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PRIMARY CHARACTERISTIC HOMOMORPHISM OF PAIRS OF LIE ALGEBROIDS AND MACKENZIE ALGEBROID

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Abstract. Connections for a pair of Lie algebroids are defined as linear homomorphisms between these Lie algebroids, commuting with the anchors. The primary characteristic homomorphism of a pair of Lie algebroids is defined and compared with other known Chern-Weil homomorphisms.

1. Introduction. By an *L-connection* in A (L and A are Lie algebroids on the same manifold B) we mean a linear homomorphism $\nabla : L \to A$ commuting with the anchors $\omega_A \circ \nabla = \omega_L$. This definition covers

— usual and partial covariant derivatives in vector bundles,
— usual and partial connections in principal bundles,
— connections in transitive or regular Lie algebroids,
— connections in extensions $\mathbf{e} : 0 \to L' \to A \to L \to 0$ of Lie algebroids, in particular, complete differentials of higher order,
— transversal connections in extensions of principal fibre bundles,
— covariant and contravariant connections known in Poisson geometry.

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L-connections in vector bundles and L-connections in principal bundles [F2] determine L-connections in suitable Lie algebroids of vector bundles and principal bundles.

In this work we construct the Chern-Weil homomorphism $h_{L,A}$ of a pair (L, A) of Lie algebroids, where A is regular, giving obstructions to the existence of flat L-connections in A. This approach generalizes known constructions by Teleman 1972 [T2], Mackenzie 1988 [M5], Kubarski 1991 [K4], Vaisman 1994 [V2], Belko 1997 [B2], Huebschmann 1999 [H2], Itskov, Karasev and Vorobjev 1999 [IKV], Fernandes (2000) [F1], [F2], Crainic (preprint 2001) [C]. For the tangential case see also Moore and Schochet 1988 [M-C], Kubarski 1993 [K5].

Next, $h_{L,A}$ is compared with the other Chern-Weil homomorphisms h_L, h_A, and $h_{\mathbf{e}}$ in the case of an extension $\mathbf{e}$. We also introduce a G-invariant Chern-Weil homomorphism for G-algebroids and use it to transversal PBG-Mackenzie algebroids.

(1) A *covariant derivative* in a vector bundle $\mathfrak{f}$ on a smooth manifold B is a mapping $\nabla : \mathfrak{X}(B) \to \mathrm{End}_{\mathbb{R}}(\mathrm{Sec}\,\mathfrak{f})$ which satisfies the known Koszul axioms [K]: $\nabla_X \sigma$ is $C^\infty(B)$-linear with respect to $X \in \mathfrak{X}(B)$ and $\mathbb{R}$-linear with respect to $\sigma \in \mathrm{Sec}\,\mathfrak{f}$, while $\nabla_X : \mathrm{Sec}\,\mathfrak{f} \to \mathrm{Sec}\,\mathfrak{f}$ is a covariant differential operator with the anchor X (i.e. $\nabla_X(f \cdot \sigma) = f \cdot \nabla_X \sigma + X(f) \cdot \sigma$). Cohomology classes from the image of the primary characteristic homomorphism of $\mathfrak{f}$ determine topological obstructions to existence of a flat covariant derivative in $\mathfrak{f}$.

If we replace $\mathfrak{X}(B)$ with $\mathrm{Sec}\,F$ (where $F \subset TB$ is a vector subbundle tangent to some regular foliation on B), then we obtain the so-called *partial covariant derivative* in the bundle $\mathfrak{f}$ [K-T]. In both cases, TB and F are trivial Lie algebroids, whereas the space of covariant differential operators in the bundle $\mathfrak{f}$ is the space of cross-sections of the Lie algebroid $A(\mathfrak{f})$ of $\mathfrak{f}$, see [K4], [K8], [M1], [K3], denoted also by $CDO(\mathfrak{f})$. An operator ∇ can be given equivalently as a linear homomorphism of vector bundles $\nabla : TB \to A(\mathfrak{f})$ $[\nabla : F \to A(\mathfrak{f})]$ such that $\omega_{\mathfrak{f}} \circ \nabla = \mathrm{id}$, where $\omega_{\mathfrak{f}} : A(\mathfrak{f}) \to TB$ is the anchor of the Lie algebroid $A(\mathfrak{f})$. The next one, very important in differential geometry generalizations, can be obtained by taking any Lie algebroid $(L, [\![\cdot,\cdot]\!], \omega_L)$ instead of TB,

$$\nabla : L \to A(\mathfrak{f}), \tag{1.1}$$

assuming additionally that ∇ commutes with anchors

$$\omega_{\mathfrak{f}} \circ \nabla = \omega_L.$$

Then the operator ∇ is said to be an *L-covariant derivative* in the vector bundle $\mathfrak{f}$ (cf. the definition of an L-connection in the more general category of Lie-Rinehart algebras [H1], [H2]).

The operator (1.1) induces the so-called linear *L-connection* in $\mathfrak{f}$, $\nabla : \mathrm{Sec}\,L \times \mathrm{Sec}\,\mathfrak{f} \to \mathrm{Sec}\,\mathfrak{f}$ (introduced in the case of Poisson manifolds B by I. Vaisman [V1] for $L = T^*B$ and known as a contravariant derivative), see also [H-M].

In the case when the operator (1.1) is a homomorphism of Lie algebroids it is known as a representation of L in $\mathfrak{f}$. Further, if we replace $A(\mathfrak{f})$ with a Lie algebroid $(A, [\![\cdot,\cdot]\!], \omega_A)$, then we obtain a quite general object (examined in this work)

$$\nabla : L \to A, \;\; \omega_A \circ \nabla = \omega_L,$$

called an *L-connection in the Lie algebroid A*. In the case when $A = TP_{/G}$ is the Lie algebroid of a principal bundle $P(B,G,p)$ and $L = TB$, considered connections correspond to connections in the principal bundle P (for $L \subset TB$—partial connections). If $A = TP_{/G}$ (as above) and $L = T^*B$ is the Lie algebroid of a Poisson manifold B, then any operator (1.1) is determined by the so-called *contravariant connection* $h : p^*T^*B \to TP$ (introduced recently by R. L. Fernandes [F1]) and vice versa.

If A is a transitive Lie algebroid and L is an arbitrary one, both over the manifold B, and the characteristic Stefan foliation $\operatorname{Im}\omega_L$ is contained in a regular foliation $F \subset TB$, $\operatorname{Im}\omega_L \subset F$, then any L-connection $\nabla : L \to A$ has values in the regular Lie algebroid $A^F = \omega_A^{-1}[F]$ so ∇ determines an L-connection in A^F, $\nabla^F : L \to A^F$, $\nabla^F(v) = \nabla(v)$. We should add here that the domain $I(A^F)$ of the Chern-Weil homomorphism h_{A^F} of A^F [K4], [K5] contains the $\Omega_b(B,F)$-module $\Omega_b(B,F)\cdot I(A)$ (i.e. linear combinations $\sum f^i \cdot \Gamma_i$, $f^i \in \Omega_b(B,F)$, $\Gamma_i \in I(A)$), where $I(A)$ is the domain of h_A and $\Omega_b(B,F)$ is the space of F-basic functions. Sometimes (and this is rather interesting), $I(A^F)$ is larger than $\Omega_b(B,F)\cdot I(A)$ [K5] so it may be a source of new "singular" characteristic classes which are obstructions to the existence of flat L-connections in A, see below.

(2) In this paper we construct a characteristic homomorphism $h_{L,A} : I(A) \to H_L(B)$ of the Chern-Weil type, which describes obstructions to the existence of flat L-connections ∇ in a regular Lie algebroid A, where $I(A)$ is the algebra of real multilinear symmetric homomorphisms of the adjoint bundle of Lie algebras $\boldsymbol{g}_A = \ker\omega_A$ and invariant with respect to the adjoint representation of A on $\boldsymbol{g}_A$ while $H_L(B)$ is a cohomology algebra of real A-forms [MR], [M1], [K4]. The domain of $h_{L,A}$ is equal to the domain of h_A. We have the equalities $h_{TB,A} = h_A$ and $h_{F,A} = h_{A^F}$ (for a regular foliation $F \subset TB$). The relation between h_A and $h_{L,A}$ is $\omega_L^\sharp \circ h_A = h_{L,A}$. Putting $L = A$ we obtain $\omega_A^\sharp \circ h_A = h_{A,A} = 0$ which gives easily that $\operatorname{Pont}(A) = \operatorname{Im} h_A \subset \ker\omega_A^\sharp$.

The homomorphism $h_{L,A}$ is a generalization of:

1. the classical Chern-Weil homomorphism of a vector bundle $\mathfrak{f}$ (if $A = A(\mathfrak{f})$ and $L = TB$),
2. the tangential Chern-Weil homomorphism of $\mathfrak{f}$ over a foliated manifold [M-C], [K5] (if $A = A(\mathfrak{f})$ and $L = F \subset TB$),
3. the characteristic homomorphism of regular Lie algebroid [K4] (if $L = F = \operatorname{Im}(\omega_A)$; in this case an F-connection in A is a splitting $\nabla : F \to A$ of the Atiyah sequence in A, $0 \to \boldsymbol{g}_A \to A \to F \to 0$),
4. the Chern-Weil homomorphism of a principal bundle, if $A = A(P) = TP_{/G}$ and $L = TB$,
5. the Fernandes [F2] and Crainic [C] L-Chern-Weil homomorphism of a real vector bundle $\mathfrak{f}$ (if $A = A(\mathfrak{f})$).

A comparison of $h_{L,A(P)}$ to the Fernandes L-Chern-Weil homomorphism of a principal bundle P [F2] is given in section 3.5 below.

(3) Recall (cf. (4) below) [K4], [B1], [B2] the relation between the classical Chern-Weil homomorphism h_P of a G-principal fibre bundle $P(B,G)$ and the Chern-Weil homomorphism $h_{A(P)}$ of the transitive Lie algebroid $A(P) = TP_{/G}$ of P: there exists a monomorphism of algebras $\nu : I(G) \to I(A(P))$ which gives commutativity of the diagram

$$\begin{tikzcd} I(A(P)) \arrow[dr, "h_A"] & \\ & H(B) \\ I(G) \arrow[uu, "\nu"] \arrow[ur, "h_P"'] & \end{tikzcd} \tag{1.2}$$

If the total space of P is a connected manifold (the structure Lie group need not be connected), then ν is an isomorphism [K4], [B1], [B2]. Notice that if the base manifold B is connected, then the connectedness of P is equivalent to the connectedness of the Ehresmann groupoid $P\times_G P$ [Lis].

Recently R. L. Fernandes [F2] has obtained an L-Chern-Weil homomorphism of a G-principal fibre bundle P which turns out to be equal to $h_{L,A(P)}$ if the domains of h_P and $h_{A(P)}$ are the same (for example, if P is connected). The Fernandes case is very important in Poisson geometry.

(4) V. Itskov, M. Karasev and Yu. Vorobjev [IKV] had constructed independently a Chern-Weil homomorphism for a transitive Lie algebroid $(A\to B,\omega_A,\{,\})$. It is a homomorphism

$$h:\bigoplus\nolimits^k \operatorname{Inv}^k(\boldsymbol{g}_{|x_0})\longrightarrow H_{dR}(B),\ \Gamma\mapsto[\Gamma(R,\dots,R)],$$

where R is the curvature of any connection in A and $\boldsymbol{g}_{|x_0}$ is the isotropy Lie algebra of A at x_0, i.e. the fibre of $\boldsymbol{g}$ at $x_0\in B$. The domain $\bigoplus^k\operatorname{Inv}^k(\boldsymbol{g}_{|x_0})$ is in general smaller than the domain $I(A)$ considered in the previous papers [T2], [K4], [B2], [H2], however sometimes equality holds. The domain $I(A)=\bigoplus^k\operatorname{Sec}(\bigvee^k\boldsymbol{g}^*)_{I(\mathrm{ad})}$ is the algebra of ad-invariant cross-sections of the symmetric powers $\bigvee^k\boldsymbol{g}^*$. Equivalently, it is the algebra of parallel sections—under the flat adjoint connection—of the vector bundles $\operatorname{inv}^k(\boldsymbol{g})$ whose fibres are the spaces of ad-invariant polynomials on $\boldsymbol{g}_{|x}$, which can be described by K. Mackenzie's formula in Th. IV.1.19 of [M1]. This formula says that an element from $\operatorname{inv}^k(\boldsymbol{g}_{|x_0})$ can be extended to a parallel cross-section if and only if it is invariant with respect to the $\pi_1(B)$-action on $\operatorname{inv}^k(\boldsymbol{g}_{|x_0})$ via the holonomy morphism for any flat adjoint connection on the vector bundle $\operatorname{inv}^k(\boldsymbol{g})$. To better understand the relation between the domains $\bigoplus^k\operatorname{Inv}^k(\boldsymbol{g}_{|x_0})$ and $I(A)$ notice first that the space $\operatorname{Inv}^k(\boldsymbol{g}_{|x_0})$ after naturally extending its elements to parallel cross-sections of $\operatorname{inv}^k(\boldsymbol{g})$, is equal to the space of invariant elements with respect to the canonical representation T of the $\operatorname{Aut}(\boldsymbol{g}_{|x_0})$-principal bundle $\operatorname{Aut}(\boldsymbol{g})$ of Lie algebra isomorphisms $\boldsymbol{g}_{|x_0}\xrightarrow{\cong}\boldsymbol{g}_{|x}$ on the vector bundle $\bigvee^k\boldsymbol{g}^*$, induced by the inclusion $\operatorname{Aut}(\boldsymbol{g})\subset L(\boldsymbol{g})$ [$L(\boldsymbol{g})$ is the full $\mathrm{GL}(\boldsymbol{g}_{|x_0})$-principal bundle of frames of $\boldsymbol{g}$, and $T(z)=\bigvee^k(z^*)^{-1}$, $z:\boldsymbol{g}_{|x_0}\to\boldsymbol{g}_{|x}$]. Next consider the sequence of inclusions

$$\operatorname{Inv}^k(\boldsymbol{g}_{|x_0})\subset\operatorname{Sec}(\bigvee\nolimits^k\boldsymbol{g}^*)_{I(d)}\subset\operatorname{Sec}(\bigvee\nolimits^k\boldsymbol{g}^*)_{I(\mathrm{ad})}$$

where the middle element consists of all invariant cross-sections with respect to the derivative of T. According to Prop. 3.3.8 of [K5] this element is equal to the space of cross-sections $\Gamma\in\operatorname{Sec}(\bigvee^k\boldsymbol{g}^*)$ invariant with respect to all covariant differential operators

