LIE ALGEBROIDS
BANACH CENTER PUBLICATIONS, VOLUME 54
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2001

LOCAL INTEGRATION OF LIE ALGEBROIDS

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Abstract. We give a complete description of the local integration of almost injective Lie algebroids. We illustrate this construction with examples.

1. Introduction. In the 70's, J. Pradines has announced a third Lie Theorem for Lie groupoids [P3]. This theorem asserts that every Lie algebroid is *integrable* which means that it is the Lie algebroid of a Lie groupoid. Some years latter R. Almeida and P. Molino provided a counter example [AM] and K. Mackenzie defined an obstruction to integrability for transitive Lie algebroids [Ma]. Nevertheless, it seems that the local Lie integration of J. Pradines remains and it is used without any proof in several articles. In the symplectic context, local integration of Poisson structure has been done by A. Coste, P. Dazord and A. Weinstein [CDW]. P. Dazord has proved that the Lie algebroid associated to a local action of a Lie group on a manifold is integrable into a Lie groupoid [Da]. More recently, several positive integrability results have been prooved [MM].
A Lie algebroid is said to be *almost injective* when its anchor is injective when restricted to a dense open subset of the base space. The purpose of this paper is to give a complete description of the local integration of almost injective Lie algebroids. This is the first step to show that these Lie algebroids are integrable as we have announced in [D].

This paper is organized as follows:
– In section 1 we recall some of the basic concepts related to the Lie theory of groupoids.
– In section 2 we make the local integration of almost injective Lie algebroids. We first do it for trivial Lie algebroids. Then we make the integration of morphisms. In the general case, we conclude by using a local trivialisation of the algebroid.
– In section 3 we give examples.

2. Survey on Lie theory for groupoids. In this part we recall the principal definitions and properties concerning Lie groupoids. For a more complete setting on this subject one can look at [Ma, CW].

2000 *Mathematics Subject Classification*: 22A22, 58H05.
The paper is in final form and no version of it will be published elsewhere.

2.1. *Lie groupoids*

DEFINITION 1. A *groupoid* over a set $G^{(0)}$ is a set G together with the following structure maps:

– An injection $u : G^{(0)} \to G$ called the *unit map*. We often identify $G^{(0)}$ with its image by u in G and call it the *space of unit*.
– A pair of surjective maps $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ such that $s \circ u = id$ and $r \circ u = id$.
  The map s is called the *source* map and r is the *range* (*target*) map.
– A *product*

$$G^{(2)} \to G, \qquad (\gamma, \eta) \mapsto \gamma \cdot \eta$$

  defined on the set of *composable pairs*: $G^{(2)} = \{(\gamma, \eta) \in G \times G \mid s(\gamma) = r(\eta)\}$.
  This product satisfies:

$$s(\gamma \cdot \eta) = s(\eta), \qquad r(\gamma \cdot \eta) = r(\gamma)$$
$$(\gamma \cdot \eta) \cdot \nu = \gamma \cdot (\eta \cdot \nu), \qquad \eta \cdot s(\eta) = \eta \text{ and } r(\eta) \cdot \eta = \eta.$$

– An *inversion* map

$$G \to G, \qquad \gamma \mapsto \gamma^{-1}$$

  such that $\gamma^{-1} \cdot \gamma = s(\gamma)$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

A *Lie groupoid* is a groupoid $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ equipped with a structure of smooth manifolds on G and $G^{(0)}$, such that all structure maps are smooth and s is a submersion. This implies that r is a submersion as well, that there is a natural smooth structure on $G^{(2)}$ and u is an embedding. We assume that $G^{(0)}$ is Hausdorff.

From now on, we shall work in the smooth context so "map" means smooth map, "manifold" means smooth manifold, etc...

EXAMPLES. 1. Let H be a Lie group and e_H its unit, then $H \rightrightarrows \{e_H\}$ is naturally endowed with a Lie groupoid structure.

2. Let $\mathcal{R}$ be a regular relation on a manifold M and consider the *graph of* $\mathcal{R}$, that is $G_{\mathcal{R}} := \{(x, y) \in M \times M \ ; \ x\mathcal{R}y\}$. Then

$$G_{\mathcal{R}} \overset{s}{\underset{r}{\rightrightarrows}} M$$

is a Lie groupoid where for all (x, y) and (y, z) in $G_{\mathcal{R}}$ we set

$$s(x, y) = y, \qquad r(x, y) = x$$
$$(x, y)^{-1} = (y, x) \quad \text{and} \quad (x, y) \cdot (y, z) = (x, z).$$

This groupoid structure on a subset of $M \times M$ is usually called the *pair* groupoid structure.

3. Let H be a Lie group and suppose that there is a differentiable left action of H on a manifold M. Then

$$M \times H \overset{s_{\rtimes}}{\underset{r_{\rtimes}}{\rightrightarrows}} M$$

is a Lie groupoid called the *groupoid of the action*, where for all x in M and h, g in H we define

$$s_{\rtimes}(x,h) = x,\ r_{\rtimes}(x,h) = h\cdot x$$

$$(x,h)^{-1} = (h\cdot x, h^{-1}) \text{ and } (h\cdot x, g)\cdot(x,h) = (x, gh).$$

NOTATIONS. If $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ is a groupoid and U is a part of $G^{(0)}$ we usually denote

$$G_U := s^{-1}(U),\ G^U := r^{-1}(U) \text{ and } G_U^U := s^{-1}(U)\cap r^{-1}(U).$$

As for Lie groups there is a notion of *local Lie groupoid* which is due to Van Est [VE]. A *local Lie groupoid* is given by:

