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COHOMOLOGY OF KOSZUL-VINBERG ALGEBROIDS AND POISSON MANIFOLDS, I

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Abstract. We introduce a cohomology theory of Koszul-Vinberg algebroids. The relationships between that cohomology and Poisson manifolds are investigated. We focus on the complex of chains of superorders [KJL1]. We prove that symbols of some sort of cycles give rise to so called bundlelike Poisson structures. In particular we show that if $E \to M$ is a transitive Koszul-Vinberg algebroid whose anchor is injective then a Koszul-Vinberg cocycle θ whose symbol has non-zero skew symmetric component defines a transversally Poissonian symplectic foliation in M.

1. Background material. Let $\mathcal{A}$ be a real algebra whose multiplication map is denoted by

$$(a, b) \to ab.$$

Given three elements a, b, c of $\mathcal{A}$ their associator in $\mathcal{A}$ is the quantity

$$(a, b, c) = a(bc) - (ab)c. \tag{1}$$

Definition 1.1. A real algebra $\mathcal{A}$ is called a Koszul-Vinberg algebra if its associator map satisfies the identity

$$(a, b, c) = (b, a, c).$$

N.B. Koszul-Vinberg algebras are also called left symmetric algebras [NB1], [PA].

Let $\mathcal{A}$ be a Koszul-Vinberg algebra and let W be a real vector space with two bilinear maps

$$\begin{aligned} \mathcal{A} \times W \to W: &\quad (a, w) \to aw; \\ W \times \mathcal{A} \to W: &\quad (w, a) \to wa. \end{aligned} \tag{2}$$

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We will set the following: given $a \in \mathcal{A}$ and $w \in W$

$$(a,b,w) = a(bw) - (ab)w,$$
$$(a,w,b) = a(wb) - (aw)b, \tag{3}$$
$$(w,a,b) = w(ab) - (wa)b.$$

DEFINITION 1.2. A vector space equiped with two bilinear maps (2) is called a Koszul-Vinberg module of $\mathcal{A}$ if the following identities hold for any $a, b \in \mathcal{A}$ and $w \in W$:

$$(a,b,w) = (b,a,w),$$
$$(a,w,b) = (w,a,b).$$

Given a Koszul-Vinberg algebra $\mathcal{A}$ and a Koszul-Vinberg module W of $\mathcal{A}$, one of the following spaces:

$$J(\mathcal{A}) = \{c \in \mathcal{A}/(a,b,c) = 0, \forall a \in \mathcal{A}, \forall b \in \mathcal{A}\};$$
$$J(W) = \{w \in W/(a,b,w) = 0, \forall a \in \mathcal{A}, \forall b \in \mathcal{A}\}.$$

The subspace $J(\mathcal{A}) \subset \mathcal{A}$ is a subalgebra of $\mathcal{A}$ and the induced multiplication map is associative. In general the vector subspace $J(W)$ is not invariant under the actions (2).

Examples of Koszul-Vinberg algebras and their modules

(e_1) Every associative algebra is a Koszul-Vinberg algebra.

(e_2) Let (M, D) be a locally flat manifold, [KJL3]; then the vector space $\Gamma(TM)$ of smooth vector fields on M is a Koszul-Vinberg algebra; its multiplication map is defined by

$$(X,Y) \to XY = D_X Y.$$

(e_3) Given a locally flat manifold (M, D) let W be the vector space of real valued smooth functions on M. For any $f \in W$ and $X \in \Gamma(TM)$ we define $Xf \in W$ and $fX \in W$ by putting

$$(Xf)(x) = \, < df, X > (x), \qquad (fX)(x) = 0 \in \mathbb{R}.$$

With the above operations W becomes a Koszul-Vinberg module of $\mathcal{A} = \Gamma(TM)$.

