorname{Im} z A_z(s) P_z(\Gamma)$.*

Proof. Without loss of generality we assume that $\operatorname{Im} z > 0$.

Let $\Gamma = \Gamma^\uparrow \cup \Gamma^\downarrow$, where $\Gamma^\uparrow \subset \mathbb{C}_+$ and $\Gamma^\downarrow \subset \mathbb{C}_-$. Further, let

$$\Upsilon^\uparrow = \operatorname{im}(P_z(\Gamma^\uparrow)) \quad \text{and} \quad \Upsilon^\downarrow = \operatorname{im}(P_z(\Gamma^\downarrow)).$$

The R-index of $A_z(s)P_z(\Gamma)$ is equal to $N_+ - N_-$, where N_+ (respectively, N_-) is the sum of the algebraic multiplicities of all points from $\Gamma^\uparrow$ (respectively, $\Gamma^\downarrow$); that is,

$$\mathcal{R}(A_z(s)P_z(\Gamma)) = N_+ - N_- := \dim \Upsilon^\uparrow - \dim \Upsilon^\downarrow.$$

For any non-zero $u \in \Upsilon^\uparrow$ we have

$$\langle u, Q_{\bar z}(\bar\Gamma) J P_z(\Gamma) u\rangle = \langle P_z(\Gamma)u, J P_z(\Gamma)u\rangle = \langle P_z(\Gamma^\uparrow)u, J P_z(\Gamma^\uparrow)u\rangle > 0,$$

where the last inequality follows from Theorem 7.1.4. Similarly, for any non-zero $u \in \Upsilon^\downarrow$,

$$\langle u, Q_{\bar z}(\bar\Gamma) J P_z(\Gamma) u\rangle = \langle P_z(\Gamma)u, J P_z(\Gamma)u\rangle = \langle P_z(\Gamma^\downarrow)u, J P_z(\Gamma^\downarrow)u\rangle < 0.$$

Therefore, by Lemma 2.1.2, the rank of the positive (respectively, negative) part of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is at least N_+ (respectively, N_-). Hence, the rank of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is at least

$$N_+ + N_- = N := \operatorname{rank}(P_z(\Gamma)),$$

and so the rank of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is N. It follows that in fact the rank of the positive (respectively, negative) part of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is N_+ (respectively, N_-). Thus, the signature of $Q_{\bar z}(\bar\Gamma) J P_z(\Gamma)$ is $N_+ - N_-$. ∎

Theorem 7.2.1 is the main ingredient of the proof of Theorem 9.2.1, which asserts that the resonance index can be treated as the signature of a certain finite-rank self-adjoint operator.

We remark that Theorems 7.1.4 and 7.2.1 also hold in a finite-dimensional Hilbert space, that is, for a pair of self-adjoint matrices H_0 and V. Still, even this special case is non-trivial. The finite-dimensional versions of Theorems 7.1.4 and 7.2.1 can be tested in numerical experiments. Such a testing was carried out by the author using MATLAB, and it confirms both theorems.

7.3. Some corollaries. Theorem 7.1.4 has the following corollaries.

COROLLARY 7.3.1. *Let z be a non-real number. For any finite set Γ of resonance up-points corresponding to z the mapping*

$$Q_{\bar z}(\bar\Gamma)\colon \Psi_z(\Gamma) \to \Psi_{\bar z}(\bar\Gamma)$$

is a linear isomorphism.

Proof. Assume the contrary. Then, since the dimensions of $\Psi_z(\Gamma)$ and $\Psi_{\bar{z}}(\bar{\Gamma})$ are finite and equal, for some non-zero $\psi \in \Psi_z(\Gamma)$ we have $Q_{\bar{z}}(\bar{\Gamma})\psi = 0$. By Lemma 3.1.4, there exists a non-zero $u \in \Upsilon_z(\Gamma)$ such that $\psi = Ju$. It follows that

$$\langle u, Q_{\bar{z}}(\bar{\Gamma})JP_z(\Gamma)u\rangle = \langle u, Q_{\bar{z}}(\bar{\Gamma})Ju\rangle = 0.$$

This contradicts Theorem 7.1.4. ∎

COROLLARY 7.3.2. *Let z be a non-real number. For any finite set Γ of resonance up-points corresponding to z the mapping*

$$P_{\bar{z}}(\bar{\Gamma})\colon \Upsilon_z(\Gamma) \to \Upsilon_{\bar{z}}(\bar{\Gamma})$$

is a linear isomorphism.

Proof. This follows from Lemma 3.1.4 and the previous corollary. ∎

These corollaries hold for a finite set of down-points too, of course. Similarly, for any finite set Γ of resonance points from $\mathbb{C}_+$ or $\mathbb{C}_-$ the mappings

$$Q_z(\Gamma)\colon \Psi_{\bar{z}}(\bar{\Gamma}) \to \Psi_z(\Gamma) \quad \text{and} \quad P_z(\Gamma)\colon \Upsilon_{\bar{z}}(\bar{\Gamma}) \to \Upsilon_z(\Gamma)$$

are linear isomorphisms.

COROLLARY 7.3.3. *For any finite set Γ of resonance up-points and for any $j = 1, 2, \ldots$ the operator*

$$\mathbf{B}^j_{\bar{z}}(\bar{\Gamma})J\mathbf{A}^j_z(\Gamma)$$

is non-negative and its rank is equal to the rank of $\mathbf{A}^j_z(\Gamma)$, where

$$\mathbf{A}^j_z(\Gamma) = \sum_{r_z \in \Gamma} \mathbf{A}^j_z(r_z).$$

A similar inequality holds with j replaced by a multi-index.

Indeed, since in this case $Q_{\bar{z}}(\bar{\Gamma})JP_z(\Gamma) \geq 0$, we have

$$\mathbf{B}^j_{\bar{z}}(\bar{\Gamma})J\mathbf{A}^j_z(\Gamma) = (\mathbf{A}^j_z(\Gamma))^*[Q_{\bar{z}}(\bar{\Gamma})JP_z(\Gamma)]\mathbf{A}^j_z(\Gamma) \geq 0.$$

7.4. An open question. In an attempt to generalize Theorem 7.1.4 one could attempt to prove that if Γ_1 and Γ_2 are finite sets of resonance up-points such that $\Gamma_1 \subset \Gamma_2$, then

$$Q_{\bar{z}}(\bar{\Gamma}_1)JP_z(\Gamma_1) \leq Q_{\bar{z}}(\bar{\Gamma}_2)JP_z(\Gamma_2);$$

but this is false, as computer experiments demonstrate.

However, we conjecture that if $\Gamma_1 \subset \Gamma_2 \subset \mathbb{C}_+$, then $Q_{\bar{z}}(\bar{\Gamma}_2)JP_z(\Gamma_2)$ spectrally dominates $Q_{\bar{z}}(\bar{\Gamma}_1)JP_z(\Gamma_1)$. Computer experiments support this conjecture.

8. Vectors of type I

In this section we study a subspace of the vector space $\Upsilon_{\lambda\pm i0}(r_\lambda)$ which consists of vectors with an additional property.

8.1. Vanishing property of resonance vectors

PROPOSITION 8.1.1. *Let λ be an essentially regular point, let $\{H_0+rV\colon r\in\mathbb{R}\}$ be a line regular at λ, let r_λ be a real resonance point of the path $\{H_0+rV\colon r\in\mathbb{R}\}$ at λ and let k be a positive integer. If $u_{\lambda\pm i0}(r_\lambda)\in\Upsilon_{\lambda\pm i0}(r_\lambda)$ is a resonance vector of order $k\geq 1$ at $\lambda\pm i0$, then for all non-resonant values of s,*

$$\langle Ju_{\lambda\pm i0}(r_\lambda), \operatorname{Im} T_{\lambda\pm i0}(H_s)Ju_{\lambda\pm i0}(r_\lambda)\rangle = \frac{c_{\pm 2}}{(s-r_\lambda)^2}+\cdots+\frac{c_{\pm k}}{(s-r_\lambda)^k}, \tag{8.1.1}$$

where, in case $k\geq 2$, for $j=2,\dots,k$,

$$\begin{aligned} c_{\pm j} &= \operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}^{j-1}_{\lambda\pm i0}(r_\lambda)u_{\lambda\pm i0}(r_\lambda)\rangle \\ &= -\operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}^{j-1}_{\lambda\mp i0}(r_\lambda)u_{\lambda\pm i0}(r_\lambda)\rangle. \end{aligned} \tag{8.1.2}$$

In particular, if $u_{\lambda\pm i0}(r_\lambda)\in\Upsilon_{\lambda\pm i0}(r_\lambda)$ is a resonance vector of order 1, then

$$\langle Ju_{\lambda\pm i0}(r_\lambda), \operatorname{Im} T_{\lambda\pm i0}(H_s)Ju_{\lambda\pm i0}(r_\lambda)\rangle = 0. \tag{8.1.3}$$

Proof. We give two proofs of (8.1.1), but only in the second proof will the formula (8.1.2) for $c_{\pm j}$ be derived. For brevity we write $u_\pm$ instead of $u_{\lambda\pm i0}(r_\lambda)$. Let

$$f_\pm(s)=\langle Ju_\pm, A_{\lambda\pm i0}(s)u_\pm\rangle = \langle Ju_\pm, T_{\lambda\pm i0}(H_s)Ju_\pm\rangle.$$

By Theorem 3.4.2, the vector $u_\pm$ satisfies (3.4.5) with $z=\lambda\pm i0$. Multiplying both sides of (3.4.5) by $\langle Ju_\pm,\cdot\rangle$, one finds that (recall that $\langle\cdot,\cdot\rangle$ is linear in the second argument)

$$\sum_{j=1}^{k}(s_j-r_\lambda)^{k-1}(\langle Ju_\pm,u_\pm\rangle+(r_\lambda-s_j)f_\pm(s_j))\prod_{i=1,\,i\neq j}^{k}(s_j-s_i)^{-1}=0$$

for all sets $s_1,\dots,s_k$ of distinct real non-resonance points. Taking the imaginary parts of both sides of this equality gives

$$\sum_{j=1}^{k}(s_j-r_\lambda)^k\operatorname{Im} f_\pm(s_j)\prod_{i=1,\,i\neq j}^{k}(s_j-s_i)^{-1}=0.$$

By Lemma 2.3.1, the left hand side is the divided difference of order $k-1$ of the function $h(s)=(s-r_\lambda)^k\operatorname{Im} f_\pm(s)$. It follows from this and Lemma 2.3.2 that $h(s)$ is a polynomial

of degree not greater than $k-2$. Hence, the function

$$\operatorname{Im} f_\pm(s) = \langle Ju_\pm, \operatorname{Im} T_{\lambda\pm i0}(H_s) Ju_\pm\rangle$$

has the form (8.1.1) with some numbers $c_{\pm 2}, \dots, c_{\pm k}$. Here it is assumed that $\operatorname{Im} f_\pm(s)$ is defined by the right hand side of the equality above for real values of s, and then continued analytically to the complex s-plane.

Second proof. We have

$$2i \operatorname{Im} f_\pm(s) = \langle Ju_\pm, T_{\lambda\pm i0}(H_s) Ju_\pm\rangle - \langle T_{\lambda\pm i0}(H_s) Ju_\pm, Ju_\pm\rangle.$$

The Laurent expansion (3.4.6) of $T_{\lambda\pm i0}(H_s)Ju_\pm$ implies that for real values of s the Laurent expansion of the function $\operatorname{Im} f_\pm(s)$ at $s = r_\lambda$ is

$$\begin{aligned}
\operatorname{Im} f_\pm(s) &= \frac{1}{2i}\Big\langle Ju_\pm, \sum_{j=0}^{k-1} \frac{1}{(s-r_\lambda)^{j+1}} \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm \Big\rangle \\
&\quad - \frac{1}{2i}\Big\langle \sum_{j=0}^{k-1} \frac{1}{(s-r_\lambda)^{j+1}} \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm, Ju_\pm \Big\rangle \\
&= \frac{1}{2i} \sum_{j=0}^{k-1} \frac{1}{(s-r_\lambda)^{j+1}} [\langle Ju_\pm, \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm\rangle - \langle \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm, Ju_\pm\rangle] \\
&= \sum_{j=1}^{k-1} \frac{1}{(s-r_\lambda)^{j+1}} \operatorname{Im}\langle Ju_\pm, \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm\rangle.
\end{aligned}$$

Comparing the coefficients of $(s-r_\lambda)^{-j}$ in this Laurent series and in (8.1.1) gives

$$c_{\pm j} = \operatorname{Im}\langle u_\pm, J\mathbf{A}^{j-1}_{\lambda\pm i0}(r_\lambda) u_\pm\rangle.$$

To derive the second formula for $c_{\pm j}$ we note that (3.3.11) and (3.3.12) imply that for all $j = 0, 1, 2, \dots$,

$$\begin{aligned}
\langle u_\pm, J\mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm\rangle &= \langle \mathbf{B}^j_{\lambda\mp i0}(r_\lambda) Ju_\pm, u_\pm\rangle = \langle J\mathbf{A}^j_{\lambda\mp i0}(r_\lambda) u_\pm, u_\pm\rangle \\
&= \overline{\langle u_\pm, J\mathbf{A}^j_{\lambda\mp i0}(r_\lambda) u_\pm\rangle}.
\end{aligned} \tag{8.1.4}$$

Hence, $-\operatorname{Im}\langle u_\pm, J\mathbf{A}^j_{\lambda\mp i0}(r_\lambda) u_\pm\rangle = \operatorname{Im}\langle u_\pm, J\mathbf{A}^j_{\lambda\pm i0}(r_\lambda) u_\pm\rangle = c_{\pm j}$. ∎

Since $\operatorname{Im} T_{\lambda-i0}(H_s) = -\operatorname{Im} T_{\lambda+i0}(H_s)$, it follows from (8.1.1) that if $u \in \Upsilon^k_{\lambda+i0}(r_\lambda)$ or $u \in \Upsilon^k_{\lambda-i0}(r_\lambda)$, then

$$\begin{aligned}
&\langle Ju_{\lambda\pm i0}(r_\lambda), \operatorname{Im} T_{\lambda+i0}(H_s) Ju_{\lambda\pm i0}(r_\lambda)\rangle \\
&\qquad = \sum_{j=2}^{k} \operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}^{j-1}_{\lambda+i0}(r_\lambda) u_{\lambda\pm i0}(r_\lambda)\rangle (s-r_\lambda)^{-j}.
\end{aligned}$$

Remark 8.1.2. Since the left hand side of (8.1.1) is non-negative (for the plus sign) or non-positive (for the minus sign), it follows from (8.1.1) that the largest j for which $c_{\pm j} \neq 0$ must be even, and

$$\operatorname{Im}\langle u_{\lambda\pm i0}(r_\lambda), J\mathbf{A}_{\lambda+i0}(r_\lambda) u_{\lambda\pm i0}(r_\lambda)\rangle \geq 0.$$

8.2. Vectors of type I

DEFINITION 8.2.1. A vector $u \in \Upsilon_{\lambda\pm i0}(r_\lambda)$ will be said to be of *type I* if for any non-resonant $s \in \mathbb{R}$,

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Ju = 0. \tag{8.2.1}$$

The equality (8.2.1) is equivalent to

$$\operatorname{Im} T_{\lambda+i0}(H_s)\, Ju = 0.$$

Since $\operatorname{Im} T_{\lambda+i0}(H_s)J = A_{\lambda+i0}(s) - A_{\lambda-i0}(s)$, this is also equivalent to

$$A_{\lambda+i0}(s)u = A_{\lambda-i0}(s)u. \tag{8.2.2}$$

PROPOSITION 8.2.2. *Every vector of order* 1 *is of type I.*

Proof. This follows from (8.1.3). ∎

LEMMA 8.2.3. *If an element u of one of the two vector spaces $\Upsilon_{\lambda\pm i0}(r_\lambda)$ is a vector of type I, then u is also an element of the other vector space, that is, $u \in \Upsilon_{\lambda\mp i0}(r_\lambda)$, and the orders of u as an element of $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$ are the same.*

Proof. If for instance $u \in \Upsilon^k_{\lambda+i0}(r_\lambda)$, then by (3.4.5) one has

$$\sum_{j=1}^{k}(s_j - r_\lambda)^{k-1}(u + (r_\lambda - s_j)A_{\lambda+i0}(s_j)u) \prod_{i=1,\, i\neq j}^{k}(s_j - s_i)^{-1} = 0,$$

where $s_1, \dots, s_k$ is any set of k distinct real non-resonance points. If u is a vector of type I then (8.2.2) holds, and therefore in the above equality $A_{\lambda+i0}(s_j)u$ can be replaced by $A_{\lambda-i0}(s_j)u$. By Theorem 3.4.2, the resulting equality implies that $u \in \Upsilon^k_{\lambda-i0}(r_\lambda)$. Similarly one shows that if $u \in \Upsilon^k_{\lambda-i0}(r_\lambda)$ is a vector of type I, then $u \in \Upsilon^k_{\lambda+i0}(r_\lambda)$. Hence, the orders of u as an element of $\Upsilon_{\lambda-i0}(r_\lambda)$ and $\Upsilon_{\lambda+i0}(r_\lambda)$ are the same. ∎

Lemma 8.2.3 combined with Proposition 8.2.2 implies

COROLLARY 8.2.4.

$$\Upsilon^1_{\lambda+i0}(r_\lambda) = \Upsilon^1_{\lambda-i0}(r_\lambda).$$

By Lemma 3.1.4, it follows that also

$$\Psi^1_{\lambda+i0}(r_\lambda) = \Psi^1_{\lambda-i0}(r_\lambda). \tag{8.2.3}$$

The vectors of type I form a vector subspace of both $\Upsilon_{\lambda\pm i0}(r_\lambda)$.

If u is a vector of type I then we deduce from (8.2.2) and (3.3.5) that for all $j = 0, 1, 2, \dots,$

$$\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = \mathbf{A}^j_{\lambda-i0}(r_\lambda)u. \tag{8.2.4}$$

On the other hand, if an element u of $\Upsilon_{\lambda-i0}(r_\lambda) \cap \Upsilon_{\lambda+i0}(r_\lambda)$ is such that for all $j = 0, 1, 2, \dots$ the equality (8.2.4) holds then, since by (3.4.12) we have

$$\tilde{A}_{\lambda+i0,r_\lambda}(r_\lambda)u = \tilde{A}_{\lambda+i0,r_\lambda}(r_\lambda)P_{\lambda+i0}(r_\lambda)u = 0$$

and similarly $\tilde{A}_{\lambda-i0,r_\lambda}(r_\lambda)u = 0$, it follows from the Laurent expansion (3.3.16) of $A_z(s)$ that (8.2.2) holds. Thus, the following lemma has been proved.

LEMMA 8.2.5. *An element u of $\Upsilon_{\lambda+i0}(r_\lambda)$ or $\Upsilon_{\lambda-i0}(r_\lambda)$ is a vector of type I if and only if* (8.2.4) *holds for all $j = 0, 1, 2, \dots$.*

COROLLARY 8.2.6. *If u is a vector of type I then so are the vectors $\mathbf{A}^j_{\lambda\pm i0}(r_\lambda)u$ for any $j = 0, 1, 2, \dots$.*

In other words, the vector space of vectors of type I is invariant under $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$.

LEMMA 8.2.7. *An element u of $\Upsilon_{\lambda+i0}(r_\lambda)$ or $\Upsilon_{\lambda-i0}(r_\lambda)$ is a vector of type I if and only if there exists a non-resonant real number r such that for all $j = 0, 1, 2, \dots$,*

$$(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = 0.$$

Proof. (Only if) If u is a vector of type I then by Corollary 8.2.6 for any $j = 0, 1, 2, \dots$ the vectors (8.2.4) are also of type I. Hence, the equality to be proved follows from (8.2.2).

(If) It follows from the premise with $j = 0$ that (8.2.2) holds for one non-resonant real number r. Let s be any other such number. Then by (2.7.11),

$$\begin{aligned}&(A_{\lambda+i0}(s) - A_{\lambda-i0}(s))u\\ &\qquad = (1 + (s-r)A_{\lambda-i0}(r))^{-1}(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))(1 + (s-r)A_{\lambda+i0}(r))^{-1}u.\end{aligned}$$

Using Proposition 3.4.5, we can expand $(1 + (s-r)A_{\lambda+i0}(r))^{-1}u$ as a linear combination of $\mathbf{A}^j_{\lambda+i0}(r_\lambda)u$. Hence, it follows from the premise that $(A_{\lambda+i0}(s) - A_{\lambda-i0}(s))u = 0$ for any non-resonant s. That is, u is a vector of type I. ∎

The vector space of vectors of type I will be denoted by $\Upsilon^{\mathrm{I}}_\lambda(r_\lambda)$. This notation is not ambiguous since, according to Lemma 8.2.3, one can omit the sign in $\Upsilon^{\mathrm{I}}_{\lambda\pm i0}(r_\lambda)$ and write $\Upsilon^{\mathrm{I}}_\lambda(r_\lambda)$. Further, the vector subspaces $\Upsilon^{k,\mathrm{I}}_\lambda(r_\lambda)$ are also correctly defined in the sense that

$$\Upsilon^{\mathrm{I}}_\lambda(r_\lambda) \cap \Upsilon^k_{\lambda+i0}(r_\lambda) = \Upsilon^{\mathrm{I}}_\lambda(r_\lambda) \cap \Upsilon^k_{\lambda-i0}(r_\lambda).$$

We summarize the results of this section in

THEOREM 8.2.8. *Let r_λ be a real resonance point of the line $\gamma = \{H_r : r \in \mathbb{R}\}$, corresponding to a real number $\lambda \in \Lambda(\gamma, F)$. Let $u \in \mathcal{K}$. The following assertions are equivalent:*

(1) *$u \in \Upsilon_{\lambda+i0}(r_\lambda)$ and for all non-resonant real numbers s,*
$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Ju = 0.$$
(2) *$u \in \Upsilon_{\lambda-i0}(r_\lambda)$ and for all non-resonant real numbers s,*
$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Ju = 0.$$
(3) *$u \in \Upsilon_{\lambda+i0}(r_\lambda)$ and for all non-resonant real numbers s,*
$$A_{\lambda+i0}(s)u = A_{\lambda-i0}(s)u.$$
(4) *$u \in \Upsilon_{\lambda-i0}(r_\lambda)$ and for all non-resonant real numbers s,*
$$A_{\lambda+i0}(s)u = A_{\lambda-i0}(s)u.$$
(5) *$u \in \Upsilon_{\lambda+i0}(r_\lambda)$ and for all $j = 0, 1, \dots, d-1$, where d is the order of r_λ,*
$$\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = \mathbf{A}^j_{\lambda-i0}(r_\lambda)u.$$
(6) *$u \in \Upsilon_{\lambda-i0}(r_\lambda)$ and for all $j = 0, 1, \dots, d-1$, where d is the order of r_λ,*
$$\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = \mathbf{A}^j_{\lambda-i0}(r_\lambda)u.$$

(7) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and there exists a non-resonant real number* r *such that for all* $j = 0, 1, 2, \dots,$
$$(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda+i0}(r_\lambda)u = 0.$$
(8) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and there exists a non-resonant real number* r *such that for all* $j = 0, 1, 2, \dots,$
$$(A_{\lambda+i0}(r) - A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda-i0}(r_\lambda)u = 0.$$
(9) $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ *and all the coefficients* c_{+j} *from* (8.1.1) *are zero.*
(10) $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ *and all the coefficients* c_{-j} *from* (8.1.1) *are zero.*

The set $\Upsilon^{\mathrm{I}}_{\lambda+i0}(r_\lambda)$ *of vectors which satisfy any of these equivalent conditions is a vector subspace of* $\Upsilon_{\lambda+i0}(r_\lambda) \cap \Upsilon_{\lambda-i0}(r_\lambda)$, *and* $\Upsilon^{\mathrm{I}}_{\lambda+i0}(r_\lambda)$ *is invariant with respect to both* $\mathbf{A}^j_{\lambda+i0}(r_\lambda)$ *and* $\mathbf{A}^j_{\lambda-i0}(r_\lambda)$.

It is an open question whether
$$\Upsilon^{\mathrm{I}}_{\lambda+i0}(r_\lambda) = \Upsilon_{\lambda+i0}(r_\lambda) \cap \Upsilon_{\lambda-i0}(r_\lambda).$$

8.3. Large depth vectors are of type I

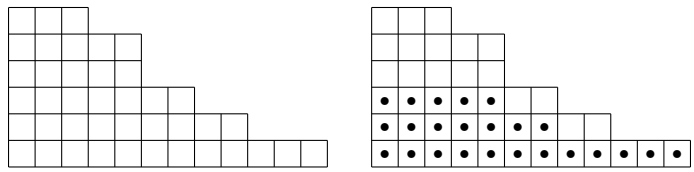
THEOREM 8.3.1. *If a resonance vector* $u^{(k)} \in \Upsilon_{\lambda\pm i0}(r_\lambda)$ *has order* k *then the vectors*
$$u^{(1)}, \dots, u^{(\lceil k/2 \rceil)}$$
are of type I, where $\lceil k/2 \rceil$ *is the smallest integer* $\geq k/2$ *and* $u^{(j)} = \mathbf{A}^{k-j}_{\lambda\pm i0}(r_\lambda)u^{(k)}$.

Proof. We prove the assertion for the plus sign.