– Two manifolds $\mathcal{L}$ and $\mathcal{L}^{(0)}$ and an embedding $u : \mathcal{L}^{(0)} \to \mathcal{L}$. The manifold $\mathcal{L}^{(0)}$ must be Hausdorff, it is called the *set of units*. We usually identify $\mathcal{L}^{(0)}$ with its image by u in $\mathcal{L}$.
– Two surjective submersions: $\mathcal{L} \overset{s}{\underset{r}{\rightrightarrows}} \mathcal{L}^{(0)}$ called the *range* and *source* map, which satisfy $s\circ u = r\circ u = id$.
– A smooth involution

$$\mathcal{L}\to\mathcal{L}, \quad l\mapsto l^{-1}$$

called the *inverse* map. It satisfies $s(l^{-1}) = r(l)$ for $l\in\mathcal{L}$.
– An open subset $\mathcal{D}^2\mathcal{L}$ of $\mathcal{L}^{(2)} = \{(l_1,l_2)\in\mathcal{L}\times\mathcal{L} \mid s(l_1) = r(l_2)\}$ called the *set of composable pairs* and a smooth product

$$\mathcal{D}^2\mathcal{L}\to\mathcal{L}, \quad (l_1,l_2)\mapsto l_1\cdot l_2.$$

The following properties must be fulfilled:

- $s(l_1\cdot l_2) = s(l_2)$ and $r(l_1\cdot l_2) = r(l_1)$ when the product $l_1\cdot l_2$ is defined.
- For all $l\in\mathcal{L}$ the products $r(l)\cdot l$, $l\cdot s(l)$, $l\cdot l^{-1}$ and $l^{-1}\cdot l$ are defined and respectively equal to l, l, $r(l)$ and $s(l)$.
- If the product $l_1\cdot l_2$ is defined then so is the product $l_2^{-1}\cdot l_1^{-1}$ and $(l_1\cdot l_2)^{-1} = l_2^{-1}\cdot l_1^{-1}$.
- If the products $l_1\cdot l_2$, $l_2\cdot l_3$ and $(l_1\cdot l_2)\cdot l_3$ are defined then so is the product $l_1\cdot(l_2\cdot l_3)$ and $(l_1\cdot l_2)\cdot l_3 = l_1\cdot(l_2\cdot l_3)$.

The only difference between Lie groupoids and local Lie groupoids is that in the second case the condition $s(l_1) = r(l_2)$ is necessary for the existence of the product $l_1\cdot l_2$ but not sufficient. The product is defined as soon as l_1 and l_2 are close enough from units.

2.2. *Lie algebroids.* Each (local) Lie groupoid admits a Lie algebroid [P2, P1]. Let us recall this construction.

Let $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ be a Lie groupoid. We denote by T^sG the bundle over G of s-vertical vector fields. That is T^sG is the kernel of the differential Ts of s.

For all γ in G let $R_\gamma : G_{r(\gamma)} \to G_{s(\gamma)}$ be the right multiplication by γ. A tangent vector field Z on G is *right invariant* if it satisfies:

– Z is s-vertical: $Ts(Z) = 0$.
– For all (γ_1,γ_2) in $G^{(2)}$, $Z(\gamma_1\cdot\gamma_2) = TR_{\gamma_2}(Z(\gamma_1))$.

Note that if Z is a right invariant vector field and h^t its flow then for all t, the local diffeomorphism h^t is a *local left translation* of G that is $h^t(\gamma_1 \cdot \gamma_2) = h^t(\gamma_1) \cdot \gamma_2$ when it makes sense.

The Lie algebroid $\mathcal{A}G$ of G is defined in the following way:

- The fibre bundle $\mathcal{A}G \to G^{(0)}$ is the restriction of T^sG to $G^{(0)}$.
- The *anchor* $p : \mathcal{A}G \to TG^{(0)}$ is the restriction of the differential Tr of r to $\mathcal{A}G$.
- If $Y : U \to \mathcal{A}G$ is a local section of $\mathcal{A}G$, where U is an open subset of $G^{(0)}$, we define the local *right invariant vector field* Z_Y *associated* with Y by
$$Z_Y(\gamma) = TR_\gamma(Y(r(\gamma))) \text{ for all } \gamma \in G^U.$$
The Lie bracket is then defined by:
$$[\,,\,] : \Gamma(\mathcal{A}G) \times \Gamma(\mathcal{A}G) \to \Gamma(\mathcal{A}G), \quad (Y_1, Y_2) \mapsto [Z_{Y_1}, Z_{Y_2}]_{G^{(0)}}$$
where $[Z_{Y_1}, Z_{Y_2}]$ denote the s-vertical vector field obtained with the usual bracket and $[Z_{Y_1}, Z_{Y_2}]_{G^{(0)}}$ is the restriction of $[Z_{Y_1}, Z_{Y_2}]$ to $G^{(0)}$.

2.3. *Coordinates of the second kind.* To describe the local integration of almost injective Lie algebroid we are going to use the model given by the *coordinates of the second kind.*

Let $G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$ be a Lie groupoid and $p : \mathcal{A}G \to TG^{(0)}$ its Lie algebroid.

Let $\{Y_1, \cdots, Y_k\}$ be a local basis of sections of $\mathcal{A}G$ defined over an open set U of $G^{(0)}$.

For $i = 1, \cdots, k$ let Z_{Y_i} be the right invariant vector field associated to Y_i. We suppose for simplicity that the Z_{Y_i} are complete tangent vector fields and we denote by h_i^t the flow of Z_{Y_i}.

In this case $X_i := p(Y_i)$ is a complete tangent vector field over U and its flow φ_i^t is equal to the restriction to U of the map $r \circ h_i^t$.

One can find an open set Ω of $U \times \mathbb{R}^k$ containing $U \times \{0\}$ such that the map
$$\Psi : \Omega \to G, \quad (x, t_1, \cdots, t_k) \mapsto h_1^{t_1} \circ \cdots \circ h_k^{t_k}(x)$$
is a diffeomorphism onto its image.