Given a Koszul-Vinberg algebra $\mathcal{A}$ and two Koszul-Vinberg modules of $\mathcal{A}$, called V and W, let $Hom(W, V)$ be the vector space of linear maps from W to V. We consider the following actions of $\mathcal{A}$ in $Hom(W, V)$: let $\theta \in Hom(W, V), a \in \mathcal{A}$, $w \in W$ then we set

$$(a\theta)(w) = a(\theta(w)) - \theta(aw), \qquad (\theta a)(w) = (\theta(w))a. \tag{4}$$

Under the actions defined in (4) the vector space $Hom(W, V)$ becomes a Koszul-Vinberg module of $\mathcal{A}$. More generally the vector space $Hom(\oplus^q W, V)$ of q-linear mappings from W to V is a Koszul-Vinberg module of $\mathcal{A}$ under the following actions: let $\theta \in Hom(\oplus^q W, V)$, $a \in \mathcal{A}$ and $w_1, ..., w_q \in W$, we set

$$(a\theta)(w_1, ..., w_q) = a(\theta(w_1, ..., w_q)) - \textstyle\sum_{1 \leq j \leq q} \theta(...aw_j, ..., w_q),$$
$$(\theta a)(w_1, ..., w_q) = (\theta(w_1, ..., w_q))a.$$

Let q be a positive integer every pair (j, w_0) where j is a non-negative integer with $j \leq q$ and $w_0 \in W$ will define a linear map from $Hom(\otimes^q, V)$ to $Hom(\otimes^{q-1} W, V)$, called $e_j(w_0)$. Let $\theta \in Hom(\otimes^q W, V)$ then $e_j(w_0)\theta \in Hom(\otimes^{q-1} W, V)$ is defined by

$$(e_j(w_0)\theta)(w_1, ..., w_{q-1}) = \theta(w_1, ..., w_{j-1}, w_0, w_j, ..w_{q-1}).$$

The linear map $e_j(w_0)$ commutes with the right action of $\mathcal{A}$, viz

$$(e_j(w_0)\theta)a = e_j(w_0)(\theta a).$$

Thus the notation $e_j(w_0)\theta a$ will be well defined.

We are now in a position to recall the definition of the complex

$$... \to C^q(\mathcal{A}, W) \overset{\delta_q}{\to} C^{q+1}(\mathcal{A}, W) \to ...$$

Let $\mathcal{A}$ be a Koszul-Vinberg algebra and let W be a Koszul-Vinberg module of $\mathcal{A}$. For each positive integer q we set

$$C^q(\mathcal{A}, W) = Hom(\otimes^q \mathcal{A}, W)$$

and for $q = 0$ we set

$$C^0(\mathcal{A}, W) = J(W).$$

Then the graded vector space

$$C(\mathcal{A}, W) = \oplus_{q \geq 0} C^q(\mathcal{A}, W)$$

is a cochain complex whose boundary operator is defined by

$$\begin{array}{c} \delta_0 : C^0(\mathcal{A}, W) \to C^1(\mathcal{A}, W), \quad (\delta_0 w)(a) = -aw + wa, \\ \delta_q : C^q(\mathcal{A}, W) \to C^{q+1}(\mathcal{A}, W), \\ (\delta\theta)(a_1, ..., a_{q+1}) = \sum_{j \leq q} (-1)^j \{(a_j\theta)(..\hat{a}_j..a_{q+1}) + (e_q(a_j)\theta a_{q+1}(..\hat{a}_j.., \hat{a}_{q+1})\} \end{array} \quad . \tag{5}$$

The family $(\delta_q)_q$ satisfies the following identity

$$\delta_{q+1}\delta_q = 0.$$

The q^{th} cohomology space of the cochain complex $C(\mathcal{A}, W)$ is denoted by $H^q(\mathcal{A}, W)$. We have

$$H^q(\mathcal{A}, W) = ker(\delta_q)/im(\delta_{q-1})$$

for $q > 0$ and

$$H^0(\mathcal{A}, W) = ker(\delta_0).$$

Example. Let (M, D) be a locally flat manifold and let $\mathcal{A} = \Gamma(TM)$ be the corresponding Koszul-Vinberg algebra. Regarding $\mathcal{A}$ as a Koszul-Vinberg module of itself the subspace $J(\mathcal{A})$ consists of affine vector fields. Thus $ker(\delta_0)$ is the subspace of locally linear vector fields. One sees that in general $H^0(\mathcal{A}, \mathcal{A})$ will be non-trivial; e.g. if (M, D) is the real flat torus then $dimH^0(\mathcal{A}, \mathcal{A}) = dimM$. On the other hand we have $H^1(\mathcal{A}, \mathcal{A}) = 0$, [NB$_3$].