We prove that $u^{(n)}$ is of type I for $n = 1, \dots, \lceil k/2 \rceil$, using induction on n. For $n = 1$ this follows from Corollary 8.2.4. Assume that all vectors $u^{(1)}, \dots, u^{(n-1)}$, where $n \leq \lceil k/2 \rceil$, are of type I. We have to prove the claim for $u^{(n)}$. Since $n \leq \lceil k/2 \rceil$, we have $2n - 1 \leq k$, so that
$$u^{(n)} = \mathbf{A}^{n-1}_{\lambda+i0}(r_\lambda)u^{(2n-1)}.$$
For any $j = 1, 2, \dots$ we have
$$\begin{aligned}\langle Ju^{(n)}, \mathbf{A}^j_{\lambda+i0}(r_\lambda)u^{(n)}\rangle &= \langle Ju^{(n)}, u^{(n-j)}\rangle = \langle J\mathbf{A}^{n-1}_{\lambda+i0}(r_\lambda)u^{(2n-1)}, u^{(n-j)}\rangle \\ &= \langle \mathbf{B}^{n-1}_{\lambda+i0}(r_\lambda)Ju^{(2n-1)}, u^{(n-j)}\rangle = \langle Ju^{(2n-1)}, \mathbf{A}^{n-1}_{\lambda-i0}(r_\lambda)u^{(n-j)}\rangle.\end{aligned}$$
By the induction assumption, all the vectors $u^{(n-j)}$, $j = 1, 2, \dots$, are of type I and therefore, according to items (5) and (6) of Theorem 8.2.8, in the expression $\mathbf{A}^{n-1}_{\lambda-i0}(r_\lambda)u^{(n-j)}$ we can replace $\mathbf{A}^{n-1}_{\lambda-i0}(r_\lambda)$ by $\mathbf{A}^{n-1}_{\lambda+i0}(r_\lambda)$; this shows that $\mathbf{A}^{n-1}_{\lambda-i0}(r_\lambda)u^{(n-j)} = 0$. Consequently, Theorem 8.2.8(9) holds for $u^{(n)}$, and therefore $u^{(n)}$ is of type I. ∎

A resonance vector $u \in \Upsilon_z(r_z)$ will be said to have *depth* k if u belongs to the image of the operator $\mathbf{A}^k_z(r_z)$, but not to the image of $\mathbf{A}^{k+1}_z(r_z)$. The depth of u will be denoted by $\gamma_z(u)$, or by $\gamma(u)$ if there is no ambiguity. In other words,
$$\gamma_z(u) = \max\{k \in \mathbb{Z}_+ \colon \exists \phi \in \mathcal{K}\ \mathbf{A}^k_z(r_z)\phi = u\}.$$
We say that a vector $u \in \Upsilon_z(r_z)$ has *property* L if
$$\mathrm{d}(u) \leq \begin{cases}\gamma(u) & \text{if } \mathrm{d}(u) + \gamma(u) \text{ is even,}\\ \gamma(u) + 1 & \text{if } \mathrm{d}(u) + \gamma(u) \text{ is odd.}\end{cases}$$
We denote by $\mathcal{L}_z(r_z)$ the linear span of all vectors u with property L.

For example, if the Young diagram of the operator $\mathbf{A}_z(r_z)$ is as in the left figure below, then one can easily prove that $\mathcal{L}_z(r_z)$ is the linear span of those vectors in the right figure which are marked by bullets.



Theorem 8.3.1 implies that every vector with property L is of type I. Hence, we have

COROLLARY 8.3.2. *The vector space $\mathcal{L}_{\lambda+i0}(r_\lambda)$ spanned by vectors with property L is a subspace of the vector space $\Upsilon^I_\lambda(r_\lambda)$ of vectors of type I:*

$$\mathcal{L}_{\lambda+i0}(r_\lambda) \subset \Upsilon^I_\lambda(r_\lambda).$$

Similarly, one can define the vector space $\mathcal{L}_{\lambda-i0}(r_\lambda)$, which is also a subspace of $\Upsilon^I_\lambda(r_\lambda)$. The vector spaces $\mathcal{L}_{\lambda+i0}(r_\lambda)$ and $\mathcal{L}_{\lambda-i0}(r_\lambda)$ coincide. A proof of this assertion will be given elsewhere. The main part of the proof is the statement that the $\lambda+i0$-depth of any vector of order 1 from $\Upsilon_\lambda(r_\lambda)$ coincides with the $\lambda - i0$-depth of that vector.

If r_λ has order $\mathbf{d} = (d_1, \dots, d_m)$, then the dimension of $\mathcal{L}_{\lambda+i0}(r_\lambda)$ is equal to

$$\lceil d_1/2 \rceil + \cdots + \lceil d_m/2 \rceil.$$

9. Resonance index and the signature of the resonance matrix

In this section we prove one of the main results of this paper: equality of the resonance index and the signature of the resonance matrix.

9.1. Non-degeneracy of $P_{\lambda\pm i0}(r_\lambda)$. The following theorem is one of the key properties of the idempotents $P_{\lambda\pm i0}(r_\lambda)$, and plays an important role in what follows. Another proof of this theorem is given in Remark 11.1.5.

THEOREM 9.1.1. *If $z = \lambda + i0 \in \partial\Pi$ or $z = \lambda - i0 \in \partial\Pi$, and if r_λ is a real resonance point corresponding to z, then the idempotents $P_{\lambda\pm i0}(r_\lambda)$ are linear isomorphisms of the vector spaces $\Upsilon_{\lambda\mp i0}(r_\lambda)$ and $\Upsilon_{\lambda\pm i0}(r_\lambda)$.*

Proof. Since by Lemma 3.1.4 the dimensions of $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$ coincide, it is enough to show that kernels of $P_{\lambda\pm i0}(r_\lambda)\colon \Upsilon_{\lambda\mp i0}(r_\lambda) \to \Upsilon_{\lambda\pm i0}(r_\lambda)$ are zero. Assume the contrary, for example, there exists a non-zero $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ such that

$$P_{\lambda-i0}(r_\lambda)u = 0. \tag{9.1.1}$$

Then it follows from (8.1.4) and (3.3.7) that for all $j = 0, 1, 2, \dots,$

$$\langle u, J\mathbf{A}^j_{\lambda+i0}(r_\lambda)u\rangle = \overline{\langle u, J\mathbf{A}^j_{\lambda-i0}(r_\lambda)u\rangle} = \overline{\langle u, J\mathbf{A}^j_{\lambda-i0}(r_\lambda)P_{\lambda-i0}(r_\lambda)u\rangle} = 0.$$

This combined with (8.1.1) implies that u is of type I. It follows from Lemma 8.2.3 that $u \in \Upsilon_{\lambda-i0}(r_\lambda)$ and therefore $u = P_{\lambda-i0}(r_\lambda)u \neq 0$, which contradicts (9.1.1). ∎

Thus, for any real resonance point r_λ,

$$P_{\lambda\pm i0}(r_\lambda)\Upsilon_{\lambda\mp i0}(r_\lambda) = \Upsilon_{\lambda\pm i0}(r_\lambda).$$

This equality is equivalent to any of the following, which therefore also hold:

$$Q_{\lambda\pm i0}(r_\lambda)\Psi_{\lambda\mp i0}(r_\lambda) = \Psi_{\lambda\pm i0}(r_\lambda), \tag{9.1.2}$$

$$\operatorname{rank}(P_{\lambda\pm i0}(r_\lambda)P_{\lambda\mp i0}(r_\lambda)) = N, \tag{9.1.3}$$

$$\operatorname{rank}(Q_{\lambda\mp i0}(r_\lambda)Q_{\lambda\pm i0}(r_\lambda)) = N, \tag{9.1.4}$$

where $N = \operatorname{rank}(P_{\lambda\pm i0}(r_\lambda)) = \operatorname{rank}(Q_{\lambda\pm i0}(r_\lambda))$.

Lemma 3.1.4 and Theorem 9.1.1 imply

PROPOSITION 9.1.2. *If $z = \lambda \pm i0 \in \partial\Pi$ and if r_λ is a real resonance point corresponding to z, then*

$$\operatorname{rank} Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda) = N,$$

where N is the rank of (any of) the operators $P_{\lambda\pm i0}(r_\lambda)$ and $Q_{\lambda\pm i0}(r_\lambda)$.

Note that Theorem 9.1.1 is similar to Corollary 7.3.2, but with an essential difference: while in Theorem 9.1.1 the number z belongs to the boundary of Π, in Corollary 7.3.2 it does not. At the same time, the methods of proof of these two assertions are completely different.

LEMMA 9.1.3. *If r_λ is a real resonance point then for all small enough $y > 0$,*

$$\operatorname{rank} Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = \operatorname{rank} Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda),$$
$$\operatorname{sign} Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = \operatorname{sign} Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda).$$

Proof. Sufficiently small (in norm) perturbations cannot decrease the rank of the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. Since the rank of $P_{\lambda+i0}(r_\lambda)$ is stable under small enough perturbations, Proposition 9.1.2 shows that the rank of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ cannot increase either. Thus, the first equality follows.

The second follows from the first and from continuity considerations, since in order to change the signature of $Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)$ some non-zero eigenvalue of this operator has to be deformed to zero, which would violate the constancy of the rank. ∎

9.2. Signature of the resonance matrix and resonance index. The following theorem is one of the main results of this paper.

THEOREM 9.2.1. *For any real resonance point r_λ the signatures* $\operatorname{sign}(Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda))$ *of the resonance matrices of r_λ are the same and are equal to the resonance index of the triple $(\lambda; H_{r_\lambda}, V)$; that is,*

$$\operatorname{sign}(Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)) = \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V).$$

Proof. By Lemma 9.1.3 for small enough $y > 0$ we have

$$\operatorname{sign}(Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)) = \operatorname{sign}(Q_{\lambda\mp iy}(r_\lambda)JP_{\lambda\pm iy}(r_\lambda)).$$

Hence, the claim follows from Theorem 7.2.1, (5.3.2) and (5.3.4). ∎

Theorem 9.2.1 implies the following corollary. Nonetheless, we give another proof.

COROLLARY 9.2.2. *For any real resonance point r_λ, the signatures of the finite-rank self-adjoint operators $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ and $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$ coincide.*

Proof. For any $y > 0$ and any real s, by Lemma 9.2.3 below,

$$(E) := \operatorname{sign}(Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)) = \mathcal{R}(T_{\lambda+iy}(H_s)Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)).$$

By stability of the R-index (Lemma 5.1.2(iv)), for small enough $y' > 0$ we get

$$(E) = \mathcal{R}(T_{\lambda+iy}(H_s)Q_{\lambda-iy'}(r_\lambda)JP_{\lambda+iy'}(r_\lambda)).$$

Since this R-index does not depend on $y > 0$, the number y in the above equality can be replaced by y'. Combining this with Lemma 5.1.2(i) and (3.2.11) and (3.2.10), we obtain

$$\begin{aligned}(E) &= \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda-iy'}(r_\lambda)JP_{\lambda+iy'}(r_\lambda)) = \mathcal{R}(P_{\lambda+iy'}(r_\lambda)T_{\lambda+iy'}(H_s)Q_{\lambda-iy'}(r_\lambda)J)\\ &= \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda+iy'}(r_\lambda)Q_{\lambda-iy'}(r_\lambda)J) = \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda+iy'}(r_\lambda)JP_{\lambda-iy'}(r_\lambda)).\end{aligned}$$

For small enough y' we also have

$$\begin{aligned}\operatorname{sign}(Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)) &= \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda))\\ &= \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda+iy'}(r_\lambda)JP_{\lambda-iy'}(r_\lambda)),\end{aligned}$$

so the proof is complete. ∎

LEMMA 9.2.3. *Let $s \in \mathbb{R}$. If r_λ is a real resonance point, then there exists $\varepsilon > 0$ such that for all $y' > 0$ and for all $y \in [0, \varepsilon)$ the operator*

$$T_{\lambda\pm iy'}(H_s)Q_{\lambda\mp iy}(r_\lambda)JP_{\lambda\pm iy}(r_\lambda)$$

belongs to the class $\mathcal{R}$, and

$$\operatorname{sign}(Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)) = \mathcal{R}(T_{\lambda+iy'}(H_s)Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)).$$

Proof. By Lemma 5.1.4, for all small enough $y > 0$ we have

$$\underline{Q}_{\lambda-iy}(r_\lambda)V\underline{P}_{\lambda+iy}(r_\lambda) = F^*Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)F.$$

Combining this with Lemmas 3.1.7 and 2.1.3 we infer that for all small enough $y > 0$,

$$\operatorname{sign}(Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)) = \operatorname{sign}(\underline{Q}_{\lambda-iy}(r_\lambda)V\underline{P}_{\lambda+iy}(r_\lambda)).$$

By Kreĭn's Theorem 5.1.6, for any real number s and for all $y' > 0$ we have

$$\operatorname{sign}(\underline{Q}_{\lambda-iy}(r_\lambda)V\underline{P}_{\lambda+iy}(r_\lambda)) = \mathcal{R}(R_{\lambda+iy'}(H_s)\underline{Q}_{\lambda-iy}(r_\lambda)V\underline{P}_{\lambda+iy}(r_\lambda)).$$

Putting together the last three equalities, we get

$$\operatorname{sign}(Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)) = \mathcal{R}(R_{\lambda+iy'}(H_s)F^*Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)F).$$

The pair of operators $R_{\lambda+iy'}(H_s)F^*Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)$ and F satisfy the premise of Lemma 5.1.3. Hence, by that lemma and the above equality,

$$\operatorname{sign}(Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)) = \mathcal{R}(FR_{\lambda+iy'}(H_s)F^*Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)).$$

Finally, Lemma 9.1.3 completes the proof. ∎

In the following theorem we gather different descriptions of the resonance index.

THEOREM 9.2.4. *Let r_λ be a real resonance point. The following numbers are all equal:*

(1) *The resonance index* $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$.
(2) *The signatures of the operators $Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)$.*
(3) *The R-index of the operator $T_{\lambda+iy}(H_s)Q_{\lambda-iy}(r_\lambda)JP_{\lambda+iy}(r_\lambda)$ for all s and for all small enough $y > 0$.*
(4) *The R-index of the operator $-T_{\lambda-iy}(H_s)Q_{\lambda+iy}(r_\lambda)JP_{\lambda-iy}(r_\lambda)$ for all s and for all small enough $y > 0$.*
(5) *The R-index of the operator $A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)$ for all s and for all small enough $y > 0$.*
(6) *The R-index of the operator $-A_{\lambda-iy}(s)P_{\lambda-iy}(r_\lambda)$ for all s and for all small enough $y > 0$.*

Proof. Equality of (1) and (2) is the statement of Theorem 9.2.1. Equality of the second and third and fourth numbers follows from Lemma 9.2.3. The equalities (1) = (5) and (1) = (6) follow from (5.3.2) and (5.3.4) respectively. ∎

10. U-turn theorem

According to Lemma 3.1.4, the four vector spaces $\Upsilon^1_{\lambda\pm i0}(r_\lambda)$ and $\Psi^1_{\lambda\pm i0}(r_\lambda)$ have the same dimension. It was noted in §4.4 that the dimension of the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$ can be interpreted as multiplicity of a point λ of the singular spectrum of a λ-resonant operator H_{r_λ}. Theorem 10.1.6 and Corollary 10.2.1, proved in this section, provide another rationale for this interpretation. Given this definition of the multiplicity of the singular spectrum, how should one interpret the case when, for example, the dimension of $\Upsilon^1_{\lambda\pm i0}(r_\lambda)$ is 1, while the dimension N of $\Upsilon_{\lambda\pm i0}(r_\lambda)$ is 2? Since $N = 2$, there are two resonance points in the group of r_λ for small y. It is reasonable to suggest that these two poles should not belong to the same half-plane $\mathbb{C}_\pm$, since this would mean that the resonance index (= jump of spectral flow) is equal to two, while the multiplicity of the point λ of the singular spectrum is one. That is, in this case we expect one up-pole and one down-pole, resulting in zero resonance index. Outside of the essential spectrum, this scenario has an obvious geometric interpretation: a point of the singular spectrum (that is, an eigenvalue) reaches λ, but instead of crossing λ it turns back. Thus, existence of vectors of order two or more should be interpreted as an indicator of the fact that some points of the singular spectrum make a "U-turn" at λ. More generally, if $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) = m$, then it is natural to suggest that the jump of spectral flow at $r = r_\lambda$ should not be greater than m, since there are only m "eigenvalues" which can cross the point λ as r crosses r_λ in the positive direction.

In other words, one may expect that

$$|N_+ - N_-| \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

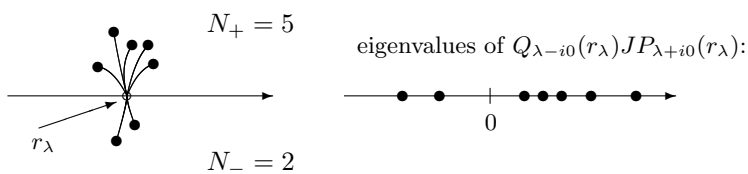
This inequality (the U-turn theorem) turns out to be true for all real resonance points r_λ, and is the main result of this section.

The U-turn theorem is non-trivial even for points λ which do not belong to the essential spectrum. For instance, a resonance with $N_+ = 5$ up-points and $N_- = 2$ down-points, depicted below, may correspond to either of the following eight possible scenarios:

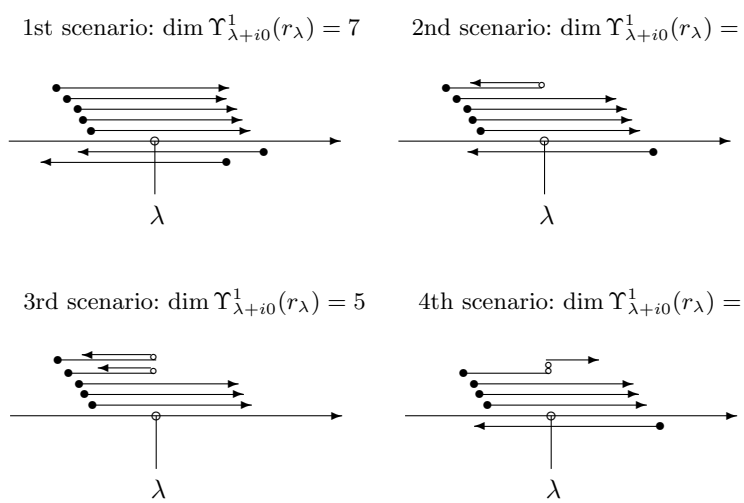
1. As r crosses a real resonance point r_λ in the positive direction, five eigenvalues of H_r cross λ in the positive direction and two eigenvalues cross λ in the negative direction. Each of the former five eigenvalues creates one up-point, and each of the latter two creates one down-point.
2. Four eigenvalues cross λ in the positive direction, one eigenvalue crosses λ in the negative direction, and one eigenvalue makes a U-turn at λ. The latter eigenvalue creates one up-point and one down-point.

3. Three eigenvalues cross λ in the positive direction and two eigenvalues make a U-turn at λ. Each of the latter two creates one up-point and one down-point.
4. Three eigenvalues cross λ in the positive direction, one eigenvalue crosses λ in the negative direction and one eigenvalue makes a double U-turn at λ. This last eigenvalue creates two up-points and one down-point.
5. Three eigenvalues cross λ in the positive direction and one eigenvalue makes a triple U-turn at λ, which results in the appearance of two up-points and two down-points.
6. One eigenvalue crosses λ in the positive direction and two eigenvalues make a double U-turn at λ.
7. Two eigenvalues cross λ in the positive direction and one eigenvalue makes a quadruple U-turn at λ. An eigenvalue making a quadruple U-turn creates three up-points and two down-points.
8. Four eigenvalues cross λ in the positive direction and one eigenvalue makes a triple U-turn at λ and crosses it in the negative direction. This last eigenvalue creates one up-point and two down-points.

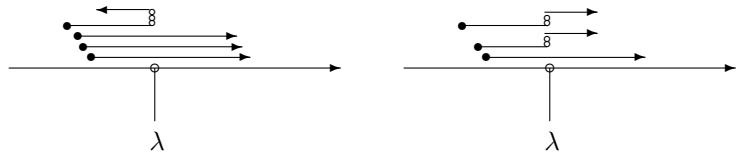
In these eight possible scenarios the dimension of the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$ is equal to, respectively, 7, 6, 5, 5, 4, 3, 3 and 5.



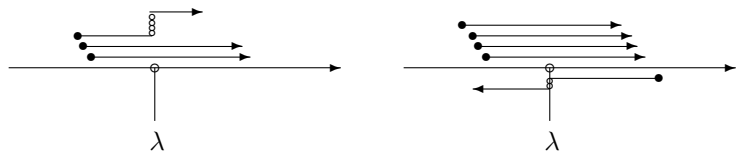
A typical motion of the eigenvalues of the operator H_r as r passes through r_λ in each of these eight possible scenarios is given below:



5th scenario: $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) = 4$ 6th scenario: $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) = 3$

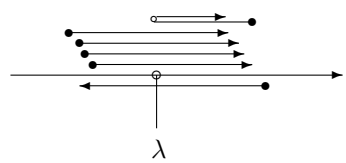


7th scenario: $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) = 3$ 8th scenario: $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) = 5$



For values of λ outside the essential spectrum these scenarios make rigorous sense, since in this case λ depends on r analytically. The U-turn theorem allows us to extrapolate this behaviour of isolated eigenvalues to points of singular spectrum inside essential spectrum.

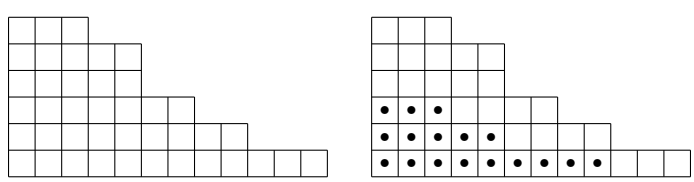
One has to note that for the resonance index $N_+ - N_-$ it does not matter which side an eigenvalue making a U-turn approaches the point λ from; in both cases the eigenvalue increases the number N_+ of up-points and the number N_- of down-points by 1. Taking this into account, we do not distinguish for example the second scenario above from the following possibility:



10.1. Proof of the U-turn theorem. Let $z \in \Pi$, let r_z be a resonance point corresponding to z, let $u \in \Upsilon_z(r_z)$ be a resonance vector and let k be a non-negative integer. We denote by $\mathcal{L}^w_z(r_z)$ the linear span of all vectors u from $\Upsilon_z(r_z)$ such that

$$\gamma(u) \geq \mathrm{d}(u). \tag{10.1.1}$$

As an example, if the Young diagram of $\mathbf{A}_z(r_z)$ is as shown in the left figure below, then the vector space $\mathcal{L}^w_z(r_z)$ is the linear span of the vectors marked by circles in the right figure.



The following lemma is trivial.

LEMMA 10.1.1. *For any $z \in \Pi$ and any resonance point r_z corresponding to z,*

$$2 \dim \mathcal{L}^w_z(r_z) + \dim \Upsilon^1_z(r_z) \geq \dim \Upsilon_z(r_z).$$

PROPOSITION 10.1.2. *If $z = \lambda \pm i0 \in \partial\Pi$ and r_λ is a real resonance point corresponding to $\lambda \pm i0$, then for any $u_1, u_2 \in \mathcal{L}^w_z(r_\lambda)$,*

$$\langle u_1, Ju_2 \rangle = 0.$$

Proof. Assume that $z = \lambda + i0$. By linearity, it is sufficient to prove the claim for the vectors u_1 and u_2 from $\mathcal{L}^w_{\lambda+i0}(r_\lambda)$ which satisfy (10.1.1). By Theorem 8.3.1, these vectors are of type I; in particular, their $\mathbf{A}_{\lambda+i0}(r_\lambda)$ and $\mathbf{A}_{\lambda-i0}(r_\lambda)$ orders are equal:

$$\mathrm{d}_+(u_1) = \mathrm{d}_-(u_1) \quad \text{and} \quad \mathrm{d}_+(u_2) = \mathrm{d}_-(u_2). \tag{10.1.2}$$

Let $k = \gamma_+(u_1)$ and $j = \gamma_+(u_2)$, and assume without loss of generality that $k \geq j$. By definition of depth, $u_1 = \mathbf{A}^k_{\lambda+i0}(r_\lambda)\phi$ for some ϕ. Since $k \geq j \geq \mathrm{d}(u_2)$, we have $\mathbf{A}^k_{\lambda+i0}(r_\lambda)u_2 = 0$. By (10.1.2), this implies that $\mathbf{A}^k_{\lambda-i0}(r_\lambda)u_2 = 0$. It now follows from (3.3.11) and (3.3.12) that

$$\langle u_1, Ju_2 \rangle = \langle \mathbf{A}^k_{\lambda+i0}(r_\lambda)\phi, Ju_2 \rangle = \langle \phi, \mathbf{B}^k_{\lambda-i0}(r_\lambda)Ju_2 \rangle = \langle \phi, J\mathbf{A}^k_{\lambda-i0}(r_\lambda)u_2 \rangle = 0. \ \blacksquare$$

PROPOSITION 10.1.3. *If $z = \lambda \pm i0 \in \partial\Pi$ and if the perturbation J is non-negative (or non-positive), then every real resonance point has order* 1.

Proof. Assume the contrary: a real resonance point r_λ has order greater than 1. In this case there exists a vector $\phi \in \Upsilon^2_{\lambda+i0}(r_\lambda)$ of order 2. Hence, by Theorem 3.4.3, the vector $u = \mathbf{A}_{\lambda+i0}(r_\lambda)\phi$ is of order 1 (and therefore is non-zero) and has depth ≥ 1. Then

$$\langle u, Ju \rangle = \langle \mathbf{A}_{\lambda+i0}(r_\lambda)\phi, Ju \rangle = \langle \phi, \mathbf{B}_{\lambda-i0}(r_\lambda)Ju \rangle = \langle \phi, J\mathbf{A}_{\lambda-i0}(r_\lambda)u \rangle = 0, \tag{10.1.3}$$

where the last equality follows from Corollary 8.2.4. Since $J \geq 0$ (or $J \leq 0$), we deduce that $Ju = 0$. But this contradicts Lemma 3.1.4. $\blacksquare$

Even if the operator J is not sign-definite, the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ may be sign-definite for some resonance points r_λ. If this is the case, one may ask whether the conclusion of Proposition 10.1.3 still holds. In fact, the same argument shows that if the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is non-negative, then r_λ is of type I.

PROPOSITION 10.1.4. *Let $z = \lambda \pm i0 \in \partial\Pi$ and let r_λ be a real resonance point corresponding to z. If the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ or $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$ is non-negative or non-positive, then r_λ has order* 1.

Proof. Let for instance $z = \lambda + i0$ and assume the contrary: r_λ has order greater than 1. Then there exists a vector u of order 1 and of depth at least 1. Since u has order 1, by Corollary 8.2.4 we have

$$P_{\lambda+i0}(r_\lambda)u = P_{\lambda-i0}(r_\lambda)u = u.$$

Further,

$$\langle u, Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)u \rangle = \langle P_{\lambda+i0}(r_\lambda)u, JP_{\lambda+i0}(r_\lambda)u \rangle = \langle u, Ju \rangle.$$

From the last two formulas, using (10.1.3), one can infer that $\langle u, Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)u \rangle = 0$. Since $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is non-negative (or non-positive), this implies that

$Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)u = 0$. On the other hand, by Proposition 9.1.2, for all real resonance points, the restriction of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ to $\Upsilon_{\lambda+i0}(r_\lambda)$ has zero kernel. This gives a contradiction. ∎

The following theorem and its corollary, Theorem 10.1.6, are among the main results of this paper.