$\xi : \mathrm{Sec}(\boldsymbol{g}) \to \mathrm{Sec}(\boldsymbol{g})$ which are differentiations of the Lie algebra $\mathrm{Sec}(\boldsymbol{g})$, i.e. cross-sections Γ such that

$$(\omega_A \circ \xi)(\Gamma(\theta_1, \ldots, \theta_k)) - \sum_i \Gamma(\theta_1, \ldots, \xi(\theta_i), \ldots, \theta_k) = 0$$

for all ξ as above. Since there are, in general, fewer adjoint differentiations than all differentiations, it can give sometimes the effect that the second inclusion is not an equality. According to Prop. 5.5.3 of [K4] the first inclusion is an equation if the $\mathrm{Aut}(\boldsymbol{g}_{|x_0})$-principal bundle $\mathrm{Aut}(\boldsymbol{g})$ is connected.

(5) In the present paper we compare the constructed Chern-Weil homomorphism of pairs of Lie algebroids with the other one given by J. Huebschmann [H2] for splittings of Lie-Rinehart algebras (and earlier, in a less general case, by N. Teleman [T1], [T2]), which in the case of Lie algebroids relates to the epimorphism of Lie algebroids $A \overset{\pi}{\to} L$ and connection interpreted as its right inverse in the class of linear homomorphisms $\nabla : L \to A$, $\pi \circ \nabla = \mathrm{id}$. Diagram (3.8) in part 3 below compares four Chern-Weil homomorphisms h_A, h_L, $h_{L,A}$, $h_{\mathbf{e}}$ for a given extension $\mathbf{e} :\ \ 0 \to L' \to A \overset{\pi}{\to} L \to 0$.

Complete differentials of higher order in the tangent bundle TB [P], [NVQ], [S], given as splittings of the exact sequence of the jet bundles $0 \to S^k(TB, TB) \to J^k(TB) \to J^{k-1}(TB) \to 0$, are important examples of right inverse connections.

Other important examples are given by K. Mackenzie [M2] in connection with extensions of principal bundles. Commutativity of diagram (3.8) on the level of transitive Lie algebroids and the epimorphism $\pi : A \to L$ given by the extension of principal bundles $Q(B, H, q) \to P(B, G, p)$ (with respect to the extension of Lie groups $0 \to N \to H \to G \to 0$)—with $h_{L,A}$ omitted—is described in [M2, Prop. 3.7]. It is done in terms of the Chern-Weil homomorphisms of principal bundles $P(B, G)$, $Q(B, H)$ and the transversal Chern-Weil homomorphism of the transversal principal bundle $Q(P, N)$, and expressed as commutativity of the following diagram

$$\begin{tikzcd}
I(G) \arrow[d, "\pi^*"'] \arrow[dr, "w"] & \\
I(H) \arrow[r, "w'"] \arrow[d, "\rho"'] & H(B) \arrow[d, "a^*"] \\
I(N)^G \arrow[r, "\overline{w}"'] & H(P)^G
\end{tikzcd} \tag{1.3}$$

(a denotes the anchor of $P(B, G)$; $\overline{w}$ is equivalent to the Chern-Weil homomorphism of the extension of transitive Lie algebroids $0 \to Q\times_H \mathfrak{n} \to TQ_{/H} \to TP_{/G} \to 0$).

(6) Next, the G-equivariant Chern-Weil homomorphism for an extension of G-algebroids is introduced (the notion of a G-algebroid comes from [K6] and generalizes the notion of a Mackenzie PBG-algebroid [M4]).

Finally we examine extension (2.3) in which L is an integrable Lie algebroid $L = TP_{/G}$. According to [M5] it produces a transversal PBG-Mackenzie algebroid p^*A (where

$p : P \to B$ is the projection). An equivalence of the Chern-Weil homomorphism $h_{\mathbf{e}}$ of $\mathbf{e}$ $: 0 \to L' \to A \xrightarrow{\pi} TP_{/G} \to 0$ and the G-equivariant Chern-Weil homomorphism $h^G_{p^*A}$ of the G-algebroid p^*A is obtained which generalizes the Mackenzie result concerning extension $Q(B,H,q) \xrightarrow{\pi} P(B,G,p) \to 0$ of principal bundles.

Some of these results were presented in the poster [K-W] during the 3rd ECM, Barcelona 2000.

2. L-connections

2.1. *Definitions and examples.* A *Lie algebroid* [Pr] on a manifold B is a triple $(L, [\![\cdot,\cdot]\!], \omega_L)$ where L is a vector bundle on B, $(\operatorname{Sec} L, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\omega_L : L \to TB$ is a linear homomorphism of vector bundles and the following Leibniz condition is satisfied

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \omega_L(\xi)(f)\cdot\eta, \quad f \in C^\infty(B),\ \xi,\eta \in \operatorname{Sec} L. \tag{2.1}$$

LEMMA 2.1 (See [H]). *The anchor is bracket-preserving, $\omega_L[\![\xi,\eta]\!] = [\omega_L\xi, \omega_L\eta]$.*

Proof. Let $\xi, \eta, \tau \in \operatorname{Sec} L$, $f \in C^\infty(B)$. Observe that

$$\begin{aligned}(\omega_L\circ\xi)((\omega_L\circ\eta)(f))\cdot\tau &= [\![\xi, ((\omega_L\circ\eta)(f))\cdot\tau]\!] - ((\omega_L\circ\eta)(f))\cdot[\![\xi,\tau]\!]\\ &= [\![\xi, [\![\eta, f\cdot\tau]\!] - f\cdot[\![\eta,\tau]\!]]\!] - ([\![\eta, f\cdot[\![\xi,\tau]\!]]\!] - f\cdot[\![\eta,[\![\xi,\tau]\!]]\!])\\ &= [\![\xi,[\![\eta,f\cdot\tau]\!]]\!] - [\![\xi, f\cdot[\![\eta,\tau]\!]]\!] - [\![\eta, f\cdot[\![\xi,\tau]\!]]\!] + f\cdot[\![\eta,[\![\xi,\tau]\!]]\!],\end{aligned}$$

and analogously

$$(\omega_L\circ\eta)((\omega_L\circ\xi)(f)) = [\![\eta,[\![\xi,f\cdot\tau]\!]]\!] - [\![\eta, f\cdot[\![\xi,\tau]\!]]\!] - [\![\xi, f\cdot[\![\eta,\tau]\!]]\!] + f\cdot[\![\xi,[\![\eta,\tau]\!]]\!].$$

Then

$$\begin{aligned}[\omega_L\xi,\omega_L\eta](f)\cdot\tau &= (\omega_L\circ\xi)((\omega_L\circ\eta)(f)) - (\omega_L\circ\eta)((\omega_L\circ\xi)(f))\\ &= [\![\xi,[\![\eta,f\cdot\tau]\!]]\!] - [\![\xi, f\cdot[\![\eta,\tau]\!]]\!] - [\![\eta, f\cdot[\![\xi,\tau]\!]]\!] + f\cdot[\![\eta,[\![\xi,\tau]\!]]\!]\\ &\quad - [\![\eta,[\![\xi,f\cdot\tau]\!]]\!] + [\![\eta, f\cdot[\![\xi,\tau]\!]]\!] + [\![\xi, f\cdot[\![\eta,\tau]\!]]\!] - f\cdot[\![\xi,[\![\eta,\tau]\!]]\!]\\ &= [\![\xi,[\![\eta,f\cdot\tau]\!]]\!] + [\![\eta,[\![f\cdot\tau,\xi]\!]]\!] + f\cdot([\![\eta,[\![\xi,\tau]\!]]\!] + [\![\xi,[\![\tau,\eta]\!]]\!])\\ &= -[\![f\cdot\tau,[\![\xi,\eta]\!]]\!] - f\cdot[\![\tau,[\![\eta,\xi]\!]]\!]\\ &= [\![[\![\xi,\eta]\!], f\cdot\tau]\!] - f\cdot[\![[\![\xi,\eta]\!],\tau]\!] = (\omega_L\circ[\![\xi,\eta]\!])(f)\cdot\tau. \ \blacksquare\end{aligned}$$

REMARK 2.2. The analogous fact for an (R, A)-Lie-Rinehart algebra (L, ω) holds provided that A-module L fulfils the axiom

- The representation $\rho : A \to End_A(L)$, $\rho(f)(\tau) = f\cdot\tau$, is faithful, i.e. $\ker\rho = 0$.

Nontrivial Lie algebroids appeared as infinitesimal objects for Lie grupoids, principal bundles, vector bundles, TC-foliations, Poisson or Jacobi manifolds, etc. and play an analogous role to the Lie algebra of a Lie group (for review see [M3] and [K7]). If ω_L is of a constant rank, then a Lie algebroid is called *regular*, moreover, $\operatorname{Im}\omega_L \subset TB$ is an involutive distribution which determines a regular foliation (called also *characteristic*) on the base manifold B. Given any Lie algebroids on the same manifold the notion of a strong homomorphism is defined which preserves structures of Lie algebras and commutes with anchors, whereas in any case we have a less obvious notion of a nonstrong homomorphism, given by K. Mackenzie [H-M] (see also [K4]).

Let $(L, [\![\cdot,\cdot]\!], \omega_L)$ and $(A, [\![\cdot,\cdot]\!], \omega_A)$ be Lie algebroids on a smooth manifold B (we do not assume regularity of these algebroids).

DEFINITION 2.3. A strong homomorphism of vector bundles $\nabla : L \longrightarrow A$ is called an *L-connection* in A if

$$\omega_A \circ \nabla = \omega_L. \tag{2.2}$$

The existence of an L-connection in A implies the inclusion of characteristic foliations $\operatorname{Im}\omega_L \subset \operatorname{Im}\omega_A$.

EXAMPLE 2.4. (1) Let A be a regular Lie algebroid with the Atiyah sequence $0 \to \boldsymbol{g}_A \to A \overset{\omega_A}{\to} F \to 0$. The splitting of this sequence $\nabla : F \to A$, $\omega \circ \nabla = \mathrm{id}_F$, is an F-connection in A. Clearly, ω_A is a unique A-connection in F. In [K4] we defined the Chern-Weil homomorphism of A, $h_A : I(A) \to H_F(B)$, where $I(A)$ is the space of invariant symmetric multilinear mappings on $\boldsymbol{g}_A$ and $H_F(B)$ is the algebra of tangential cohomologies. This homomorphism is trivial if there exists a flat F-connection in A. The transitive case was applied to TC-foliations in [K4] where we give an interpretation of the Chern-Weil homomorphism for the Lie algebroid $A(B,E)$ of a TC-foliation (B,E); in particular, for foliations of left cosets of nonclosed Lie subgroups in a given Lie group. We have also given a family of examples of TC-foliations (B,E) such that the homomorphism $h_{A(B,E)}$ is nontrivial and the algebroid $A(B,E)$ of this foliation is nonintegrable. As yet these examples have not been studied by the classical theory.

(2) If L and A are Lie algebroids over the same manifold B, A is regular over (B,F), $\operatorname{Im}\omega_L \subset F$ and $\lambda : F \to A$ is any connection in A, then $\nabla := \lambda \circ \omega_L : L \to A$ is an L-connection in A. Indeed, $\omega_A \circ \nabla = (\omega_A \circ \lambda) \circ \omega_L = \omega_L$.