2. Koszul-Vinberg algebroids and coalgebroids. Let M be a smooth manifold and E a vector bundle over M. The space of smooth sections of E is denoted by $\Gamma(E)$.

Definition 2.1. A Koszul-Vinberg algebroid over M is a vector bundle E over M with a bundle map $a : E \to TM$, called the anchor map, such that

(P_1) $\Gamma(E)$ is a Koszul-Vinberg algebra;

(P_2) The anchor $a : \Gamma(E) \to \Gamma(TM)$ satisfies the following identities: $\forall f \in C^\infty(M, \mathbb{R})$, $\forall s \in \Gamma(E)$, $\forall s' \in \Gamma(E)$

$$(fs)s' = f(ss'), \quad s(fs') = f(ss') + < df, a(s) > s'.$$

Remark. It follows from conditions (P1) and (P2) that the anchor map is a homomorphism of the associated Lie algebras.

Examples of Koszul-Vinberg algebroids

(e_1) The tangent bundle of a locally flat manifold (M, D) is a Koszul-Vinberg algebroid. Its anchor is Identity map; given two sections of TM, called X, Y then

$$XY = D_X Y.$$

(e_2) Let $\mathcal{F}$ be an affine foliation in a smooth manifold M and let $E_{\mathcal{F}}$ be the tangent bundle of $\mathcal{F}$ in TM. Since each leaf of $\mathcal{F}$ is a locally flat manifold $E_{\mathcal{F}}$ is a Koszul-Vinberg algebroid over M.

(e_3) Each completely integrable system in an m-dimensional symplectic manifold (M, ω) gives rise to an action of $\mathbb{R}^m$ in M. The orbits of that action are locally flat manifolds; thus every completely integrable system will generate a Koszul-Vinberg algebroid.

(e_4) Given a lagrangian foliation $\mathcal{F}$ in a symplectic manifold (M, ω) one defines a Koszul-Vinberg algebroid E as in (e_2). If $s, s' \in \Gamma(E)$ then ss' is defined by the relation

$$\iota(ss')\omega = L_s \iota(s')\omega \tag{6}$$

where $\iota(s')$ is the inner product by s' and L_s is the Lie derivation w.r.t. s. The multiplication in $\Gamma(E)$ given by (6) induces a locally flat structure in each leaf of $\mathcal{F}$.

Now given a Koszul-Vingberg algebroid E whose anchor map is injective, it is natural to ask whether the locally flat structure of leaves of E extends to a locally flat structure in M. The notion of Koszul-Vinberg co-algebroid together with cochain complex (5) help to study the extension that we just raised, [NBW] (see also [KI] for the notion of partial connection).

Definition 2.2. Given a Koszul-Vinberg algebroid $E \to M$, a Koszul-Vinberg coalgebroid of E is a vector bundle $N \to M$ together with a bundle map $\alpha : N \to TM$ satisfying the following conditions:

(c_1) $\Gamma(N)$ is a Koszul-Vinberg algebra.

(c_2) There exists a linear map $j : \Gamma(TM) \to \Gamma(N)$ such that the sequence

$$\Gamma(E) \xrightarrow{a} \Gamma(TM) \xrightarrow{j} \Gamma(N) \to 0$$

is exact and $j \circ \alpha(s) = s, \forall s \in \Gamma(N)$.

(c_3) Let s, s' be elements of $\Gamma(N)$ and $f \in C^\infty(M, \mathbb{R})$; then

$$(fs)s' = f(ss')$$

and if $< df, a(\sigma) >= 0$ for every $\sigma \in \Gamma(E)$ then

$$s(fs') = f(ss') + < df, \alpha(s) > s'.$$

Example. Let $\mathcal{F}$ be a locally flat foliation which is a transversally affine foliation at the same time. Then the Koszul-Vinberg algebroid $E_{\mathcal{F}}$ corresponding to $\mathcal{F}$ admits a Koszul-Vinberg coalgebroid, [NBW].