THEOREM 10.1.5. *If r_λ is a real resonance point corresponding to $z = \lambda \pm i0$, then the absolute value of the signature of the resonance matrices $Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)$ is not greater than the dimension of the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$:*

$$|\operatorname{sign} Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)| \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

Proof. We prove this for the operator $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. Let μ_+, respectively μ_-, be the rank of the positive, respectively negative, part of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. Assume, contrary to the claim, that

$$|\mu_+ - \mu_-| > m,$$

where $m = \dim \Upsilon^1_{\lambda+i0}(r_\lambda)$. By Proposition 9.1.2,

$$\mu_+ + \mu_- = N = \dim \Upsilon_{\lambda+i0}(r_\lambda).$$

This equality combined with the previous inequality imply that either μ_+ or μ_- is less than $(N-m)/2$. Since by Lemma 10.1.1,

$$(N-m)/2 \le \dim \mathcal{L}^w_{\lambda+i0}(r_\lambda),$$

we conclude that either μ_+ or μ_- is less than

$$\dim \mathcal{L}^w_{\lambda+i0}(r_\lambda) =: p.$$

Without loss of generality it can be assumed that $\mu_+ < p$. Let $u_1, \dots, u_p$ be a basis of $\mathcal{L}^w_{\lambda+i0}(r_\lambda)$. Since $\mu_+ < p$, there exists a non-zero linear combination

$$u = \alpha_1 u_1 + \cdots + \alpha_p u_p \in \mathcal{L}^w_{\lambda+i0}(r_\lambda)$$

such that the positive part of u with respect to $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is zero. Since $u_1, \dots, u_p \in \mathcal{L}^w_{\lambda+i0}(r_\lambda)$, it follows from Proposition 10.1.2 that

$$\langle u, Ju\rangle = \sum_{i=1}^{p}\sum_{j=1}^{p} \bar\alpha_i \alpha_j \langle u_i, Ju_j\rangle = 0. \tag{10.1.4}$$

Since, by Proposition 9.1.2, the restriction of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ to $\Upsilon_{\lambda+i0}$ has zero kernel and since the positive part of u with respect to $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is zero, the negative part of u with respect to $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is non-zero. Hence,

$$\langle u, Ju\rangle = \langle P_{\lambda+i0}(r_\lambda)u, JP_{\lambda+i0}(r_\lambda)u\rangle = \langle u, Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)u\rangle < 0.$$

This contradicts (10.1.4). ∎

The following result is immediate from Theorems 10.1.5 and 9.2.1.

THEOREM 10.1.6 (U-turn theorem). *For any real resonance point r_λ,*

$$|\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)| \le \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

10.2. Some corollaries of the U-turn theorem

COROLLARY 10.2.1. *If the perturbation V is non-negative or non-positive, then* $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \dim \Upsilon^1_{\lambda+i0}(r_\lambda)$.

Proof. By Theorem 9.2.1, $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ is equal to the signature of the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. By Proposition 9.1.2, the dimension N of $\Upsilon_{\lambda+i0}(r_\lambda)$ is equal to the rank of the resonance matrix. Since the resonance matrix is also non-negative or non-positive, it follows that its signature is N or $-N$. Finally, since V is non-negative or non-positive, by Proposition 10.1.3 the vector space $\Upsilon_{\lambda+i0}(r_\lambda)$ coincides with $\Upsilon^1_{\lambda+i0}(r_\lambda)$, and therefore

$$|\operatorname{sign} Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)| = N = \dim \Upsilon_{\lambda+i0}(r_\lambda) = \dim \Upsilon^1_{\lambda+i0}(r_\lambda). \ \blacksquare$$

Theorem 10.1.5 and Proposition 9.1.2 imply

COROLLARY 10.2.2. *Let $z = \lambda \pm i0 \in \partial\Pi$. Assume that a real resonance point r_λ corresponding to z has geometric multiplicity $m = 1$. If the order of r_λ is even, then the signature of the resonance matrix $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is zero; if the order of r_λ is odd, then this signature is $+1$ or -1.*

COROLLARY 10.2.3. *Let r_λ be a real resonance point. If one of the numbers N_+ or N_- from the definition* (5.3.3) *of the resonance index is zero, then r_λ has order* 1.

Proof. By Theorem 9.2.1, the resonance index $N_+ - N_-$ of r_λ is equal to the signature of the self-adjoint operator $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. If N_+ or N_- is zero, this signature is N or $-N$, where N is the rank of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$. Hence, $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is either non-positive or non-negative. Therefore, Proposition 10.1.4 implies that r_λ has order 1. $\blacksquare$

11. Order-preserving property of $P_{\lambda\pm i0}(r_\lambda)\colon \Upsilon_{\lambda\mp i0}(r_\lambda) \to \Upsilon_{\lambda\pm i0}(r_\lambda)$

The main result of this section is Theorem 11.2.7, which asserts in particular that if the geometric multiplicity of a real resonance point is equal to 1, then the mappings

$$P_{\lambda\pm i0}(r_\lambda)\colon \Upsilon_{\lambda\mp i0}(r_\lambda) \to \Upsilon_{\lambda\pm i0}(r_\lambda)$$

preserve the order of resonance vectors. Along the way we prove some properties of the operators $P_{\lambda\pm i0}(r_\lambda)$ and $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$ which seem to be interesting on their own.

11.1. Some properties of $P_{\lambda\pm i0}(r_\lambda)$ and $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$

PROPOSITION 11.1.1. *For any non-resonance point $r \in \mathbb{R}$ and any real resonance point r_λ,*

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, J P_{\lambda\pm i0}(r_\lambda) \sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)} = 0, \tag{11.1.1}$$

and for all $j = 1, 2, \dots,$

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, J \mathbf{A}^j_{\lambda\pm i0}(r_\lambda) \sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)} = 0. \tag{11.1.2}$$

Proof. We prove these equalities for the plus sign. The equalities for the minus sign can then be derived by taking adjoint and using (3.2.5), (3.2.10), (3.3.11), (3.3.12).

It is well-known (see e.g. [Pu$_2$]) that the operator

$$\begin{aligned}&\tilde S(\lambda + i0; H_s, H_r)\\ &\quad = 1 - 2i\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\,(s-r) J (1 + (s-r) T_{\lambda+i0}(H_r) J)^{-1} \sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\end{aligned} \tag{11.1.3}$$

is unitary for real non-resonant r and s; the proof is a direct calculation. Since the right hand side of (11.1.3) makes sense for complex values of s, we will treat the operator $\tilde S(\lambda + i0; H_s, H_r)$ as a function of the complex variable s. By the analytic Fredholm alternative (Theorem 2.2.1) the operator-function $\tilde S(\lambda + i0; H_s, H_r)$ is a meromorphic function of s. Since this function is also unitary and therefore is bounded for real s, it cannot have poles on the real axis $\mathbb{R}$. Hence, $\tilde S(\lambda + i0; H_s, H_r)$ as a function of s is holomorphic in a neighbourhood of $\mathbb{R}$. Making the change of variables $\sigma = (r-s)^{-1}$ one infers that the function of σ given by

$$\tilde S(\lambda + i0; H_{s(\sigma)}, H_r) = 1 + 2i\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, J (\sigma - T_{\lambda+i0}(H_r) J)^{-1} \sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}$$

is holomorphic in a neighbourhood of $\mathbb{R}$. Hence, the residue of this function at

$$\sigma_0 := (r - r_\lambda)^{-1}$$

is equal to zero. By the definition (3.2.1) of the idempotent $P_z(r_z)$, this residue is equal (up to a non-zero constant) to the left hand side of (11.1.1), which is therefore zero too. This completes proof of (11.1.1).

Further, since the function $\tilde{S}(\lambda+i0; H_{s(\sigma)}, H_0)$ of σ is holomorphic in a neighbourhood of $\mathbb{R}$, all the other terms $(\sigma-\sigma_0)^{-j}$ with negative powers in the Laurent expansion of $\tilde{S}(\lambda+i0; H_{s(\sigma)}, H_0)$ at $\sigma=\sigma_0$ also vanish. Combining this with (3.3.19) implies

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, J\mathbf{A}^{d-1}_{\lambda+i0}(r_\lambda)\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)} = 0.$$

Further, using this equality and (3.3.20) with $k=d-2$, we infer that

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, J\mathbf{A}^{d-2}_{\lambda+i0}(r_\lambda)\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)} = 0.$$

Continuing in this way gives (11.1.2) for all $j=d-1, d-2, \dots, 1$. ∎

Proposition 11.1.1 implies that for all $j=0,1,2,\dots$ and all s,

$$(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))\mathbf{A}^j_{\lambda+i0}(r_\lambda)(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))=0. \tag{11.1.4}$$

This equality itself is not useful but its modification which follows is.

Lemma 11.1.2. *For any non-resonant real numbers r and s and all $j=0,1,2,\dots$,*

$$(A_{\lambda+i0}(r)-A_{\lambda-i0}(r))\mathbf{A}^j_{\lambda+i0}(r_\lambda)(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))=0. \tag{11.1.5}$$

Proof. Using (2.7.11) we have

$$\begin{aligned}
&A_{\lambda+i0}(r)-A_{\lambda-i0}(r)\\
&\quad= 2i\operatorname{Im} T_{\lambda+i0}(r)J\\
&\quad= 2i(1+(r-s)A_{\lambda-i0}(s))^{-1}\operatorname{Im} T_{\lambda+i0}(H_s)(1+(r-s)B_{\lambda+i0}(s))^{-1}J\\
&\quad= 2i(1+(r-s)A_{\lambda-i0}(s))^{-1}\operatorname{Im} T_{\lambda+i0}(H_s)J(1+(r-s)A_{\lambda+i0}(s))^{-1}\\
&\quad= (1+(r-s)A_{\lambda-i0}(s))^{-1}(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))(1+(r-s)A_{\lambda+i0}(s))^{-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&[A_{\lambda+i0}(r)-A_{\lambda-i0}(r)]P_{\lambda+i0}(r_\lambda)\\
&\qquad= (1+(r-s)A_{\lambda-i0}(s))^{-1}(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))(1+(r-s)A_{\lambda+i0}(s))^{-1}P_{\lambda+i0}(r_\lambda).
\end{aligned}$$

Expanding the factor $(1+(r-s)A_{\lambda+i0}(s))^{-1}P_{\lambda+i0}(r_\lambda)$ by (3.4.13) and multiplying both sides on the right by $\mathbf{A}^j_{\lambda+i0}(r_\lambda)(A_{\lambda+i0}(s)-A_{\lambda-i0}(s))$, one can see from (11.1.4) that the left hand side of (11.1.5) is zero. ∎

The left hand side of (11.1.5) is a meromorphic function of the two variables r and s. Using (3.3.16), one can expand this function in a Laurent series at $r=r_\lambda$, $s=r_\lambda$. Since the function is zero, all coefficients of the terms $(r-r_\lambda)^k(s-r_\lambda)^l$, $k,l=0,\pm1,\pm2,\dots$, in the Laurent expansion are also zero. This gives some relations between the operators $\tilde{A}_{\lambda\pm i0, r_\lambda}(r_\lambda)$, $P_{\lambda\pm i0}(r_\lambda)$ and $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$, such as

$$(\mathbf{A}^k_{\lambda+i0}(r_\lambda)-\mathbf{A}^k_{\lambda-i0}(r_\lambda))\mathbf{A}^j_{\lambda+i0}(r_\lambda)(\mathbf{A}^l_{\lambda+i0}(r_\lambda)-\mathbf{A}^l_{\lambda-i0}(r_\lambda))=0. \tag{11.1.6}$$

The one which will be used shortly is obtained by setting to zero the coefficient of $(r-r_\lambda)^{-1}(s-r_\lambda)^{-1}$ from the Laurent expansion of the left hand side of (11.1.5). Taking $j=0$ in the resulting relation gives the following equality.

Lemma 11.1.3. *For any real resonance point r_λ,*

$$(P_{\lambda+i0}(r_\lambda)-P_{\lambda-i0}(r_\lambda))P_{\lambda+i0}(r_\lambda)(P_{\lambda+i0}(r_\lambda)-P_{\lambda-i0}(r_\lambda))=0. \tag{11.1.7}$$

THEOREM 11.1.4. *For any real resonance point r_λ the spectrum of the product $P_{\lambda+i0}(r_\lambda)$ $P_{\lambda-i0}(r_\lambda)$ consists of only 0 and 1. Moreover, the algebraic multiplicity of 1 is equal to $N = \dim \Upsilon_{\lambda+i0}(r_\lambda)$.*

Proof. For brevity, we write P_+ instead of $P_{\lambda+i0}(r_\lambda)$ and P_- instead of $P_{\lambda-i0}(r_\lambda)$. Expanding (11.1.7) we obtain

$$P_+ - P_-P_+ - P_+P_- + P_-P_+P_- = 0. \tag{11.1.8}$$

Taking traces of both sides and using $\operatorname{Tr}(P_+P_-) = \operatorname{Tr}(P_-P_+P_-)$ gives

$$\operatorname{Tr}(P_-P_+) = \operatorname{Tr}(P_+) = N. \tag{11.1.9}$$

Multiplying both sides of (11.1.8) by P_+ on the right leads to

$$P_+ - P_-P_+ - P_+P_-P_+ + P_-P_+P_-P_+ = 0. \tag{11.1.10}$$

Taking traces and using (11.1.9) one gets

$$\operatorname{Tr}(P_-P_+P_-P_+) = N.$$

Multiplying (11.1.10) on the right by P_-P_+ and taking traces we get

$$\operatorname{Tr}((P_-P_+)^3) = N.$$

Continuing in this manner, it can be shown that for any $k = 1, 2, \dots,$

$$\operatorname{Tr}((P_-P_+)^k) = N. \tag{11.1.11}$$

Since P_-P_+ has rank $\le N$ (in fact this rank is N by Theorem 9.1.1, but we do not need this), if $x_1, \dots, x_N$ is the list of all non-zero eigenvalues of P_-P_+ counting multiplicities, then it follows from the spectral mapping theorem, the Lidskiĭ theorem (2.1.6) and (11.1.11) that for all $k = 1, 2, \dots,$

$$x_1^k + \cdots + x_N^k = N.$$

This is only possible if all the N numbers $x_1, \dots, x_N$ are equal to 1. ∎

REMARK 11.1.5. Theorem 11.1.4 implies that the ranks of $P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda)$ are the same as that of $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$, which gives another proof of Theorem 9.1.1.

11.2. Proof of the order-preserving property of $P_{\lambda\pm i0}(r_\lambda)$ for property C points

DEFINITION 11.2.1. We say that a real resonance point r_λ of geometric multiplicity m has *property C* if the vector spaces $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$ admit Jordan decompositions (see p. 34 for definition)

$$\Upsilon_{\lambda+i0}(r_\lambda) = \Upsilon^{[1]}_{\lambda+i0}(r_\lambda) \dotplus \cdots \dotplus \Upsilon^{[m]}_{\lambda+i0}(r_\lambda), \tag{11.2.1}$$

$$\Upsilon_{\lambda-i0}(r_\lambda) = \Upsilon^{[1]}_{\lambda-i0}(r_\lambda) \dotplus \cdots \dotplus \Upsilon^{[m]}_{\lambda-i0}(r_\lambda) \tag{11.2.2}$$

such that for all $j = 1, \dots, m$,

$$P_{\lambda+i0}(r_\lambda)\Upsilon^{[\nu]}_{\lambda-i0}(r_\lambda) = \Upsilon^{[\nu]}_{\lambda+i0}(r_\lambda) \quad \text{and} \quad P_{\lambda-i0}(r_\lambda)\Upsilon^{[\nu]}_{\lambda+i0}(r_\lambda) = \Upsilon^{[\nu]}_{\lambda-i0}(r_\lambda). \tag{11.2.3}$$

The goal of this subsection is to prove Theorem 11.2.7. The proof starts with the following lemma.

LEMMA 11.2.2. *Let r_λ be a real resonance point with property C and let j, k, l be non-negative integers. If the operator*

$$\mathbf{A}^k_{\lambda\pm i0}(r_\lambda)\mathbf{A}^j_{\lambda\mp i0}(r_\lambda)\mathbf{A}^l_{\lambda\pm i0}(r_\lambda)$$

sends all vectors from $\Upsilon_{\lambda\pm i0}(r_\lambda)$ to vectors of type I and if it sends all vectors of type I to zero, then it decreases the order of vectors from $\Upsilon_{\lambda\pm i0}(r_\lambda)$.

Proof. We prove this only for the upper signs.

Since r_λ has property C, the vector spaces $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$ admit the decompositions (11.2.1) and (11.2.2) into direct sums of vector spaces $\Upsilon^{[\nu]}_{\lambda\pm i0}(r_\lambda)$ such that for any $k \geq 0$,

$$\mathbf{A}^k_{\lambda\pm i0}(r_\lambda)\Upsilon^{[\nu]}_{\lambda\pm i0}(r_\lambda) \subset \Upsilon^{[\nu]}_{\lambda\pm i0}(r_\lambda)$$

and the relations (11.2.3) hold.

Each vector space $\Upsilon^{[\nu]}_{\lambda\pm i0}(r_\lambda)$ has a basis

$$u^{(1)}_{\nu\pm}, \dots, u^{(d_\nu)}_{\nu\pm}$$

such that $\mathbf{A}^k_{\lambda\pm i0}(r_\lambda)u^{(j)}_{\nu\pm} = u^{(j-1)}_{\nu\pm}$. Therefore, it is enough to show that the operator $\mathbf{A}^k_{\lambda+i0}(r_\lambda)\mathbf{A}^j_{\lambda-i0}(r_\lambda)\mathbf{A}^l_{\lambda+i0}(r_\lambda)$ decreases the order of each of the vectors $u^{(j)}_{\nu+}$. We shall prove this assertion.

For each $\nu = 1, \dots, m$, there exists the largest index α such that $u^{(\alpha)}_{\nu+}$ is a vector of type I. Corollary 8.2.6 implies that

$$\underbrace{u^{(1)}_{\nu+},\ \dots,\ u^{(\alpha)}_{\nu+}}_{\text{are of type I}},\quad \underbrace{u^{(\alpha+1)}_{\nu+},\ \dots,\ u^{(d_\nu)}_{\nu+}}_{\text{are not of type I}}.$$

The operator $\mathbf{A}^k_{\lambda+i0}(r_\lambda)\mathbf{A}^j_{\lambda-i0}(r_\lambda)\mathbf{A}^l_{\lambda+i0}(r_\lambda)$ decreases the order of the vectors $u^{(1)}_{\nu+}, \dots, u^{(\alpha)}_{\nu+}$, since by assumption they belong to its kernel. Now we show that the image of each of $u^{(\alpha+1)}_{\nu+}, \dots, u^{(d_\nu)}_{\nu+}$ is a linear combination of $u^{(1)}_{\nu+}, \dots, u^{(\alpha)}_{\nu+}$, and this will complete the proof. By (11.2.3), it is enough to show that any vector of type I from $\Upsilon^{[\nu]}_{\lambda+i0}(r_\lambda)$ is a linear combination of $u^{(1)}_{\nu+}, \dots, u^{(\alpha)}_{\nu+}$. Assume the contrary. Then there exists a vector f of type I and of order $> \alpha$. Using Corollary 8.2.6, we can assume that this vector has order $\alpha+1$. Since f is a linear combination of $u^{(1)}_{\nu+}, \dots, u^{(\alpha+1)}_{\nu+}$, it follows that $u^{(\alpha+1)}_{\nu+}$ is a vector of type I. This contradicts the definition of α. ∎

Let

$$D_{\lambda+i0}(r_\lambda) = P_{\lambda+i0}(r_\lambda) - P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda),$$
$$D_{\lambda-i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda) - P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda).$$

LEMMA 11.2.3. $D_{\lambda+i0}(r_\lambda) = D_{\lambda-i0}(r_\lambda)$.

Proof. By Lemma 11.1.3 we have $P_-D_+ = D_+$ and similarly $D_-P_+ = D_-$. It remains to note that $P_-D_+ = D_-P_+$. ∎

This lemma allows us to write $D_\lambda(r_\lambda)$ instead of $D_{\lambda-i0}(r_\lambda)$ and $D_{\lambda+i0}(r_\lambda)$.

LEMMA 11.2.4. *The operator $D_\lambda(r_\lambda)$ has the following properties:*

(1) $D_\lambda^2(r_\lambda) = 0$.
(2) *The image of $D_\lambda(r_\lambda)$ consists of vectors of type I.*
(3) *The kernel of $D_\lambda(r_\lambda)$ contains all vectors of type I.*

Proof. Multiplying the left hand side of (11.1.7) on both sides by $P_{\lambda+i0}(r_\lambda)$ gives $D_\lambda^2(r_\lambda) = 0$. It follows from (11.1.6) with $j = l = 0$ that for all $k = 0, 1, 2, \dots$,

$$(\mathbf{A}_+^k - \mathbf{A}_-^k)D_\lambda(r_\lambda) = (\mathbf{A}_+^k - \mathbf{A}_-^k)P_+(P_+ - P_-)P_+ = 0.$$

Hence, by Lemma 8.2.5, the image of $D_\lambda(r_\lambda)$ consists only of vectors of type I. The third assertion is obvious from Theorem 8.2.8. ∎

LEMMA 11.2.5. *If a real resonance point r_λ has property C then the operator*

$$P_{\lambda\pm i0}(r_\lambda)P_{\lambda\mp i0}(r_\lambda)P_{\lambda\pm i0}(r_\lambda)$$

preserves the order of vectors from $\Upsilon_{\lambda\pm i0}(r_\lambda)$, that is, for all $j = 1, 2, \dots$,

$$P_{\lambda\pm i0}(r_\lambda)P_{\lambda\mp i0}(r_\lambda)\Upsilon^j_{\lambda\pm i0}(r_\lambda) = \Upsilon^j_{\lambda\pm i0}(r_\lambda).$$

Proof. We prove this for the upper signs. In the proof we will use the properties of the operator $D = D_\lambda(r_\lambda)$ from the previous lemma.

If $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ is of type I, then

$$P_+P_-u = P_+P_-P_+u = (P_+ - D)u = u - Du = u,$$

so $P_+P_-P_+$ preserves order of type I vectors. For any $u \in \Upsilon_{\lambda+i0}(r_\lambda)$ the vector Du is of type I, and therefore by Lemma 11.2.2 the order of Du is less than the order of u. Hence, $P_+P_-P_+ = P_+ - D$ preserves order. ∎

LEMMA 11.2.6. *If a real resonance point r_λ has property C then the operator $P_{\lambda\pm i0}(r_\lambda)$ $\mathbf{A}_{\lambda\mp i0}(r_\lambda)P_{\lambda\pm i0}(r_\lambda)$ decreases the order of vectors from $\Upsilon_{\lambda\pm i0}(r_\lambda)$.*

Proof. We prove this for the upper signs. Let

$$E_+ := \mathbf{A}_+ - P_+\mathbf{A}_-P_+.$$

It follows from (11.1.6) with $k = l = 1$ and $j = 0$ that

$$E_+^2 = (\mathbf{A}_+ - P_+\mathbf{A}_-P_+)(\mathbf{A}_+ - P_+\mathbf{A}_-P_+) = P_+(\mathbf{A}_+ - \mathbf{A}_-)P_+(\mathbf{A}_+ - \mathbf{A}_-)P_+ = 0.$$

It follows from (11.1.6) with $l = 1$ and $j = 0$ that for all $k = 0, 1, 2, \dots$,

$$(\mathbf{A}_+^k - \mathbf{A}_-^k)E_+ = (\mathbf{A}_+^k - \mathbf{A}_-^k)P_+(\mathbf{A}_+ - \mathbf{A}_-)P_+ = 0.$$

Lemma 8.2.5 now shows that the image of E_+ is a subspace of $\Upsilon^I_\lambda(r_\lambda)$. So, on the one hand, the operator E_+ obviously maps all vectors of type I to zero; on the other hand, the image of E_+ consists of vectors of type I only. By Lemma 11.2.2, this implies that E_+ decreases order. Since $P_+\mathbf{A}_-P_+ = \mathbf{A}_+ - E_+$ and since $\mathbf{A}_+$ also decreases order, it follows that $P_+\mathbf{A}_-P_+$ decreases order too. ∎

THEOREM 11.2.7. *For any $z = \lambda \pm i0 \in \partial\Pi$, for any real resonance point r_λ with property C corresponding to z and for any $j = 1, 2, \dots$ the restriction of $P_{\lambda\pm i0}(r_\lambda)$ to $\Upsilon^j_{\lambda\mp i0}(r_\lambda)$ is a linear isomorphism of $\Upsilon^j_{\lambda\mp i0}(r_\lambda)$ and $\Upsilon^j_{\lambda\pm i0}(r_\lambda)$.*

Proof. As usual, only the statement for the upper signs is proved. Since by Theorem 9.1.1, P_+ is a linear isomorphism of Υ_- and Υ_+, the claim is equivalent to $P_+(\Upsilon_-^j) \subset \Upsilon_+^j$ for all j. Since $u \in \Upsilon_\pm^j$ if and only if $\mathbf{A}_\pm^j u = 0$, the last assertion is equivalent to

$$\forall u \in \Upsilon_- \quad \mathbf{A}_-^j u = 0 \Rightarrow \mathbf{A}_+^j u = 0.$$

For $j = 1$ this follows from Corollary 8.2.4. Assume that the claim holds for $j = k$, and let $u \in \Upsilon_-^{k+1}$. Then, since by Lemma 11.2.6 the operator $P_-\mathbf{A}_+P_-$ decreases order, we have $P_-\mathbf{A}_+P_-u \in \Upsilon_-^k$, which implies

$$\mathbf{A}_-^k(P_-\mathbf{A}_+P_-u) = 0.$$

By induction assumption, this gives

$$\mathbf{A}_+^k(P_-\mathbf{A}_+P_-u) = 0.$$

The left hand side can be written as $\mathbf{A}_+^k(P_+P_-P_+)\mathbf{A}_+u$, and so

$$\mathbf{A}_+^k(P_+P_-P_+)\mathbf{A}_+u = 0.$$

It follows that $(P_+P_-P_+)\mathbf{A}_+u$ is a $+$-vector of order $\le k$. Since by Lemma 11.2.5 the operator $P_+P_-P_+$ preserves order, $\mathbf{A}_+u$ is a $+$-vector of order $\le k$ too. Hence, $\mathbf{A}_+^{k+1}u = 0$. ∎

THEOREM 11.2.8. *For any $z = \lambda \pm i0 \in \partial\Pi$ and for any real resonance point r_λ with property C corresponding to z, the idempotent $Q_{\lambda\pm i0}(r_\lambda)$ is a linear isomorphism of $\Psi_{\lambda\mp i0}^j(r_\lambda)$ and $\Psi_{\lambda\pm i0}^j(r_\lambda)$ for all $j = 1, 2, \dots$.*

Proof. We prove this assertion for the upper signs. Using successively Lemma 3.1.4, (3.2.10), Theorem 11.2.7 and Lemma 3.1.4 again, one has

$$\begin{aligned} Q_{\lambda+i0}(r_\lambda)\Psi_{\lambda-i0}^j(r_\lambda) &= Q_{\lambda+i0}(r_\lambda)J\Upsilon_{\lambda-i0}^j(r_\lambda) \\ &= JP_{\lambda+i0}(r_\lambda)\Upsilon_{\lambda-i0}^j(r_\lambda) \simeq J\Upsilon_{\lambda+i0}^j(r_\lambda) = \Psi_{\lambda+i0}^j(r_\lambda). \end{aligned}$$ ∎

For real resonance points with property C the following two commutative diagrams of linear isomorphisms summarize Theorems 11.2.7, 11.2.8 and Lemma 3.1.4:

$$\begin{array}{ccc} \Psi_{\lambda+i0}^j(r_\lambda) & \xleftarrow{J} & \Upsilon_{\lambda+i0}^j(r_\lambda) \\ \uparrow{\scriptstyle Q_{\lambda+i0}(r_\lambda)} & & \uparrow{\scriptstyle P_{\lambda+i0}(r_\lambda)} \\ \Psi_{\lambda-i0}^j(r_\lambda) & \xleftarrow[J]{} & \Upsilon_{\lambda-i0}^j(r_\lambda) \end{array} \qquad \begin{array}{ccc} \Psi_{\lambda+i0}^j(r_\lambda) & \xleftarrow{J} & \Upsilon_{\lambda+i0}^j(r_\lambda) \\ \downarrow{\scriptstyle Q_{\lambda-i0}(r_\lambda)} & & \downarrow{\scriptstyle P_{\lambda-i0}(r_\lambda)} \\ \Psi_{\lambda-i0}^j(r_\lambda) & \xleftarrow[J]{} & \Upsilon_{\lambda-i0}^j(r_\lambda) \end{array}$$

We say that a real resonance point r_λ has *property U* if the operators $P_{\lambda\pm i0}(r_\lambda)\colon \Upsilon_{\lambda\mp i0}(r_\lambda) \to \Upsilon_{\lambda\pm i0}(r_\lambda)$ preserve the order of vectors. Thus, Theorem 11.2.7 asserts that property C implies property U.