(3) Let $\pi : A \to L$ be a given epimorphism of Lie algebroids. We can always put it into an extension of Lie algebroids

$$\mathbf{e}: \quad 0 \to L' \to A \overset{\pi}{\to} L \to 0 \tag{2.3}$$

where $L' = \ker \pi$ has a structure of a Lie algebroid with the zero anchor. The right inverse of π, i.e. a linear homomorphism $\nabla : L \to A$, $\pi \circ \nabla = \mathrm{id}_L$ (called an $\mathbf{e}$-*connection* or a *connection in* $\mathbf{e}$) is an L-connection in A since $\omega_L = \omega_L \circ \pi \circ \nabla = \omega_A \circ \nabla$. The sequence of global cross-sections induced by (2.3) gives the so-called extension of Lie-Rinehart algebras which was examined by J. Huebschmann in [H1], [H2] (and earlier by N. Teleman [T1], [T2]). Note that they were also examined from different points of view by other authors and by the use of different terminology. For instance, these algebras were called Lie pseudoalgebras, Lie modules, etc., see [M3]. In [H2], [T2] the Chern-Weil homomorphism for extensions of Lie-Rinehart algebras is defined.

(4) Consider the short exact sequence of the jet bundles of a tangent bundle

$$0 \to S^k(TB, TB) \to J^k(TB) \to J^{k-1}(TB) \to 0. \tag{2.4}$$

$J^k(TB)$ has a Lie algebroid structure [L] defined as follows: the anchor $\omega^k : J^k(TB) \to TB$ is given by $\omega^k(j^k_x X) = X_x$, in $\operatorname{Sec} J^k(TB)$ there exists a unique structure of a Lie algebra $[\![\cdot,\cdot]\!]$ with $[\![j^k X, j^k Y]\!] = j^k\,[X,Y]$. Hence we obtain the algebroid of the Lie grupoid of invertible holonomic k-jets of local diffeomorphisms of B; connections in this algebroid are called connections of rank k on B. Splittings of the sequence (2.4) are called complete differentials of order k on B. If we take $k = 1$, we obtain the normal covariant derivative in the tangent bundle TB. Let us explain this phenomenon. The sequence (2.4) has the form $0 \to \mathrm{End}(TB) \xrightarrow{\iota} J^1(TB) \to TB \to 0$, $\iota(df \otimes X) = j^1(f \cdot X) - f \cdot j^1 X$, while a covariant derivative induced by the splitting $\nabla : TB \to J^1(TB)$ is defined by $\nabla_X Y = (j^1 Y - \nabla Y)(X)$ (notice that $j^1 Y - \nabla Y \in \mathrm{End}(TB)$), and conversely we have $\nabla Y = j^1 Y - \nabla_{(\cdot)} Y$.

(5) [K. Mackenzie, [M4], [M5]] If

$$N \rightarrowtail Q(B, H, q) \xrightarrow{\pi} P(B, G, p) \to 0 \tag{2.5}$$

is an extension of principal bundles (with respect to an extension of Lie groups $0 \to N \to H \xrightarrow{\pi} G \to 0$) and $\mathbf{e} : 0 \to K \to TQ_{/H} \to TP_{/G} \to 0$ is an extension which corresponds to Lie algebroids, then a connection in $\mathbf{e}$ is called a transversal connection in (2.5). There exists a one-to-one correspondence between these connections and G-equivariant connections in the Lie algebroid $TQ_{/N}$ of the transversal principal N-bundle $Q(P, N, \pi)$.

(6) If Lie algebroids L and A are regular over the same foliated manifold (B, F), then for their Whitney product $A \boxplus L$ we can observe the following fact: there exists a one-to-one correspondence between L-connections in A and connections of the induced extension $\mathbf{e}_{L,A} : 0 \to \boldsymbol{g}_A \to A \boxplus L \xrightarrow{\pi} L \to 0$. In fact, $\nabla : L \to A \boxplus L$, $v \mapsto (\nabla^1(v), v)$, is a connection in this splitting if and only if $\omega_A \circ \nabla^1 = \omega_L$, i.e. if ∇^1 is an L-connection in A. Recall that the Whitney product (see [K1]) of regular algebroids L and A over the same foliated manifold (B, F) is defined as a subbundle $A \boxplus L$ of the direct sum $A \oplus L$ such that fibres consist of pairs of vectors with equal anchors; a Lie bracket in the space of cross-sections is defined by coordinates and the anchor is given in an evident manner. This example shows that for a regular Lie algebroid the homomorphism $h_{L,A}$ constructed in our work can be expressed in terms of the Chern-Weil homomorphism of the extension $\mathbf{e}_{L,A}$ (generally given by N. Teleman and J. Huebschmann).

(7) Let $\nabla : L \to A$ be any L-connection in a regular Lie algebroid A (with the Atiyah sequence $0 \to \boldsymbol{g} \to A \xrightarrow{\omega_A} F \to 0$) and let Ω_∇ be its curvatute form defined below. It is easy to check that in the direct sum $\boldsymbol{g} \oplus L$ there exists a Lie algebroid structure such that $\omega_L \circ \mathrm{pr}_2 : \boldsymbol{g} \oplus L \to \mathrm{Im}(\omega_L) \subset TB$ serves as the anchor and the Lie bracket in $\mathrm{Sec}\, \boldsymbol{g} \oplus L$ is defined via the formula

$$[\![(\nu_1, \xi_1), (\nu_2, \xi_2)]\!] = ([\nu_1, \nu_2] + [\![\nabla \xi_1, \nu_2]\!] + [\![\nu_1, \nabla \xi_2]\!] + \Omega_\nabla(\xi_1, \xi_2), [\![\xi_1, \xi_2]\!]).$$

The following is an extension of Lie algebroids

$$\mathbf{e}_\nabla :\ 0 \to \boldsymbol{g} \to \boldsymbol{g} \oplus L \to L \to 0,$$

$\boldsymbol{g} \oplus L$ and L are of course over the same foliation $\mathrm{Im}(\omega_L)$.

(a) Clearly, L-connections in A are in 1-1 correspondence to splittings of $\mathbf{e}_\nabla$:

$$\nabla + r \overset{\cong}{\mapsto} \lambda^r,$$

where $r : L \to \boldsymbol{g}$ and $\lambda^r : L \to \boldsymbol{g} \oplus L$, $\xi \mapsto (r\xi, \xi)$. It is important that flat connections correspond to flat splittings. (Remark: for a *connection form* of an L-connection $\nabla^2 = \nabla + r : L \to A$ we can take $\boldsymbol{g} \oplus L \to \boldsymbol{g}$, $(\nu, \xi) \mapsto \nu - r(\xi)$, the connection form corresponding to the suitable splitting λ^r of $\mathbf{e}_\nabla$.)

(b) If $\nabla^2 = \nabla + r$ is another L-connection in A and $\mathbf{e}_{\nabla^2}$ is the extension defined via ∇^2 then the extensions are equivalent: $\rho : \boldsymbol{g} \oplus L \to (\boldsymbol{g} \oplus L)^{(2)}$, $(\nu, \xi) \mapsto (\nu - r\xi, \xi)$, is an isomorphism of extensions of Lie algebroids.

The above results generalize the standard case of $L = \operatorname{Im}(\omega_A) = F$ for regular Lie algebroids [K8] (and the earlier one corresponding to transitive Lie algebroids [M1], [K3]).

2.2. *The curvature of an L-connection and exterior covariant derivative.* With a Lie algebroid L and a vector bundle $\mathfrak{f}$ on the same manifold B we can connect the space of alternating forms $\Omega_L(B;\mathfrak{f}) = \oplus^n \Omega_L^n(B;\mathfrak{f})$, where $\Omega_L^n(B;\mathfrak{f}) = \operatorname{Sec}\bigwedge^n L^* \otimes \mathfrak{f}$. For $\Theta_i \in \Omega_L^{q_i}(B;\mathfrak{f}^i)$ and a multilinear homomorphism of vector bundles $\varphi : \mathfrak{f}^1 \times \ldots \times \mathfrak{f}^k \to \mathfrak{f}$ we assume that $\varphi_*(\Theta_1,\ldots,\Theta_k) \in \Omega_L^{q_1+\ldots+q_k}(B;\mathfrak{f})$ is given by the well-known formula

$$\begin{aligned}&\varphi_*(\Theta_1,\ldots,\Theta_k)(x;v_1,\ldots,v_m)\\ &= \frac{1}{q_1!\cdot\ldots\cdot q_k!}\sum_\sigma \operatorname{sgn}\sigma\cdot\varphi(x;\Theta_1(x;v_{\sigma(1)}\wedge\ldots),\ldots,\Theta_k(x;\ldots\wedge v_{\sigma(m)})).\end{aligned}$$

In the case of standard homomorphisms φ of the form $\bigvee^k : \mathfrak{f}\times\ldots\times\mathfrak{f} \to \bigvee^k \mathfrak{f}$ (symmetric power), $\langle\cdot,\cdot\rangle : \bigvee^k \mathfrak{f}^* \times \bigvee^k \mathfrak{f} \to \mathbb{R}$ (duality) etc., it is better to use the notation from [G-H-V], the form $\varphi_*(\Theta_1,\ldots,\Theta_k)$ will be denoted by $\Theta_1 \vee \ldots \vee \Theta_k$, $\langle\Theta_1,\Theta_2\rangle$, etc. For real forms $\Omega_L(B)$ there exists a derivative d_L defined by the known formula [MR], [K2], [M1]

$$\begin{aligned}(d_L\theta)(\xi_0,\ldots,\xi_k) = &\sum_{j=0}^k (-1)^j(\omega_L\circ\xi_i)(\theta(\xi_0,\ldots\hat{\jmath}\ldots,\xi_k))\\ &+\sum_{i<j}(-1)^{i+j}\theta([\![\xi_i,\xi_j]\!],\xi_0,\ldots\hat{\imath}\ldots\hat{\jmath}\ldots,\xi_k).\end{aligned} \tag{2.6}$$

Definition 2.5. Assume that L and A are Lie algebroids on B, and A is regular with Atiyah sequence $0 \to \boldsymbol{g}_A \to A \to F \to 0$. By the *curvature form* of an L-connection $\nabla : L \to A$ we shall mean the 2-form $\Omega_\nabla \in \Omega_L^2(B;\boldsymbol{g}_A)$ defined by

$$\Omega_\nabla(\xi,\eta) = [\![\nabla\circ\xi,\nabla\circ\eta]\!] - \nabla\circ[\![\xi,\eta]\!], \qquad \xi,\eta\in\operatorname{Sec}L,$$

($\Omega_\nabla(\xi,\eta)$ is a cross-section of the bundle $\boldsymbol{g}_A$ since $\omega_A\circ\Omega_\nabla(\xi,\eta) = 0$).

With a given L-connection $\nabla : L \to A$ we can connect the so-called *exterior covariant derivative* $\tilde\nabla : \Omega_L^*(B;\boldsymbol{g}_A) \to \Omega_L^{*+1}(B;\boldsymbol{g}_A)$ defined for a k-form Θ by the formula ($\xi_i \in \operatorname{Sec} L$)

$$\begin{aligned}(\tilde\nabla\Theta)(\xi_0,\ldots,\xi_k) = &\sum_{j=0}^k(-1)^j[\![\nabla\circ\xi_j,\Theta(\xi_0,\ldots\hat{\jmath}\ldots,\xi_k)]\!]\\ &+\sum_{i<j}(-1)^{i+j}\Theta([\![\xi_i,\xi_j]\!],\xi_0,\ldots\hat{\imath}\ldots\hat{\jmath}\ldots,\xi_k).\end{aligned} \tag{2.7}$$

It is standard to check the so-called *Bianchi identity*

$$\tilde\nabla(\Omega_\nabla) = 0. \tag{2.8}$$

Immediately from definition (2.7) we obtain $\tilde\nabla\nu \in \Omega_L^1(B;\boldsymbol{g}_A)$, $(\tilde\nabla\nu)(\xi) = [\![\nabla\circ\xi,\nu]\!]$, for $\nu\in\operatorname{Sec}\boldsymbol{g}_A$ and $\xi\in\operatorname{Sec}L$. If $\theta\in\Omega_L(B)$ and $\nu\in\operatorname{Sec}\boldsymbol{g}_A$, then

$$\tilde\nabla(\nu\cdot\theta) = \tilde\nabla\nu\wedge\theta + \nu\cdot d_L\theta. \tag{2.9}$$

Indeed, according to (2.2) and (2.1), for $\theta \in \Omega^k_L(B)$ and $\xi_i \in \operatorname{Sec} L$, we have

$$\begin{aligned}
&(\tilde{\nabla}\nu \wedge \theta + \nu \cdot d_L\theta)(\xi_0, \ldots, \xi_k) \\
&= \sum_{j=0}^{k} (-1)^j [\![\nabla \circ \xi_j, \nu]\!] \cdot \theta(\xi_0, \ldots \hat{\jmath} \ldots, \xi_k) \\
&+ \sum_{j=0}^{k} (-1)^j (\omega_L \circ \xi_j)(\theta(\xi_0, \ldots \hat{\jmath} \ldots, \xi_k)) \cdot \nu + \sum_{i<j} (-1)^j \theta([\![\xi_i, \xi_j]\!], \xi_0, \ldots \hat{\imath} \ldots \hat{\jmath} \ldots, \xi_k) \cdot \nu \\
&= \sum_{j=0}^{k} (-1)^j [\![\nabla \circ \xi_j, \theta(\xi_0, \ldots \hat{\jmath} \ldots, \xi_k) \cdot \nu]\!] + \sum_{i<j} (-1)^{i+j} \theta([\![\xi_i, \xi_j]\!], \xi_0, ..\hat{\imath} \ldots \hat{\jmath} \ldots, \xi_k) \cdot \nu \\
&= \nabla(\nu \cdot \theta)(\xi_0, \ldots, \xi_k).
\end{aligned}$$

A trivial verification again shows that

$$\varphi_*(\nu_1 \cdot \theta_1, \ldots, \nu_k \cdot \theta_k) = \varphi(\nu_1, \ldots, \nu_k) \cdot \theta_1 \wedge \ldots \wedge \theta_k, \tag{2.10}$$

$$\begin{aligned}
&\varphi_*(\nu_1, \ldots, \tilde{\nabla}\nu_i, \ldots, \nu_k) \wedge \theta_1 \wedge \ldots \wedge \theta_k \\
&= (-1)^{q_1 + \ldots + q_{i-1}} \varphi_*(\nu_1 \cdot \theta_1, \ldots, \nabla\nu_i \wedge \theta_i, \ldots, \nu_k \cdot \theta_k)
\end{aligned} \tag{2.11}$$

for $\nu_i \in \operatorname{Sec} \boldsymbol{g}_A,\ \theta_i \in \Omega^{q_i}_L(B),\ \varphi : \boldsymbol{g}_A \times \ldots \times \boldsymbol{g}_A \to \mathbb{R}$.