Indeed let $\mathcal{L}$ be the sheaf of locally linear sections of $E_{\mathcal{F}}$, i.e. $s \in \mathcal{L}$ iff $s's = 0$, $\forall s' \in \Gamma(E_{\mathcal{F}})$. We consider the quotient vector bundle $TM/E_{\mathcal{F}}$. Since $\mathcal{F}$ is transversally affine the space of smooth sections of $N = TM/E_{\mathcal{F}}$ admits a structure of Koszul-Vinberg algebra (every germ of submanifold which is transverse to $\mathcal{F}$ is a germ of affine manifold). Thus $\Gamma(N)$ admits a Koszul-Vinberg algebra structure. Let us write

$$J(N) = J(\Gamma(N)).$$

Then $C^\infty(M,\mathbb{R})J(N) = \Gamma(N)$. Using a riemannian metric on M one constructs a section

$$\alpha : N \to TM$$

of the exact sequence

$$0 \to E_{\mathcal{F}} \xrightarrow{a} TM \xrightarrow{j} N \to 0$$

where j is the canonical projection.

In [NBW] we have used the Lie algebra

$$\mathcal{A} = norm(\mathcal{L}) \cap j^{-1}(J(N))$$

to study the extension problem of the locally flat structure of $\mathcal{F}$; $norm(\mathcal{L})$ is the normalizer of $\mathcal{L}$ in the Lie algebra $\Gamma(TM)$.

Remark that every Koszul-Vinberg algebroid E gives rise to a Lie algebroid E_L; the total space of E_L is E; for s and s' in $\Gamma(E_L)$ the bracket is defined by

$$[s, s'] = ss' - s's. \tag{7}$$

The anchor map of E satisfies the identity

$$a([s, s']) = [a(s), a(s')].$$

Indeed let s, s', s'' be elements of $\Gamma(E)$ and $f \in C^\infty(M,\mathbb{R})$, then

$$[s, s'](fs'') = (ss')(fs'') - (s's)(fs'') = s(s'(fs'')) - s'(s(fs''))$$

and property (P_2) of definition 2.1 implies that

$$< df, a([s, s']) >= a(s)(a(s')f) - a(s')(a(s)f),$$

where $a(s)f =< df, a(s) >$.

3. Real cohomology of Koszul-Vinberg algebroids. Let $E \to M$ be a Koszul-Vinberg algebroid. The vector space $W = C^\infty(M,\mathbb{R})$ is a Koszul-Vinberg module of $\mathcal{A} = \Gamma(E)$. The left action and the right action are defined by

$$(sf)(x) =< df, a(s) >, \quad (f.s)(x) = 0,$$

where a is the anchor map of E.

We will focus on the cochain complex

$$... \to C^q(\mathcal{A}, W) \xrightarrow{\delta_q} C^{q+1}(\mathcal{A}, W) \to ... \tag{8}$$

The q^{th} cohomology space of (8) is denoted by $H^q(E,\mathbb{R})$, i.e. $H^q(E,\mathbb{R}) = H^q(\mathcal{A}, W)$.

Definition 2.3. The vector space $H^q(E,\mathbb{R})$ is called the q^{th} cohomology space of the Koszul-Vinberg algebroid $E \to M$.

Example. Let E be a regular Koszul-Vinberg algebroid whose anchor map is denoted by a. Then $a(E)$ defines a foliation on M. A function f belongs to $J(W)$ iff $L_{a(s)} \circ$

$L_{a(s')}(f) = 0$ for arbitrary sections s, s' of E. Thus if the anchor is injective then $J(W)$ consists of smooth functions which are affine along each leaf of $a(E)$. Since $H^0(E, \mathbb{R}) = ker(\delta_0)$ we see that $H^0(E, \mathbb{R})$ is just the vector space of first integrals of $a(E)$.

THEOREM 3.1. *If a regular Koszul-Vinberg algebroid $E \to M$ admits a dense leaf then* $dim H^0(E, \mathbb{R}) = 1$.