12. Independence from the rigging operator

Here we discuss independence from the rigging operator F for some of the notions studied so far.

12.1. Independence of resonance index from the rigging operator

LEMMA 12.1.1. *The R-indices of $A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)$ and $\underline{A}_{\lambda+iy}(s)\underline{P}_{\lambda+iy}(r_\lambda)$ coincide for all s and for all small enough $y > 0$.*

Proof. By Lemma 5.1.4 the eigenvalue counting measures of $A_{\lambda+iy}(s)P_{\lambda+iy}(r_\lambda)$ and $\underline{A}_{\lambda+iy}(s)\underline{P}_{\lambda+iy}(r_\lambda)$ coincide. Hence, their R-indices are also equal. ∎

THEOREM 12.1.2. *The resonance index* $\operatorname{ind}_{\mathrm{res}}(\lambda; H, V)$ *does not depend on the choice of the rigging operator F as long as λ is essentially regular for the pair $(\mathcal{A}, F)$, where $\mathcal{A} = \{H + rV : r \in \mathbb{R}\}$ and V is a regularizing direction for an operator H which is resonant at λ.*

Proof. Since $\underline{A}_z(s) = R_z(H_s)V$ and $\underline{P}_z(r_z)$ do not depend on F, this follows immediately from Theorem 9.2.4 and Lemma 12.1.1. ∎

Theorem 12.1.2 raises natural questions of independence of the notions of essentially regular points and regularizing directions from the rigging operator F.

12.2. Independence of $\dim \Upsilon^1_{\lambda+i0}(r_\lambda)$ from the rigging operator

COROLLARY 12.2.1. *If the perturbation V is non-negative (or non-positive) then the dimension of $\Upsilon^1_{\lambda+i0}(H_{r_\lambda}, V)$ does not depend on the choice of F.*

Proof. Since V is non-negative, $\dim \Upsilon^1_{\lambda+i0}(H_{r_\lambda}, V) = \dim \Upsilon_{\lambda+i0}(H_{r_\lambda}, V)$ by Proposition 10.1.3. Since by $V \geq 0$ there are no resonance down-points, this dimension is equal to $\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$, which is independent of F by Theorem 12.1.2. ∎

By Lemma 9.1.3, for small enough y the signatures of $Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)$ and $\underline{Q}_{\lambda\mp iy}(r_\lambda)V\underline{P}_{\lambda\pm iy}(r_\lambda)$ coincide. Hence, another way to prove Theorem 12.1.2 is to observe that the latter operator does not depend on F.

Combining Corollary 12.2.1 with Theorem 4.4.1 we obtain

THEOREM 12.2.2. *If the real vector space $\mathcal{A}_0$ of self-adjoint perturbation operators contains at least one non-negative operator V, then the dimension of $\Upsilon^1_{\lambda+i0}(r_\lambda)$ is independent of F.*

13. Resonance points of type I

It turns out that real resonance points have a certain generic property, which admits many equivalent reformulations. A real resonance point with this property will be called a point of type I. As will be shown, if a point λ on the spectral line lies outside the essential spectrum, then all real resonance points corresponding to $\lambda \pm i0$ are of type I. Further, if the perturbation V is non-negative, then all points are also of type I for any essentially regular point λ. For a resonance point to be of type I is a generic property since, as will be shown, all resonance points of order 1 are of type I. Resonance points which are not of type I exist: examples will be given in Subsection 14.4.2.

At the end of this section we introduce the class of real resonance points with the so-called property S which is strictly larger than the class of real resonance points of type I.

Initially, results of Section 10 were proved for points of type I. At that stage of preparation of this paper I did not know whether there were real resonance points not of type I. In fact, a significant time was spent in an effort to prove a conjecture that all real resonance points are of type I. This conjecture was supported by the fact that it holds in several special cases mentioned at the beginning of this section. However, later an example of a resonance point not of type I was found. This example is given in Section 14. A similar story was repeated with resonance points with property S. To prove the main results of Section 10 in the case of arbitrary real resonance points took another year.

13.1. Resonance points of type I. By definition, a real resonance point r_λ is *of type I* if for some non-resonance point $s \in \mathbb{R}$,

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, JP_{\lambda+i0}(r_\lambda) = 0. \tag{13.1.1}$$

This is a strengthened version of (11.1.1), and while (11.1.1) holds for all resonance points r_λ, it will be shown that not all resonance points are of type I. One can also see that the definition of a point of type I is equivalent to requiring that all resonance vectors corresponding to $\lambda + i0$ are of type I.

Lemma 13.1.1. *A real resonance point r_λ is of type I if and only if for some non-resonant $s \in \mathbb{R}$,*

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, Q_{\lambda+i0}(r_\lambda) = 0. \tag{13.1.2}$$

Proof. By Lemma 3.1.4, the range of the operator $Q_{\lambda+i0}(r_\lambda)$ coincides with the range of the product $JP_{\lambda+i0}(r_\lambda)$. The assertion follows. ∎

In what follows it is assumed for convenience that the point s for which (13.1.1) holds is $s = 0$.

LEMMA 13.1.2. *A real resonance point r_λ is a point of type I if and only if the function*

$$\mathbb{C} \ni s \mapsto w(s) := \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,(1 + sJT_{\lambda+i0}(H_0))^{-1} \tag{13.1.3}$$

is holomorphic at r_λ.

Proof. Let $\sigma = -s^{-1}$ and let

$$\tilde{w}(\sigma) = \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,(\sigma - JT_{\lambda+i0}(H_0))^{-1} = -\frac{1}{s}w(s).$$

Then $w(s)$ is holomorphic at r_λ if and only if $\tilde{w}(\sigma)$ is holomorphic at $\sigma_\lambda(0) = -r_\lambda^{-1}$.

($\Rightarrow$) By the analytic Fredholm alternative, the function $\tilde{w}(\sigma)$ is meromorphic with a possible pole at $\sigma_\lambda(0)$. It follows from the definition (3.2.3) of $Q_{\lambda+i0}(r_\lambda)$ and Lemma 13.1.1 that

$$\begin{aligned}\oint_{C(\sigma_\lambda(0))} \tilde{w}(\sigma)\,d\sigma &= \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)} \oint_{C(\sigma_\lambda(0))} (\sigma - JT_{\lambda+i0}(H_0))^{-1}\,d\sigma \\ &= 2\pi i\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,Q_{\lambda+i0}(r_\lambda) = 0,\end{aligned} \tag{13.1.4}$$

where $C(\sigma_\lambda(0))$ is a small closed contour enclosing $\sigma_\lambda(0) = -r_\lambda^{-1}$. Hence, the coefficient of $(\sigma - \sigma_\lambda(0))^{-1}$ in the Laurent series of $\tilde{w}(\sigma)$ is 0. Now Proposition 3.3.4 and equality (3.3.15) imply that the coefficients of the terms $(\sigma - \sigma_\lambda(0))^{-n}$ with $n > 1$ also vanish.

($\Leftarrow$) If $\tilde{w}(\sigma)$ is holomorphic at $\sigma_\lambda(0)$, then $\oint_C \tilde{w}(\sigma)\,d\sigma$ vanishes. On the other hand, this integral is equal to $2\pi i\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,Q_{\lambda+i0}(r_\lambda)$. It now follows from Lemma 13.1.1 that r_λ has type I. ∎

The function $w(s)$ is holomorphic, but the adjoint function $w^*(s)$ is not. For this reason, instead of $w^*(s)$, the meromorphic continuation $w^\dagger(s)$ of the restriction of $w^*(s)$ to the real axis will be used:

$$\mathbb{C} \ni s \mapsto w^\dagger(s) := (1 + sT_{\lambda-i0}(H_0)J)^{-1}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}.$$

LEMMA 13.1.3. *If $w(s)$ is a meromorphic operator-valued function in some domain $G \subset \mathbb{C}$ which is symmetric with respect to the real axis, then $w(s)$ is holomorphic at a real point $r_0 \in G$ if and only if the function $w(s)w^\dagger(s)$ is.*

Proof. If $(s-r_0)^{-k}X_k$ is the lowest order term in the Laurent series of $w(s)$ at $s = r_0$, then the lowest order term in the Laurent series of $w(s)w^\dagger(s)$ at $s = r_0$ is $(s - r_0)^{-2k}X_kX_k^*$. Since $X_k = 0$ if and only if $X_kX_k^* = 0$, the claim follows. ∎

PROPOSITION 13.1.4. *Let $w(s)$ be given by* (13.1.3). *The following assertions are equivalent:*

(i) *r_λ is of type I.*
(ii) *$\mathbb{C} \ni s \mapsto w(s)$ is holomorphic at r_λ.*
(iii) *$\mathbb{C} \ni s \mapsto w^\dagger(s)w(s)$ is holomorphic at r_λ.*
(iv) *$\mathbb{C} \ni s \mapsto w^\dagger(s)$ is holomorphic at r_λ.*
(v) *$\mathbb{C} \ni s \mapsto \operatorname{Im} T_{\lambda+i0}(H_s)$ is holomorphic at r_λ.*

Proof. The equivalence (i)⇔(ii) is the content of Lemma 13.1.2; (ii)⇔(iv) is obvious; (iii)⇔(v) follows from (2.7.11); and (ii)⇔(iii) follows from Lemma 13.1.3. ∎

Observation 13.1.5. The equality (13.1.2) is plainly equivalent to

$$P_{\lambda-i0}(r_\lambda)\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)} = 0,$$

which gives another characterization of points of type I.

Lemma 13.1.6. *A real resonance point r_λ is of type I if and only if*

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, JP_{\lambda-i0}(r_\lambda) = 0.$$

That is, the definition (13.1.1) *of a resonance point r_λ of type I does not depend on the choice of sign in $P_{\lambda\pm i0}(r_\lambda)$.*

Proof. Since $\operatorname{Im} T_{\lambda-i0}(H_s) = -\operatorname{Im} T_{\lambda+i0}(H_s)$, the function $\operatorname{Im} T_{\lambda+i0}(H_s)$ is holomorphic at some point s if and only if $\operatorname{Im} T_{\lambda-i0}(H_s)$ is. Since, by (2.7.11),

$$\operatorname{Im} T_{\lambda-i0}(H_s) = (1 + sT_{\lambda+i0}(H_0)J)^{-1} \operatorname{Im} T_{\lambda-i0}(H_0)(1 + sJT_{\lambda-i0}(H_0))^{-1},$$

it follows from Proposition 13.1.4(v) and Lemma 13.1.3 that a real resonance point r_λ is a point of type I if and only if the function

$$h(s) = \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,(1 + sJT_{\lambda-i0}(H_0))^{-1}$$

is holomorphic at r_λ. Hence, making the change of variables $\sigma = -s^{-1}$ and taking the contour integral of the function $s \cdot h(s)$ over a small circle C enclosing $-r_\lambda^{-1}$ shows that if r_λ is a point of type I, then

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, Q_{\lambda-i0}(r_\lambda) = 0.$$

It follows from Lemma 3.1.4 that

$$\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, JP_{\lambda-i0}(r_\lambda) = 0.$$

Now, the argument of Lemma 13.1.2 shows that the last equality implies that $h(s)$ is holomorphic at r_λ; hence, the "if" direction is also proved. ∎

Lemma 13.1.7. *The equality* (13.1.1) *holds if and only if*

$$\operatorname{Im} T_{\lambda+i0}(H_s)\, JP_{\lambda+i0}(r_\lambda) = 0. \tag{13.1.5}$$

Proof. Plainly, (13.1.1) implies (13.1.5). If (13.1.5) holds, then by the C^*-equality $\|T\|^2 = \|T^*T\|$, we have

$$\|\sqrt{\operatorname{Im} T_{\lambda+i0}(H_s)}\, JP_{\lambda+i0}(r_\lambda)\|^2 = \|Q_{\lambda-i0}(r_\lambda)J \operatorname{Im} T_{\lambda+i0}(H_s)\, JP_{\lambda+i0}(r_\lambda)\| = 0. \ ∎$$

Lemma 13.1.8. *If* (13.1.1) *holds for one real non-resonant value of s, then it holds for any other such value.*

Proof. Assume that (13.1.1) holds for $s = r$. By Lemma 13.1.7, the square root in (13.1.1) can be removed, so that

$$T_{\lambda+i0}(H_r)JP_{\lambda+i0}(r_\lambda) = T_{\lambda-i0}(H_r)JP_{\lambda+i0}(r_\lambda). \tag{13.1.6}$$

Hence, the restrictions of $A_{\lambda+i0}(r) = T_{\lambda+i0}(H_r)J$ and $A_{\lambda-i0}(r) = T_{\lambda-i0}(H_r)J$ to $\Upsilon_{\lambda+i0}(r_\lambda) = \operatorname{im} P_{\lambda+i0}(r_\lambda)$ coincide. By Corollary 3.1.6, $\Upsilon_{\lambda+i0}(r_\lambda)$ is invariant under $A_{\lambda+i0}(r)$, and therefore, by (13.1.6), $\Upsilon_{\lambda+i0}(r_\lambda)$ is invariant under $A_{\lambda-i0}(r)$ too. It now

follows from (2.7.4) that the restrictions of $A_{\lambda+i0}(s)$ and $A_{\lambda-i0}(s)$ to $\Upsilon_{\lambda+i0}(r_\lambda)$ coincide for all non-resonance s. Hence, $A_{\lambda+i0}(s)P_{\lambda+i0}(r_\lambda) = A_{\lambda-i0}(s)P_{\lambda+i0}(r_\lambda)$ for all such s, as required. ∎

These results are summarized in the following theorem.

THEOREM 13.1.9. *Let λ be an essentially regular point for the pair $(\mathcal{A}, F)$. Let $H_0 \in \mathcal{A}$ be an operator regular at λ and let $V \in \mathcal{A}_0$. Let $r_\lambda \in \mathbb{R}$ be a resonance point of the path $\{H_0 + rV : r \in \mathbb{R}\}$. The following assertions are all equivalent to r_λ being of type I:*

- (i$_\pm$) *For any regular point r, $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, JP_{\lambda\pm i0}(r_\lambda) = 0$.*
- (i$^*_\pm$) *There exists a regular point r such that $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, JP_{\lambda\pm i0}(r_\lambda) = 0$.*
- (ii$_\pm$) *For any regular point r, $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, Q_{\lambda\pm i0}(r_\lambda) = 0$.*
- (ii$^*_\pm$) *There exists a regular point r such that $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, Q_{\lambda\pm i0}(r_\lambda) = 0$.*
- (iii$_\pm$) *The meromorphic function*
$$w_\pm(s) := \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\,[1 + sJT_{\lambda\pm i0}(H_0)]^{-1}$$
is holomorphic at $s = r_\lambda$.
- (iii$'_\pm$) *The meromorphic function*
$$w_\pm(s)J = \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, J[1 + sT_{\lambda\pm i0}(H_0)J]^{-1}$$
is holomorphic at $s = r_\lambda$.
- (iv$_\pm$) *The meromorphic function*
$$w^\dagger_\pm(s) = [1 + sT_{\lambda\mp i0}(H_0)J]^{-1}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}$$
is holomorphic at $s = r_\lambda$.
- (v$_\pm$) *The residue of $w_\pm(s)$ at $s = r_\lambda$ is zero.*
- (vi$_\pm$) *For all $\pm$-resonance vectors the real numbers c_{-j} from Proposition* 8.1.1 *are all zero.*
- (vii) *The function $s \mapsto \operatorname{Im} T_{\lambda+i0}(H_s)$ is holomorphic at $s = r_\lambda$.*
- (viii) *The function $s \mapsto J \operatorname{Im} T_{\lambda+i0}(H_s)J$ is holomorphic at $s = r_\lambda$.*

Moreover, the assertions obtained from (i$_\pm$)–(ii$_\pm$) *and* (i$^*_\pm$)–(ii$^*_\pm$) *by removing the square root are also equivalent to the above ones.*

Proof. Equivalence of (i$_\pm$), (i$^*_\pm$), (ii$_\pm$), (ii$^*_\pm$), (iii$_\pm$), (iv$_\pm$), (v$_\pm$) and (vii) has already been proved.

It is not difficult to see that (iii$_\pm$) implies (iii$'_\pm$). Now it will be shown that (iii$'_\pm$) implies (i$_\pm$). Making the change of variables $\sigma = -s^{-1}$ and taking the contour integral over $C(\sigma_\lambda(0))$ (where $\sigma_\lambda(0) = -r_\lambda^{-1}$) of the function $sw_\pm(s)J$ gives

$$0 = \oint_{C(\sigma_\lambda(0))} \sigma^{-1} w_\pm(-\sigma^{-1})J\, d\sigma = \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\, JP_{\lambda+i0}.$$

Item (vii) obviously implies (viii). Item (viii) combined with Lemma 13.1.3 and (2.7.11) implies (iii′).

Finally, (vii) implies (vi$_\pm$), and (vi$_\pm$) implies (i$_\pm$). ∎

COROLLARY 13.1.10. *If the right hand side of* (8.1.1) *is non-zero, then it is strictly positive for all non-resonance points s.*

Proof. If the right hand side of (8.1.1) vanishes at some point s, then by the implication $(\mathrm{i}_\pm^*)\Rightarrow(\mathrm{i}_\pm)$ of Theorem 13.1.9 it vanishes at all points s. ∎

Remark 13.1.11. Properties $(\mathrm{iii}_\pm)$ and $(\mathrm{iv}_\pm)$ have something in common with the fact that the scattering matrix and the $\tilde{S}$-function, defined by (11.1.3), are holomorphic in a neighbourhood of $\mathbb{R}$. One can see this from the stationary formula for the scattering matrix, recalling the relation (1.4.4) between $\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}$ and $\mathcal{E}_\lambda(H_0)$.

In addition to the equivalent conditions of Theorem 13.1.9, one has the equivalent conditions

$$A_{\lambda+i0}(s) = A_{\lambda-i0}(s) \qquad \text{on } \Upsilon_{\lambda\pm i0}(r_\lambda), \tag{13.1.7}$$

$$B_{\lambda+i0}(s) = B_{\lambda-i0}(s) \qquad \text{on } \Psi_{\lambda\pm i0}(r_\lambda), \tag{13.1.8}$$

$$\mathbf{A}_{\lambda+i0}(r_\lambda) = \mathbf{A}_{\lambda-i0}(r_\lambda) \qquad \text{on } \Upsilon_{\lambda\pm i0}(r_\lambda), \tag{13.1.9}$$

$$\mathbf{B}_{\lambda+i0}(r_\lambda) = \mathbf{B}_{\lambda-i0}(r_\lambda) \qquad \text{on } \Psi_{\lambda\pm i0}(r_\lambda). \tag{13.1.10}$$

The equality (13.1.7) and Lemma 3.1.4 imply that the restrictions of $T_{\lambda+i0}(H_s)$ and $T_{\lambda-i0}(H_s)$ to $\Psi_{\lambda\pm i0}(r_\lambda)$ coincide. Hence, the restrictions of $B_{\lambda+i0}(s) = JT_{\lambda+i0}(H_s)$ and $B_{\lambda-i0}(s) = JT_{\lambda-i0}(H_s)$ to $\Psi_{\lambda\pm i0}(r_\lambda)$ also coincide. Therefore, (13.1.7) implies (13.1.8).

Further, (13.1.8) and Lemma 3.1.4 imply that $B_{\lambda+i0}(s)J = B_{\lambda-i0}(H_s)J$ on $\Upsilon_{\lambda\pm i0}(r_\lambda)$. Hence, $JA_{\lambda+i0}(s) = JA_{\lambda-i0}(H_s)$ on $\Upsilon_{\lambda\pm i0}(r_\lambda)$, and so, by Lemma 3.1.4, $A_{\lambda+i0}(s) = A_{\lambda-i0}(H_s)$ on $\Upsilon_{\lambda\pm i0}(r_\lambda)$. Consequently, (13.1.8) implies (13.1.7).

Further, the definition (3.3.1) of $\mathbf{A}_{\lambda+i0}$ and (3.4.11) imply that (13.1.7) and (13.1.9) are equivalent. Similarly, (13.1.8) and (13.1.10) are equivalent. Finally, (13.1.7) is just a reformulation of Theorem $13.1.9(\mathrm{i}_\pm)$.

By Corollary 8.2.4 the vector spaces $\Upsilon^1_{\lambda+i0}(r_\lambda)$ and $\Upsilon^1_{\lambda-i0}(r_\lambda)$ of $+$-resonance and $-$-resonance vectors of order 1 coincide for any real resonance point r_λ. For $k > 1$ the vector spaces $\Upsilon^k_{\lambda+i0}(r_\lambda)$ and $\Upsilon^k_{\lambda-i0}(r_\lambda)$ are different in general, but if r_λ is a type I point, then they coincide for all $k = 1, 2, \dots$ as the following proposition shows.

Proposition 13.1.12. *In the conditions of Proposition* 8.1.1*, if r_λ is a real resonance point of type I, then for all $k = 1, 2, \dots$ solutions of the resonance equations*

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)^k u = 0$$

and

$$(1 + (r_\lambda - r)T_{\lambda-i0}(H_r)J)^k u = 0$$

coincide, that is,

$$\Upsilon^k_{\lambda+i0}(r_\lambda) = \Upsilon^k_{\lambda-i0}(r_\lambda). \tag{13.1.11}$$

Proof. This follows directly from Lemma 8.2.3. Nevertheless, we give another proof.

The case $k = 1$ follows from Proposition 8.2.2 (and holds for all resonance points). Assume that the claim holds for $k - 1$. If u is a solution of

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)^k u = 0,$$

then $(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)u$ is a solution of

$$(1 + (r_\lambda - r)T_{\lambda+i0}(H_r)J)^{k-1} f = 0. \tag{13.1.12}$$

Since the real resonance point r_λ is of type I, we have $\operatorname{Im} T_{\lambda+i0}(H_r)Ju = 0$. It follows that $(1 + (r_\lambda - r)T_{\lambda-i0}(H_r)J)u$ is also a solution of (13.1.12). From the induction assumption we see that $(1 + (r_\lambda - r)T_{\lambda-i0}(H_r)J)u$ is a solution of

$$(1 + (r_\lambda - r)T_{\lambda-i0}(H_r)J)^{k-1} f = 0.$$

It follows that u is a solution of $(1 + (r_\lambda - r)T_{\lambda-i0}(H_r)J)^k u = 0$. ∎

The same argument shows that for points r_λ of type I we have

$$\Psi^k_{\lambda+i0}(r_\lambda) = \Psi^k_{\lambda-i0}(r_\lambda).$$

This equality also follows from (13.1.11) and Lemma 3.1.4.

Proposition 13.1.12 implies, in particular, that for points r_λ of type I the ranges of the idempotent operators $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$ coincide. In fact, for points of type I these idempotents coincide, as the following theorem shows.

THEOREM 13.1.13. *Let H_0 be a self-adjoint operator from $\mathcal{A}$, let λ be an essentially regular point and let V be a regularizing direction. If a real number r_λ is a resonance point of type I, then $P_{\lambda-i0}(r_\lambda) = P_{\lambda+i0}(r_\lambda)$.*

Proof. Let y be a small positive number. Proposition 5.2.1 implies that

$$\frac{1}{\pi} \oint_{C(r_\lambda)} \operatorname{Im} T_{\lambda+iy}(H_s)J\, ds = P_{\lambda+iy}(r_\lambda) - P_{\lambda-iy}(r_\lambda),$$

where $C(r_\lambda)$ is a contour which encloses all poles $r^1_{\lambda+iy}, \dots, r^N_{\lambda+iy}$ of the group of r_λ and their conjugates $\bar{r}^1_{\lambda+iy}, \dots, \bar{r}^N_{\lambda+iy}$ (see Subsection 5.2 for the definition of the poles of the group of r_λ). By Lemmas 2.7.2 and 5.2.2, taking the limit as $y \to 0$ in the above equality gives

$$\frac{1}{\pi} \oint_{C(r_\lambda)} \operatorname{Im} T_{\lambda+i0}(H_s)J\, ds = P_{\lambda+i0}(r_\lambda) - P_{\lambda-i0}(r_\lambda). \tag{13.1.13}$$

By Proposition 13.1.4(v), the integrand is holomorphic in a neighbourhood of r_λ, and therefore the integral vanishes. Hence, $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$. ∎

Theorem 13.1.13 and (13.1.9) provide another proof of Proposition 13.1.12.