2.3. *Representations and their invariant cross-sections.* Let $\mathfrak{f}$ be a vector bundle on a manifold B. By a *representation (action)* of a Lie algebroid A on $\mathfrak{f}$ we mean a strong homomorphism of Lie algebroids $R : A \to A(\mathfrak{f})$ [M1]. A cross-section $\nu \in \operatorname{Sec} \mathfrak{f}$ is called *R-invariant* (or briefly *invariant*—unless it leads to confusion) if $(R \circ \xi)(\nu) = 0$ for all $\xi \in A$. If two invariant cross-sections are equal at a point, then they are equal along the leaf of the characteristic foliation which contains this point. An important example of a representation for a regular Lie algebroid A is the so-called *adjoint representation* $\mathrm{ad}_A : A \to A(\boldsymbol{g}_A)$ defined so that $(\mathrm{ad}_A \circ \xi)(\eta) = [\![\xi, \eta]\!]$, $\xi \in \operatorname{Sec} A$, $\eta \in \operatorname{Sec} \boldsymbol{g}_A$.

A single representation R determines (as in the case of a representation of a Lie algebra on a vector space) new representations on associated bundles [K4], for example the contragredient representation $R^{\natural}$ on the dual bundle $\mathfrak{f}^*$, the symmetric power $\bigvee^k R$ on the symmetric product $\bigvee^k \mathfrak{f}$, $\mathrm{Hom}^k(R)$ on the bundle of k-linear homomorphisms $\mathrm{Hom}^k(\mathfrak{f}; \mathbb{R})$. The last one is given by

$$\begin{aligned}
&(\mathrm{Hom}^k(R) \circ \xi)(\varphi)(v_1, \ldots, v_k) \\
&\qquad = (\omega_L \circ \xi)(\varphi(v_1, \ldots, v_k)) - \sum_{i=1}^{k} \varphi(v_1, \ldots, (R \circ \xi)(\nu_i), \ldots, v_k),
\end{aligned} \tag{2.12}$$

$\varphi : \mathfrak{f} \times \ldots \times \mathfrak{f} \to \mathbb{R}$, $\nu_i \in \operatorname{Sec} \mathfrak{f}$.

LEMMA 2.6. *Let $\varphi : \boldsymbol{g}_A \times \ldots \times \boldsymbol{g}_A \to \mathbb{R}$ be a $\mathrm{Hom}^k(\mathrm{ad}_A)$-invariant cross-section. Then for $\Theta_i \in \Omega^{q_i}_A(B; \boldsymbol{g}_A)$ the following equality holds:*

$$d_A(\varphi_*(\Theta_1, \ldots, \Theta_k)) = \sum_{i=1}^{k} (-1)^{q_1 + \ldots + q_{i-1}} \varphi_*(\Theta_1, \ldots, \tilde{\nabla}\Theta_i, \ldots, \Theta_k). \tag{2.13}$$

Proof. Observe that for forms of degree zero, i.e. cross-sections $\nu_i \in \operatorname{Sec} \boldsymbol{g}_A$, the formula (2.13) can be written as follows:

$$d_A(\varphi(\nu_1,\dots,\nu_k)) = \sum_{i=1}^{k} \varphi_*(\nu_1,\dots,\tilde{\nabla}\nu_i,\dots,\nu_k), \quad \nu_i \in \operatorname{Sec} \boldsymbol{g}_A. \tag{2.14}$$

In view of (2.12) (for $R = \mathrm{ad}_A$), from definition (2.6) we have

$$\begin{aligned} d_A(\varphi(\nu_1,\dots,\nu_k))(\xi) = (\omega_A \circ \xi)(\varphi(\nu_1,\dots,\nu_k)) &= \sum_{i=1}^{k} \varphi(\nu_1,\dots,[\![\nabla\circ\xi,\nu_i]\!],\dots,\nu_k) \\ &= \sum_{i=1}^{k} \varphi(\nu_1,\dots,\tilde{\nabla}\nu_i(\xi),\dots,\nu_k) = \sum_{i=1}^{k} \varphi_*(\nu_1,\dots,\tilde{\nabla}\nu_1,\dots,\nu_k)(\xi) \end{aligned}$$

for $\xi \in \operatorname{Sec} A$.

It is sufficient to prove the equality (2.13) for forms $\Theta_i = \nu_i \cdot \theta_i$ where $\nu_i \in \operatorname{Sec} \boldsymbol{g}_A$, $\theta_i \in \Omega_A^{q_i}(B)$. From the equalities (2.9)–(2.11) and (2.14) we obtain

$$\begin{aligned} &d_A(\varphi_*(\nu_1\cdot\theta_1,\dots,\nu_k\cdot\theta_k)) \\ &= d_A(\varphi(\nu_1,\dots,\nu_k)) \wedge \theta_1 \wedge \ldots \wedge \theta_k + \varphi(\nu_1,\dots,\nu_k)\cdot d_A(\theta_1\wedge\ldots\wedge\theta_k) \\ &= \sum_{i=1}^{k} \varphi_*(\nu_1,\dots,\tilde{\nabla}\nu_k,\dots,\nu_k)\wedge\theta_1\wedge\ldots\wedge\theta_k \\ &\quad + \varphi(\nu_1,\dots,\nu_k)\sum_{i=1}^{k}(-1)^{q_1+\ldots+q_{i-1}}\theta_1\wedge\ldots\wedge d_A\theta_i\wedge\ldots\wedge\theta_k \\ &= \sum_{i=1}^{k}(-1)^{q_1+\ldots+q_{-1}}\varphi_*(\nu_1\cdot\theta_1,\dots,\tilde{\nabla}\nu_1\wedge\theta_i+\nu_1\cdot d_A\theta_i,\dots,\nu_k\cdot\theta_k) \\ &= \sum_{i=1}^{k}(-1)^{q_1+\ldots+q_{i-1}}\varphi_*(\nu_1\cdot\theta_1,\dots,\tilde{\nabla}(\nu_i\theta_i),\dots,\nu_k\cdot\theta_k). \ \blacksquare \end{aligned}$$

By definition, the contragredient representation $R^\natural$ of $R : A \longrightarrow A(\mathfrak{f})$ is such that $R^\natural\circ\xi$ is a covariant differential operator in the dual bundle $\mathfrak{f}^*$, satisfying $(R^\natural\circ\xi)(\varphi)(\nu) = (\omega_A\circ\xi)\langle\varphi,\nu\rangle - \langle\varphi,(R\circ\xi)\nu\rangle$. The symmetric product $\bigvee^k R$ on $\bigvee^k \mathfrak{f}$ is defined as the one for which $((\bigvee^k R)\circ\xi)(\nu^1\vee\ldots\vee\nu^k) = \sum_{i=1}^{k}\nu^1\vee\ldots\vee(R\circ\xi)(\nu^i)\vee\ldots\vee\nu^k$, $\nu^i\in\operatorname{Sec}\mathfrak{f}$. If $\gamma^1\in\operatorname{Sec}\bigvee^k\mathfrak{f}$ and $\gamma^2\in\operatorname{Sec}\bigvee^l\mathfrak{f}$ are invariant cross-sections with respect to representations $\bigvee^k R$ and $\bigvee^l R$, respectively, then their symmetric product $\gamma^1\vee\gamma^2\in\operatorname{Sec}\bigvee^{k+l}\mathfrak{f}$ is also invariant [K4]. This implies that the space of invariant cross-sections of all symmetric powers of $\mathfrak{f}$ forms an algebra. In the cited work it is also proved (using the permanent of a matrix) that the symmetric product of the contragredient representation of R, $\bigvee^k R^\natural$, is given by the formula

$$\begin{aligned} &\Big\langle\Big(\Big(\bigvee^k R^\natural\Big)\circ\xi\Big)(\Gamma),\nu^1\vee\ldots\vee\nu^k\Big\rangle \\ &= (\omega_A\circ\xi)\langle\Gamma,\nu^1\vee\ldots\vee\nu^k\rangle - \sum_{i=1}^{k}\langle\Gamma,\nu^1\vee\ldots\vee\nu^k(R\circ\xi)(\nu^i)\vee\ldots\vee\nu^k\rangle, \end{aligned} \tag{2.15}$$

$\Gamma\in\operatorname{Sec}\bigvee^k\mathfrak{f}^*$, $\nu^i\in\operatorname{Sec}\mathfrak{f}$.

3. The Chern-Weil homomorphism

3.1. *Classical theory for principal bundles.* The Chern-Weil homomorphism for a G-principal bundle $P(B,G,\pi)$ is a homomorphism of algebras

$$h_P : (\bigvee \mathfrak{g}^*)_I \to H_{dR}(B).$$

The domain

$$I(G) := (\bigvee \mathfrak{g}^*)_I$$

is the space of polynomials $\Gamma : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$ invariant with respect to the adjoint representation Ad_G of G on $\bigvee \mathfrak{g}^*$. The homomorphism h_P is defined by the formula

$$h_P(\Gamma) = [\chi_P(\Gamma)]$$

where

$$\pi^*(\chi_P(\Gamma)) = \frac{1}{k!}\left[\langle \Gamma, \Omega \vee \ldots \vee \Omega\rangle\right] \qquad (\text{for } \Gamma \in (\textstyle\bigvee^k \mathfrak{g}^*)_I)$$

and $\Omega \in \Omega^2(P,\mathfrak{g})$ is the curvature form of some connection $H \subset TP$ and $\Omega \vee \ldots \vee \Omega$ denotes the standard skew-product of forms with symmetric multiplication of the values [G-H-V].

Equivalently, we can define h_P directly on B without using the lifting to P; we can use the Atiyah sequence

$$0 \to P \times_G \mathfrak{g} \to TP_{/G} \overset{\omega}{\to} TB \to 0 \tag{3.1}$$

of P instead the lifting. Let $\nabla : TB \to TP_{/G}$ be the splitting of this sequence corresponding to a connection H,

$$0 \to P \times_G \mathfrak{g} \to TP_{/G} \underset{\nabla}{\overset{\omega}{\rightleftarrows}} TB \to 0.$$

Recall that $P \times_G \mathfrak{g}$ is a Lie algebra bundle (LAB); for $z \in P_{|x}$ we have an isomorphism of Lie algebras

$$\hat{z} : \mathfrak{g} \to (P \times_G \mathfrak{g})_{|x}, \quad v \mapsto (A_z)_{*e}(v),$$

$A_z : G \to P,\ a \mapsto za$, provided that $\mathfrak{g}$ is the right Lie algebra of G (not left!). Next, we make the following observations:

- each invariant polynomial $\Gamma \in (\bigvee^k \mathfrak{g}^*)_I$ determines a symmetric k-linear homomorphism

  $$\tilde{\Gamma} : P \times_G \mathfrak{g} \times \ldots \times P \times_G \mathfrak{g} \to B \times \mathbb{R}$$

  (i.e. a cross-section of $\bigvee^k (P \times_G \mathfrak{g})^*$) defined by

  $$\tilde{\Gamma}_x = \textstyle\bigvee^k (\hat{z}^{-1})^*(\Gamma), \quad z \in P_{|x},$$

  (the formula is independent of the choice of z since Γ is invariant).
- the curvature form $\Omega = \pi^*(\Omega_b)$, where $\Omega_b \in \Omega^2(B; P \times_G \mathfrak{g})$ is defined by

  $$\Omega_b(X,Y) = [\![\nabla X, \nabla Y]\!] - \nabla\,[X,Y]\,.$$

We have the equality

$$\langle \Gamma, \Omega \vee \ldots \vee \Omega \rangle = \pi^* \langle \tilde{\Gamma}, \Omega_b \vee \ldots \vee \Omega_b \rangle,$$

and

$$\chi_P(\Gamma) = \frac{1}{k!} \langle \tilde{\Gamma}, \Omega_b \vee \ldots \vee \Omega_b \rangle,$$

therefore

$$h_P(\Gamma) = \frac{1}{k!} \left[\langle \tilde{\Gamma}, \Omega_b \vee \ldots \vee \Omega_b \rangle \right].$$