Such a map is called a *coordinate map of the second kind.* It has the following remarkable properties:

- Ψ maps the zero section $U \times \{0\}$ onto the units U.
- $s \circ \Psi = pr_1$.
- For all $(x, t_1, \cdots, t_k) \in \Omega$, $r \circ \Psi(x, t_1, \cdots, t_k) = \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(x)$.

Thus, the coordinate maps of the second kind enable one to identify locally a neighborhood of the zero section of $\mathcal{A}G$ with a neighborhood of the set of units of G. Furthermore Ω inherits a structure of local Lie groupoid over U whose source map and range map are respectively given by
$$s : (y, t_1, \cdots, t_k) \mapsto y \text{ and } r : (y, t_1, \cdots, t_k) \mapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(y).$$
The local product is given by:
$$(x, s_1, \cdots, s_k) \cdot (y, t_1, \cdots, t_k) = \Psi^{-1}(h_1^{s_1} \circ \cdots \circ h_k^{s_k} \circ h_1^{t_1} \circ \cdots \circ h_k^{t_k}(y))$$
when $x = \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(y)$ and $h_1^{s_1} \circ \cdots \circ h_k^{s_k} \circ h_1^{t_1} \circ \cdots \circ h_k^{t_k}(y)$ belongs to the image of Ψ.

Note that the source map s is simply the canonical projection over U and the range map r only depends on the vector fields $X_1, \cdots, X_k$ over U.

3. Local integration. Let $\mathcal{A} = (p : \mathcal{A} \to TM \ ; \ [,])$ be a Lie algebroid. We denote by $\tilde{p} : \Gamma_{loc}(\mathcal{A}) \to \Gamma_{loc}(TM)$ the morphism induced by p from the set of differentiable local sections of $\mathcal{A}$ to the set of local differentiable tangent vector fields over M. Let M_0 denote the set of points of M over which p is injective. We will say that $\mathcal{A}$ is *almost injective* if $\tilde{p}$ is injective or equivalently if M_0 is a dense open subset of M.

The purpose here is to give a complete description of the local integration of such Lie algebroids.

If we are given a Lie algebroid $\mathcal{A}$ over M the coordinate maps of the second kind provide a model for the local integration of $\mathcal{A}$. Using this model there is no choice for the source and range maps. The difficulty is to define the product. To do so one must be able to recover the right invariant vector fields. That's what we are going to do in this section.

3.1. *Baker-Campbell-Hausdorff type formula.* Suppose you are given a manifold M and k tangent vector fields $X_1, \cdots, X_k$ over M such that for all $i, j = 1, \cdots, k$ there exist k maps $f_l^{ij} \in \mathcal{C}^\infty(M)$ which satisfy

$$[X_i, X_j] = \sum_{l=1}^{k} f_l^{ij} \cdot X_l.$$

For $i = 1, \cdots, k$ we denote by $A_i \in M_k(\mathcal{C}^\infty(M))$ the matrix with entries $(A_i)_{jl} = f_j^{il}$. Thus, the equality $([X_i, X_1], \cdots, [X_i, X_k]) = (X_1, \cdots, X_k) \cdot A_i$ holds.

We denote by φ_i^t the flow of X_i.

Let Ω_0 be an open subset of $M \times \mathbb{R}^k$ containing $M \times \{0\}$ and which is in the domain of the map $(x, t_1, \cdots, t_k) \mapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(x)$.

We define

$$s : M \times \mathbb{R}^k \to M, \quad (x, t_1, \cdots, t_k) \mapsto x,$$
$$r : \Omega_0 \to M, \quad (x, t_1, \cdots, t_k) \mapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(x).$$

As previously, a tangent vector field Y over $M \times \mathbb{R}^k$ is *s-vertical* when $Ts(Y) = 0$. If Ω is an open subset of $M \times \mathbb{R}^k$, we will denote by $T^s(\Omega)$ the bundle of s-vertical vector fields defined on Ω. In other words $T^s(\Omega)$ is the restriction of $T^s(M \times \mathbb{R}^k) := \mathrm{Ker}(Ts)$ to Ω.

The purpose of this part is to show the following proposition.

PROPOSITION 1. *There exist an open subset Ω_1 of Ω_0 which contains $M \times \{0\}$, a family $\{Z_1, \cdots, Z_k\}$ of s-vertical vector fields defined on Ω_1 and a real number $\eta > 0$ such that*

1. *The family $\{Z_1, \cdots, Z_k\}$ forms a base of sections of $T^s\Omega_1$.*
2. *The equality $Tr(Z_i) = X_i \circ r$ holds for $i = 1, \cdots, k$.*
3. *For $i, j = 1, \cdots, k$ and $(x, \xi) \in \Omega_1 \subset M \times \mathbb{R}^k$ we have*

$$[Z_i, Z_j](x, \xi) = \sum_{l=1}^{k} f_l^{ij}(r(x, \xi)) \cdot Z_l(x, \xi).$$

4. *For all $t \in]-\eta,\eta[$ and $i=1,\cdots,k$, the flow h_i^t of the vector field Z_i is defined on Ω_1 and satisfies*
   (a) $s \circ h_i^t = s$,
   (b) $r \circ h_i^t = \varphi_i^t \circ r$,
   (c) *for all $x \in M$ and $(t_1,\cdots,t_k) \in \mathbb{R}^k$ such it makes sense we have*
$$h_1^{t_1} \circ \cdots \circ h_k^{t_k}(x,0) = (x,t_1,\cdots,t_k).$$

To prove this proposition we need the following lemma.