PROPOSITION 13.1.14. *A point r_λ is of type I if and only if for some and thus for any non-resonant r we have*

$$\mathfrak{h}_\lambda(H_r) \perp \Psi_{\lambda+i0}(r_\lambda),$$

where $\mathfrak{h}_\lambda(H_r)$ is the fibre Hilbert space as defined by (1.4.1).

Proof. This follows from items (ii_+) and (ii^*_+) of Theorem 13.1.9 and the equality $\Psi_{\lambda+i0}(r_\lambda) = \operatorname{im} Q_{\lambda+i0}(r_\lambda)$. ∎

By Proposition 8.2.2 for any real resonance point r_λ the relation $\mathfrak{h}_\lambda(H_r) \perp \Psi^1_{\lambda+i0}(r_\lambda)$ holds. The vector space $\Psi_{\lambda+i0}(r_\lambda)$ is in fact also the image of the resonance matrix $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$, as shown by (9.1.2) and Lemma 3.1.4. This gives another characterization of points of type I.

PROPOSITION 13.1.15. *A point r_λ is of type I if and only if for some and thus for any non-resonant r,*

$$\operatorname{Im} T_{\lambda+i0}(H_r)Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = 0.$$

13.2. Examples of points of type I. In this subsection we give several conditions which ensure that a resonance point has type I.

THEOREM 13.2.1. *Let λ be an essentially regular point, let $H_0 \in \mathcal{A}$ and let $V \in \mathcal{A}_0$ be a regularizing direction at λ. If λ does not belong to the (necessarily common) essential spectrum of operators from $\mathcal{A}$, then every resonance point of the triple $(\lambda; H_0, V)$ is of type I.*

Proof. In this case the function $\mathbb{R} \ni r \mapsto \operatorname{Im} T_{\lambda+i0}(H_r)$ is zero. Thus, the claim follows from, for example, Theorem 13.1.9(vii). ∎

The next assertion is immediate from Proposition 8.2.2 and the definition (13.1.1) of resonance points of type I.

THEOREM 13.2.2. *Let λ be an essentially regular point, let $H_0 \in \mathcal{A}$ and let $V \in \mathcal{A}_0$ be a regularizing direction at λ. All resonance points of the triple $(\lambda; H_0, V)$ which have order 1 are of type I.*

Since resonance points generically have order 1, Theorem 13.2.2 shows that points of type I are in abundance. An example of a resonance point not of type I will be given in Section 14.

THEOREM 13.2.3. *Let λ be an essentially regular point, let $H_0 \in \mathcal{A}$ and let $V \in \mathcal{A}_0$ be a regularizing direction at λ. If the perturbation V is non-negative (or non-positive), then every resonance point of the triple $(\lambda; H_0, V)$ is of type I.*

Proof. This follows from Proposition 10.1.3 and Theorem 13.2.2. ∎

COROLLARY 13.2.4. *If r_λ is not a point of type I, then $\lambda \in \sigma_{\mathrm{ess}}$ and the order of r_λ is greater than 1. Moreover, in this case the perturbation J is not sign-definite.*

PROPOSITION 13.2.5. *Let r_λ be a real resonance point corresponding to $\lambda \pm i0 \in \partial\Pi$. If the resonance matrix $Q_{\lambda\mp i0}(r_\lambda)JP_{\lambda\pm i0}(r_\lambda)$ is either non-negative or non-positive, then r_λ is of type I.*

Proof. This follows from Proposition 10.1.4 and Theorem 13.2.2. ∎

13.3. Resonance points with property S. In this subsection a class of real resonance points is introduced which is strictly larger than the class of points of type I. Let λ be an essentially regular point. A real resonance point r_λ will be said to have *property S* if

$$\ker P_{\lambda+i0}(r_\lambda) = \ker P_{\lambda-i0}(r_\lambda).$$

PROPOSITION 13.3.1. *Let λ be an essentially regular point and let r_λ be a real resonance point. The following assertions are equivalent:*

(i) *r_λ has property S.*
(ii) *$P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) = P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$.*
(iii) *$\operatorname{im} Q_{\lambda+i0}(r_\lambda) = \operatorname{im} Q_{\lambda-i0}(r_\lambda)$, that is, $\Psi_{\lambda+i0}(r_\lambda) = \Psi_{\lambda-i0}(r_\lambda)$.*
(iv) *$Q_{\lambda+i0}(r_\lambda)Q_{\lambda-i0}(r_\lambda) = Q_{\lambda-i0}(r_\lambda)$ and $Q_{\lambda-i0}(r_\lambda)Q_{\lambda+i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)$.*
(v) *$Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = JP_{\lambda+i0}(r_\lambda)$.*
(vi) *$Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = JP_{\lambda-i0}(r_\lambda)$.*

(vii) $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = Q_{\lambda-i0}(r_\lambda)J$.
(viii) $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)J$.
(ix) $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$.

Proof. (ii)⇒(i). If $P_{\lambda-i0}f = 0$, then $P_{\lambda+i0}f = P_{\lambda+i0}P_{\lambda-i0}f = 0$. Similarly, if $P_{\lambda+i0}f = 0$, then $P_{\lambda-i0}f = P_{\lambda-i0}P_{\lambda+i0}f = 0$.

(i)⇒(ii). Let $f \in \mathcal{K}$ and let $f = f' + f''$, where f' and f'' satisfy $P_{\lambda-i0}f' = f'$ and $P_{\lambda-i0}f'' = 0$. Then $\mathrm{t}P_{\lambda+i0}f''$ is also zero, and therefore

$$P_{\lambda+i0}P_{\lambda-i0}f = P_{\lambda+i0}f' = P_{\lambda+i0}(f' + f'') = P_{\lambda+i0}f.$$

By the same argument, $P_{\lambda-i0}P_{\lambda+i0}f = P_{\lambda-i0}f$.

(i)⇔(iii) follows from $\overline{\operatorname{im} A^*} = (\ker A)^\perp$ and (3.2.5).

(ii)⇔(iv) follows from (3.2.5).

(iii)⇔(v) follows from Lemma 3.1.4, (3.2.12) and (9.1.2). The equivalence (iii)⇔(vi) is proved by the same argument.

(v)⇔(vii) and (vi)⇔(viii) follow from self-adjointness of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ and from (3.2.5).

(ii)⇒(ix). We have

$$\begin{aligned} Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) &= Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) \\ &= Q_{\lambda-i0}(r_\lambda)Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda), \end{aligned}$$

where the first equality follows from (ii), the second from (3.2.10) and the third from (iv).

(ix)⇒(iii). By Propoition 9.1.2, the ranks of $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ and $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$ are both equal to $N = \operatorname{rank} Q_{\lambda\pm i0}(r_\lambda)$. Hence,

$$\operatorname{im} Q_{\lambda-i0}(r_\lambda) = \operatorname{im} Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = \operatorname{im} Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) = \operatorname{im} Q_{\lambda+i0}(r_\lambda). \quad \blacksquare$$

According to Theorem 11.1.4, the operators

$$P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) - P_{\lambda+i0}(r_\lambda) \quad \text{and} \quad P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda) - P_{\lambda-i0}(r_\lambda)$$

are nilpotent. Hence, a real resonance point has property S if and only if the nilpotent parts of $P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)P_{\lambda+i0}(r_\lambda)$ are zero.

Proposition 13.3.2. *Every resonance point of type I has property S. There are resonance points which do not have property S, and there are points with property S which are not of type I.*

The first part of this proposition is trivial; to prove it one can note that by Theorem 13.1.13 for points r_λ of type I we have $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$, and therefore r_λ has property S. Examples of resonance points with the required properties will be given in Subsection 14.4.2.

Propositions 13.3.1 and 13.3.2 give answers to some natural questions, such as whether the operators $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ and $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$ always coincide or not.

Proposition 13.3.3. *If $\Upsilon_{\lambda+i0}(r_\lambda) = \Upsilon_{\lambda-i0}(r_\lambda)$, then $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$.*

Proof. If $\Upsilon_{\lambda+i0}(r_\lambda) = \Upsilon_{\lambda-i0}(r_\lambda)$, then since $\Upsilon_{\lambda\pm i0}(r_\lambda) = \operatorname{im} Q_{\lambda\pm i0}(r_\lambda)$, it follows from Proposition 13.3.1(iii) that the kernels of the idempotents $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$ coincide. Since the ranges $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$ of these idempotents are also equal by assumption, it follows that $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$. ∎

Plainly, the equality $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$ is also equivalent to $Q_{\lambda+i0}(r_\lambda) = Q_{\lambda-i0}(r_\lambda)$, but these equalities are not equivalent to $\Psi_{\lambda+i0}(r_\lambda) = \Psi_{\lambda-i0}(r_\lambda)$, which is property S.

14. Perturbation of an embedded eigenvalue

In this section we study the behaviour of an eigenvalue of a self-adjoint operator embedded into the essential spectrum as the operator undergoes a perturbation. This is a classical problem, but in this section some new results will be given. Not only is the behaviour of embedded eigenvalues under perturbations interesting on its own, but this investigation will also provide examples and counter-examples to many possible relations which may be contemplated in regard to the material of previous sections. In fact, from the point of view of the deductive structure, this section is quite independent of the previous ones; on the other hand, this section was written almost in parallel with previous sections, and it is this study of embedded eigenvalues that gave many suggestions about possible properties of resonance points.

14.1. Two lemmas

LEMMA 14.1.1. *Let N be a positive integer and let $\mathcal{H} = \hat{\mathcal{H}} \oplus \mathbb{C}^N$ be a decomposition of a Hilbert space $\mathcal{H}$ into the orthogonal direct sum of another Hilbert space $\hat{\mathcal{H}}$ and $\mathbb{C}^N$. If*

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \qquad n = 1, 2, \dots,$$

is a sequence of operators on $\mathcal{H}$ which converges to an operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the uniform norm, then this convergence also holds in p-norm if and only if the sequence A_n, $n = 1, 2, \dots$, converges to A in p-norm.

Proof. The "only if" part is trivial. Since the ranks of the operators $B, B_1, B_2, \dots$ and $C, C_1, C_2, \dots$ are bounded by N, the "if" part follows from Lemma 2.1.4. ∎

Usually we denote by r_z a resonance point corresponding to z. In the following two lemmas we divert from this convention. The reason for this is that later in this section we are going to embed the Hilbert space on which the operators H_0 and V act in a slightly larger Hilbert space, where a non-resonance point r_z will become a resonant one.

LEMMA 14.1.2. *Let r_z be a non-resonance point for z. For any regular points s and t the operator $(1 + (r_z - s)A_z(s))^{-1}$ is a linear combination of 1 and $(1 + (r_z - t)A_z(t))^{-1}$, namely,*

$$(1 + (r_z - s)A_z(s))^{-1} = \frac{t - s}{t - r_z} + \frac{s - r_z}{t - r_z}\,(1 + (r_z - t)A_z(t))^{-1}.$$

Proof. This is a direct calculation based on (2.7.5) and (2.7.3). ∎

COROLLARY 14.1.3. *Let r_z be a non-resonance point for z. For any integer $k \geq 1$ and for any regular points s and t the operator $(1 + (r_z - s)A_z(s))^{-k}$ is a linear combination of*

$$1,\ (1 + (r_z - t)A_z(t))^{-1},\ \ldots,\ 1 + (r_z - t)A_z(t))^{-k}.$$

Proof. This follows from the previous lemma and induction. ∎

14.2. The vector spaces $\Upsilon^j_{\lambda \pm i0}(r_\lambda)$. Let H_{r_λ} be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with an eigenvalue λ of multiplicity one. No assumptions are made about the location of this eigenvalue yet: it can be outside of the essential spectrum or inside of it. Let χ be the corresponding eigenvector:

$$H_{r_\lambda}\chi = \lambda\chi. \tag{14.2.1}$$

The orthogonal complement of χ will be denoted by $\hat{\mathcal{H}}$. The subspace $\hat{\mathcal{H}}$ reduces H_{r_λ}, and the reduction will be denoted by $\hat{H}_{r_\lambda}$. Thus, $\mathcal{H}$ becomes split into an orthogonal sum $\hat{\mathcal{H}} \oplus \mathbb{C}$, and in this representation the operator H_{r_λ} has the form

$$H_{r_\lambda} = \begin{pmatrix} \hat{H}_{r_\lambda} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{14.2.2}$$

We have to choose a rigging operator F. To simplify calculations, we choose

$$F = \begin{pmatrix} \hat{F} & 0 \\ 0 & 1 \end{pmatrix}, \tag{14.2.3}$$

where $\hat{F}\colon \hat{\mathcal{H}} \to \hat{\mathcal{K}}$ is a rigging operator in $\hat{\mathcal{H}}$, so that F itself acts from $\mathcal{H}$ to $\mathcal{K} = \hat{\mathcal{K}} \oplus \mathbb{C}$. Since λ is a non-degenerate eigenvalue of H_{r_λ}, it cannot be an eigenvalue of $\hat{H}_{r_\lambda}$, but it is still possible that $\lambda \notin \Lambda(\hat{H}_{r_\lambda}, \hat{F})$. By Proposition 2.6.3, since λ is an eigenvalue of H_{r_λ}, we have $\lambda \notin \Lambda(H_{r_\lambda}, F)$, but we assume that

$$\lambda \in \Lambda(\hat{H}_{r_\lambda}, \hat{F}). \tag{14.2.4}$$

This means that there are no other singularities of H_{r_λ} at λ except that (14.2.1) holds.

Let V be a self-adjoint operator on $\mathcal{H}$. Then

$$V = \begin{pmatrix} \hat{V} & \hat{v} \\ \langle \hat{v}, \cdot \rangle & \alpha \end{pmatrix},$$

where $\hat{V}$ is a self-adjoint operator in $\hat{\mathcal{H}}$. We assume that there exists a bounded self-adjoint operator J on $\mathcal{K}$ such that V has a well-defined factorization

$$V = F^* J F.$$

Let

$$J = \begin{pmatrix} \hat{J} & \hat{\psi} \\ \langle \hat{\psi}, \cdot \rangle & \alpha \end{pmatrix} \tag{14.2.5}$$

be the representation of J in the orthogonal sum $\hat{\mathcal{K}} \oplus \mathbb{C}$, where $\hat{J}$ is a bounded self-adjoint operator on $\hat{\mathcal{K}}$, $\hat{\psi} \in \hat{\mathcal{K}}$ and $\alpha \in \mathbb{R}$. Then one can see that

$$\hat{V} = \hat{F}^* \hat{J} \hat{F} \tag{14.2.6}$$

and

$$\hat{v} = \hat{F}^* \hat{\psi}.$$

In particular, $\hat{v} \in \hat{\mathcal{H}}_+(\hat{F})$ and $\hat{V} \in \mathcal{A}_0(\hat{F})$. The eigenvector χ of H_{r_λ} in $\hat{\mathcal{H}} \oplus \mathbb{C}$ has the form $\text{const} \cdot \binom{0}{1}$. The matrix components $\hat{\psi}$ and α of J can be recovered from the equalities

$$\alpha = \langle \chi, V\chi \rangle = \langle F\chi, JF\chi \rangle \quad \text{and} \quad \hat{\psi} \oplus 0 = JF\chi - \alpha F\chi.$$

LEMMA 14.2.1. *If $\alpha = 0$, then $\hat{\psi} \oplus 0$ is a co-resonance vector of order 1.*

Proof. By Theorem 4.1.1, $F\chi$ is a resonance vector of order 1. Hence, by Lemma 3.1.4, $\hat{\psi} \oplus 0 = JF\chi$ is a co-resonance vector of order 1. ∎

For a real number s define

$$H_s := H_{r_\lambda} + (s - r_\lambda)V = \begin{pmatrix} \hat{H}_s & (s - r_\lambda)\hat{v} \\ (s - r_\lambda)\langle \hat{v}, \cdot \rangle & \lambda + (s - r_\lambda)\alpha \end{pmatrix}, \tag{14.2.7}$$

where

$$\hat{H}_s = \hat{H}_{r_\lambda} + (s - r_\lambda)\hat{V}. \tag{14.2.8}$$

A direct but lengthy calculation shows that the operator $T_z(H_s) = FR_z(H_s)F^*$ is given by

$$T_z(H_s) = \begin{pmatrix} T_z(\hat{H}_s) + (s - r_\lambda)^2 \mathcal{D}_z(s)\langle \hat{u}_{\bar{z}}(s), \cdot \rangle \hat{u}_z(s) & (r_\lambda - s)\mathcal{D}_z(s)\hat{u}_z(s) \\ (r_\lambda - s)\mathcal{D}_z(s)\langle \hat{u}_{\bar{z}}(s), \cdot \rangle & \mathcal{D}_z(s) \end{pmatrix}, \tag{14.2.9}$$

where

$$\hat{u}_z(s) = T_z(\hat{H}_s)\hat{\psi}, \tag{14.2.10}$$

$$\mathcal{D}_z(s) = (\lambda - z + (s - r_\lambda)\alpha - (s - r_\lambda)^2 \langle \hat{\psi}, \hat{u}_z(s) \rangle)^{-1}. \tag{14.2.11}$$

The condition (14.2.4) means that the operator $\hat{H}_{r_\lambda}$ is regular at λ, and thus any perturbation operator $\hat{V}$ of the form (14.2.6) is a regularizing direction at λ for $\hat{H}_{r_\lambda}$. We wish to find conditions which ensure that V is a regularizing direction at λ for H_{r_λ}. Recall that V is a regularizing direction at λ for H_{r_λ} if for some real number s the operator $T_z(H_s)$ has the norm limit $T_{\lambda+i0}(H_s)$. Since the norm limit $T_{\lambda+i0}(\hat{H}_s)$ of $T_{\lambda+iy}(\hat{H}_s)$ exists for some s (namely, for $s = r_\lambda$) by (14.2.4), it follows from (14.2.9) and Lemma 14.1.1 that the norm limit $T_{\lambda+i0}(H_s)$ exists for some real s if and only if the limit $\mathcal{D}_{\lambda+i0}(s)$ exists for some real s. From the definition (14.2.11) it is easy to see that $\mathcal{D}_{\lambda+i0}(s)$ exists if and only if either $\alpha \neq 0$ or both $\alpha = 0$ and $\langle \hat{\psi}, \hat{u}_{\lambda+i0}(s) \rangle \neq 0$. Thus, we have proved

LEMMA 14.2.2. *The operator $V = F^*JF$, where F and J are defined by* (14.2.3) *and* (14.2.5)*, is a regularizing direction for the λ-resonant operator H_{r_λ} given by* (14.2.2) *if and only if $\alpha \neq 0$ or both $\alpha = 0$ and*

$$\langle \hat{\psi}, \hat{u}_{\lambda+i0}(s) \rangle \neq 0 \quad \textit{for some real number } s. \tag{14.2.12}$$

From now on we shall assume that V is a regularizing direction for H_{r_λ}.

Let

$$\hat{A}_z(s) = T_z(\hat{H}_s)\hat{J} \quad \text{and} \quad \hat{B}_z(s) = \hat{J}\, T_z(\hat{H}_s). \tag{14.2.13}$$

The operator $A_z(s) = T_z(H_s)J$ is equal to

$$A_z(s) = \begin{pmatrix} T_z(\hat{H}_s) + (s-r_\lambda)^2\mathcal{D}_z(s)\langle\hat{u}_{\bar{z}}(s),\cdot\rangle\hat{u}_z(s) & (r_\lambda - s)\mathcal{D}_z(s)\hat{u}_z(s) \\ (r_\lambda - s)\mathcal{D}_z(s)\langle\hat{u}_{\bar{z}}(s),\cdot\rangle & \mathcal{D}_z(s) \end{pmatrix}\begin{pmatrix} \hat{J} & \hat{\psi} \\ \langle\hat{\psi},\cdot\rangle & \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \hat{A}_z(s) + (r_\lambda - s)X_{2,1}\hat{u}_z(s) & [1 + (s-r_\lambda)X_{2,2}]\hat{u}_z(s) \\ X_{2,1} & -X_{2,2} \end{pmatrix},$$

where

$$X_{2,1} = \mathcal{D}_z(s)\langle\hat{\psi} + (r_\lambda - s)\hat{J}\hat{u}_{\bar{z}}(s),\cdot\rangle,$$
$$X_{2,2} = \mathcal{D}_z(s)\big((s-r_\lambda)\langle\hat{u}_{\bar{z}}(s),\hat{\psi}\rangle - \alpha\big). \tag{14.2.14}$$

In what follows, $1 + (r_\lambda - s)\hat{A}_z(s)$ will be encountered very often. Therefore, we introduce a special notation for this operator:

$$\mathcal{F}_z(s) = 1 + (r_\lambda - s)\hat{A}_z(s). \tag{14.2.15}$$

Note that

$$\mathcal{F}_z^*(s) = 1 + (r_\lambda - s)\hat{B}_{\bar{z}}(s). \tag{14.2.16}$$

Lemma 14.2.3.

$$\mathcal{F}_{\lambda+i0}^{-1}(s) = 1 + (s - r_\lambda)\hat{A}_{\lambda+i0}(r_\lambda).$$

Proof. This follows from (2.7.3). ∎

Since by (14.2.10) and (14.2.13),

$$\hat{\psi} + (r_\lambda - s)\hat{J}\hat{u}_{\bar{z}}(s) = [1 + (r_\lambda - s)\hat{B}_{\bar{z}}(s)]\hat{\psi} = \mathcal{F}_z^*(s)\hat{\psi},$$

the following lemma has been proved.

Lemma 14.2.4. *In the above notation, the operator $A_z(s) = T_z(H_s)J$ is equal to*

$$\begin{pmatrix} \hat{A}_z(s) + (r_\lambda - s)\mathcal{D}_z(s)\langle\mathcal{F}_z^*(s)\hat{\psi},\cdot\rangle\hat{u}_z(s) & [1 + (s-r_\lambda)X_{2,2}]\hat{u}_z(s) \\ \mathcal{D}_z(s)\langle\mathcal{F}_z^*(s)\hat{\psi},\cdot\rangle & -X_{2,2} \end{pmatrix}. \tag{14.2.17}$$

Now we study the operator (14.2.17) when z belongs to the boundary of Π, that is, $z = \lambda \pm i0$. It will be assumed that $z = \lambda + i0$, but all the equalities and assertions have analogues for $z = \lambda - i0$ too. If $z = \lambda + i0$, then, using the definition (14.2.11) of $\mathcal{D}_z(s)$ and noting that

$$\langle\hat{u}_{\bar{z}}(s),\hat{\psi}\rangle = \langle\hat{\psi},\hat{u}_z(s)\rangle,$$

one can see that the $(1,2)$-entry of (14.2.17) vanishes and therefore

$$A_{\lambda+i0}(s) = \begin{pmatrix} \hat{A}_{\lambda+i0}(s) + (r_\lambda - s)\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}_{\lambda+i0}^*(s)\hat{\psi},\cdot\rangle\hat{u}_{\lambda+i0}(s) & 0 \\ \mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}_{\lambda+i0}^*(s)\hat{\psi},\cdot\rangle & (s-r_\lambda)^{-1} \end{pmatrix}. \tag{14.2.18}$$

Hence, the resonance equation of order k (see (3.1.1))

$$[1 + (r_\lambda - s)A_{\lambda+i0}(s)]^k u = 0$$

takes the form

$$\begin{pmatrix} \mathcal{F}_{\lambda+i0}(s) + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}_{\lambda+i0}^*(s)\hat{\psi},\cdot\rangle\hat{u}_{\lambda+i0}(s) & 0 \\ (r_\lambda - s)\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}_{\lambda+i0}^*(s)\hat{\psi},\cdot\rangle & 0 \end{pmatrix}^k u = 0. \tag{14.2.19}$$

Hence, the vector space $\Upsilon^1_{\lambda+i0}(r_\lambda)$ of solutions of this equation when $k=1$ consists of all vectors of the form

$$\begin{pmatrix}\hat{u}\\ b\end{pmatrix},$$

where $b\in\mathbb{C}$ and $\hat{u}$ is a solution of

$$\mathcal{F}_{\lambda+i0}(s)\hat{u}=0 \quad\text{and}\quad \langle\mathcal{F}^*_{\lambda+i0}(s)\hat{\psi},\hat{u}\rangle=0. \tag{14.2.20}$$

The vector space of resonance vectors of order $\le k$ for the pair $(\hat{H}_s,\hat{V})$ at $s=r_\lambda$ will be denoted by $\hat{\Upsilon}^k_{\lambda\pm i0}(r_\lambda)$. In particular, $\hat{u}\in\hat{\Upsilon}^1_{\lambda+i0}(r_\lambda)$ if and only if $\mathcal{F}_{\lambda+i0}(s)\hat{u}=0$. Since the second equality of (14.2.20) follows from the first one, we have

$$\Upsilon^1_{\lambda+i0}(r_\lambda)=\hat{\Upsilon}^1_{\lambda+i0}(r_\lambda)\oplus\mathbb{C}.$$

In fact, the condition (14.2.4), which says that λ is a regular point of $(\hat{H}_{r_\lambda},\hat{F})$, is equivalent to $\Upsilon^1_{\lambda+i0}(\hat{H}_{r_\lambda},\hat{V})=\{0\}$, and therefore

$$\Upsilon^1_{\lambda+i0}(r_\lambda)=\{0\}\oplus\mathbb{C}. \tag{14.2.21}$$

We introduce the following notation for convenience.

NOTATION. Let $j=-1,0,1,2,\dots$. We define

$$\hat{u}^{(j)}_{\lambda+i0}(s)=\mathcal{F}^{-j}_{\lambda+i0}(s)\hat{u}_{\lambda+i0}(r_\lambda). \tag{14.2.22}$$

The operator $\hat{A}_{\lambda+i0}(s)$ is compact, and the assumption (14.2.4) means that the operator

$$\mathcal{F}_{\lambda+i0}(s)=1+(r_\lambda-s)\hat{A}_{\lambda+i0}(s)$$

has zero kernel. Hence, it is invertible and therefore the vectors (14.2.22) are well-defined.

LEMMA 14.2.5. *We have*

$$\mathcal{F}^{-1}_{\lambda+i0}(s)\hat{u}_{\lambda+i0}(s)=\hat{u}_{\lambda+i0}(r_\lambda). \tag{14.2.23}$$

That is,

$$\hat{u}^{(-1)}_{\lambda+i0}(s)=\hat{u}_{\lambda+i0}(s).$$

In particular, the vector $\mathcal{F}^{-1}_{\lambda+i0}(s)\hat{u}_{\lambda+i0}(s)$ does not depend on s.