4. Koszul-Vinberg algebroids and Poisson manifolds. To every Koszul-Vinberg algebroid E we attach the following new Koszul-Vinberg algebroid

$$\mathcal{E} = E \times \mathcal{R}$$

where $\mathcal{R}$ is the trivial vector bundle $\mathcal{R} =: M \times \mathbb{R}$. We identify $\Gamma(\mathcal{R})$ with the associative algebra $C^\infty(M, \mathbb{R})$ of smooth real valued functions on M. Thus we will identify $\Gamma(\mathcal{E})$ with $\Gamma(E) \times C^\infty(M, \mathbb{R})$ as well.

Henceforth $\Gamma(\mathcal{E})$ is an algebra whose multiplication is

$$(s, f)(s', f') = (ss', ff' + < df', a(s) >). \tag{9}$$

It is easy to see that (9) endows $\Gamma(\mathcal{E})$ with a structure of Koszul-Vinberg algebra. Morever if $g \in C^\infty(M, \mathbb{R})$ then we have

$$(g(s, f))(s', f') = g((s, f)(s', f'))$$

and

$$(s, f)(g(s', f')) = g((s, f)(s', f')) + < dg, a(s) > (s', f').$$

Naturally the anchor map of $\mathcal{E}$ is defined by

$$a_\epsilon(s, f) = a(s)$$

where a is the anchor of $E \to M$. The Koszul-Vinberg algebra $\Gamma(\mathcal{E})$ is the semi-product

$$\Gamma(E) \times C^\infty(M, \mathbb{R}).$$

Now let V be a vector space, let r be a non-negative integer; we will put

$$T^r(V) = \otimes^r V.$$

Henceforth we are concerned with the cochain complex

$$... \to C^q(\mathcal{G}, W) \overset{\delta_q}{\to} C^q(\mathcal{G}, W) \to ...$$

where $\mathcal{G}$ is the Koszul-Vinberg algebra (9) and $W = C^\infty(M, \mathbb{R})$. For each non-negative integer q the vector space $C^q(\mathcal{G}, W)$ is bigraded

$$C^q(\mathcal{G}, W) = \oplus_{r+s=q} C^{r,s}(\mathcal{G}, W)$$

with

$$C^{r,s}(\mathcal{G}, W) = Hom(T^r A \otimes T^s W, W),$$

r and s being non-negative integers.

The boundary operator δ_q goes from $C^{r,s}(\mathcal{G}, W)$ to the direct sum $C^{r+1,s}(\mathcal{G}, W) \oplus C^{r,s+1}(\mathcal{G}, W)$. Thus we will equip the cohomology space $H^q(\mathcal{G}, W)$ with the bigradation

$$H^q(\mathcal{G}, W) = \oplus_{r+s=q} H^{r,s}(\mathcal{G}, W)$$

with
$$H^{r,s}(\mathcal{G},W)=\frac{ker(\delta_q : C^{r,s}(\mathcal{G},W)\to C^{r+1,s}(\mathcal{G},W)\oplus C^{r,s+1}(\mathcal{G},W))}{\delta_{q-1}(C^{q-1}(\mathcal{G},W))\cap C^{r,s}(\mathcal{G},W)}.$$

Naturally one sees that
$$\delta_{q-1}(C^{q-1}(\mathcal{G},W))\cap C^{r,s}(\mathcal{G},W)=\delta_{q-1}(C^{r-1,s}(\mathcal{G},W)+C^{r,s-1}(\mathcal{G},W))\cap C^{r,s}(\mathcal{G},W).$$

We will develop the analogue of the complex of differential forms of superorder introduced by Jean-Louis Koszul, [KJL2].

To begin with, let $\xi\in\mathcal{G}$, for a non-negative integer k and $x\in M$ $j^k_x\xi$ is the k^{th} jet at x of $\xi\in\mathcal{G}$. We will present $j^k_x\xi$ by
$$j^k_x\xi=(d^1_x\xi,..,d^l_x\xi,..,d^k_x\xi)$$
where $d^l_x\xi$ is the l^{th} differential at x of the section $\xi\in\Gamma(\mathcal{E})$.

Definition [KJL2]. A cochain $\theta\in C^q(\mathcal{G},W)$ is of order $\le k$ if at every $x\in M$ and for $\xi_1,..,\xi_q\in\mathcal{G}$ the value at x of $\theta(\xi_1,..\xi_q)$ depends on, $j^k_x\xi_1,..j^k_x\xi_q$.