Now we describe the space

$$\{\tilde{\Gamma};\ \Gamma \in (\bigvee^k \mathfrak{g}^*)_I\}$$

in terms of the Lie algebroid $TP_{/G}$. We notice that $(\bigvee^k \mathfrak{g}^*)_I$ depends on G, not only on its Lie algebra $\mathfrak{g} = \mathfrak{g}l(G)$ (unless G is a connected Lie group) but in $TP_{/G}$ we have no information about the structural Lie group G. We need to use the adjoint representation of the Lie algebroid $TP_{/G}$ in the vector bundle $P \times_G \mathfrak{g}$

$$\mathrm{ad}_{TP_{/G}} : TP_{/G} \to A(P \times_G \mathfrak{g})$$

defined by

$$\mathrm{ad}_{TP_{/G}}(\xi)(\nu) = [\![\xi, \nu]\!],\ \xi \in \mathrm{Sec}(TP_{/G}),\ \nu \in \mathrm{Sec}(P \times_G \mathfrak{g}).$$

The representation $\mathrm{ad}_{TP_{/G}}$ induces representations of $TP_{/G}$ on associated vector bundles, among others, on $\bigvee^k (P \times_G \mathfrak{g})^*$, by

$$\mathrm{ad}_{TP_{/G}}(\xi)(\varphi)(\nu_1, \ldots, \nu_k) = (\omega\xi)(\varphi(\nu_1, \ldots, \nu_k)) - \sum_i \varphi(\nu_1, \ldots, [\![\xi, \nu_i]\!], \ldots, \nu_k),$$

where $\varphi \in \mathrm{Sec} \bigvee^k (P \times_G \mathfrak{g})^*$, equivalently φ is a symmetric tensor $\varphi : (P \times_G \mathfrak{g}) \times \ldots \times (P \times_G \mathfrak{g}) \to B \times \mathbb{R}$. φ is called *invariant* if $\mathrm{ad}_{TP_{/G}}(\xi)(\varphi) = 0$ for all $\xi \in \mathrm{Sec}(TP_{/G})$. We define the algebra of invariant tensors

$$I(TP_{/G}) = \bigoplus^k (\mathrm{Sec} \bigvee^k (P \times_G \mathfrak{g})^*)_I.$$

THEOREM 3.1 (Kubarski [K4]; Belko [B1], [B2]). *The homomorphism*

$$\rho : (\bigvee \mathfrak{g}^*)_I \to I(TP_{/G}),\ \Gamma \mapsto \tilde{\Gamma},$$

is a monomorphism of algebras and is an isomorphism if P is connected (G may be disconnected!).

By the *Chern-Weil homomorphism of the Lie algebroid* $TP_{/G}$ we mean the homomorphism

$$h_{TP_{/G}} : I(TP_{/G}) \to H_{dR}(M),\ \ \varphi \mapsto \frac{1}{k!} [\langle \varphi, \Omega_b \vee \ldots \vee \Omega_b \rangle]$$

(for φ of degree k) where $\Omega_b \in \Omega^2(B; P \times_G \mathfrak{g})$ is the curvature of any connection $\nabla : TB \to TP_{/G}$ in the Lie algebroid $TP_{/G}$.

We have the commuting diagram

$$\begin{tikzcd} I(TP_{/G}) \arrow[dr, "h_{TP_{/G}}"] & \\ & H_{dR}(B) \\ (\bigvee \mathfrak{g}^*)_I \arrow[uu, "\rho"] \arrow[ur, "h_P"'] & \end{tikzcd}$$

Therefore, if P is connected, its Chern-Weil homomorphism is equivalent to the Chern-Weil homomorphism of the Lie algebroid $TP_{/G}$.

3.2. *The Chern-Weil homomorphism for a pair of Lie algebroids.* Assume that L and A are Lie algebroids on a manifold B, and A is regular (assumptions as in Definition 2.5). We shall construct a characteristic homomorphism which will somehow measure obstructions to existence of a flat L-connection in A. The Chern-Weil homomorphism in the case of an integrable Lie algebroid $A = TP_{/G}$ is closely connected with the one obtained recently by R. L. Fernandes [F2].

3.2.1. *Construction.* Let $x \in B$ be a fixed point. Since $\bigwedge^{\mathrm{ev}} L^*_{|x} := \bigoplus_{k\geq 0} \bigwedge^{2k} L^*_{|x}$ is a symmetric algebra, there exists precisely one homomorphism of algebras (see [G, p. 192])

$$\chi_{\nabla,x} : \bigvee(\boldsymbol{g}_A)^*_{|x} \to \bigoplus_{k\geq 0} \bigwedge^{2k} L^*_{|x}$$

such that

1. $\chi_{\nabla,x}(1) = 1$,
2. $\chi_{\nabla,x}(\Gamma) = \langle \Gamma, \Omega_{\nabla x}\rangle, \quad \Gamma \in (\boldsymbol{g}_A)^*_{|x}$.

LEMMA 3.2. *For $\Gamma \in \bigvee^k(\boldsymbol{g}_A)^*_{|x}$ we have the equality*

$$\chi_{\nabla,x}(\Gamma) = \frac{1}{k!}\cdot \langle \Gamma, \underbrace{\Omega_{\nabla x} \vee \ldots \vee \Omega_{\nabla x}}_{k \text{ times}}\rangle. \tag{3.2}$$

Proof. We give an elementary proof using properties of permanents of matrices. Another proof, using a tensor algebra $\bigotimes(\boldsymbol{g}_A)^*_{|x}$ and its relation to a symmetric algebra [G, pp.91, 193], is similar to the proof of Lemma 4.1.1 in [K4]. It is sufficient to show the equality (3.2) for $\Gamma = \Gamma_1 \vee \ldots \vee \Gamma_k$, $\Gamma_i \in (\boldsymbol{g}_A)^*_{|x}$:

$$\begin{aligned}
&\frac{1}{k!}\cdot \langle \Gamma_1 \vee \ldots \vee \Gamma_k, \Omega_{\nabla x} \vee \ldots \vee \Omega_{\nabla x}\rangle(v_1 \wedge \ldots \wedge v_{2k}) \\
&\qquad = \frac{1}{k!}\cdot \langle \Gamma_1 \vee \ldots \vee \Gamma_k, \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma \cdot \Omega_{\nabla x}(v_{\sigma_1} \wedge v_{\sigma_2}) \vee \ldots \vee \Omega_{\nabla x}(v_{\sigma_{2k-1}} \wedge v_{\sigma_{2k}})\rangle \\
&\qquad = \frac{1}{k!}\cdot \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma \cdot \operatorname{perm}[\langle \Gamma_i, \Omega_{\nabla x}(v_{\sigma_{2j-1}} \wedge v_{\sigma_{2j}})\rangle]_{i,j=1,\ldots,k} \\
&\qquad = \frac{1}{k!}\cdot \frac{1}{2^k}\sum_\sigma \operatorname{sgn}\sigma \cdot k! \cdot \langle \Gamma_1, \Omega_{\nabla x}(v_{\sigma_1} \wedge v_{\sigma_2})\rangle \cdot \ldots \cdot \langle \Gamma_k, \Omega_{\nabla x}(v_{\sigma_{2k-1}} \wedge v_{\sigma_{2k}})\rangle \\
&\qquad = \langle \Gamma_1, \Omega_{\nabla x}\rangle \wedge \ldots \wedge \langle \Gamma_k, \Omega_{\nabla x}\rangle(v_1 \wedge \ldots \wedge v_{2k}) \\
&\qquad = \chi_{\nabla,x}(\Gamma_1 \vee \ldots \vee \Gamma_k)(v_1 \wedge \ldots \wedge v_{2k}). \blacksquare
\end{aligned}$$

For a fixed integer number $k \geq 0$, the family of homomorphisms $\chi_{\nabla,x}$ determines a strong homomorphism of vector bundles

$$\chi^k_\nabla : \bigvee^k \boldsymbol{g}^*_A \to \bigwedge^{2k} L^*$$

such that, for $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}^*_A$, we obtain the equality $\chi^k_\nabla \circ \Gamma = \frac{1}{k!} \cdot \langle \Gamma, \Omega_\nabla \vee \ldots \vee \Omega_\nabla \rangle$, which, in particular, implies smoothness of the constructed homomorphism. Finally, we have the homomorphism of $C^\infty(B)$-modules

$$\chi_\nabla : \bigoplus_{k\geq 0} \operatorname{Sec} \bigvee^k \boldsymbol{g}^*_A \to \Omega^{\mathrm{ev}}_L(B),$$

which restricted to the algebra of invariant cross-sections $I(A)$ gives the homomorphism $\chi_{\nabla,I}$. The forms from its image are closed. In fact, a cross-section Γ determines, in a standard way, a multilinear map $\tilde{\Gamma} : \boldsymbol{g}_A \times \ldots \times \boldsymbol{g}_A \to \mathbb{R}$. Then $\langle \Gamma, \Theta_1 \vee \ldots \vee \Theta_k \rangle = \tilde{\Gamma}_*(\Theta_1, \ldots, \Theta_k)$. If Γ is invariant, then so is $\tilde{\Gamma}$. Hence, in view of the Bianchi identity (2.8) and formula (2.13), the form $\chi^k_\nabla \circ \Gamma$ is closed. This gives a homomorphism of algebras which acts into cohomologies $H_L(B)$

$$h_\nabla : I(A) \to \ker d_L \to H_L(B).$$

All we need is to show that this homomorphism is independent of the choice of an L-connection ∇. This will imply that h_∇ is a characteristic object for a pair of Lie algebroids (L, A) on the same manifold.

3.2.2. *Independence of the choice of an L-connection* ∇. One of the ways to show that the characteristic homomorphism h_∇ does not depend on the choice of the connection needs the use of nonstrong homomorphisms between two Lie algebroids, the notion of homotopy joining such homomorphisms, and the fact that homotopic homomorphisms induce the same mappings on cohomology. Let $(A, [\![\cdot,\cdot]\!], \omega_A)$ and $(A', [\![\cdot,\cdot]\!]', \omega_{A'})$ be two Lie algebroids (not necessarily regular) on manifolds B and B', respectively. By a *homomorphism* between them [M1], [K4], [K6], we mean a homomorphism of vector bundles $H : A \to A'$, say, over $f : B \to B'$, such that (1) $\omega_{A'} \circ H = f_* \circ \omega_A$, (2) for arbitrary cross-sections $\xi, \eta \in \operatorname{Sec} A$ with H-decompositions $H \circ \xi = \sum_i f^i \cdot (\sigma_i \circ f)$, $H \circ \eta = \sum_j g^j \cdot (\tau_j \circ f)$, $f^i, g^j \in C^\infty(B)$, $\sigma_i, \tau_j \in \operatorname{Sec} A'$, we have

$$\begin{aligned} &H \circ [\![\xi, \eta]\!] \\ &= \sum_{i,j} f^i \cdot g^j \cdot [\![\sigma_i, \tau_j]\!]' \circ f + \sum_j (\omega_A \circ \xi)(g^j) \cdot \tau_j \circ f - \sum_i (\omega_A \circ \eta)(f^i) \cdot \sigma_i \circ f. \end{aligned}$$

Although this notion seems to be complicated it is quite obvious from the Lie groupoid point of view (a nonstrong homomorphism of groupoids preserves source and target and partial multiplication; after infinitesimal linearization we obtain a homomorphism of algebroids). Notice that in the case of Lie algebroids on the same manifold B, a strong homomorphism is just a homomorphism of bundles which commutes with anchors and Lie brackets. Let H and H' be homomorphisms $A \to A'$. By a *homotopy* [K6] joining H to H' we mean a homomorphism of Lie algebroids $\bar{H} : T\mathbb{R} \times A \to A'$ such that $\bar{H}(\theta_0, \cdot) = H$ and $\bar{H}(\theta_1, \cdot) = H'$ where θ_0 and θ_1 are null vectors tangent to $\mathbb{R}$ at 0 and 1, respectively. The homotopy $\bar{H}$ induces a chain homotopy operator $k : \Omega^*_{A'}(B') \to \Omega^{*-1}_A(B)$ by the formula

$$k(\Phi)_x(v_1 \wedge \ldots \wedge v_s) = \int_0^1 (\bar{H}^*\Phi)_{(t,x)} \left(\frac{\partial}{\partial t}|_{(t,x)} \wedge v_1 \wedge \ldots \wedge v_s \right) dt$$

which implies that H and H' induce the same mappings on cohomology.

If $(A, [\![\cdot,\cdot]\!], \omega_A)$ and $(A', [\![\cdot,\cdot]\!]', \omega_{A'})$ are regular Lie algebroids over manifolds B and B', respectively, and $H : A \to A'$ is a homomorphism of these algebroids over $f : B \to B'$, then [K4] the induced homomorphism of Lie algebra bundles $H^+ : \boldsymbol{g}_A \to \boldsymbol{g}_{A'}$ induces pull-backs of cross-sections of dual bundles and their symmetric powers $H^{+*} : \mathrm{Sec} \bigvee^k \boldsymbol{g}^*_{A'} \to \mathrm{Sec} \bigvee^k \boldsymbol{g}^*_A$ given by the evident formula. The pull-back of an invariant section is again an invariant section. Hence we obtain a homomorphism of algebras $H^{+*} : I(A') \to I(A)$ [K4]. Assume that L and L' are Lie algebroids on B and B', respectively. First we check the functoriality of the homomorphism h_∇.