LEMMA 2. *For all $i=1,\cdots,k$ there exists a map $C_i : \mathcal{O} \to M_k(\mathbb{R})$, where $\mathcal{O}$ is an open subset of $M \times \mathbb{R}$ containing $M \times \{0\}$, which satisfies:*

– $C_i(\cdot,0) = Id$;
– *For all (x,t) in $\mathcal{O} \subset M \times \mathbb{R}$ and $(v_1,\cdots,v_k)$ in $\mathbb{R}^k$,*

$$(\varphi_i^t)_*(v_1 \cdot X_1 + \cdots + v_k \cdot X_k)(\varphi_i^{-t}(x)) = (X_1,\cdots,X_k)(x) \cdot C_i(x,t) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

Proof. Fix an x in M and $i=1,\cdots,k$. Let $H_i(x,\cdot)$ be the path in $(T_xM)^k$ given by
$$H_i(x,\cdot) :]-\epsilon,\epsilon[\to (T_xM)^k, \qquad t \mapsto (\varphi_i^t)_*(X_1,\cdots,X_k)(\varphi_i^{-t})(x)$$
where ϵ is a strictly positive real number such that x is in the domain of φ_i^t for all t in $]-\epsilon,\epsilon[$. Using [S] we can show that
$$H_i(x,t)' = -(\varphi_i^t)_*([X_i,X_1],\cdots,[X_i,X_k])(\varphi_i^{-t})(x) = -H_i(x,t) \cdot A_i(\varphi_i^{-t}(x)).$$
Consider now the vector field
$$Y_i : M \times M_k(\mathbb{R}) \to T(M) \times M_k(\mathbb{R}) \times M_k(\mathbb{R}) \simeq T(M \times M_k(\mathbb{R})),$$
$$(x,m) \mapsto (x,-X_i(x); m, -m \cdot A_i(x)).$$
Denote by T_i^t the flow of Y_i. One can find an open subset $\mathcal{O}_i$ of $M \times \mathbb{R}$ containing $M \times \{0\}$ such that the following map is defined
$$C_i : \mathcal{O}_i \to M_k(\mathbb{R}), \qquad (x,t) \mapsto pr_2 \circ T_i^t(x,Id).$$
Then $C_i(\cdot,0) = Id$ and $C_i'(x,t) = -C_i(x,t) \cdot A_i(\varphi_i^{-t}(x))$.

Thus for all x in M the path in $(T_xM)^k$ defined by $t \mapsto (X_1,\cdots,X_k)(x) \cdot C_i(x,t)$ fulfils the same differentiable equation as $H_i(x,\cdot)$ and takes the same value at $t=0$.

So the equality $H_i(x,t) = (X_1,\cdots,X_k)(x) \cdot C_i(x,t)$ holds for all (x,t) in $\mathcal{O}_i$. ∎

Proof of Proposition 1. We denote by $\{\frac{\partial}{\partial t_i}\ ;\ i=1,\cdots,k\}$ the canonical base of the bundle of s-vertical fields on $M \times \mathbb{R}^k$ and by $\{e_1,\cdots,e_k\}$ the canonical base of $\mathbb{R}^k$.

We can find an open subset $\tilde{\Omega} \subset M \times \mathbb{R}^k$ which contains $M \times \{0\}$ such that for all $(x,\xi) = (x,t_1,\cdots,t_k)$ in $\tilde{\Omega}$ and $i=1,\cdots,k$

$$\begin{aligned} T_{(x,\xi)}r(\frac{\partial}{\partial t_i}) &= (\varphi_1^{t_1})_* \circ \cdots \circ (\varphi_i^{t_i})_* \cdot X_i(\varphi_i^{-t_i} \circ \cdots \circ \varphi_1^{-t_1}(r(x,\xi))) \\ &= (X_1,\cdots,X_k)(r(x,\xi)) \cdot B(x,\xi) \cdot e_i \end{aligned}$$

where $B \in M_k(\mathcal{C}^\infty(\tilde{\Omega}))$ is the matrix which the i-th column is the i-th column of the product

$$C_1(r(x,\xi),t_1)\cdot C_2(\varphi_2^{t_2}\circ\cdots\circ\varphi_k^{t_k}(x),t_2)\cdots C_i(\varphi_i^{t_i}\circ\cdots\circ\varphi_k^{t_k}(x),t_i).$$

Furthermore the matrix $B(x,0)$ is equal to Id for all $x \in M$ so we can find an open subset Ω_1 of $\tilde{\Omega}$ containing $M\times\{0\}$ such that $B(x,\xi)$ admits an inverse for all $(x,\xi)\in\Omega_1$. We define $Z_i(x,\xi) = (x,0,\xi,B^{-1}(x,\xi)\cdot e_i) \in T_{(x,\xi)}(M\times\mathbb{R}^k) = T_xM\times T_\xi\mathbb{R}^k$. It is then straightforward to check that conditions 1), 2) and 3) are fulfilled.

The equalities $s\circ h_i^t = s$ and $r\circ h_i^t = \varphi_i^t\circ r$ come from $s_*(Z_i)=0$ and $r_*(Z_i) = X_i\circ r$, respectively.

Take $i = 1,\cdots,k$ and $(x,\xi_t^i) = (x,0,\cdots,0,t,t_{i+1},\cdots,t_k)\in\Omega_1$. Then the matrix $B(x,\xi_t^i)$ is of the form $\begin{pmatrix} Id_i & D_0 \\ 0 & D_1\end{pmatrix}$, thus $Z_i(x,\xi_t^i) = \frac{\partial}{\partial t_i}$.

Let c be the path in Ω_1 defined by $t\mapsto h_i^{-t}(x,\xi_t^i)$ then $c(0) = (x,\xi_0^i)$ and the equality $Z_i(x,\xi_t^i) = \frac{\partial}{\partial t_i}$ implies that $c'(t) = 0$. Finally $(x,0,\cdots,0,t,t_{i+1},\cdots,t_k) = h_i^t(x,0,\cdots,0,0,t_{i+1},\cdots,t_k)$ for all t. This proves the last point. ∎

3.2. *Local integration of trivial Lie algebroids.* Let $(p:\mathcal{A}\to TM,[,])$ be a Lie algebroid. We denote by n the dimension of the manifold M and by k the dimension of the bundle $\mathcal{A}$. We suppose that $0<k\le n$ and that the bundle $\mathcal{A}$ is trivialisable.