Let $I=(i_1,..,i_q)$ be a q-tuple of non-negative integers such that $i_l\le k$. Given a q-cochain $\theta\in C^q(\mathcal{G},W)$ of order $\le k$, we set

$$\theta^I(\xi_1,..,\xi_q)(x)=\theta(d^{i_1}_x\xi_1,..,d^{i_q}_x\xi_q). \tag{11}$$

Since θ is q-multilinear (11) makes sense.

Thus every $\theta\in C^q(\mathcal{G},W)$ which is of order $\le k$ will be decomposed as follows
$$\theta(\xi_1,..,\xi_q)=\sum_I\theta^I(\xi_1,..,\xi_q)$$
where $I=(i_1,..,i_q)$ with $0\le i_1,..,i_q\le k$.

We call θ^I the component of type I of θ.

The following definition is crucial for the forthcoming applications.

Definition 4.1 Given a cochain of order $\le k$, say $\theta\in C^q(\mathcal{G},W)$, then its component of type $(k,..,k)$ is called the symbol of θ.

Notice that the symbol of θ may be zero.

Proposition [NB4]. *The symbol $\sigma(\theta)$ of every q-cocycle $\theta\in C^{0,q}(\mathcal{G},W)$ is $\delta_q$closed and satisfies the identity*
$$s\sigma(\theta)=0$$
for any arbitrary element $s\in\Gamma(E)$.

We recall that
$$(s(\sigma(\theta))(\xi_1,..,\xi_q)=a(s)(\sigma(\theta)(\xi_1,..,\xi_q))-\sum_{j\le q}\sigma(\theta)(...s\xi_j,..\xi_q).$$

For every non-negative interger r, $H^{r,0}(\mathcal{G},W)=0$. (That phenomenon may be explained by using an appropriate spectral sequence.)

We are going now to relate symbols of so called Koszul-Vinberg cocycle to Poisson manifolds structures.

We will deal with the vector spaces $C^{r,s}(\mathcal{G},W)$ such that $rs=0$. For instance $C^{0,2}(\mathcal{G},W)$ may contain Poisson tensors as well as Jacobi tensors.

On the other hand let us suppose that the Koszul-Vinberg algebroid $E \to M$ has an injective anchor map. Then Riemannian metrics or symplectic structures on the vector bundle $E \to M$ give rise to elements of $C^{2,0}(\mathcal{G}, W)$.

DEFINITION 4.2. (i) A cochain $\theta \in C^2(\mathcal{G}, W)$ is called a Koszul-Vinberg cochain if for arbitrary elements ξ_1, ξ_2, ξ_3 of $\mathcal{G}$ one has

$$(\xi_1, \xi_2, \xi_3)_\theta = (\xi_2, \xi_1, \xi_3)_\theta$$

where

$$(\xi_1, \xi_2, \xi_3)_\theta = \theta(\xi_1, \theta(\xi_2, \xi_3)) - \theta(\theta(\xi_1, \xi_2), \xi_3).$$

(ii) $\theta \in C^2(\mathcal{G}, W)$ is a Koszul-Vinberg cocycle if $\delta\Pi_\theta = \delta.\theta = 0$ and $(\xi_1, \xi_2, \xi_3)_\theta = (\xi_2, \xi_1, \xi_3)_\theta$.

Definition 4.2 makes sense because W may be regarded as a subspace of $\mathcal{G}$.

Every Koszul-Vinberg cochain $\theta \in C^2(\mathcal{G}, W)$ defines a Koszul-Vinberg algebra structure whose multiplication is given by

$$\xi_1\xi_2 = \theta(\xi_1, \xi_2).$$

Therefore we define in $\mathcal{G}$ a new Lie algebra structure called $\mathcal{G}_\theta$, whose bracket is given by

$$[\xi_1, \xi_2]_\theta = \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1).$$

Before continuing we will recall some differential geometry structures related to the cohomology of Koszul-Vinberg algebroids.

DEFINITION 4.3. (i) A Poisson foliation in a manifold M is a foliation $\mathcal{F}$ whose leaves are Poisson manifolds.