Theorem 3.3. *If $\nabla : L \to A$ and $\nabla' : L' \to A'$ are connections, $H_L : L \longrightarrow L'$ and $H_A : A \longrightarrow A'$ are homomorphisms of algebroids such that the diagram*

$$\begin{tikzcd} L \arrow[r, "\nabla"] \arrow[d, "H_L"'] & A \arrow[d, "H_A"] \\ L' \arrow[r, "\nabla'"'] & A' \end{tikzcd}$$

commutes, then

$$h_\nabla \circ H_A^{+*} = H_L^{\#} \circ h_{\nabla'}. \tag{3.3}$$

Proof. Observe that

$$\Omega_{\nabla'}(H_L(v), H_L(w)) = H_A^+(\Omega_\nabla(v,w)), \quad v, w \in L_{|x},\ x \in B. \tag{3.4}$$

Indeed, let ξ, η be two cross-sections of L such that $\xi(x) = v$ and $\eta(x) = w$, and consider their (local) H_L-decompositions $H_L \circ \xi = \sum_i f^i \cdot (\xi'_i \circ f)$ and $H_L \circ \eta = \sum_i g^i \cdot (\xi'_i \circ f)$, $\xi'_i \in \mathrm{Sec}\, A'$. The equality (3.4) is an immediate consequence of the fact that the diagram below is commutative

$$\begin{tikzcd} M \times M \arrow[r, "\xi \times \eta"] \arrow[d, "\xi \times \eta"'] & L \times L \arrow[r, "H_L \times H_L"] & L' \times L' \arrow[d, "\Omega_{\nabla'}"] \\ L \times L \arrow[r, "\Omega_\nabla"'] & \boldsymbol{g}_A \arrow[r, "H_A^+"'] & \boldsymbol{g}_{A'}. \end{tikzcd}$$

To show this, take H_A-decompositions of $\nabla \circ \xi$ and $\nabla \circ \eta$, $H_A \circ (\nabla \circ \xi) = \nabla' \circ H_L \circ \xi = \sum_i f^i \cdot (\nabla' \circ \xi'_i \circ f)$ (similarly for η). Hence

$$\begin{aligned} H_A^+ \circ \Omega_\nabla(\xi, \eta) &= H_A \circ [\![\nabla \circ \xi, \nabla \circ \eta]\!] - H_A \circ \nabla [\![\xi, \eta]\!] \\ &= H_A \circ [\![\nabla \circ \xi, \nabla \circ \eta]\!] - \nabla' \circ H_L \circ [\![\xi, \eta]\!] \\ &= \sum_{i,j} f^i \cdot g^j \cdot [\![\nabla' \circ \xi'_i, \nabla' \circ \xi'_j]\!] + \sum_j \omega_A \circ (\nabla \circ \xi)(g^j) \cdot \nabla' \circ \xi'_j \circ f \\ &\quad - \sum_i \omega_A \circ (\nabla \circ \eta)(f^i) \cdot \nabla' \circ \xi'_i \circ f - \sum_{i,j} f^i \cdot g^j \cdot \nabla' \circ [\![\xi'_i, \xi'_j]\!] \circ f \end{aligned}$$

$$
\begin{aligned}
&- \sum_j (\omega_L \circ \xi)(g^j) \cdot \nabla' \circ \xi'_j \circ f + \sum_i (\omega_L \circ \eta)(f^i) \cdot \nabla' \circ \xi'_i \circ f \\
&= \sum_{i,j} f^i \cdot g^j \cdot ([\![\nabla' \circ \xi'_i, \nabla' \circ \xi'_j]\!] - \nabla' \circ [\![\xi'_i, \xi'_j]\!]) \circ f \\
&= \Omega_{\nabla'} \circ (H_L \circ \xi, H_L \circ \eta) = \Omega_{\nabla'} \circ (H_L \times H_L)(\xi, \eta).
\end{aligned}
$$

Observe that it suffices to show that (3.3) is satisfied by forms. Let $\Gamma \in \mathrm{Sec} \bigvee^k \boldsymbol{g}^*_{A'}$, $x \in B$ and $v_1, \ldots, v_{2k} \in L_{|x}$. Then

$$
\begin{aligned}
&(\chi_\nabla \circ H_A^{+*})(\Gamma)(x; v_1, \ldots, v_{2k}) \\
&\quad = \frac{1}{k!} \cdot \langle H_A^{+*}(\Gamma), \Omega_\nabla \vee \ldots \vee \Omega_\nabla \rangle (x; v_1, \ldots, v_{2k}) \\
&\quad = \frac{1}{k!} \cdot \langle \Gamma_{|fx}, \frac{1}{2^k} \sum_\sigma \mathrm{sgn}\, \sigma \cdot H_A^+ \circ \Omega_\nabla(x; v_{\sigma_1}, v_{\sigma_2}) \vee \ldots \vee H_A^+ \circ \Omega_\nabla(x; v_{\sigma_{2k-1}}, v_{\sigma_{2k}}) \rangle \\
&\quad = \frac{1}{k!} \cdot \langle \Gamma_{|fx}, \frac{1}{2^k} \sum_\sigma \mathrm{sgn}\, \sigma \cdot \Omega_{\nabla'}(f(x); H_L(v_{\sigma_1}), H_L(v_{\sigma_2})) \vee \ldots \\
&\qquad\qquad\qquad\qquad \ldots \vee \Omega_{\nabla'}(f(x); H_L(v_{\sigma_{2k-1}}), H_L(v_{\sigma_{2k}})) \rangle \\
&\quad = \frac{1}{k!} \cdot \langle \Gamma, \Omega_{\nabla'} \vee \ldots \vee \Omega_{\nabla'} \rangle (f(x); H_L(v_1), \ldots, H_L(v_{2k})) \\
&\quad = \chi_{\nabla'}(\Gamma)(f(x); H_L(v_1), \ldots, H_L(v_{2k})) \\
&\quad = H_L^*(\chi_{\nabla'}(\Gamma))(x; v_1, \ldots, v_{2k}). \quad \blacksquare
\end{aligned}
$$

THEOREM 3.4. *Let L and A be Lie algebroids and assume that A is regular. If $\nabla_0, \nabla_1 : L \longrightarrow A$ are L-connections, then $h_{\nabla_0} = h_{\nabla_1}$.*

Proof. Define $\nabla : T\mathbb{R} \times L \longrightarrow T\mathbb{R} \times A$ by

$$\nabla(v_t, w) = (v_t, (1-t)\nabla_0(w) - t\nabla_1(w)), \qquad v_t \in T_t\mathbb{R},\ w \in L.$$

Observe that ∇ is a $T\mathbb{R} \times L$-connection, i.e. $(\mathrm{id}, \omega_A) \circ \nabla(v_t, w) = (\mathrm{id}, \omega_L)(v_t, w)$. Define now $H_{L,t} : L \longrightarrow T\mathbb{R} \times L$, $H_{A,t} : A \longrightarrow T\mathbb{R} \times A$ by $H_{L,t}(w) = (\theta_t, w)$, $H_{A,t}(v) = (\theta_t, v)$, $w \in L$, $v \in A$ and $G : T\mathbb{R} \times A \longrightarrow A$ by $G(v, w) = w$, $v \in T\mathbb{R}$, $w \in A$. The mappings above are non-strong homomorphisms of Lie algebroids [K4] such that $H_{A,t} \circ \nabla_i = \nabla \circ H_{L,t}$, $G \circ H_{A,i} = \mathrm{id}$, $i = 0, 1$. Hence by Theorem 3.3 we have

$$
\begin{aligned}
h_{\nabla_0} &= h_{\nabla_0} \circ (H_{A,0}^{+*} \circ G^{+*}) = (h_{\nabla_0} \circ H_{A,0}^{+*}) \circ G^{+*} \\
&= (H_{L,0}^\sharp \circ h_\nabla) \circ G^{+*} = (H_{L,1}^\sharp \circ h_\nabla) \circ G^{+*} \\
&= (h_{\nabla_1} \circ H_{A,1}^{+*}) \circ G^{+*} = h_{\nabla_1} \circ (H_{A,1}^{+*} \circ G^{+*}) = h_{\nabla_1}. \quad \blacksquare
\end{aligned}
$$

According to this theorem, the Chern-Weil homomorphism defined in terms of a given L-connection does not depend on the choice of the connection. Thus, it will be denoted simply by $h_{L,A}$. The image of $h_{L,A}$, $\mathrm{Im}\, h_{L,A} \subset H_L(B)$, is a subalgebra of $H_L(B)$; it is called the *Pontryagin algebra* of (L, A) and is denoted by

$$\mathrm{Pont}(L, A).$$

Cohomology classes of $\mathrm{Pont}(L, A)$ are called *Pontryagin classes* of the pair (L, A).

Let A and L be Lie algebroids over a foliated manifold (B, F) and consider the splitting $\mathbf{e}_{L,A}$ given by their Whitney product $A \boxplus L$. Clearly $h_{L,A} = h_{\mathbf{e}_{L,A}}$; therefore the Chern-Weil homomorphism of a pair of Lie algebroids can be obtained by using J. Huebschmann's theory. However, constructions and proofs given in our paper are simpler, and in fact, they are adapted from the case of one regular Lie algebroid, see [K4].

In general, there is no relation between $h_{L,A}$ and $h_{A,L}$: one of them can be zero and the other one not. For example, if A is a regular Lie algebroid over (B, F), then $h_{A,L} = 0$ and $h_{L,A}$ is not always zero.

3.2.3. *G-equivariant Chern-Weil homomorphisms*

Definition 3.5 [K6]. Let A be a Lie algebroid on a manifold B and let G be a Lie group. A is called a *Lie G-algebroid* if there is a right action of G on the manifold A, $\widetilde{R} : A \times G \to A$, and a right action of G on the manifold B, $R : B \times G \to B$, such that $\widetilde{R}_g : A \to A$ is an automorphism of A over $R_g : B \to B$ for each $g \in G$.

According to this definition, the diagram

$$\begin{tikzcd} A \arrow[r, "\widetilde{R}_g"] \arrow[d, "\omega_A"'] & A \arrow[d, "\omega_A"] \\ TB \arrow[r, "R_{g*}"'] & TB \end{tikzcd}$$

is commutative and $\widetilde{R}_g \circ [\![\xi, \eta]\!] = [\![\bar{\xi}, \bar{\eta}]\!] \circ R_g$, where $\bar{\xi}, \bar{\eta} \in \operatorname{Sec} A$ are sections such that $\widetilde{R}_g \circ \xi = \bar{\xi} \circ R_g$ and $\widetilde{R} \circ \eta = \bar{\eta} \circ R_g$. This definition of a Lie G-algebroid is more general than the definition of a Mackenzie PBG-algebroid [M4] (recall that a transitive G-algebroid A is a PBG-algebroid if the base manifold is the total space of a principal bundle $P(B, G, p)$ and the action of G on P is the structural action of this bundle).

Assume that L and A are Lie G-algebroids on a base B.

Definition 3.6. An L-connection χ in A, $\chi : L \to A$, is called *G-equivariant* if χ is a G-equivariant homomorphism.

Assume additionally that A is regular and $0 \to \boldsymbol{g}_A \to A \to F \to 0$ is its Atiyah sequence. Clearly, the bundle $\boldsymbol{g}_A$ is G-invariant. Let $\Omega : L \times L \to \boldsymbol{g}_A$ be the curvature form of a G-equivariant L-connection χ in A. It is standard to check that

$$\Omega(R_g v, R_g w) = R_g \Omega(v, w)$$

which means that the $2k$-form $\langle \Gamma, \Omega \vee \ldots \vee \Omega \rangle$ is G-equivariant for a G-equivariant section $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}_A^*$. Hence we obtain the G-equivariant Chern-Weil homomorphism for the pair (L, A)

$$h^G_{L,A} : I(A)^G \longrightarrow H^G_L(B),$$

where the upper index ^{G} denotes the space of G-equivariant elements of $I(A)$ and the cohomology of G-equivariant forms respectively. The proof that $h^G_{L,A}$ does not depend on the choice of the G-equivariant connection is nearly the same as the proof of Theorem 3.4. The difference is that we introduce the Lie G-algebroids $T\mathbb{R} \times L$ and $T\mathbb{R} \times A$ given by $R_g(v_t, w) = (v_t, R_g w)$, and next we observe that the joining connection is G-equivariant and the homomorphisms $H_{L,t}$, $H_{A,t}$ and T are G-equivariant as well. Moreover, $H_{L,0}$ and $H_{L,1}$ are homotopic via a G-equivariant homotopy.