Proof. This follows from (2.7.4) (or rather its proof) and the definition (14.2.10) of $\hat{u}_{\lambda+i0}(s)$. ∎

Plainly, also $\hat{u}^{(0)}_{\lambda+i0}(s)=\hat{u}_{\lambda+i0}(r_\lambda)$.

LEMMA 14.2.6. *Let H_s, V and F be as above. For each $j=1,2,\dots$ the resonance vector space $\Upsilon^j_{\lambda+i0}(r_\lambda)$ is the linear span of the j vectors*

$$\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}\hat{u}_{\lambda+i0}(r_\lambda)\\0\end{pmatrix},\ \begin{pmatrix}\hat{u}^{(1)}_{\lambda+i0}(s)\\0\end{pmatrix},\ \dots,\ \begin{pmatrix}\hat{u}^{(j-2)}_{\lambda+i0}(s)\\0\end{pmatrix}. \tag{14.2.24}$$

In particular, $\dim\Upsilon^j_{\lambda+i0}(r_\lambda)\le j$.

Proof. For $j=1$ this has already been observed (see (14.2.21)). Assume that $\left(\begin{smallmatrix}\hat{\phi}\\ \hat{a}\end{smallmatrix}\right)$ is a vector of order two, that is, $\left(\begin{smallmatrix}\hat{\phi}\\ \hat{a}\end{smallmatrix}\right)$ is a solution of (14.2.19) with $k=2$ and $\hat{\phi}\ne 0$. Applying to this vector the operator $[1+(r_\lambda-s)A_{\lambda+i0}(s)]$ gives a vector of order 1.

Since by (14.2.21) such a vector has the form $\binom{0}{b}$ with non-zero b, the first component of $[1+(r_\lambda - s)A_{\lambda+i0}(s)]\binom{\hat\phi}{a}$ must be zero:

$$\mathcal{F}_{\lambda+i0}(s)\hat\phi + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}^*_{\lambda+i0}(s)\hat\psi,\hat\phi\rangle\hat u_{\lambda+i0}(s) = 0, \tag{14.2.25}$$

and the second component must be non-zero:

$$\langle\mathcal{F}^*_{\lambda+i0}(s)\hat\psi,\hat\phi\rangle \neq 0.$$

Applying $\mathcal{F}^{-1}_{\lambda+i0}(s)$ to (14.2.25) and using (14.2.23) gives

$$\hat\phi + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}^*_{\lambda+i0}(s)\hat\psi,\hat\phi\rangle\hat u_{\lambda+i0}(r_\lambda) = 0. \tag{14.2.26}$$

It follows that if $\binom{\hat\phi}{a}$ is a vector of order two, then $\hat\phi$ has to be collinear with $\hat u_{\lambda+i0}(r_\lambda)$. Hence $\Upsilon^2_{\lambda+i0}(r_\lambda)$ has dimension ≤ 2 and $\Upsilon^2_{\lambda+i0}(r_\lambda)$ is a subspace of the linear span of $\binom{0}{1}$ and

$$\begin{pmatrix}\mathcal{F}^{-1}_{\lambda+i0}(s)\hat u_{\lambda+i0}(s)\\ 0\end{pmatrix} = \begin{pmatrix}\hat u_{\lambda+i0}(r_\lambda)\\ 0\end{pmatrix}.$$

This proves the assertion for $j=2$.

Now, assuming that the assertion holds for $j=k$, we will show it for $j=k+1$. Let $\binom{\hat\phi}{a}$ be a vector of order $\leq k+1$. Then

$$[1+(r_\lambda - s)A_{\lambda+i0}(s)]\begin{pmatrix}\hat\phi\\ a\end{pmatrix}$$

has order $\leq k$. By the induction assumption, the first component of this vector, given by the left hand side of (14.2.25), is a linear combination of

$$\hat u_{\lambda+i0}(r_\lambda),\ \hat u^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat u^{(k-2)}_{\lambda+i0}(s).$$

Thus,

$$\hat\phi + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}^*_{\lambda+i0}(s)\hat\psi,\hat\phi\rangle\mathcal{F}^{-1}_{\lambda+i0}(s)\hat u_{\lambda+i0}(s)$$

is a linear combination of

$$\mathcal{F}^{-1}_{\lambda+i0}(s)\hat u_{\lambda+i0}(r_\lambda) = u^{(1)}_{\lambda+i0}(s),\ \hat u^{(2)}_{\lambda+i0}(s),\ \ldots,\ \hat u^{(k-1)}_{\lambda+i0}(s).$$

It now follows from (14.2.23) that $\hat\phi$ is a linear combination of

$$\hat u_{\lambda+i0}(r_\lambda),\ \hat u^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat u^{(k-1)}_{\lambda+i0}(s).\ \blacksquare$$

LEMMA 14.2.7. *The order of the real resonance point r_λ is greater than* 1 *if and only if $\alpha = 0$. In that case the vector space $\Upsilon^2_{\lambda+i0}(r_\lambda)$ is two-dimensional and is generated by*

$$F\chi = \begin{pmatrix}0\\1\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}\hat u_{\lambda+i0}(r_\lambda)\\ 0\end{pmatrix},$$

which have orders 1 *and* 2 *respectively.*

Proof. By Lemma 14.2.6, a resonance vector of order ≤ 2 has the form

$$\begin{pmatrix}\hat u_{\lambda+i0}(r_\lambda)\\ b\end{pmatrix}.$$

The vector

$$\begin{pmatrix}\hat u_{\lambda+i0}(r_\lambda)\\ 0\end{pmatrix}$$

is a resonance vector of order 2 if and only if

$$[1+(r_\lambda - s)A_{\lambda+i0}(s)]\begin{pmatrix}\hat{u}_{\lambda+i0}(r_\lambda)\\ 0\end{pmatrix}$$

is a vector of order 1, and thus has the form $\binom{0}{b}$. That is, the first component of this vector is zero:

$$\mathcal{F}_{\lambda+i0}(s)\hat{u}_{\lambda+i0}(r_\lambda) + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}^*_{\lambda+i0}(s)\hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle\hat{u}_{\lambda+i0}(s) = 0.$$

Applying the operator $\mathcal{F}^{-1}_{\lambda+i0}(s)$ and using Lemma 14.2.5 we infer that this equality is equivalent to

$$1 + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle = 0.$$

The definition (14.2.11) of $\mathcal{D}_{\lambda+i0}(s)$ implies that this is equivalent to

$$(s-r_\lambda)^2\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle = (s-r_\lambda)^2\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle - \alpha(s-r_\lambda).$$

Hence the order d of r_λ is greater than 1 if and only if $\alpha = 0$. ∎

Since throughout this section we are assuming that V is a regularizing direction, Lemma 14.2.7 combined with Lemma 14.2.2 imply

COROLLARY 14.2.8. *If the order d of the real resonance point r_λ is greater than 1, then for some real number s,*

$$\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle \neq 0.$$

Since the vector spaces $\Upsilon^j_z(r_z)$ have the stability property $\Upsilon^j_z(r_z) = \Upsilon^{j+1}_z(r_z) \Rightarrow \Upsilon^j_z(r_z) = \Upsilon_z(r_z)$, Lemma 14.2.6 leads to

THEOREM 14.2.9. *Let d be an integer ≥ 2. The following assertions are equivalent:*

(1) *The order of the real resonance point r_λ is equal to d.*
(2) *The dimension of the vector space $\Upsilon_{\lambda+i0}(r_\lambda)$ is equal to d.*
(3) *The vectors*

$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-2)}_{\lambda+i0}(s)$$

are linearly independent and $\hat{u}^{(d-1)}_{\lambda+i0}(s)$ is their linear combination.

Further, if the order of r_λ is equal to d, then for all $j = 1, \ldots, d$ the vector space $\Upsilon^j_{\lambda+i0}(r_\lambda)$ is j-dimensional and is generated by

$$\begin{pmatrix}0\\1\end{pmatrix},\ \begin{pmatrix}\hat{u}_{\lambda+i0}(r_\lambda)\\0\end{pmatrix},\ \begin{pmatrix}\hat{u}^{(1)}_{\lambda+i0}(s)\\0\end{pmatrix},\ \ldots,\ \begin{pmatrix}\hat{u}^{(j-2)}_{\lambda+i0}(s)\\0\end{pmatrix},$$

which have orders $1, \ldots, d$ respectively.

This theorem gives a criterion for the order of r_λ to be equal to d but is not very tangible. To get a better criterion, one needs to find out when a vector

$$\begin{pmatrix}\hat{u}^{(j-2)}_{\lambda+i0}(s)\\0\end{pmatrix}, \qquad j = 1, 2, \ldots,$$

is a resonance vector of order j. Lemma 14.2.7 gives an answer in the case of $j = 2$.

THEOREM 14.2.10. *Let d be an integer ≥ 2. The order of the real resonance point r_λ is d if and only if for some real s, and thus for any real s, the vectors*

$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-3)}_{\lambda+i0}(s) \tag{14.2.27}$$

are all orthogonal to $\hat{\psi}$ but the vector $\hat{u}^{(d-2)}_{\lambda+i0}(s)$ is not.

Proof. It can be seen that it is enough to prove the following assertion: the order of the real resonance point r_λ is not less than d if and only if for some s the vectors

$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-3)}_{\lambda+i0}(s)$$

are all orthogonal to $\hat{\psi}$. We prove this using induction on $d = 2, 3, \ldots$.

According to Theorem 14.2.9, the real resonance point r_λ has order ≥ 3 if and only if the vector

$$\begin{pmatrix} \hat{u}^{(1)}_{\lambda+i0}(s) \\ 0 \end{pmatrix}$$

is a resonance vector of order 3. This happens if and only if

$$[1 + (r_\lambda - s)A_{\lambda+i0}(s)] \begin{pmatrix} \hat{u}^{(1)}_{\lambda+i0}(s) \\ 0 \end{pmatrix} \tag{14.2.28}$$

is a vector of order 2, which by Theorem 14.2.9 is collinear to a vector of the form $\left(\begin{smallmatrix} \hat{u}_{\lambda+i0}(r_\lambda) \\ b \end{smallmatrix}\right)$. We calculate the first component of the vector (14.2.28):

$$\begin{aligned} \mathcal{F}_{\lambda+i0}(s)\hat{u}^{(1)}_{\lambda+i0}(s) &+ (s - r_\lambda)^2 \mathcal{D}_{\lambda+i0}(s)\langle \mathcal{F}^*_{\lambda+i0}(s)\hat{\psi}, \hat{u}^{(1)}_{\lambda+i0}(s)\rangle \hat{u}_{\lambda+i0}(s) \\ &= \hat{u}_{\lambda+i0}(r_\lambda) + (s - r_\lambda)^2 \mathcal{D}_{\lambda+i0}(s)\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle \hat{u}_{\lambda+i0}(s) \\ &= \hat{u}_{\lambda+i0}(r_\lambda) - \frac{\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle}{\langle \hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle} \hat{u}_{\lambda+i0}(s), \end{aligned}$$

where the second equality follows from (14.2.11) and $\alpha = 0$. Hence, v is a resonance vector of order 3 if and only if

$$\hat{u}_{\lambda+i0}(r_\lambda) - \frac{\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle}{\langle \hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle} \hat{u}_{\lambda+i0}(s)$$

is non-zero and collinear to $\hat{u}_{\lambda+i0}(r_\lambda)$. On the other hand, by Theorem 14.2.9 the order of v is 3 if and only if $\hat{u}_{\lambda+i0}(r_\lambda)$ and $\hat{u}^{(1)}_{\lambda+i0}(s)$ are linearly independent. Since the operator $\mathcal{F}_{\lambda+i0}(s)$ is invertible, this holds if and only if

$$\mathcal{F}_{\lambda+i0}(s)\hat{u}_{\lambda+i0}(r_\lambda) = \hat{u}_{\lambda+i0}(s) \quad \text{and} \quad \mathcal{F}_{\lambda+i0}(s)\hat{u}^{(1)}_{\lambda+i0}(s) = \hat{u}_{\lambda+i0}(r_\lambda)$$

are linearly independent. We conclude that v has order 3 if and only if

$$\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle = 0.$$

If this is the case then the vector space $\Upsilon^3_{\lambda+i0}(r_\lambda)$ is 3-dimensional and is generated by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \hat{u}^{(1)}_{\lambda+i0}(s) \\ 0 \end{pmatrix},$$

which have orders 1, 2 and 3 respectively. We have also proved that $d = 2$ if and only if

$$\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda)\rangle \neq 0.$$

This gives the induction base.

Now assuming that the assertion holds for the order of r_λ less than d we will prove it for the order of r_λ equal to d. According to Theorem 14.2.9, the real resonance point r_λ has order $\geq d$ iff the vector

$$\begin{pmatrix} \hat{u}^{(d-2)}_{\lambda+i0}(s) \\ 0 \end{pmatrix}$$

is a resonance vector of order d. This in turn is equivalent to

$$[1 + (r_\lambda - s)A_{\lambda+i0}(s)]\begin{pmatrix} \hat{u}^{(d-2)}_{\lambda+i0}(s) \\ 0 \end{pmatrix} \tag{14.2.29}$$

being a vector of order $d-1$; by Lemma 14.2.6 it is a linear combination of

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \hat{u}_{\lambda+i0}(r_\lambda) \\ 0 \end{pmatrix},\ \begin{pmatrix} \hat{u}^{(1)}_{\lambda+i0}(s) \\ 0 \end{pmatrix},\ \ldots,\ \begin{pmatrix} \hat{u}^{(d-3)}_{\lambda+i0}(s) \\ 0 \end{pmatrix}.$$

The first component of (14.2.29) is

$$\begin{aligned} &\mathcal{F}_{\lambda+i0}(s)\hat{u}^{(d-2)}_{\lambda+i0}(s) + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\mathcal{F}^*_{\lambda+i0}(s)\hat{\psi}, \hat{u}^{(d-2)}_{\lambda+i0}(s)\rangle\hat{u}_{\lambda+i0}(s) \\ &\qquad = \hat{u}^{(d-3)}_{\lambda+i0}(s) + (s-r_\lambda)^2\mathcal{D}_{\lambda+i0}(s)\langle\hat{\psi}, \hat{u}^{(d-3)}_{\lambda+i0}(s)\rangle\hat{u}_{\lambda+i0}(s) \\ &\qquad = \hat{u}^{(d-3)}_{\lambda+i0}(s) - \frac{\langle\hat{\psi}, \hat{u}^{(d-3)}_{\lambda+i0}(s)\rangle}{\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle}\hat{u}_{\lambda+i0}(s). \end{aligned} \tag{14.2.30}$$

Thus, the order of r_λ is $\geq d$ iff this vector is a linear combination of

$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-3)}_{\lambda+i0}(s).$$

By Theorem 14.2.9, the order r_λ is $\geq d$ iff the vectors

$$\hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-2)}_{\lambda+i0}(s)$$

are linearly independent. Since the operator $\mathcal{F}_{\lambda+i0}(s)$ is invertible, this holds iff

$$\hat{u}_{\lambda+i0}(s),\ \hat{u}_{\lambda+i0}(r_\lambda),\ \hat{u}^{(1)}_{\lambda+i0}(s),\ \ldots,\ \hat{u}^{(d-3)}_{\lambda+i0}(s)$$

are linearly independent. Hence the order of r_λ is $\geq d$ iff the coefficient of $\hat{u}_{\lambda+i0}(s)$ in (14.2.30) is zero, that is, iff $\langle\hat{\psi}, \hat{u}^{(d-3)}_{\lambda+i0}(s)\rangle = 0$. Combined with the induction assumption, this completes the proof. ∎

THEOREM 14.2.11. *Let d be an integer greater than 1 and let*

$$\hat{u}_+ = \hat{u}_{\lambda+i0}(r_\lambda).$$

The order of the real resonance point r_λ is equal to d if and only if the vectors

$$\hat{u}_+,\ \hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_+,\ \ldots,\ \hat{A}^{d-3}_{\lambda+i0}(r_\lambda)\hat{u}_+ \tag{14.2.31}$$

are orthogonal to $\hat{\psi}$ but $\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_+$ is not. If this is the case, then for all $j = 1, \ldots, d$ the vector space $\Upsilon^j_{\lambda+i0}(r_\lambda)$ is j-dimensional and is generated by the vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \hat{u}_+ \\ 0 \end{pmatrix},\ \begin{pmatrix} \hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_+ \\ 0 \end{pmatrix},\ \ldots,\ \begin{pmatrix} \hat{A}^{j-2}_{\lambda+i0}(r_\lambda)\hat{u}_+ \\ 0 \end{pmatrix}, \tag{14.2.32}$$

which have orders $1, \ldots, j$ respectively.

Proof. By Lemma 14.2.3 and the definition (14.2.22) of $\hat{u}^{(j)}_{\lambda+i0}(s)$ we have

$$\hat{u}^{(j)}_{\lambda+i0}(s) = [1 + (s - r_\lambda)\hat{A}_{\lambda+i0}(r_\lambda)]^j \hat{u}_{\lambda+i0}(r_\lambda).$$

Hence, the assertion is a direct consequence of Theorem 14.2.10. ∎

Corollary 14.2.12. *Under the conditions of Theorem* 14.2.11*, if r_λ has order d then the vector space $\Psi_{\lambda+i0}(r_\lambda)$ is d-dimensional and is generated by the vectors*

$$\begin{pmatrix}\hat{\psi}\\0\end{pmatrix},\ \begin{pmatrix}\hat{J}\hat{u}_+\\0\end{pmatrix},\ \dots,\ \begin{pmatrix}\hat{J}\hat{A}^{d-3}_{\lambda+i0}(r_\lambda)\hat{u}_+\\0\end{pmatrix},\ \begin{pmatrix}\hat{J}\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_+\\\langle\hat{\psi},\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_+\rangle\end{pmatrix},$$

which have orders $1,\dots,d$ respectively. Further, the second component of the last vector is non-zero.

Proof. By Lemma 3.1.4, $\Psi_{\lambda+i0}(r_\lambda)$ is the image of $\Upsilon_{\lambda+i0}(r_\lambda)$ under the mapping J given by (14.2.5). Applying J to the vectors (14.2.32), which by Theorem 14.2.11 generate $\Upsilon^j_{\lambda+i0}(r_\lambda)$, one infers that the d vectors

$$\begin{pmatrix}\hat{\psi}\\0\end{pmatrix},\ \begin{pmatrix}\hat{J}\hat{u}_+\\\langle\hat{\psi},\hat{u}_+\rangle\end{pmatrix},\ \begin{pmatrix}\hat{J}\hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_+\\\langle\hat{\psi},\hat{A}_{\lambda+i0}(r_\lambda)\hat{u}_+\rangle\end{pmatrix},\ \dots,\ \begin{pmatrix}\hat{J}\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_+\\\langle\hat{\psi},\hat{A}^{d-2}_{\lambda+i0}(r_\lambda)\hat{u}_+\rangle\end{pmatrix}$$

form a basis of $\Psi_{\lambda+i0}(r_\lambda)$. It remains to note that by Theorem 14.2.11 the second components of all these vectors except the last one are zero. ∎

14.3. Type I vectors for an embedded eigenvalue. In order to simplify formulas, we write

$$\hat{u}_\pm = \hat{u}_{\lambda\pm i0}(r_\lambda), \qquad \hat{A}_\pm = \hat{A}_{\lambda\pm i0}(r_\lambda), \qquad \hat{B}_\pm = \hat{B}_{\lambda\pm i0}(r_\lambda). \tag{14.3.1}$$

For convenience we set

$$a_{j,\pm} := \langle\hat{\psi}, \hat{A}^j_{\lambda\pm i0}(r_\lambda)\hat{u}_{\lambda\pm i0}(r_\lambda)\rangle. \tag{14.3.2}$$

In what follows, a vector $f \in \hat{\mathcal{K}}$ will often be identified with $\binom{f}{0} \in \mathcal{K}$. Also, $\binom{0}{1} \in \mathcal{K}$ will be written as 1. By Theorem 14.2.11, the vectors

$$\hat{A}^{d-2}_+\hat{u}_+,\ \hat{A}^{d-3}_+\hat{u}_+,\ \dots,\ \hat{A}_+\hat{u}_+,\ \hat{u}_+,\ 1$$

form a basis of $\Upsilon_{\lambda+i0}(r_\lambda)$. By Corollary 14.2.12, the vectors

$$\hat{B}^{d-1}_-\hat{\psi} + a_{d-2,-},\ \hat{B}^{d-2}_-\hat{\psi},\ \dots,\ \hat{B}_-\hat{\psi},\ \hat{\psi}$$

form a basis of $\Psi_{\lambda-i0}(r_\lambda)$.

The following lemma is a direct consequence of Theorem 8.3.1, but still we give another proof.

Lemma 14.3.1. *Let k be a positive integer. If $d \ge 2k+1$, then $\hat{u}_+ = \hat{u}_-$, $\hat{A}_+\hat{u}_+ = \hat{A}_-\hat{u}_-$, $\dots$, $\hat{A}^{k-1}_+\hat{u}_+ = \hat{A}^{k-1}_-\hat{u}_-$.*

Proof. If $k = 1$ then $d \ge 3$, and therefore, by Theorem 14.2.11, $a_{0,+}$ and $a_{0,-}$ are zero, that is,

$$0 = \langle\hat{\psi}, \hat{u}_\pm\rangle = \langle\hat{\psi}, \hat{T}_\pm\hat{\psi}\rangle.$$

It follows that $\sqrt{\operatorname{Im}\hat{T}_+}\,\hat{\psi} = 0$, and therefore $\hat{T}_+\hat{\psi} = \hat{T}_-\hat{\psi}$, that is, $\hat{u}_+ = \hat{u}_-$.

Assume that the assertion is true for $k = n$ and let $k = n + 1$. Then $d \geq 2n + 3$ and therefore, by Theorem 14.2.11, $a_{2n,\pm} = 0$, that is,

$$\langle\hat{\psi}, (\hat{T}_+ J)^{2n}\hat{T}_+\hat{\psi}\rangle = 0.$$

This implies that

$$0 = \langle(J\hat{T}_-)^n\hat{\psi}, (\hat{T}_+ J)^n\hat{T}_+\hat{\psi}\rangle = \langle J\hat{A}_-^{n-1}\hat{u}_+, \hat{T}_+ J\hat{A}_+^{n-1}\hat{u}_+\rangle = \langle J\hat{A}_+^{n-1}\hat{u}_+, \hat{T}_+ J\hat{A}_+^{n-1}\hat{u}_+\rangle,$$

where the last equality follows from the induction assumption. So $(\operatorname{Im}\hat{T}_+)J\hat{A}_+^{n-1}\hat{u}_+ = 0$, that is,

$$\hat{A}_+\hat{A}_+^{n-1}\hat{u}_+ = \hat{A}_-\hat{A}_+^{n-1}\hat{u}_+ = \hat{A}_-\hat{A}_-^{n-1}\hat{u}_-,$$

where the last equality follows from the induction assumption. ∎

14.4. The idempotents $P_{\lambda\pm i0}(r_\lambda)$ and $Q_{\lambda\pm i0}(r_\lambda)$. In this subsection we calculate $P_{\lambda\pm i0}(r_\lambda)$. Since by (14.2.4) the operator-function $T_{\lambda+i0}(\hat{H}_s)$ is holomorphic at $s = r_\lambda$, the functions $T_{\lambda+i0}(\hat{H}_s)$ and $\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle$ can be expanded into a Taylor series convergent in some neighbourhood of $s = r_\lambda$ as follows:

$$T_{\lambda+i0}(\hat{H}_s) = \sum_{k=0}^{\infty}(-1)^k(s - r_\lambda)^k\hat{A}_{\lambda+i0}^k(r_\lambda)T_{\lambda+i0}(\hat{H}_{r_\lambda}).$$

Hence

$$\begin{aligned}\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle &= \langle\hat{\psi}, T_{\lambda+i0}(\hat{H}_s)\hat{\psi}\rangle \\ &= a_{0,+} - a_{1,+}(s - r_\lambda) + a_{2,+}(s - r_\lambda)^2 - a_{3,+}(s - r_\lambda)^3 + \cdots . \qquad (14.4.1)\end{aligned}$$

If d is the order of r_λ, then by Theorem 14.2.11, we have

$$a_{0,\pm} = a_{1,\pm} = \cdots = a_{d-3,\pm} = 0$$

and the number $a_{d-2,\pm}$ is non-zero.

We shall need a Taylor series expansion for the function

$$(r_\lambda - s)\mathcal{D}_{\lambda+i0}(s) = -\frac{1}{\alpha + (r_\lambda - s)\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle}.$$

For this, we write the first few terms of the Taylor expansion of the inverse function:

$$\begin{aligned}&(c_0 - c_1(s - r_\lambda) + c_2(s - r_\lambda)^2 - c_3(s - r_\lambda)^3 + \cdots)^{-1} \\ &\quad = \frac{1}{c_0} + \frac{c_1}{c_0^2}(s - r_\lambda) + \frac{c_1^2 - c_0c_2}{c_0^3}(s - r_\lambda)^2 + \frac{c_1^3 - 2c_0c_1c_2 + c_3c_0^2}{c_0^4}(s - r_\lambda)^3 \\ &\qquad + \frac{c_1^4 - 3c_0c_1^2c_2 + 2c_0^2c_1c_3 + c_0^2c_2^2 - c_0^3c_4}{c_0^5}(s - r_\lambda)^4 + \cdots . \qquad (14.4.2)\end{aligned}$$

Using (14.2.18) for $A_{\lambda+i0}(s)$ and Proposition 3.2.3, one can calculate the idempotent $P_{\lambda+i0}(r_\lambda)$ for points of small order.

14.4.1. Order $d = 1$. By Lemma 14.2.6, the order d of the real resonance point r_λ is 1 if and only if $\alpha \neq 0$. If $\alpha \neq 0$, then the $(1,1)$-entry of the matrix (14.2.18) is holomorphic at $s = r_\lambda$, and therefore its residue vanishes. Hence, in this case the $(1,1)$-entries of

$P_{\lambda\pm i0}(r_\lambda)$ are also zero, and as a result these idempotents have rank one:

$$P_{\lambda\pm i0}(r_\lambda) = \begin{pmatrix} 0 & 0 \\ \alpha^{-1}\langle\hat{\psi},\cdot\rangle & 1 \end{pmatrix}.$$

It follows that

$$Q_{\lambda\pm i0}(r_\lambda) = \begin{pmatrix} 0 & \alpha^{-1}\hat{\psi} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = \begin{pmatrix} \alpha^{-1}\langle\hat{\psi},\cdot\rangle\hat{\psi} & \hat{\psi} \\ \langle\hat{\psi},\cdot\rangle & \alpha \end{pmatrix}.$$

Hence, $Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda)$ is a rank one operator with range generated by $\left(\begin{smallmatrix}\hat{\psi}\\ \alpha\end{smallmatrix}\right)$. Also, in this case the operators $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$ are zero.