Let $\{Y_1,\cdots,Y_k\}$ be a base of sections of $\mathcal{A}$ and $X_1 = p(Y_1),\cdots,X_k = p(Y_k)$ the corresponding tangent vector fields over M. Because $\mathcal{A}$ is trivial, for all $i,\ j$, we can find k maps $f_l^{ij}\in\mathcal{C}^\infty(M)$ such that

$$[Y_i,Y_j] = \sum_{l=1}^k f_l^{ij}\cdot Y_l,$$

and then

$$[X_i,X_j] = p([Y_i,Y_j]) = \sum_{l=1}^k f_l^{ij}\cdot X_l.$$

So we can apply Proposition 1.

As previously, we denote by φ_i^t the flow of X_i. We consider the maps

$$s:(x,\xi)\in M\times\mathbb{R}^k\mapsto x\in M \quad\text{and}\quad r:(x,t_1,\cdots,t_k)\in M\times\mathbb{R}^k\mapsto\varphi_1^{t_1}\circ\cdots\circ\varphi_k^{t_k}(x)\in M.$$

We take an open subset Ω_1 of $M\times\mathbb{R}^k$ which contains $M\times\{0\}$, a base $\{Z_1,\cdots,Z_k\}$ of sections of $T^s\Omega_1$ and a real number $\eta>0$ such that

$$Z_i(x,0) = Y_i(x),\ Tr(Z_i) = Tr(Z_i\circ r) = p(Y_i\circ r) = X_i\circ r;$$

$$[Z_i,Z_j](x,\xi) = \sum_{l=1}^k f_l^{ij}(r(x,\xi))\cdot Z_l(x,\xi);$$

for all $t\in]-\eta,\eta[$ the set Ω_1 is in the domain of the flow h_i^t of the vector field Z_i and the following assertions are fulfilled:

$$s\circ h_i^t = s,\qquad r\circ h_i^t = \varphi_i^t\circ r \tag{1}$$

and for all $x\in M$ and $(t_1,\cdots,t_k)\in\mathbb{R}^k$ such it makes sense we have

$$h_1^{t_1}\circ\cdots\circ h_k^{t_k}(x,0) = (x,t_1,\cdots,t_k). \tag{2}$$

Let

$$\Omega_2 = \{(x, t_1, \cdots, t_k) \in \Omega_1 \; ; \; (r(x, t_1, \cdots, t_k), 0) \in \mathrm{Im}(h_1^{t_1} \circ \cdots \circ h_k^{t_k}) \\ \text{and } \; (h_1^{t_1} \circ \cdots \circ h_k^{t_k})^{-1}(r(x, t_1, \cdots, t_k), 0) \in \Omega_1\}.$$

The set Ω_2 is an open subset of Ω_1 which contains the zero section $M \times \{0\}$. we define the differentiable map

$$i : \Omega_2 \to \Omega_1, \quad (x, t_1, \cdots, t_k) \mapsto h_k^{-t_k} \circ \cdots \circ h_1^{-t_1}(r(x, t_1, \cdots, t_k), 0).$$

Using the equalities (1) and (2), we check that $s \circ i = r$, $r \circ i = s$ and that the restriction of i to the zero section $M \times \{0\}$ is the identity map.

Theorem 3. *Suppose that the anchor p is almost injective. There exists an open subset G of Ω_2 which contains the zero section such that $G \underset{r}{\overset{s}{\rightrightarrows}} M$ equipped with the following maps is a local Lie groupoid over M.*

- *The source map is s and the range map is r.*
- *The unit map is the zero section.*
- *The inverse map is the map i:*

$$(x, t_1, \cdots, t_k) \mapsto h_k^{-t_k} \circ \cdots \circ h_1^{-t_1}(r(x, t_1, \cdots, t_k), 0).$$

- *The local product is given by*

$$(y, s_1, \cdots, s_k) \cdot (x, t_1, \cdots, t_k) = h_1^{s_1} \circ \cdots \circ h_k^{s_k}(x, t_1, \cdots, t_k)$$

and is defined over

$$\mathcal{D}^2 G = \{((y, s_1, \cdots, s_k), (x, t_1, \cdots, t_k)) \in G \times G \; ; \; y = r(x, t_1, \cdots, t_k), \\ (x, t_1, \cdots, t_k) \in dom(h_1^{s_1} \circ \cdots \circ h_k^{s_k}) \text{ and } h_1^{s_1} \circ \cdots \circ h_k^{s_k}(x, t_1, \cdots, t_k) \in G\}.$$

Moreover, the Lie algebroid of G is $\mathcal{A}$.

The non-straightforward points to check in order to prove this theorem are the associativity of the product and that the inverse map is an involutive diffeomorphism.

3.2.1. *Preliminary work.* We are going to show here the following proposition:

Proposition 4. *There exists a neighborhood Ω_3 of the zero section in $M \times \mathbb{R}^k$ such that the only local section of both s and r with values in Ω_3 is the zero section.*

For each $\tau = (t_1, \cdots, t_k) \in \mathbb{R}^k$, we define the s-vertical vector field

$$Z_\tau : \Omega_1 \to T\Omega_1, \quad (x, \xi) \mapsto \sum_{i=1}^k t_i \cdot Z_i(x, \xi).$$

We denote by ψ_τ^λ the flow of Z_τ. Remark that $\psi_\tau^\lambda = \psi_{\lambda\tau}^1$ and that the equality $Tr(Z_t) = Tr(Z_t \circ r)$ implies that $r \circ \psi_\tau^\lambda = r \circ \psi_\tau^\lambda \circ r$.