Similarly, if we consider an extension of Lie algebroids

$$\mathbf{e} : 0 \longrightarrow K \longrightarrow A \xrightarrow{\pi} L \longrightarrow 0,$$

where A and L are Lie G-algebroids and π is a G-equivariant homomorphism, then we obtain the G-equivariant Chern-Weil homomorphism for the extension $\mathbf{e}$ in terms of G-equivariant connections

$$h^G_{\mathbf{e}} : I(K)^G \longrightarrow H^G_L(B).$$

3.2.4. *Comparison of $h_{L,A}$ with h_A*

THEOREM 3.7. *If L and A are Lie algebroids over a manifold B, A is regular over (B,F) and $\operatorname{Im}\omega_L \subset F$, then the diagram*

$$\begin{tikzcd} I(A) \arrow[r, "h_A"] \arrow[dr, "h_{L,A}"'] & H_F(B) \arrow[d, "\omega_L^\sharp"] \\ & H_L(B) \end{tikzcd} \tag{3.5}$$

commutes. In particular

$$\omega_L^\sharp\,[\mathrm{Pont}(A)] = \mathrm{Pont}(L,A).$$

Proof. Consider the L-connection $\nabla : L \to A$ given by $\nabla = \lambda \circ \omega_L$, where $\lambda : F \to A$ is a connection in A. It is easy to show that the curvature tensors $\Omega_\lambda \in \Omega^2_F(B, \boldsymbol{g}_A)$ and $\Omega_\nabla \in \Omega^2_L(B, \boldsymbol{g}_A)$ satisfy

$$\Omega_\nabla = \omega_L^*(\Omega_\lambda).$$

Hence for an invariant section Γ of the symmetric polynomials bundle we obtain

$$\omega_L^* \circ \chi_\lambda(\Gamma) = \frac{1}{k!}\left\langle \Gamma, \omega_L^*\Omega_\lambda \vee \ldots \vee \omega_L^*\Omega_\lambda \right\rangle = \chi_\nabla(\Gamma). \ \blacksquare$$

An immediate result is the implication

$$h_A = 0 ongrightarrow h_{L,A} = 0,$$

for each pair (L,A) of Lie algebroids over a foliated manifold (B,F). Hence, if A is a Lie algebroid with Atiyah sequence $0 \to \boldsymbol{g}_A \to A \to F \to 0$ and if we consider an L-connection in A for another Lie algebroid L, then it can turn out that the amount of obstructions for the existence of a flat L-connection will decrease. In the special case $L = A$ we obtain no such obstructions: $h_{A,A} = 0$ since $\mathrm{id}_A : A \to A$ is a flat A-connection in A. Hence the diagram below is commutative

$$\begin{tikzcd} I(A) \arrow[r, "h_A"] \arrow[dr, "h_{A,A} = 0"'] & H_F(B) \arrow[d, "\omega_A^\sharp"] \\ & H_A(B) \end{tikzcd}$$

which means that

$$\mathrm{Pont}(A) \subset \ker \omega_A^\sharp,$$

i.e. the images of Pontryagin classes under the anchor ω_A are zero: for an invariant section $\Gamma \in I(A)$ of rank k and any connection χ in A

$$\omega_A^*(\langle \Gamma, \Omega_\lambda \vee \ldots \vee \Omega_\lambda \rangle) = d\Phi,$$

for some form Φ. One of that forms is given in [K10]; let $\eta : A \to \boldsymbol{g}_A$ be the form of λ. Then

$$\Phi = k! \sum_{i+j=k-1} \frac{1}{k+j} \cdot \left\langle \Gamma, \eta \vee \frac{1}{i!}(d_A\eta)^i \vee \frac{1}{j!}\left(-\frac{1}{2}[\![\eta,\eta]\!]\right)^j \right\rangle.$$

Corollary 3.8. *If* $\omega_L^\sharp : H_F(B) \to H_L(B)$ *is a monomorphism, then*

$$h_A = 0 \iff h_{L,A} = 0.$$

In the class of transitive unimodular invariantly oriented Lie algebroids L on an oriented manifold B [K8], $\omega_L^\sharp$ is a monomorphism if and only if the fibre integral of forms with fibre-compact carrier $\oint^\sharp_{L,c} : H_{L,c}(B) \to H_c(B)$ [K9] is an epimorphism. In the smaller class of so-called s-Lie algebroids [K10], $\omega_L^\sharp$ is a monomorphism if and only if the Euler class of a Lie algebroid L, χ_E, vanishes (for example, if $H^{n+1}(B) = 0$, where $n = \operatorname{rank} \boldsymbol{g}_L$).

3.2.5. *Comparison of* $h_{L,A}$ *with Fernandes* L*-Chern-Weil homomorphism.* Fernandes [F2] has constructed an L-Chern-Weil homomorphism $h_{L,P} : I(G) \to H_L(B)$ of a G-principal fibre bundle $P(B,G)$ in such a way that $h_{L,P} = \omega_L^\sharp \circ h_P$ (see Proposition 4.3 in [F2]). According to the diagrams (1.2) and (3.5) the Chern-Weil homomorphisms $h_{L,A(P)}$ and $h_{L,P}$ are joined via the following diagram

$$\begin{tikzcd} I(A(P)) \arrow[dr, "h_{L,A(P)}"] & \\ & H_L(B) \\ I(G) \arrow[uu, "\nu"] \arrow[ur, "h_{L,P}"'] & \end{tikzcd}$$

If P is connected then identifying $I(G) = I(A(P))$ we obtain $h_{L,P} = h_{L,A(P)}$ which proves the algebroid nature of the Fernandes L-Chern-Weil homomorphism.

3.2.6. *Comparison of* $h_{L,A}$ *with the Chern-Weil homomorphism of an extension.* Let A and L be two Lie algebroids on a manifold B and let $\pi : A \to L$ be a given epimorphism of Lie algebroids. Then $L' := \ker \pi$ is a subbundle of A.

Lemma 3.9. *There is precisely one structure of a Lie algebroid in* L' *for which*

$$\mathbf{e}(\pi) : \quad 0 \longrightarrow L' \xrightarrow{i} A \xrightarrow{\pi} L \longrightarrow 0$$

is a short exact sequence of Lie algebroids. The anchor of L' *is* $\omega_{L'} = 0$.

Proof. The uniqueness is a consequence of the fact that the anchor and Lie bracket must be given by $\omega_{L'} = \omega_A \circ i$ and $[\![\xi,\eta]\!]^{L'} = [\![\xi,\eta]\!]^A$ for $\xi, \eta \in \operatorname{Sec} L'$. Observe now that

$$\omega_{L'}(\xi') = \omega_A \circ i(\xi') = (\omega_L \circ \pi) i(\xi') = \omega_L \circ (\pi \circ i)(\xi') = 0. \ \blacksquare$$

Given an epimorphism $\pi : A \to L$ and the induced extension $\mathbf{e}(\pi)$ consider the structural diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \boldsymbol{g}_A \arrow[r, "\pi^+"] \arrow[d] & \boldsymbol{g}_L \arrow[d] & \\
\mathbf{e}(\pi) : 0 \arrow[r] & L' \arrow[r, "i"] \arrow[ur, "j"] & A \arrow[r, "\pi"] \arrow[d, "\omega_A"'] & L \arrow[r] \arrow[d, "\omega_L"] & 0 \\
 & & F_A \arrow[r, "\iota"'] \arrow[d] & F_L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The existence of the $\mathbf{e}(\pi)$-connection implies the equality $F_A = F_L$.

There is a representation $\mathrm{ad}_{\mathbf{e}(\pi)}$ of the Lie algebroid A in the bundle L' satisfying $\mathrm{ad}_{\mathbf{e}(\pi)}(\xi)(\eta') = [\![\xi, \eta']\!]^A$, $\eta' \in \mathrm{Sec}\, L'$, $\xi \in \mathrm{Sec}\, A$. Indeed, $[\![\xi, \eta']\!]^A \in \mathrm{Sec}\, L'$ and $\mathrm{ad}_{\mathbf{e}(\pi)} : \mathrm{Sec}\, L' \to \mathrm{Sec}\, L'$ is a covariant differential operator with the anchor $\omega_A(\xi)$. $\mathrm{ad}_{\mathbf{e}(\pi)}$ and the adjoint representation ad_A of A induces representations in $\bigvee^k L'$ and $\bigvee^k \boldsymbol{g}_A^*$, $k = 1, 2, \ldots$ given by (2.15) where the domains are the algebras of invariant sections $I(\mathbf{e})$ and $I(A)$, respectively. If $\Gamma \in \mathrm{Sec} \bigvee^k \boldsymbol{g}_A^*$ is ad_A-invariant, then the restriction $\Gamma_{|L'} \in \mathrm{Sec} \bigvee^k L'^*$ is $\mathrm{ad}_{\mathbf{e}(\pi)}$-invariant and

$$I(A) \longrightarrow I(\mathbf{e}), \quad \Gamma \mapsto \Gamma_{|L'},$$

is a homomorphism of algebras. Characteristic classes from the image of the Chern-Weil homomorphims of an extension $\mathbf{e}(\pi)$ of Lie-Rinehart algebras (see [H2] and [T2]) give obstructions to the existence of a flat $\mathbf{e}(\pi)$-connection. In brief, we give the construction of the Chern-Weil homomorphism of an extension $\mathbf{e}(\pi)$. The curvature tensor of an $\mathbf{e}(\pi)$-connection $\nabla : L \to A$ is a 2-form $\Omega_\nabla^{\mathbf{e}(\pi)} \in \Omega_L^2(B; L')$ such that

$$\Omega_\nabla^{\mathbf{e}(\pi)} = [\![\nabla\xi, \nabla\eta]\!]^A - \nabla[\![\xi, \eta]\!]^L. \tag{3.6}$$

Since ∇ is an L-connection in A as well, it possesses its curvature tensor $\Omega_\nabla \in \Omega_L^2(B; \boldsymbol{g}_A)$ which is given by (3.6). Hence

$$\Omega_\nabla = j\, \Omega_\nabla^{\mathbf{e}(\pi)}. \tag{3.7}$$

The Chern-Weil homomorphism $h_{\mathbf{e}(\pi)}$ of the extension $\mathbf{e}(\pi)$ satisfies

$$h_{\mathbf{e}(\pi)}(\Gamma) = \frac{1}{k!} \cdot [\langle \Gamma, \Omega_\nabla^{\mathbf{e}(\pi)} \vee \ldots \vee \Omega_\nabla^{\mathbf{e}(\pi)} \rangle].$$

The four Chern-Weil homomorphisms h_A, h_L, $h_{L,A}$, $h_{\mathbf{e}(\pi)}$ (A and L are regular Lie algebroids over a foliated manifold $(B;F)$) are connected by the commutative diagram

$$\begin{tikzcd} I(L) \arrow[r, "\pi^{+*}"] \arrow[dr, "h_L"'] & I(A) \arrow[r, "j^{+*}"] \arrow[d, "h_A"'] \arrow[dr, "h_{L,A}"] & I(\mathbf{e}(\pi)) \arrow[d, "h_{\mathbf{e}(\pi)}"] \\ & H_F(B) \arrow[r, "\omega_L^\sharp"'] & H_L(B). \end{tikzcd} \tag{3.8}$$

Commutativity of the left-hand side triangle is a consequence of the fact that the Chern -Weil homomorphism of regular Lie algebroids is functorial. The next triangle is commutative by Theorem 3.7. To see that the triangle on the right-hand side is commutative we compute

$$\begin{aligned} h_{\mathbf{e}(\pi)}(j^{+*}(\Gamma)) &= \frac{1}{k!}\cdot[\langle j^{+*}\Gamma, \Omega_\nabla^{\mathbf{e}(\pi)}\vee\ldots\vee\Omega_\nabla^{\mathbf{e}(\pi)}\rangle] \\ &= \frac{1}{k!}\cdot[\langle \Gamma, j\Omega_\nabla^{\mathbf{e}(\pi)}\vee\ldots\vee j\Omega_\nabla^{\mathbf{e}(\pi)}\rangle] \\ &= \frac{1}{k!}\cdot[\langle \Gamma, \Omega_\nabla\vee\ldots\vee\Omega_\nabla\rangle] = h_A(\Gamma). \end{aligned}$$

$h_{L,A}\circ\pi^{+*}=0$ since $\operatorname{Im} h_L\subset\ker\omega_L^\sharp$. Hence, if π^{+*} is an epimorphism, then $h_{L,A}=0$.

In the case of G-Lie algebroids we can put the superscript ^{G} in the diagram (3.8).