14.4.2. Order $d = 2$. By Theorem 14.2.11, in this case $\alpha = 0$ and $\langle\hat{\psi}, \hat{u}_+\rangle \neq 0$. Since $\alpha = 0$, it follows from (14.2.11), (14.4.1) and (14.4.2) that

$$\begin{aligned} -\mathcal{D}_{\lambda+i0}(s) &= \frac{1}{(s-r_\lambda)^2\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle} \\ &= \frac{1}{(s-r_\lambda)^2}(a_{0,+} - a_{1,+}(s-r_\lambda) + a_{2,+}(s-r_\lambda)^2 - \cdots)^{-1} \\ &= \frac{1}{a_{0,+}}(s-r_\lambda)^{-2} + \frac{a_{1,+}}{a_{0,+}^2}(s-r_\lambda)^{-1} + \frac{a_{1,+}^2 - a_{0,+}a_{2,+}}{a_{0,+}^3} + \cdots. \end{aligned}$$

Hence the coefficient of $(s-r_\lambda)^{-1}$ in the $(1,1)$-entry of $A_{\lambda+i0}(s)$ is $a_{0,+}^{-1}\langle\hat{\psi},\cdot\rangle\hat{u}_+$ and the coefficient of $(s-r_\lambda)^{-1}$ in the $(2,1)$-entry of $A_{\lambda+i0}(s)$ is

$$-\frac{a_{1,+}}{a_{0,+}^2}\langle\hat{\psi},\cdot\rangle + \frac{1}{a_{0,+}}\langle\hat{B}_-\hat{\psi},\cdot\rangle.$$

Therefore,

$$P_{\lambda+i0}(r_\lambda) = \begin{pmatrix} \frac{1}{a_{0,+}}\langle\hat{\psi},\cdot\rangle\hat{u}_+ & 0 \\ -\frac{a_{1,+}}{a_{0,+}^2}\langle\hat{\psi},\cdot\rangle + \frac{1}{a_{0,+}}\langle\hat{B}_-\hat{\psi},\cdot\rangle & 1 \end{pmatrix}. \tag{14.4.3}$$

Similarly,

$$P_{\lambda-i0}(r_\lambda) = \begin{pmatrix} \frac{1}{a_{0,-}}\langle\hat{\psi},\cdot\rangle\hat{u}_- & 0 \\ -\frac{a_{1,-}}{a_{0,-}^2}\langle\hat{\psi},\cdot\rangle + \frac{1}{a_{0,-}}\langle\hat{B}_+\hat{\psi},\cdot\rangle & 1 \end{pmatrix}. \tag{14.4.4}$$

Using these equalities one can check that in general

$$P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) \neq P_{\lambda+i0}(r_\lambda).$$

It follows from Proposition 13.3.1 that the real resonance point r_λ in general does not have property S. Further, since by (3.3.16) the operator $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$ is the coefficient of $(s-r_\lambda)^{-2}$ in the Laurent expansion of $A_{\lambda+i0}(s)$ at $r = r_\lambda$, it follows from (14.2.18) that

$$\mathbf{A}_{\lambda\pm i0}(r_\lambda) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\langle\hat{\psi},\hat{u}_\pm\rangle}\langle\hat{\psi},\cdot\rangle & 0 \end{pmatrix}.$$

One can calculate that

$$Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) = \begin{pmatrix} (*) & \hat{\psi} \\ \langle\hat{\psi},\cdot\rangle & 0 \end{pmatrix},$$

where

$$(*) = \frac{1}{|a_{0,+}|^2}\langle \hat{u}_+, \hat{B}_+\hat{\psi}\rangle\langle\hat{\psi},\cdot\rangle\hat{\psi} - 2\operatorname{Re}\frac{a_{1,+}}{a_{0,+}^2}\langle\hat{\psi},\cdot\rangle\hat{\psi} + \frac{1}{a_{0,+}}\langle\hat{B}_-\hat{\psi},\cdot\rangle\hat{\psi} + \frac{1}{a_{0,-}}\langle\hat{\psi},\cdot\rangle\hat{B}_-\hat{\psi}.$$

A similar equality holds for $Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda)$, which shows that in general

$$Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) \neq Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda).$$

It is another way to see that in general a real resonance point r_λ does not have property S.

From (14.4.3) and (14.4.4) one can see that the kernel of the idempotent $P_{\lambda\pm i0}(r_\lambda)$ is

$$\ker P_{\lambda\pm i0}(r_\lambda) = \{\hat{\phi} - \langle\hat{\psi},\hat{u}_\pm\rangle^{-1}\langle\hat{\psi},\hat{A}_\pm\hat{\phi}\rangle\cdot 1\colon \hat{\phi}\perp\hat{\psi}\}.$$

If $\hat{J}=0$, then

$$\ker P_{\lambda\pm i0}(r_\lambda) = \operatorname{span}\{1,\hat{\psi}\}^\perp.$$

Thus, in this case $\ker P_{\lambda-i0}(r_\lambda) = \ker P_{\lambda-i0}(r_\lambda)$, which is the definition of resonance points with property S. Since the vector spaces $\Upsilon_{\lambda\pm i0}(r_\lambda) = \operatorname{span}\{1,\hat{u}_\pm\}$ are in general different, there exist real resonance points with property S for which $\Upsilon_{\lambda+i0}(r_\lambda) \neq \Upsilon_{\lambda-i0}(r_\lambda)$. By Theorem 13.1.13, it follows that in this case r_λ is not a resonance point of type I. Hence, this gives an example of a resonance point of order two with property S which is not of type I.

These examples give a proof of the second part of Proposition 13.3.2.

14.4.3. Order $d=3$. By Theorem 14.2.11, in this case the vectors $\hat{u}_+$ and $\hat{\psi}$ are orthogonal while $\hat{A}_+\hat{u}_+$ and $\hat{\psi}$ are not, so that

$$a_{0,+} = \langle\hat{\psi},\hat{u}_+\rangle = 0 \quad\text{and}\quad a_{1,+} = \langle\hat{\psi},\hat{A}_+\hat{u}_+\rangle \neq 0. \tag{14.4.5}$$

The first of these two equalities implies that $\langle\hat{\psi},\operatorname{Im}T_+\hat{\psi}\rangle = 0$, and therefore $\operatorname{Im}T_+\hat{\psi}=0$. It follows that

$$\hat{u}_+ = \hat{u}_-,\quad \hat{B}_+\hat{\psi} = \hat{B}_-\hat{\psi} \quad\text{and}\quad a_{1,+} = \langle\hat{\psi},\hat{A}_-\hat{u}_-\rangle = a_{1,-}. \tag{14.4.6}$$

Further, (14.4.5) implies that the first term of the Neumann series (14.4.1) for $\langle\hat{\psi},\hat{u}_{\lambda+i0}(s)\rangle$ vanishes:

$$\langle\hat{\psi},\hat{u}_{\lambda+i0}(s)\rangle = -a_{1,+}(s-r_\lambda) + a_{2,+}(s-r_\lambda)^2 + \cdots$$

and we get

$$\begin{aligned}\mathcal{D}_{\lambda+i0}(s) &= -\frac{1}{(s-r_\lambda)^2\langle\hat{\psi},\hat{u}_{\lambda+i0}(s)\rangle}\\ &= \frac{1}{(s-r_\lambda)^3}\left(\frac{1}{a_{1,+}} + \frac{a_{2,+}}{a_{1,+}^2}(s-r_\lambda) + \frac{a_{2,+}^2 - a_{1,+}a_{3,+}}{a_{1,+}^3}(s-r_\lambda)^2 + \cdots\right).\end{aligned}$$

Also,

$$1 + (r_\lambda - s)\hat{B}_{\lambda-i0}(s) = 1 + (r_\lambda - s)\hat{B}_- + (s-r_\lambda)^2\hat{B}_-^2 + (r_\lambda - s)^3\hat{B}_-^3 + \cdots$$

and

$$\hat{u}_{\lambda+i0}(s) = \hat{u}_+ + (r_\lambda - s)\hat{A}_+\hat{u}_+ + (s-r_\lambda)^2\hat{A}_+^2\hat{u}_+ - \cdots.$$

From this we find the coefficient of $1/(s-r_\lambda)$ in the $(1,1)$-entry of $A_{\lambda+i0}(s)$, that is, the $(1,1)$-entry of $P_{\lambda+i0}(r_\lambda)$:

$$\hat{P}_+ := -\frac{a_{2,+}}{a_{1,+}^2}\langle\hat{\psi},\cdot\rangle\hat{u}_+ + \frac{1}{a_{1,+}}(\langle\hat{\psi},\cdot\rangle\hat{A}_+\hat{u}_+ + \langle\hat{B}_-\hat{\psi},\cdot\rangle\hat{u}_+),$$

and similarly we find the $(2,1)$-entry of $P_{\lambda+i0}(r_\lambda)$:

$$\frac{a_{2,+}^2 - a_{1,+}a_{3,+}}{a_{1,+}^3}\langle\hat{\psi},\cdot\rangle - \frac{a_{2,+}}{a_{1,+}^2}\langle\hat{B}_-\hat{\psi},\cdot\rangle + \frac{1}{a_{1,+}}\langle\hat{B}_-^2\hat{\psi},\cdot\rangle.$$

Hence,

$$P_{\lambda+i0}(r_\lambda) = \begin{pmatrix} -\frac{a_{2,+}}{a_{1,+}^2}\langle\hat{\psi},\cdot\rangle\hat{u}_+ + \frac{1}{a_{1,+}}(\langle\hat{\psi},\cdot\rangle\hat{A}_+\hat{u}_+ + \langle\hat{B}_-\hat{\psi},\cdot\rangle\hat{u}_+) & 0 \\ \frac{a_{2,+}^2-a_{1,+}a_{3,+}}{a_{1,+}^3}\langle\hat{\psi},\cdot\rangle - \frac{a_{2,+}}{a_{1,+}^2}\langle\hat{B}_-\hat{\psi},\cdot\rangle + \frac{1}{a_{1,+}}\langle\hat{B}_-^2\hat{\psi},\cdot\rangle & 1 \end{pmatrix}.$$

The structure of this operator becomes a bit more transparent if it is written as a matrix in the basis $(\hat{B}_-^2\hat{\psi}+a_{1,-}, \hat{B}_-\hat{\psi}, \hat{\psi})$ of the range of $Q_{\lambda-i0}(r_\lambda)$ and in the basis $(\hat{A}_+\hat{u}_+, \hat{u}_+, 1)$ of the range of $P_{\lambda+i0}(r_\lambda)$ as follows:

$$P_{\lambda+i0}(r_\lambda) = \begin{pmatrix} 0 & 0 & \frac{1}{a_{1,+}} \\ 0 & \frac{1}{a_{1,+}} & -\frac{a_{2,+}}{a_{1,+}^2} \\ \frac{1}{a_{1,+}} & -\frac{a_{2,+}}{a_{1,+}^2} & \frac{a_{2,+}^2-a_{1,+}a_{3,+}}{a_{1,+}^3} \end{pmatrix}.$$

Similarly, one can find $P_{\lambda-i0}(r_\lambda)$. Further, one can calculate that

$$\mathbf{A}_{\lambda+i0}(r_\lambda) = \begin{pmatrix} -\frac{1}{a_{1,+}}\langle\hat{\psi},\cdot\rangle\hat{u}_+ & 0 \\ \frac{a_{2,+}}{a_{1,+}^2}\langle\hat{\psi},\cdot\rangle - \frac{1}{a_{1,+}}\langle\hat{B}_-\hat{\psi},\cdot\rangle & 0 \end{pmatrix}.$$

In the pair of bases $(\hat{B}_-^2\hat{\psi}+a_{1,-}, \hat{B}_-\hat{\psi}, \hat{\psi})$ and $(\hat{A}_+\hat{u}_+, \hat{u}_+, 1)$ this operator takes the form

$$\mathbf{A}_{\lambda+i0}(r_\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{a_{1,+}} \\ 0 & -\frac{1}{a_{1,+}} & \frac{a_{2,+}}{a_{1,+}^2} \end{pmatrix}.$$

One can check that the $(1,1)$-entries $\hat{P}_+$ and $\hat{P}_-$ of $P_{\lambda+i0}(r_\lambda)$ and $P_{\lambda-i0}(r_\lambda)$ satisfy $\hat{P}_+\hat{P}_- = \hat{P}_+$. This implies that the image of $P_{\lambda+i0}(r_\lambda)P_{\lambda-i0}(r_\lambda) - P_{\lambda+i0}(r_\lambda)$ consists of vectors of order 1.

These examples also show how to calculate $P_{\lambda\pm i0}(r_\lambda)$ and $\mathbf{A}_{\lambda\pm i0}(r_\lambda)$ in the case of arbitrary order.

14.5. Example of calculation of resonance index. The function $A_{\lambda+i0}(s)$ of the coupling constant s has an eigenvalue $\sigma_\lambda(s) = (s-r_\lambda)^{-1}$. When $\lambda+i0$ is shifted to $\lambda+iy$ with small positive y, the eigenvalue $\sigma_\lambda(s)$ in general splits into $N_\pm$ non-real eigenvalues in $\mathbb{C}_\pm$ respectively. The difference $N_+ - N_-$ is the resonance index. To calculate it we need to find the eigenvalues of $A_{\lambda+iy}(s)$ which belong to the group of the eigenvalue $\sigma_\lambda(s)$, that is, which converge to $\sigma_\lambda(s)$ as $y \to 0^+$. The eigenvalue equation

$$A_z(s)u = \sigma u,$$

where $u = \binom{\hat{u}}{1}$ and where $A_z(s)$ is given in (14.2.17), leads to the equations

$$\hat{A}_z(s)\hat{u} + (r_\lambda - s)\mathcal{D}_z(s)\langle\mathcal{F}_z^*(s)\hat{\psi}, \hat{u}\rangle\hat{u}_z(s) + [1 + (s - r_\lambda)\mathcal{D}_z(s)\big((s - r_\lambda)\langle\hat{u}_{\bar{z}}(s), \hat{\psi}\rangle - \alpha\big)]\hat{u}_z(s) = \sigma\hat{u}$$

and

$$\mathcal{D}_z(s)\langle\mathcal{F}_z^*(s)\hat{\psi}, \hat{u}\rangle - \mathcal{D}_z(s)\big((s - r_\lambda)\langle\hat{u}_{\bar{z}}(s), \hat{\psi}\rangle - \alpha\big) = \sigma.$$

From the latter it follows that the former is equivalent to

$$\hat{A}_z(s)\hat{u} + \sigma(r_\lambda - s)\hat{u}_z(s) + \hat{u}_z(s) = \sigma\hat{u}.$$

We consider the case $\hat{J} = 0$. Then $\hat{A}_z(s) = 0$, $\mathcal{F}_z(s) = 1$, and $\langle\hat{\psi}, \hat{u}_z(s)\rangle$ does not depend on s and is equal to $\langle\hat{\psi}, \hat{u}_z(r_\lambda)\rangle$. The first equation becomes

$$\sigma(r_\lambda - s)\hat{u}_z(s) + \hat{u}_z(s) = \sigma\hat{u},$$

while the second turns into (using $\langle\hat{u}_{\bar{z}}(s), \hat{\psi}\rangle = \langle\hat{\psi}, \hat{u}_z(s)\rangle$)

$$\mathcal{D}_z(s)\big(\langle\hat{\psi}, \hat{u}\rangle + (r_\lambda - s)\langle\hat{\psi}, \hat{u}_z(s)\rangle + \alpha\big) = \sigma.$$

If we exclude the vector $\hat{u}$ from these two equations we obtain the following quadratic equation for σ:

$$\sigma^2 - \sigma\mathcal{D}_z(s)(2(r_\lambda - s)\langle\hat{\psi}, \hat{u}_z(s)\rangle + \alpha) - \mathcal{D}_z(s)\langle\hat{\psi}, \hat{u}_z(s)\rangle = 0. \tag{14.5.1}$$

We consider first the case of $\alpha = 0$. Then by the definition (14.2.11) of $\mathcal{D}_z(s)$ we have

$$\mathcal{D}_z(s) = -(iy + (s - r_\lambda)^2\langle\hat{\psi}, \hat{u}_z(s)\rangle)^{-1},$$

where as usual $z = \lambda + iy$. Let

$$w(y) = -\mathcal{D}_z(s)\langle\hat{\psi}, \hat{u}_z(s)\rangle = \langle\hat{\psi}, \hat{u}_z(s)\rangle(iy + (s - r_\lambda)^2\langle\hat{\psi}, \hat{u}_z(s)\rangle)^{-1}. \tag{14.5.2}$$

The equation (14.5.1) for σ then becomes

$$\sigma^2 - 2(s - r_\lambda)w\sigma + w = 0.$$

Its roots are

$$\sigma_{1,2}(y) = (s - r_\lambda)w \pm \sqrt{(s - r_\lambda)^2w^2 - w},$$

where we agree that the complex square root belongs to either the upper half-plane or the positive semi-axis. From (14.5.2) one can find that as $y \to 0^+$,

$$w(y) = (s - r_\lambda)^{-2} - \frac{iy}{\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle}(s - r_\lambda)^{-4} + O(y^2).$$

Consequently,

$$(s - r_\lambda)^2w^2 - w = -\frac{iy}{\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle}(s - r_\lambda)^{-4} + O(y^2).$$

Let $\rho e^{i2\theta}$ be the polar form of $i/\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle$. Then one can see that

$$\sigma_{1,2}(y) = (s - r_\lambda)^{-1} \pm \sqrt{\rho y}\, e^{i\theta}(s - r_\lambda)^{-2} + O(y).$$

Since $\langle\hat{\psi}, \hat{u}_{\lambda+i0}(s)\rangle \neq 0$, it follows that the roots approach $(s - r_\lambda)^{-1}$ from different half-planes $\mathbb{C}_\pm$. Therefore, in the case of $\alpha = 0$ the resonance index is equal to $1 - 1 = 0$.

Now let $\alpha \neq 0$. Since the resonance index does not depend on s, to simplify calculations we choose $s = 1 + r_\lambda$. In this case

$$\mathcal{D}_z^{-1}(s) = \alpha - iy - \langle \hat{\psi}, \hat{u}_z \rangle$$

and the eigenvalue $\sigma_\lambda(s) = (s - r_\lambda)^{-1}$ is equal to 1. One can calculate the roots of (14.5.1):

$$\begin{aligned}\sigma_{1,2} &= \frac{\alpha - 2\langle \hat{\psi}, \hat{u}_z \rangle \pm \sqrt{\alpha^2 - 4iy\langle \hat{\psi}, \hat{u}_z \rangle}}{2(\alpha - iy - \langle \hat{\psi}, \hat{u}_z \rangle)} \\ &= \frac{\alpha - 2\langle \hat{\psi}, \hat{u}_z \rangle \pm |\alpha|(1 - 2iy\langle \hat{\psi}, \hat{u}_z \rangle/\alpha^2 + O(y^2))}{2(\alpha - iy - \langle \hat{\psi}, \hat{u}_z \rangle)}.\end{aligned}$$

Hence the root which approaches the eigenvalue $\sigma_\lambda(s) = 1$ as $y \to 0^+$ is

$$\sigma(y) = \frac{\alpha - \langle \hat{\psi}, \hat{u}_z \rangle - iy\langle \hat{\psi}, \hat{u}_z \rangle/\alpha + O(y^2)}{\alpha - iy - \langle \hat{\psi}, \hat{u}_z \rangle} = 1 + \frac{iy - iy\langle \hat{\psi}, \hat{u}_z \rangle/\alpha + O(y^2)}{\alpha - iy - \langle \hat{\psi}, \hat{u}_z \rangle}.$$

Since

$$\sigma'(0) = \frac{i - i\langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda) \rangle/\alpha}{\alpha - \langle \hat{\psi}, \hat{u}_{\lambda+i0}(r_\lambda) \rangle} = \frac{i}{\alpha},$$

the root $\sigma(y)$ approaches 1 from above (and moreover, at the right angle) if $\alpha > 0$, and from below if $\alpha < 0$. It follows that, in the case of $\hat{J} = 0$,

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \operatorname{sign} \alpha.$$

14.6. Examples of resonance points of orders three and four (in finite dimensions). One feature of the resonance index theory is that it makes sense and gives non-trivial results for spectral points λ outside of the essential spectrum (that is, for the classical spectral flow) and even in finite dimensions. For example, assume that there is a straight path of self-adjoint matrices $H_r = H_0 + rV$; then the eigenvalues of H_r are analytic functions of r which may have extrema, or critical points. Critical points of eigenvalues of H_r may have different orders. A natural question is: how to construct a path of self-adjoint matrices such that an eigenvalue of the path has a critical point of a given order? Theorem 14.2.11 gives an answer.

14.6.1. Example 1. Let

$$H_0 = \begin{pmatrix} \lambda + \varepsilon & 0 & 0 \\ 0 & \lambda - \varepsilon & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Since λ is an eigenvalue of H_0, the point $r = 0$ is a λ-resonant point of the path $H_0 + rV$ for any perturbation V. The direction

$$V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is not regularizing for the matrix H_0, since λ is a common eigenvalue of all operators H_r. That V_1 is not regularizing can also be seen from the fact that (14.2.12) fails.

The following direction is regularizing:

$$V_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since $\alpha = 0$, the order of the resonance point $r = 0$ is at least two. The resonance index of $(\lambda; H_0, V_2)$ is $2 - 1 = 1$.

14.6.2. Example 2. For the matrix

$$H_0 = \begin{pmatrix} \lambda+1 & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda-1/2 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

the direction

$$V_1 = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

is not regularizing at λ for the same reason as above: (14.2.12) fails.

If V_2 is chosen as

$$V_2 = \begin{pmatrix} -4 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

then (14.2.31) holds with $d = 3$. As a result, $r = 0$ has order at least three. According to Theorem 14.2.11, the order of $r_\lambda = 0$ is in fact three, since for the perturbation V_2 the following condition fails:

$$\langle \hat{\psi}, T_{\lambda+i0}(\hat{H}_0)\hat{J}T_{\lambda+i0}(\hat{H}_0)\hat{\psi}\rangle = 0.$$

But the regularizing direction

$$V_3 = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

satisfies (14.2.31) for $d = 4$, and therefore the corresponding resonance point $r_\lambda = 0$ has order 4. Computer shows that

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V_2) = 2 - 1 = 1 \quad \text{and} \quad \operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V_3) = 2 - 2 = 0.$$

15. Open problems

15.1. On points λ which are not essentially regular. According to Theorem 4.2.1, if a real number λ is an eigenvalue of infinite multiplicity of an operator from the affine space $\mathcal{A}$, then λ is not essentially regular. Is there a real number λ which is not essentially regular and such that some $H \in \mathcal{A}$ has finite multiplicity in a neighbourhood of λ?

15.2. Some questions about the resonance matrix. In Section 9 it was shown (Theorem 9.2.1) that the finite-rank self-adjoint operators

$$Q_{\lambda+i0}(r_\lambda)JP_{\lambda-i0}(r_\lambda) \quad \text{and} \quad Q_{\lambda-i0}(r_\lambda)JP_{\lambda+i0}(r_\lambda) \tag{15.2.1}$$

have equal signatures. In Subsection 13.3 it was shown that if a real resonance point r_λ has the generic property S then these operators are in fact equal and vice versa, but points without property S also exist.

Do the eigenvalue counting measures of the operators (15.2.1) coincide? What meaning do eigenvalues of self-adjoint operators (15.2.1) have?

15.3. Some questions about type I points. In Section 13 it was shown that if r_λ is a real resonance point of type I, then $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$. This equality is equivalent to $\Upsilon_{\lambda+i0}(r_\lambda) = \Upsilon_{\lambda-i0}(r_\lambda)$ (see Subsection 13.3).

Conjecture 1.

(a) *If $P_{\lambda+i0}(r_\lambda) = P_{\lambda-i0}(r_\lambda)$ then r_λ is of type I.*
(b) *More generally, if a vector u belongs to both $\Upsilon_{\lambda+i0}(r_\lambda)$ and $\Upsilon_{\lambda-i0}(r_\lambda)$, then u is of type I.*

15.4. On multiplicity of H_0. Recall that a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$ has *multiplicity* m if m is the smallest positive integer k such that for some k vectors $f_1, \dots, f_k$ the linear span of the vectors $H_0^i f_j$, $i = 1, 2, \dots$ and $j = 1, \dots, k$, is dense in the Hilbert space $\mathcal{H}$.

Conjecture 2. *If a self-adjoint operator $H_0 \in \mathcal{A}$ has multiplicity m, then for every essentially regular number λ at which H_0 is resonant,* $\dim \Upsilon^1_{\lambda+i0}(r_\lambda) \le m$.

Combined with the U-turn Theorem 10.1.6, this conjecture would imply that the resonance index cannot be greater than the multiplicity of H_0 for any regularizing perturbation V. This is a reasonable conjecture, since one would not expect the multiplicity of the singular spectrum to be greater than the multiplicity of H_0.

15.5. Resonance index as a function of perturbation. In this paper a fixed perturbation V has been considered. An open matter of study is the dependence of the resonance index $\mathrm{ind}_{\mathrm{res}}(\lambda; H_0, V)$ on the perturbation V.

Let H_0 be resonant at an essentially regular point λ. A regularizing direction V will be called *simple* if the resonance point $r_\lambda = 0$ has order 1. In this case

$$\Upsilon^1_{\lambda+i0}(r_\lambda) = \Upsilon_{\lambda+i0}(r_\lambda),$$

and therefore by Theorem 4.4.1 for simple directions V the vector space $\Upsilon_{\lambda+i0}(r_\lambda)$ does not depend on V.