Thus the restriction of the map $r \circ \psi_\tau^\lambda$ to $M \times \{0\}$ is the flow of the tangent vector field X_τ over M defined by

$$x \in M \mapsto T_{(x,0)} r(Z_t) = \sum_{i=1}^k t_i \cdot X_i(x) \in TM.$$

LEMMA 5. *There exists an open subset* $\tilde{\Omega}_1$ *of* Ω_1 *which contains the zero section such that the map*

$$\Theta : \tilde{\Omega}_1 \to \Omega_1, \qquad (x, \tau) \mapsto \psi^1_\tau(x, 0)$$

is a diffeomorphism onto its image.

Proof. The existence of an open subset of Ω_1 which contains the zero section and which is in the domain of the map Θ comes from the equality $\psi^\lambda_\tau = \psi^1_{\lambda\tau}$.

We first remark that the restriction of Θ to $M \times \{0\}$ is equal to identity. Furthermore $T_{(x,0)}\Theta(\frac{\partial}{\partial t_i}) = Z_i(x, 0)$ for all $x \in M$ and $i = 1, \cdots, k$. So for all x in M the differential of Θ is an isomorphism. ∎

Proof of Proposition 4. Let M_0 be the dense open subset of M over which p is injective. Let K be a compact subset of M and $t \in \mathbb{R}^n \setminus \{0\}$.

As previously we denote by $X_t = Tr(Z_t|_M)$ the tangent vector field on M whose flow is the restriction of the map $r \circ \psi^\lambda_t$ to units.

The Period bounding Lemma [AR] asserts that there exists a real number $\eta_{K,t} > 0$ such that if the orbit of X_t passing through $x \in K$ is periodic with a non zero period τ_x then $\tau_x \geq \eta_{K,t}$. So one can find a real number $0 < \varepsilon_{K,t} < 1$ such that $K \times] - \varepsilon_{K,t}, \varepsilon_{K,t}[$ is a subset of Ω_0 and for all $x \in K \cap M_0$ the map

$$\pi_{x,t} :] - \varepsilon_{K,t}, \varepsilon_{K,t}[\to M, \qquad \lambda \mapsto r \circ \Theta(x, \lambda t) = r \circ \ \psi^\lambda_t(x)$$

is injective. When $t = 0$, we let $\pi_{x,0}(\lambda) = x$ for all $\lambda \in \mathbb{R}$.

We define $A_K = \inf_{t \in \mathbb{R}^n,\ \|t\|=1}\{\varepsilon_{K,t}\}$, then $0 < A_K < 1$.

Note that for all $x \in K \cap M_0$, $t \in] - A_K, A_K[\setminus \{0\}$ and $\lambda \in [-1, 1]$ the following equality holds:

$$\pi_{x,t}(\lambda) = \pi_{x,\frac{t}{\|t\|}}(\lambda\|t\|).$$

So for all $x \in K \cap M_0$ and $t \in] - A_K, A_K[\setminus \{0\}$ the map $\pi_{x,t}$ is injective when restricted to $[-1, 1]$. In particular $\pi_{x,t}(\lambda) = x$ if and only if $\lambda = 0$ or $t = 0$.

Let $\{K_i,\ i \in I\}$ be a covering of M by compact balls. We proceed as before and define $W = \bigcup_{i \in I} K_i \times] - A_{K_i}, A_{K_i}[$ and $\hat{\Omega}_1 = \overset{\circ}{W}$. Then $\hat{\Omega}_1$ is an open subset of Ω_1 containing the zero section. We define $\Omega_3 = \Theta(\hat{\Omega}_1) \cap \Omega_1$, it is also an open subset of Ω_1 containing the zero section.

Let $\nu : O \to \Omega_3$ be a local section of both s and r with values in Ω_3. We consider $\tilde{\nu} := \Theta^{-1} \circ \nu : O \to \hat{\Omega}_1$; $x \mapsto (x, t_x)$. Then for all $x \in O$,

$$r \circ \nu(x) = r \circ \Theta(x, t_x) = \pi_{x,t_x}(1) = x.$$

So $t_x = 0$ when x belongs to $O \cap M_0$. The continuity of $\tilde{\nu}$ implies that $\tilde{\nu}$ is the restriction to O of the zero section. Furthermore, Θ is a diffeomorphism which sends the zero section onto the zero section, so ν is the restriction to O of the zero section as well. ∎

If $\xi = (t_1, \cdots, t_k) \in \mathbb{R}^k$ we denote $h^\xi := h^{t_1}_1 \circ \cdots \circ h^{t_k}_k$. It is a local diffeomorphism of Ω_1. We denote by Υ the pseudo-group of local diffeomorphisms of Ω_3 generated by the h^ξ restricted to Ω_3. The equations (1) and (2) imply that for all $\varphi \in \Upsilon$ the equalities $s \circ \varphi = s$ and $r \circ \varphi = r \circ \varphi \circ r$ hold.

We deduce from the previous proposition

Corollary 6. *For all $\varphi \in \Upsilon$ and $x \in M$ the following equality holds when it makes sense*

$$\varphi \circ i \circ \varphi(r \circ \varphi^{-1}(x,0)) = (x,0).$$

Moreover if $\varphi(x,0) = (x,0)$ then $\varphi(z) = z$ for all $z \in r^{-1}(x) \cap dom(\varphi)$.

3.2.2. *Proof of Theorem 3.* We define $G := (\Omega_2 \cap \Omega_3) \cap i(\Omega_2 \cap \Omega_3)$. It is an open subset of Ω_2 which contains the zero section. We have already noticed that in order to prove the theorem we must show that the inverse map is involutive and that the product is associative.

Let's show that the restriction of i to G is an involutive diffeomorphism.
Let $(x,\xi) = h^{\xi}(x) \in G$. By applying the first assertion of the corollary 6 to $(h^{\xi})^{-1}$ we get $(h^{\xi})^{-1} \circ i \circ i(h^{\xi}(x)) = x$ for all x in M. Thus $i \circ i = i$.