PROBLEM 3.10. *Find an example of an extension* $\mathbf{e}$ *with* $h_{\mathbf{e}}\neq 0$ *(then there is no flat* $\mathbf{e}$*-connection) and such that there is a flat* L*-connection in* A *(in particular* $h_{L,A}=0$*). In this example* j^{+*} *cannot be a surjective homomorphism. A wide class of objects with an epimorphism* j^{+*} *(in the non-transitive case where* $\pi: A\to L$ *comes from an extension of principal bundles* $Q(B,H,q)\xrightarrow{\pi}P(B,G,p)$ *with respect to an extension of Lie groups* $0\to N\to H\to G\to 0$*) was given by K. Mackenzie [M5]. It is true when* N*,* H *and* G *are compact and connected. The problem is to study the non-compact case which can be extended to the case of Lie algebroids.*

3.2.7. *Induced tangential case.* If A is a transitive Lie algebroid and $\operatorname{Im}\omega_L$ is contained in a regular foliation $F\subset TB$, then together with an L-connection $\nabla: L\to A$ we can consider an induced tangential L-connection $\nabla^F: L\to A^F$ defined by the same formula (see Introduction). According to Theorem 3.7 we have the following diagram joining the Chern-Weil homomorphisms $h_{L,A}$ and h_{L,A^F}.

$$\begin{tikzcd} I(A) \arrow[rr, "h_A"] \arrow[dr, "h_{L,A}"] \arrow[dd] & & H(B) \arrow[dl, "\omega_L^\sharp"'] \arrow[dd, "i_F^\sharp"] \\ & H_L(B) & \\ I(A^F) \arrow[rr, "h_{A^F}"'] \arrow[ur, "h_{L,A^F}"] & & H_F(B) \arrow[ul, "\omega_L^\sharp"] \end{tikzcd}$$

Since—in general—the domain $I(A^F)$ is not generated as an $\Omega_b(B,F)$-module by $I(A)$ (see [K5]) we can pose the problem:

PROBLEM 3.11. *Find an example in which* $h_{L,A}=0$ *and* $h_{L,A^F}\neq 0$.

4. Transversal Lie algebroids and transversal lifting of connections

4.1. *Mackenzie algebroid.* Consider an extension $\mathbf{e}: 0\to K\to A\xrightarrow{\pi} TP_{/G}\to 0$ of transitive Lie algebroids where $TP_{/G}$ is the Lie algebroid of a principal fibre bundle P. Recall the construction (given by K. Mackenzie) of a lifting of $\mathbf{e}$ via a projection $p:P\to B$ to some PBG-Lie algebroid p^*A on P with the Atiyah sequence

$$0\longrightarrow p^*K\longrightarrow p^*A\xrightarrow{\widetilde{q}} TP\longrightarrow 0,$$

where $\widetilde{q}$ is a lifting of π (under the identification $TP\cong p^*(TP_{/G})$). We identify also $\operatorname{Sec}(p^*A)\cong C^\infty(P)\otimes_{C^\infty(B)}\operatorname{Sec}A$. The reader can find the next theorems in [M5].

THEOREM 4.1 (K. Mackenzie). *There is a structure of a Lie algebroid in the bundle* p^*A. *The anchor is* $\widetilde{q}$ *and sections of* p^*A *satisfy*

$$[\![f\otimes\xi, f\otimes\eta]\!]=f\cdot g\cdot[\![\xi,\eta]\!]+f\cdot(\pi\circ\xi)^{\rightarrow}(g)\otimes\eta-g\cdot(\pi\circ\eta)^{\rightarrow}(f)\otimes\xi,$$

where $X^{\rightarrow}$ *denotes the right-invariant vector field on* P *induced by* $X\in\operatorname{Sec}(TP_{/G})$.

The canonical homomorphisms $\overline{p}$ and $\overline{\overline{p}}$ in the diagram

$$\begin{tikzcd}
0 \arrow[r] & p^*K \arrow[r] \arrow[d] & p^*A \arrow[r, "\widetilde{q}"] \arrow[d, "\overline{\overline{p}}"] & TP \arrow[r] \arrow[d, "\overline{p}"] & 0 \\
0 \arrow[r] & K \arrow[r] & A \arrow[r, "\pi"'] & TP_{/G} \arrow[r] & 0
\end{tikzcd}$$

are non-strong homomorphisms of Lie algebroids (over $p:P\to B$). Notice that p^*A is not a pull-back of A according to the definitions in [H-M] and [K4].

THEOREM 4.2 (K. Mackenzie). *A right action R of G on p^*A is given by*

$$R_g(z,v)=(zg,v).$$

$R_g: p^*A\to p^*A$ *is an automorphism of the Lie algebroid* p^*A *over the right translation* $R_g:P\to P$ *and hence* p^*A *is a* PBG*-algebroid.*

DEFINITION 4.3. The transitive algebroid p^*A on P (constructed above) will be called the *Mackenzie algebroid* of an extension $\mathbf{e}$ or the algebroid transversal to $\mathbf{e}$ (by analogy with the notion a transversal principal bundle when $\mathbf{e}$ comes from an extension of principal bundles [M5]).

PROPOSITION 4.4. *If* $\chi: TP_{/G}\to A$ *is an* $\mathbf{e}$*-connection, then its lifting* $\widetilde{\chi}=p^*(\chi)$ *is a* G*-equivariant connection in the transversal algebroid* p^*A. *Conversely, any* G*-equivariant connection in* p^*A *is of this form.*

The last proposition corresponds to Prop. 3.2 in [M5] for extensions of principal bundles. $\widetilde{\chi}$ will be called the transversal lifting of the $\mathbf{e}$-connection χ.

4.2. *Some generalizations of connection.* We now study lifting of any $TP_{/G}$-connection in A.

PROPOSITION 4.5. *Let $\chi : TP_{/G} \to A$ be an arbitrary $TP_{/G}$-connection in A (i.e. a linear homomorphism with $\omega_A \circ \chi = \omega$). Then, the lifting $\widetilde{\chi} : TP \to p^*A$ is a G-equivariant homomorphism such that $p_*(\widetilde{q} \circ \widetilde{\chi}) = p_*$ (i.e. the difference between $\widetilde{q} \circ \widetilde{\chi}(v)$ and v is a vertical vector, one which is tangent to an orbit of the action of G on P). Conversely, if $\gamma : TP \to p^*A$ is a G-equivariant homomorphism with $p_*(\widetilde{q} \circ \widetilde{\chi}) = p_*$, then $\gamma = \widetilde{\chi}$ for some $TP_{/G}$-connection χ in A.*

Proof. "*ongrightarrow*" It is easy to show that $\widetilde{\chi}$ is G-equivariant. To see that $p_*(\widetilde{q} \circ \widetilde{\chi}) = p_*$ consider the diagram

$$\begin{tikzcd}
p^*A \arrow[r, shift left, "\widetilde{q}"] \arrow[d, "\overline{\overline{p}}"'] & TP \arrow[l, shift left, "\widetilde{\chi}"] \arrow[d, "\overline{p}"] \arrow[dr, "p_*"] & \\
A \arrow[r, shift left, "\pi"] \arrow[rr, bend right, "\omega_A"'] & TP_{/G} \arrow[l, shift left, "\chi"] \arrow[r, "\omega"] & TB
\end{tikzcd}$$

Then

$$p_* \circ (\widetilde{q} \circ \widetilde{\chi}) = \omega_A \circ \overline{\overline{p}} \circ \widetilde{\chi} = \omega_A \circ \chi \circ \overline{p} = \omega \circ \overline{p} = p_*.$$

"*ongleftarrow*" Let $\gamma : TP \to p^*A$ be a G-equivariant linear homomorphism with $p_*(\widetilde{q} \circ \widetilde{\chi}) = p_*$. Since γ is G-equivariant, there is a linear homomorphism of vector bundles $\chi : TP_{/G} \to A$, and its lifting is γ. It remains to check that $\omega_A \circ \chi = \omega$. Since $\overline{p}$ is a linear homomorphism on the fibres, we have

$$\omega_A \circ \chi \circ \overline{p} = \omega_A \circ \overline{\overline{p}} \circ \gamma = p_* \circ \widetilde{q} \circ \gamma = p_* = \omega \circ \overline{p}. \ \blacksquare$$

The proposition above suggests that we should generalize the notion of a connection in a regular G-Lie algebroid $A \xrightarrow{\omega_A} F$ on B as follows: a linear homomorphism $\lambda : F \to A$ is called a G-connection if the difference between $\omega_A \circ \lambda(v)$ and v, $v \in F$, is a vector tangent to an orbit of the action G on B (if the group G is trivial, then we obtain the standard connection in A, $\omega_A \circ \lambda = \mathrm{id}$). Under this definition we can say that the transversal lifting of a $TP_{/G}$-connection in A is a G-connection in the G-Mackenzie algebroid p^*A and vice versa.

4.3. *Geometric interpretation of a Mackenzie algebroid.* Recall the geometric interpretation of a Mackenzie algebroid [M5]. Let (2.5) be an extension of principal bundles (with respect to an extension of Lie groups $0 \to N \to H \xrightarrow{\pi} G \to 0$) and consider the corresponding exact sequence (an extension of transitive Lie algebroids)

$$\mathbf{e} : 0 \to Q \times_H \mathfrak{n} \to TQ_{/H} \xrightarrow{\pi_*^G} TP_{/G} \to 0.$$

Let $Q(P, N, \pi)$ be an N-principal bundle transversal to the extension of principal bundles and let $TQ_{/N}$ be its Lie algebroid with the Atiyah sequence

$$0 \to Q \times_N \mathfrak{n} \to TQ_{/N} \xrightarrow{\pi_*^N} TP \to 0.$$

The Lie group G acts on $TQ_{/N}$ by

$$\widetilde{R}_g : TQ_{/N} \longrightarrow TQ_{/N}, \quad [X_v]^N \mapsto [R_{h*}X_v]^N,$$

where $X_v \in T_vQ$ and $h \in \pi^{-1}(g)$. $\widetilde{R}_g$ is an automorphism of the Lie algebroid $TQ_{/N}$ over the right translation $R_g : P \to P$. Under this action, $TQ_{/N}$ is a PBG-Lie algebroid. The homomorphism $\widetilde{R}_g : TQ_{/N} \to TQ_{/N}$ has a (not necesarily unique) lifting to an automorphism of the N-principal bundle $\pi : Q \to Q$, $z \mapsto zh$, with respect to an automorphism of Lie group $\tau_{h^{-1}} : N \to N$, $n \mapsto h^{-1}nh$. Given $g \in G$ denote by $\alpha(g)$ any element from $\pi^{-1}(g)$. The family of these automorphisms gives a smooth mapping $Q \times G \to Q$ if and only if α is a smooth mapping, which is equivalent to the fact that the quotient bundle $H \to H_{/N} = G$ is trivial. Hence there is usually no action of the Lie group G on the transversal bundle Q which gives $\widetilde{R}$.

Situation like this seems to be more elegant from the Lie groupoids point of view: given the Lie-Ehresmann groupoid $T = Q \times Q_{/N}$ of a transversal bundle $Q(P, N, \pi)$ there is a right action (Mackenzie) $R : T \times G \to T$ given by $R_g(< v_2, v_1 >) =< v_2h, v_1h >$, $h \in \pi^{-1}(g)$. The differential of R_g defines an automorphism of the Lie algebroid $A(T)$ of T and under the canonical identification $A(T) \cong TQ_{/N}$ we obtain an action $\tilde{R}$ which is given as above. The advantage of the groupoid and algebroid approach in comparison with principal bundles is now clear (see [M5, Remark (1)]).

A connection $\chi : TP_{/G} \to TQ_{/H}$ in the extenision $\mathbf{e}$ has a unique lifting to a connection $\widetilde{\chi} : TP \to TQ_{/N}$ in the algebroid $TQ_{/N}$ (and next to a connection in the transversal bundle). Indeed, if $\chi([u_z]^G) = [X_v]^H$ $(X_v \in T_vQ,\ \pi z = h)$, then $\widetilde{\chi}(u_z) = [X_v]^N$. Observe that $\widetilde{R}_g$ induces an automorphism of $p^*(TQ_{/H})$, $(z, \alpha) \mapsto (zg, \alpha)$. The connection $\widetilde{\chi}$, we have obtained is G-equivariant and conversely, each G-equivariant connection $\gamma : TP \to TQ_{/N}$ has the form of $\widetilde{\chi}$ for some connection χ in $\mathbf{e}$. Under the identification $TQ_{/N} \cong p^*(TQ_{/H})$, we can introduce the Lie algebroid structure in $p^*(TQ_{/H})$ obtaining a Mackenzie algebroid.

4.4. *The equivariant Chern-Weil homomorphism of a Mackenzie algebroid.* Consider an extension of Lie algebroids $\mathbf{e} : 0 \to K \to A \to TP_{/G} \to 0$ and the corresponding transversal PBG-Mackenzie algebroid $0 \to p^*K \to p^*A \to TP \to 0$. According to Section 3.2.3, we have the well defined G-equivariant Chern-Weil homomorphism $h^G_{p^*A} : I(p^*A)^G \to H^G(P)$ and this homomorphism is equivalent to the Chern-Weil homomorphism of the extension $\mathbf{e}$ under the canonical isomorphisms

$$\begin{tikzcd} I(p^*A)^G \arrow[r, "h^G_{p^*A}"] & H^G(P) \\ I(A) \arrow[u, "\cong"] \arrow[r, "h_{\mathbf{e}}"'] & H_{TP_{/G}}(B) \arrow[u, "\cong"'] \end{tikzcd}$$

If manifolds P and Q in the extension $Q(B, H, q) \xrightarrow{\pi} P(B, G, p) \to 0$ are connected (the Lie groups H and G need not to be connected), then $I(G) \cong I(TP_{/G})$, $I(H) \cong I(TQ_{/H})$, $I(N) \cong I(TQ_{/N})$ and in consequence, $I(N)^G \cong I(TQ_{/N})^G$. Hence, given an extension

e which comes from an extension of principal bundles $Q \to P$, the diagram (3.8) (if we omit $h_{L,A}$) is equivalent to the diagram (1.3) given by K. Mackenzie.

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