Conjecture 3. *If H_0 is resonant at an essentially regular point λ, then the set of simple directions is open in the norm of $\mathcal{A}_0$, given by $\|F^*JF\|_{\mathcal{A}_0} = \|J\|$. Moreover, the set of non-simple directions is a meager subset of $\mathcal{A}_0$. Finally,* $\mathrm{ind}_{\mathrm{res}}(\lambda; H_0, V)$ *is stable under small perturbations of a simple direction V.*

15.6. Resonance lines and eigenvalues. Recall that a pair of self-adjoint operators H and V is called *reducible* if there exists a non-zero proper (closed) subspace $\mathcal{L}$ of the Hilbert space $\mathcal{H}$ such that $H\mathcal{L} \subset \mathcal{L}$ and $V\mathcal{L} \subset \mathcal{L}$.

By Proposition 2.6.3, for every essentially regular point $\lambda \in \Lambda(\mathcal{A}, F)$, the resonance set $R(\lambda; \mathcal{A}, F)$ is analytic, in the sense that every analytic curve either intersects $R(\lambda; \mathcal{A}, F)$ in a discrete set of points or is entirely contained in this set. There is a distinguished class of analytic curves—the straight lines. We suggest that the straight lines $\{H_0 + rV : r \in \mathbb{R}\}$ in the resonance set $R(\lambda; \mathcal{A}, F)$ have a special meaning.

Conjecture 4. *If $\{H_0 + rV : r \in \mathbb{R}\}$ is a line which is resonant at λ, then λ is a common eigenvalue of all operators $H_0 + rV$.*

This is motivated by the fact that embedded eigenvalues are highly unstable, and there has to be a reason for them not to get dissolved under perturbations rV for all $r \in \mathbb{R}$.

If $\{H_0 + rV : r \in \mathbb{R}\}$ is a line which is resonant at λ, then as simple finite-dimensional examples show, the eigenvectors corresponding to λ may not in general be common for all operators $H_0 + rV$, $r \in \mathbb{R}$.

15.7. On resonance points r_z as functions of z

15.7.1. On the analytic continuation of resonance points r_z. A resonance point r_z corresponding to z is a holomorphic function of z. Here we write $r(z)$ instead of r_z and call $r(z)$ a *resonance function*. This function is in general multi-valued and it can have continuous branching points of a finite period; examples can easily be constructed even in a finite-dimensional Hilbert space $\mathcal{H}$. A point z_0 of the complement of the essential spectrum will be called an *absorbing point* if $r(z) \to \infty$ as z approaches z_0 along some half-interval γ_1 from the domain of holomorphy of $r(z)$. It can be shown that if z_0 is an absorbing point, then $r(z) \to \infty$ as z approaches z_0 along any half-interval γ_2 from the domain of holomorphy of $r(z)$ which is homotopic to γ_1 in the domain of holomorphy. Recall that the domain of holomorphy of $r(z)$ is in general a multi-sheet Riemannian surface.

Conjecture 5.

(1) *If* $r_{\lambda+i0} := \lim_{y\to 0^+} r_{\lambda+iy}$ *exists and is a real number, then as* $y \to 0^+$ *the number* $r_{\lambda+iy}$ *approaches* $r_{\lambda+i0}$ *at a non-zero angle.*
(2) *The derivative of a resonance function* $r(z)$ *at a continuous branching point* z_0 *is equal to* ∞.
(3) *Let* $r(z)$ *be a resonance function. If* $r(z)$ *is holomorphic at a point* z_0 *(and does not branch at* z_0*) then* $r'(z_0) \neq 0$.
(4) *If* z_0 *is a continuous branching point of a resonance function* $r(z)$*, then the inverse* $z(r)$ *of* $r(z)$ *is a single-valued function in a neighbourhood of* $r_0 = r(z_0)$.
(5) *On any compact subset of* $\mathbb{C} \setminus \sigma_{\mathrm{ess}}$ *a resonance function* r_z *can have only a finite number of isolated continuous branching points. In general, what can be said about the distribution of branching points of* r_z?
(6) *A resonance function* $r(z)$ *has a cycle of largest period* d *at a continuous branching point* $z = z_0$ *if and only if* r_{z_0} *has order* d.
(7) *Resonance functions do not have* (a) *non-real* (b) *real absorbing points, including isolated absorbing points.*
(8) *Any resonance function* $r(z)$ *admits analytic continuation, possibly multi-valued, to the complement of the essential spectrum with only one possible type of isolated singularity: continuous branching points of finite period.*

Clearly, (8) implies (7). It can be shown that these two statements are equivalent. Further, it is not difficult to prove that an isolated absorbing point z_0, if it exists, must have infinite period; in particular, a resonance point $r(z)$ cannot be single-valued in a neighbourhood of an isolated absorbing point.

15.7.2. On the splitting property of resonance points r_z. Let λ be an essentially regular point, let H_0 be a self-adjoint operator from $\mathcal{A}$ and let V be a regularizing direction at λ. Let r_λ be a real resonance point of the line $H_r = H_0 + rV$ and let $r_z^1, \dots, r_z^N$ be resonance points of the group of r_λ.

Conjecture 6.

(1) *If the pair* (H_0, V) *is irreducible, then all resonance points* $r_z^1, \dots, r_z^N$ *of the group of* r_λ *considered as functions of* z *are non-degenerate. More generally, for an irreducible pair every resonance point* r_z *as a function of* z *is non-degenerate.*
(2) *All resonance points* $r_z^1, \dots, r_z^N$ *of the group of* r_λ *considered as functions of* z *have order* 1*. More generally, every resonance point* r_z *as a function of* z *has order* 1.

15.7.3. Analytic continuation through gaps in the essential spectrum. Assume that there is an island I in the essential spectrum, that is, I is a closed interval such that for some $\varepsilon > 0$ the intersection of σ_{ess} and $(a - \varepsilon, b + \varepsilon)$ is equal to $[a, b]$. Assume that a resonance function $r(z)$ can be continued analytically over the island. The analytic continuation back to the initial point may differ from the original function, of course. What can be said about the period of this analytic continuation?

What can be said about an integral of r_z over a contour which encloses an island of essential spectrum?

15.8. Mittag–Leffler representation of $A_z(s)$. Is it true that the function $A_z(s)$ satisfies the equality

$$A_z(s) = \sum_{r_z} A_z(s)P_z(r_z),$$

where the sum is taken over all resonance points r_z, and where the product $A_z(s)P_z(r_z)$ is the Laurent series (3.4.11)? Note that this assertion holds for finite-rank perturbations V, in which case the sum above is finite. In general, though, this seems to be unlikely.

15.9. On regular resonance vectors. Theorem 4.1.1 asserts that if $\chi \in \mathcal{H}$ is an eigenvector of a λ-resonant operator H_0, then $F\chi$ is a resonance vector of order 1. The resonance vector $F\chi$ is regular by definition.

Conjecture 7. *If $F\chi$ is a resonance vector of order* 1*, then χ is an eigenvector of H_0.*

This is proved in Theorem 4.3.2 under the additional condition that λ does not belong to the essential spectrum.

15.10. On singular ssf for trace class perturbations. Similarly to the definition of the singular spectral shift function one can define pure point and singular continuous spectral shift functions as distributions by

$$\xi^{(\mathrm{pp})}(\phi) = \int_0^1 \mathrm{Tr}(V\phi(H_r^{(\mathrm{pp})})\,dr, \qquad \phi \in C_c(\mathbb{R}),$$

$$\xi^{(\mathrm{sc})}(\phi) = \int_0^1 \mathrm{Tr}(V\phi(H_r^{(\mathrm{sc})})\,dr, \qquad \phi \in C_c(\mathbb{R}),$$

where $H_r^{(\mathrm{pp})}$ and $H_r^{(\mathrm{sc})}$ are the pure point and singular continuous parts of H_r respectively. Clearly, $\xi^{(s)} = \xi^{(\mathrm{pp})} + \xi^{(\mathrm{sc})}$.

The density of the distributions $\xi^{(\mathrm{pp})}$ and $\xi^{(\mathrm{sc})}$ will be denoted by the same symbols.

Conjecture 8. *Let H_0 be an arbitrary self-adjoint operator. If V is trace class then the restriction of the pure point spectral shift function for the pair $(H_0, H_0 + V)$ to the essential spectrum of H_0 is zero.*

That is, for trace class perturbations the restriction of $\xi^{(s)}$ to σ_{ess} coincides with $\xi^{(\mathrm{sc})}$.

We say that for an irreducible pair (H_0, V) of self-adjoint operators, where V is trace class, the singular spectral shift function $\xi^{(s)}(\lambda; H_0+V, H_0)$ is *non-trivial* if the restriction of this function to $\sigma_{\mathrm{ess}}(H_0)$ is a non-zero element of $L_1(\sigma_{\mathrm{ess}}(H_0), d\lambda)$, where $d\lambda$ is the restriction of the Lebesgue measure to $\sigma_{\mathrm{ess}}(H_0)$.

There is a question of existence of non-trivial singular spectral shift functions. If the essential spectrum of H_0 contains an open interval I which has no absolutely continuous spectrum, then it is not difficult to prove the existence of non-trivial singular spectral shift functions. Indeed, the absolutely continuous spectrum is stable under trace class perturbations, and therefore, on the interval I the singular spectral shift function coincides with the spectral shift function.

For this reason, in discussing the problem of existence of non-trivial singular spectral shift functions we shall assume that the absolutely continuous spectrum is everywhere dense in the essential spectrum.

An example of a non-trivial singular spectral shift function was constructed in [Az$_4$]. This example relies on tools such as Cantor sets and Luzin–Privalov's theorem on the boundary behaviour of analytic functions. A more natural approach to constructing a non-trivial singular spectral shift function would be to take a self-adjoint operator H_0 with an eigenvalue λ_0 embedded into the essential spectrum of H_0, and to try to perturb H_0 by a trace class self-adjoint operator V so that the eigenvalue λ_0 would change and not dissolve in essential spectrum. My numerous attempts to construct such an example of moving and stable embedded eigenvalue did not succeed, and as a result I developed a "gut-feeling" that this is not possible at all. This is the reason for Conjecture 8.

At the same time it is quite possible that non-trivial pure point spectral shift functions can be constructed if we allow relatively trace class perturbations V.

An embedded eigenvalue moving continuously inside the essential spectrum without dissolving in it is a peculiar phenomenon; in particular, by the Schwarz reflection principle, the existence of a moving embedded eigenvalue would imply that the corresponding resonance point $r_z = r(z)$ admits analytic continuation through an interval inside essential spectrum.

Speculating further on this topic, it is possible that moving embedded eigenvalues might have some connections with physical phenomena such as superconductivity.

15.11. On the pure point and singular continuous parts of the resonance index. The material of this subsection and motivation for it are based on Section 4.

In addition to our usual assumptions about H_0, F and V we assume that V is positive. Let r_λ be a resonance point of $(\lambda; H_0, V)$. Since V is positive, we have

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \dim \Upsilon_{\lambda+i0}(r_\lambda) = \dim \Upsilon^1_{\lambda+i0}(r_\lambda).$$

We define the pure point and singular continuous parts of the resonance index by

$$\operatorname{ind}^{(\mathrm{pp})}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \dim \mathcal{V}_\lambda \quad \text{and} \quad \operatorname{ind}^{(\mathrm{sc})}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) - \dim \mathcal{V}_\lambda,$$

where $\mathcal{V}_\lambda$ is the vector space of eigenvectors of H_{r_λ} corresponding to the eigenvalue λ.

Conjecture 9. *For a.e. λ,*

$$\xi^{(\mathrm{pp})}(\lambda; H_1, H_0) = \sum_{r\in[0,1]} \operatorname{ind}^{(\mathrm{pp})}_{\mathrm{res}}(\lambda; H_r, V).$$

By Theorem 6.3.2, this equality is equivalent to

$$\xi^{(\mathrm{sc})}(\lambda; H_1, H_0) = \sum_{r\in[0,1]} \operatorname{ind}^{(\mathrm{sc})}_{\mathrm{res}}(\lambda; H_r, V).$$

For non-sign-definite operators V it is not clear how one can define the pure point and singular parts of the resonance index.

15.12. On the singular spectral shift function for relatively compact perturbations. The singular spectral shift function is well-defined for relatively trace class perturbations. For such perturbations it admits three other descriptions as the singular μ-invariant, the total resonance index and the total signature of the resonance matrix.

These three descriptions are well-defined for relatively compact perturbations too, provided that the limiting absorption principle holds, and in this case they are all equal (see Theorem 9.2.1 and [Az_5]). While for relatively compact perturbations the spectral shift function in general is not defined, it is quite possible that in this case the singular spectral shift function still makes sense and is equal to the other three integer-valued functions. Indeed, while VE_Δ^H for a bounded Borel set Δ may fail to be trace class, it is still possible that $VE_\Delta^{H^{(s)}}$ is trace class for a sufficiently large class of Borel sets Δ (for example for compact subsets Δ of $\Lambda(H_0, F)$). This would allow us to use a modification of the Birman–Solomyak formula to define the singular spectral shift function. The second step would be to show that this function is integer-valued and equal to the other three.

Conjecture 10.

(a) *Assume that for an affine space $\mathcal{A}$ of self-adjoint operators with rigging operator F the assumptions of Section 2 are satisfied including the Limiting Absorption Principle. Let H_0 be a self-adjoint operator from the affine space $\mathcal{A}$, and V be a relatively compact (and not necessarily relatively trace class) self-adjoint operator from the corresponding vector space $\mathcal{A}_0$. For any compact subset K of the set $\Lambda(H_0, F)$ the operator $FE_K^{H^{(s)}}$ is Hilbert–Schmidt, where $H^{(s)}$ is the singular part of $H = H_0 + V$.*

(b) *The measure*

$$K \mapsto \int_0^1 \mathrm{Tr}(VE_K^{H_r^{(s)}})\, dr$$

is well-defined on compact subsets of $\Lambda(H_0, F)$ and is absolutely continuous, where $H_r = H_0 + rV$.

(c) *The density of this measure is a.e. integer-valued and coincides with the total resonance index*

$$\sum_{r\in[0,1]} \mathrm{ind}_{\mathrm{res}}(\lambda; H_r, V)$$

of the pair (H_0, H_1).

For relatively trace class perturbations the only non-trivial part of this conjecture is (c). A proof of this conjecture for relatively trace class perturbations with applications to Schrödinger operators will be given in [AzD].

References

[Ag] Sh. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 2 (1975), 151–218.

[AS] W. O. Amrein and K. B. Sinha, *On pairs of projections in a Hilbert space*, Linear Algebra Appl. 208/209 (1994), 425–435.

[Ar] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm–Liouville equations*, Amer. J. Math. 79 (1957), 597–610.

[AD] N. Aronszajn and W. F. Donoghue, *On exponential representations of analytic functions in the upper half-plane with positive imaginary part*, J. Anal. Math. 5 (1956), 321–388.

[APS] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69.

[APS$_2$] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 71–99.

[ASS] J. Avron, R. Seiler and B. Simon, *The index of a pair of projections*, J. Funct. Anal. 120 (1994), 220–237.

[Az] N. A. Azamov, *Infinitesimal spectral flow and scattering matrix*, arXiv:0705.3282v4 (2007).

[Az$_2$] N. A. Azamov, *Pushnitski's μ-invariant and Schrödinger operators with embedded eigenvalues*, arXiv:0711.1190v1 (2007).

[Az$_3$] N. A. Azamov, *Absolutely continuous and singular spectral shift functions*, Dissertationes Math. 480 (2011), 102 pp.

[Az$_4$] N. A. Azamov, *Resonance index and singular spectral shift function*, arXiv:1104.1903 (2011).

[Az$_5$] N. A. Azamov, *Resonance index and singular μ-invariant*, arXiv:1501.07673 (2015).

[Az$_6$] N. A. Azamov, *A constructive approach to stationary scattering theory*, arXiv:1302.4142 (2013).

[ACS] N. A. Azamov, A. L. Carey and F. A. Sukochev, *The spectral shift function and spectral flow*, Comm. Math. Phys. 276 (2007), 51–91.

[AzD] N. A. Azamov and Th. Daniels, *Singular spectral shift function for Schrödinger operators*, in preparation.

[AzS] N. A. Azamov and F. A. Sukochev, *Spectral averaging for trace compatible operators*, Proc. Amer. Math. Soc. 136 (2008), 1769–1778.

[Ba] N. S. Bakhvalov, *Numerical Methods*, Nauka, Moscow, 1973 (in Russian).

[BW] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Birkhäuser, Basel, 1983.

[BCPRSW] M.-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev and K. P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*, in: Analysis, Geometry and Topology of Elliptic Operators, World Sci., Hackensack, NJ, 2006, 297–352.

[BeSh] F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, Kluwer, Dordrecht, 1991.

[BÈ] M. Sh. Birman and S. B. Èntina, *The stationary approach in abstract scattering theory*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 401–430 (in Russian); English transl.: Math. USSR Izv. 1 (1967), 391–420.

[BK] M. Sh. Birman and M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR 144 (1962), 475–478 (in Russian).

[BS] M. Sh. Birman and M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht, 1987.

[BS_2] M. Sh. Birman and M. Z. Solomyak, *Remarks on the spectral shift function*, J. Soviet Math. 3 (1975), 408–419.

[BY] M. Sh. Birman and D. R. Yafaev, *The spectral shift function. The work of M. G. Kreĭn and its further development*, Algebra i Analiz 4 (1992), no. 5, 1–44 (in Russian).

[BY_2] M. Sh. Birman and D. R. Yafaev, *Spectral properties of the scattering matrix*, Algebra i Analiz 4 (1992), no. 6, 1–27 (in Russian); English transl.: St. Petersburg Math. J. 4 (1993), 1055–1079.

[Bl] B. Blackadar, *K-theory for Operator Algebras*, Math. Sci. Res. Inst. Publ. 5, Springer, New York, 1986.

[BoSh] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley, New York, 1980.

[Bö] A. Böhm, *Quantum Mechanics*, Springer, New York, 1979.

[BF] B. Booss-Bavnbek and K. Furutani, *The Maslov index: a functional analytical definition and the spectral flow formula*, Tokyo J. Math. 21 (1998), 1–34.

[BLP] B. Booss-Bavnbek, M. Lesch and J. Phillips, *Unbounded Fredholm operators and spectral flow*, Canad. J. Math. 57 (2005), 225–250.

[Br] L. de Branges, *Perturbation of self-adjoint transformations*, Amer. J. Math. 84 (1962), 543–560.

[BR] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer, New York, 1979.

[BuFa] V. S. Buslaev and L. D. Faddeev, *On trace formulas for a singular Sturm–Liouville differential operator*, Dokl. Akad. Nauk SSSR 132 (1960), 13–16 (in Russian); English transl.: Soviet Math. Dokl. 3 (1962), 451–454.

[CP] A. L. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math. 50 (1998), 673–718.

[CP_2] A. L. Carey and J. Phillips, *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle*, K-Theory 31 (2004), 135–194.

[CPRS] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, *The Hochschild class of the Chern character of semifinite spectral triples*, J. Funct. Anal. 213 (2004), 111–153.

[$CPRS_2$] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, *The local index formula in semifinite von Neumann algebras I: spectral flow*, Adv. Math. 202 (2006), 451–516.

[$CPRS_3$] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, *The local index formula in semifinite von Neumann algebras II: even case*, Adv. Math. 202 (2006), 517–554.

[C] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, 1994.

[CM] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. 5 (1995), 174–243.

[Fa] L. D. Faddeev, *On a model of Friedrichs in the theory of perturbations of the continuous spectrum*, Trudy Mat. Inst. Steklov. 73 (1964), 292–313 (in Russian).

[Far] Yu. B. Farforovskaya, *Example of a Lipschitz function of self-adjoint operators that gives a nonnuclear increment under a nuclear perturbation*, J. Soviet Math. 4 (1975), 426–433.

[Fr] K. O. Friedrichs, *Über die Spektralzerlegung eines Integraloperators*, Math. Ann. 115 (1938), 249–272.

[Fr$_2$] K. O. Friedrichs, *On the perturbation of continuous spectra*, in: Comm. Appl. Math. 1, Interscience, New York, 1948, 361–406.

[Fr$_3$] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space*, Amer. Math. Soc., Providence, RI, 1965.

[GG] M. Gell-Mann and M. L. Goldberger, *The formal theory of scattering*, Phys. Rev. 91 (1953), 398–408.

[GM] F. Gesztesy and K. A. Makarov, *The Ξ operator and its relation to Krein's spectral shift function*, J. Anal. Math. 81 (2000), 139–183.

[GM$_2$] F. Gesztesy and K. A. Makarov, $\mathrm{SL}_2(\mathbb{R})$*, exponential Herglotz representations, and spectral averaging*, Algebra i Analiz 15 (2003), no. 3, 104–144 (in Russian); English transl.: St. Petersburg Math. J. 15 (2004), 393–418.

[Ge] E. Getzler, *The odd Chern character in cyclic homology and spectral flow*, Topology 32 (1993), 489–507.

[GK] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Non-Selfadjoint Operators*, Transl. Math. Monogr. 18, Amer. Math. Soc., Providence, RI, 1969.

[Gor] A. Ya. Gordon, *Pure point spectrum under 1-parameter perturbations and instability of Anderson model*, Comm. Math. Phys. 164 (1994), 489–505.

[HPh] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc., Providence, RI, 1957.

[I] T. Ikebe, *Eigenfunction expansions associated with the Schrödinger operators and their application to scattering theory*, Arch. Ration. Mech. Anal. 5 (1960), 1–34; Erratum Sect. I, *Remarks on the orthogonality of eigenfunctions for the Schrödinger operator in* $\mathbb{R}^n$, J. Fac. Sci. Univ. Tokyo 17 (1970), 355–361.

[K] N. J. Kalton, *A note on pairs of projections*, Bol. Soc. Mat. Mexicana (3) 3 (1997), 309–311.

[Ka] T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad. 33 (1957), 260–264.

[Ka$_2$] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1980.

[Ka$_3$] T. Kato, *Notes on projections and perturbation theory*, Tech. Rep. No. 9, Univ. California at Berkeley, 1955.

[Ka$_4$] T. Kato, *Some results on potential scattering*, in: Proc. Int. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), Univ. of Tokyo Press, Tokyo, 1970, 206–215.

[KK] T. Kato and S. T. Kuroda, *The abstract theory of scattering*, Rocky Mountain J. Math. 1 (1971), 127–171.

[KN] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience, New York, 1969.

[Kr] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb. 33 (75) (1953), 597–626 (in Russian).

[Ku] S. T. Kuroda, *An Introduction to Scattering Theory*, Lecture Notes Ser. 51, Math. Inst., Århus Univ., Århus, 1976.

[Ku$_2$] S. T. Kuroda, *Scattering theory for differential operators, I: Operator theory*, J. Math. Soc. Japan 25 (1973), 75–104.

[Ku$_3$] S. T. Kuroda, *Scattering theory for differential operators, II: Self-adjoint elliptic operators*, J. Math. Soc. Japan 25 (1973), 222–234.

[LF] O. A. Ladyzhenskaya and L. D. Faddeev, *On continuous spectrum perturbation theory*, Dokl. Akad. Nauk SSSR 120 (1958), 1187–1190 (in Russian).

[L] I. M. Lifshitz, *On a problem in perturbation theory connected with quantum statistics*, Uspekhi Mat. Nauk 7 (1952), no. 1, 171–180 (in Russian).

[LSch] B. A. Lippmann and J. Schwinger, *Variational principles for scattering processes, I*, Phys. Rev. 79 (1950), 469–480.

[Mu] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, Boston, 1990.

[Pel] V. V. Peller, *Hankel operators in perturbation theory of unitary and self-adjoint operators*, J. Funct. Anal. 19 (1985), 37–51.

[Ph] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. 39 (1996), 460–467.

[Ph$_2$] J. Phillips, *Spectral flow in type I and type II factors—a new approach*, Fields Inst. Comm. 17 (1997), 137–153.

[Po] A. Ya. Povzner, *On the expansion of arbitrary functions in terms of eigenfunctions of the operator $-\Delta u+cu$*, Mat. Sb. 32 (1953), 109–156 (in Russian); English transl.: Amer. Math. Soc. Transl. (2) 60 (1967), 1–49.

[Po$_2$] A. Ya. Povzner, *On expansions in functions which are solutions of a scattering problem*, Dokl. Akad. Nauk SSSR 104 (1955), 360–363 (in Russian).

[Pu] A. B. Pushnitski, *Spectral shift function of the Schrödinger operator in the large coupling constant limit*, Comm. Partial Differential Equations 25 (2000), 703–736.

[Pu$_2$] A. B. Pushnitski, *The spectral shift function and the invariance principle*, J. Funct. Anal. 183 (2001), 269–320.

[Pu$_3$] A. B. Pushnitski, *Representation for the spectral shift function for perturbations of fixed sign*, Algebra i Analiz 9 (1997), no. 6, 197–213 (in Russian).

[Pu$_4$] A. B. Pushnitski, *The spectral flow, the Fredholm index and the spectral shift function*, in: Spectral Theory of Differential Operators, Amer. Math. Soc. Transl. (2) 225 (2008), 141–155.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: 1. Functional Analysis*, Academic Press, New York, 1972.

[RS$_2$] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: 2. Fourier Analysis*, Academic Press, New York, 1975.

[RS$_3$] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: 3. Scattering Theory*, Academic Press, New York, 1979.

[RS$_4$] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: 4. Analysis of Operators*, Academic Press, New York, 1978.

[RJMS] R. del Rio, S. Jitomirskaya, N. Makarov and B. Simon, *Singular continuous spectrum is generic*, Bull. Amer. Math. Soc. 31 (1994), 208–212.

[RMS] R. del Rio, N. Makarov and B. Simon, *Operators with singular continuous spectrum: II. Rank one operators*, Comm. Math. Phys. 165 (1994), 59–67.

[R] M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence*, Pacific J. Math. 7 (1957), 997–1010.

[S] B. Simon, *Spectral averaging and the Krein spectral shift*, Proc. Amer. Math. Soc. 126 (1998), 1409–1413.

[S$_2$] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Math. Surveys Monogr. 120, Amer. Math. Soc., Providence, RI, 2005.

[SW] B. Simon and T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Comm. Pure Appl. Math. 39 (1986), 75–90.

[T] J. R. Taylor, *Scattering Theory*, Wiley, New York, 1972.

[We] R. O. Wells, *Differential Analysis on Complex Manifolds*, 2nd ed., Springer, New York, 1980.

[Y] D. R. Yafaev, *Mathematical Scattering Theory: General Theory*, Amer. Math. Soc., Providence, RI, 1992.

[Y$_2$] D. R. Yafaev, *Lectures on scattering theory*, in: Surveys in Analysis and Operator Theory (Canberra, 2001), A. Hassell Centre Math. Appl., Austral. Nat. Univ., Canberra, 2002, 61–88.

Index