Let's show the associativity of the product.

Suppose that the following products are defined:

$$(r(x,\xi_3),\xi_2) \cdot (x,\xi_3) = h^{\xi_2} \circ h^{\xi_3}(x),$$
$$(r(h^{\xi_2} \circ h^{\xi_3}(x)),\xi_1) \cdot (r(x,\xi_3),\xi_2) = h^{\xi_1} \circ h^{\xi_2}(r(x,\xi_3)) \quad \text{and}$$
$$((r(h^{\xi_2} \circ h^{\xi_3}(x)),\xi_1) \cdot (r(x,\xi_3),\xi_2)) \cdot (x,\xi_3).$$

Fix $\tau \in \mathbb{R}^k$ such that $h^{\xi_1} \circ h^{\xi_2}(r(x,\xi_3)) = h^{\tau}(r(x,\xi_3))$. According to the corollary we have the equality $h^{\xi_1} \circ h^{\xi_2} = h^{\tau}$ on $r^{-1}(r(x,\xi_3))$. So we obtain

$$\begin{aligned}((r(h^{\xi_2} \circ h^{\xi_3}(x)),\xi_1) \cdot (r(x,\xi_3),\xi_2)) \cdot (x,\xi_3) &= h^{\xi_1} \circ h^{\xi_2} \circ h^{\xi_3}(x) \\ &= (r(h^{\xi_2} \circ h^{\xi_3}(x)),\xi_1) \cdot ((r(x,\xi_3),\xi_2) \cdot (x,\xi_3)). \quad \blacksquare\end{aligned}$$

Remark 1. – The elements of Υ restricted to G are local left translations of G.
– The almost injectivity of $\mathcal{A}$ implies that the local groupoid $G \overset{s}{\underset{r}{\rightrightarrows}} M$ is such that the only local section of both s and r is the unit map. A (local) groupoid having this property is called a (local) *quasi-graphoid* or an *essentially principal* groupoid [B, R].

3.3. *Local integration of morphisms.* We describe here the (local) integration of morphisms between Lie algebroids. One can find different proofs of this in [Ma, MM, MX].

For $* = 1,2$, let $(p_* : \mathcal{A}_* \to M, [,]_*)$ be an almost injective trivialisable Lie algebroid and $\{Y_1^*, \cdots, Y_{k_*}^*\}$ a base of sections of $\mathcal{A}_*$. Let $G_* \overset{s_*}{\underset{r_*}{\rightrightarrows}} M$ be a local Lie groupoid associated to $\mathcal{A}_*$ as in theorem 3.

We suppose that we have a morphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ of Lie algebroid over identity. In particular we have $p_2 \circ f = p_1$.

Proposition 7. *There exists a local Lie sub-groupoid V_1 of G_1 with space of units M and a morphism $F : V_1 \to G_2$ which integrates f.*

Proof. For $* = 1,2$, let $\{Z_1^*, \cdots, Z_{k_*}^*\}$ be the base of s_*-vertical vector fields associated to the base of sections $\{Y_1^*, \cdots, Y_{k_*}^*\}$.

For $i = 1, \cdots, k_1$, we denote by q_i the s_2-vertical vector field on $\mathcal{A}_2$ associated to $f(Y_i^1) = \sum_{l=1}^{k_2} a_l Y_l^2$, that is $q_i(x,\xi) = \sum_{l=1}^{k_2} a_l(r_2(x,\xi)) Z_l^2(x,\xi)$.

We denote by Φ_i^t the flow of q_i and by h_i^t the flow of Z_i^1 as we did previously. We can find a local Lie sub-groupoid V_1 of G_1 having M as space of units such that the map

$$F : V_1 \to G_2, \quad (x, t_1, \cdots, t_{k_1}) = h_1^{t_1} \circ \cdots \circ h_{k_1}^{t_{k_1}}(x) \mapsto \Phi_1^{t_1} \circ \cdots \circ \Phi_{k_1}^{t_{k_1}}(x)$$

is defined and smooth. One can easily check that F is a morphism of local groupoids which integrates f. ∎

When f is an isomorphism, F is an isomorphism onto its image as well.

Remark 2. If G is a (local) groupoid over M and G' a (local) quasi-graphoid over M then there is at most one morphism from G to G' over identity. So if $f : V \to G'$ and $g : W \to G'$ are morphisms where V and W are two local sub-groupoids of G having M as space of units then $f = g$ when restricted to $V \cap W$. So there exists a unique morphism from $V \cup W$ to G' which is equal to f when restricted to V and equal to g when restricted to W. We will say that a morphism $f : V \to G'$, where V is a local sub-groupoid of G having M as space of units, is *maximal* if for any morphism $g : W \to G'$ where W is a local sub-groupoid of G having M as space of units, we have $W \subset V$.

3.4. *General case.* Let $(p : \mathcal{A} \to TM, [,])$ be an almost injective Lie algebroid. We denote by π the projection of $\mathcal{A}$ onto M. Let $\{\tau_i : \mathcal{A}_i = \pi^{-1}(O_i) \xrightarrow{\sim} O_i \times \mathbb{R}^k,\ i \in I\}$ be a local trivialisation of the bundle $\mathcal{A}$, $\{O_i,\ i \in I\}$ being an open covering of M.

For each $i \in I$, $(p_i : \mathcal{A}_i \to TO_i, [,]_{\mathcal{A}_i})$ is an almost injective trivialisable Lie algebroid and we can apply theorem 3.

We obtain in this way a family $\mathcal{U} = \{G_i \underset{r_i}{\overset{s_i}{\rightrightarrows}} O_i,\ i \in I\}$ of local Lie groupoids such that for all $i \in I$, $G_i \rightrightarrows O_i$ is a local quasi-graphoid which integrates $\mathcal{A}_i$.

For all $i, j \in I$, $\mathcal{A}|_{O_i \cap O_j} = \mathcal{A}_i|_{O_i \cap O_j} = \mathcal{A}_j|_{O_i \cap O_j}$ so using the integration of morphisms, there exist a local Lie sub-groupoid V_{ij} of G_i, V_{ji} of G_j having $O_i \cap O_j$ as space of units and an isomorphism $F_{ij} : V_{ij} \to V_{ji}$ which integrates identity. We can require F_{ij} to be maximal and because we are dealing with local quasi-graphoids, the cocycle conditions are fulfilled:

$$F_{ii} = Id_{G_i},\ F_{ij}^{-1} = F_{ji} \text{ and } F_{jk} \circ F_{ij} = F_{ik} \text{ when it makes sense.}$$

So there is a natural equivalence relation on $\bigsqcup_{i \in I} G_i$ given by:

$$\text{if } \gamma_i \in G_i \text{ and } \eta_j \in G_j,\ \gamma_i \sim \eta_j \text{ if and only if } \begin{cases} \gamma_i \in V_{ij},\ \eta_j \in V_{ji} \\ F_{ij}(\gamma_i) = \eta_j \end{cases}.$$

Let $G = \bigsqcup_{i \in I} G_i/\sim$. If γ_i belongs to G_i we denote by $\overline{\gamma_i}$ its image in G.

One can easily check that $G \underset{r}{\overset{s}{\rightrightarrows}} M$ equipped with the following maps is a local Lie groupoid over M.

- The source map is $s : \overline{\gamma_i} \mapsto s_i(\gamma_i)$ and the range map is $r : \overline{\gamma_i} \mapsto r_i(\gamma_i)$.
- The unit map is given by $x \in O_i \mapsto \overline{u_i(x)}$.
- The inverse map is given by $\overline{\gamma_i} \mapsto \overline{\gamma_i^{-1}}$.
- The local product is given by $\overline{\gamma_i} \cdot \overline{\eta_i} = \overline{\gamma_i \cdot \eta_i}$ and is defined over

$$\mathcal{D}^2 G = \{(\overline{\gamma_i}, \overline{\eta_i}) \in G \times G \mid (\gamma_i, \eta_i) \in \mathcal{D}^2 G_i,\ i \in I\}.$$

Furthermore, G is a quasi-graphoid and it integrates $\mathcal{A}$ by construction.

Let $\mathcal{U}$ and $\mathcal{U}'$ be two local trivialisations of the bundle $\mathcal{A}$. Let G and G' the local groupoid obtained from $\mathcal{U}$ and $\mathcal{U}'$ as previously. Using the integration of morphisms and the fact that the groupoids G and G' are local quasi-graphoids one can easily show that there exist a local sub-groupoid V of G over M, a local sub-groupoid V' of G' over M and an isomorphism between V and V'. Thus the "germs" of the groupoid obtained do not depend on any choice.

In particular, let G be a (local) Lie groupoid over $G^{(0)}$ and $\mathcal{A}G$ its Lie algebroid. Suppose that $\mathcal{A}G$ is almost injective. If H is a local Lie groupoid obtained as in Theorem 3 then H and G are isomorphic around units, that is, there is a local sub-groupoid of H having $G^{(0)}$ for unit space which is isomorphic to a local sub-groupoid of G.

4. Examples. *1. Injective Lie algebroid.* Let $(p : \mathcal{A} \to TM, [,])$ be a Lie algebroid of dimension k over a manifold M of dimension n. We suppose that p is injective and that $0 < k \leq n$. So $\mathcal{A}$ is isomorphic to the tangent bundle of a regular foliation $\mathcal{F}$ on M. If O is a distinguished open set for $\mathcal{F}$, one can find a local basis $\{Y_1, \cdots, Y_k\}$ of sections of $\mathcal{A}$ defined on O such that $[Y_i, Y_j] = 0$ for all $i,\ j$. In other word, there is a local free action of $\mathbb{R}^k$ on O. The local integration of the restriction of $\mathcal{A}$ to O gives rise to the local groupoid Ω of this local action of $\mathbb{R}^k$ on O. So Ω is isomorphic (around the units) to the groupoid of the regular equivalence relation induced by the foliation $\mathcal{F}$ on O.

2. Lie algebra of tangent vector fields. Let $\mathcal{H}$ be a Lie algebra of tangent vector fields over a manifold M. Suppose that the dimension of $\mathcal{H}$ is finite. We consider the trivial bundle $\mathcal{A} = M \times \mathcal{H}$ over M and we define $p : \mathcal{A} \to TM$, $(x, X) \mapsto X(x)$. We suppose that p is injective over a dense open subset of M. The bundle $\mathcal{A}$ is naturally endowed with a structure of trivial Lie algebroid over M of anchor p. The local integration gives rise to a local Lie group H which integrates $\mathcal{H}$ and a local action of H on M. The local Lie groupoid we obtain is then the local groupoid of this action.

3. Action of $\mathbb{R}$. Let N be a manifold and $M = N \times \mathbb{R}$. We equip the bundle $\mathcal{A} = TM \simeq TN \times T\mathbb{R}$ over M with the structure of Lie algebroid which anchor is the map

$$p : \mathcal{A} = TM \simeq TN \times T\mathbb{R} \to TM \simeq TN \times T\mathbb{R}, \qquad ((x, v), (t, \lambda)) \mapsto ((x, v)(t, t\lambda)).$$

The local integration of $\mathcal{A}$ over an open set $O \times \mathbb{R}$ of $M = N \times \mathbb{R}$ gives rise to the groupoid $O \times O \times \mathbb{R}^+_* \times \mathbb{R} \rightrightarrows O \times \mathbb{R}$ which is the product of the pair groupoid over O with the groupoid of the action of $\mathbb{R}^+_*$ on $\mathbb{R}$ by multiplication.

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