(ii) A transversally Poisson foliation in M is a foliation whose sheaf of basic functions is a sheaf of Poisson algebra.

Part (ii) in definition 4.3 has the following meaning: the sheaf of local first integrals of $\mathcal{F}$ admits a Lie algebra bracket

$$(f, g) \to \{f, g\}$$

such that

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

Let us go back to considerations regarding the complex

$$.. \to C^q(\mathcal{G}, W) \xrightarrow{\delta_q} C^{q+1}(\mathcal{G}, W) \to ..$$

which is defined by a Koszul-Vinberg algebroid $E \to M$. The following claim is easily verified [KM]. Let A be an associative and commutative algebra and let $C(A, A)$ be its Hochschild complex. Given any 2-cocycle $\theta \in C^2(A, A)$ and any $\xi \in A$, then the linear map say θ_ξ

$$\zeta \to \theta(\xi, \zeta) - \theta(\zeta, \xi)$$

is a derivation of the algebra A. This elementary result has deep consequences; for example given a smooth manifold M with a start product in $C^\infty(M,\mathbb{R})$, say

$$f * g = fg + \sum_{k>0} h^k B_k(f,g)$$

the bilinear map $B_1 : C^\infty(M,\mathbb{R})^2 \to C^\infty(M,\mathbb{R})$ is a cocycle of the Hochschild complex of $C^\infty(M,\mathbb{R})$. One deduces that B_1 is a bidifferential operator of order 1 whose skew symmetric component defines a Poisson manifold structure on M, [KM]. The same claim doesn't hold in the cohomology theory of Koszul-Vinberg algebras. For instance in a Koszul-Vinberg algebra $\mathcal{A}$ the multiplication map

$$(a,b) \to ab$$

is an exact cocycle of $C(A,A)$, but the linear map

$$b \to ab - ba$$

for a fixed a is a derivation of $\mathcal{A}$ iff $a \in J(\mathcal{A})$. That makes relevant the theorem which is stated below.

Let $E \to M$ be a Koszul-Vinberg algebroid and let $C(\mathcal{G},W)$ be the complex associated to $\mathcal{G} = \Gamma(\mathcal{E})$.

Theorem I [NB_4]. *Let $\theta \in C^{0,2}(\mathcal{G},W)$ be a cocycle of order $\leq k$. If the skew symmetric component of the symbol $\sigma(\theta)$ is non-zero, then $k=1$.*

An important consequence of theorem is the following statement:

Theorem II [NB4]. *The skew symmetric component of the symbol $\sigma(\theta)$ of every Koszul-Vinberg cocycle $\theta \in C^{0,2}(\mathcal{G},W)$ is a Poisson tensor.*

Now let us assume the Koszul-Vinberg algebroid $E \to M$ to be regular. Then E defines a foliation $E_{\mathcal{F}}$ in M. Given any Koszul-Vinberg cocycle $\theta \in C^{0,2}(\mathcal{G},W)$ of order $\leq k$ we denote by Π_θ the skew symmetric component of $\sigma(\theta)$. The following corollary follows directly from theorem II.

Corollary 4.4. *Every germ of submanifold in M which is normal to $\mathcal{F}_E$ is a germ of Poisson submanifold of (M,Π_θ). In particular if $\mathcal{F}_E$ is simple then the quotient manifold $M/\mathcal{F}_E$ admits a Poisson manifold structure $(M/\mathcal{F}_E,\tilde{\Pi}_\theta)$ such that the canonical projection from M to $M/\mathcal{F}_E$ is a Poisson morphism from (M,Π_θ) to $(M/\mathcal{F}_E,\tilde{\Pi}_\theta)$.*

Considering the case of Koszul-Vinberg algebroids with injective anchor maps, we see that such algebroids define locally flat foliations in their base manifolds. Thus we can state the following

Theorem III [NB4]. *Let $E \to M$ be a Koszul-Vinberg algebroid whose anchor map is injective. If E is transitive, then every Koszul-Vinberg cocycle $\theta \in C^{0,2}(\mathcal{G},W)$ defines a regular Poisson structure on M.*

Remark that W being a Koszul-Vinberg submodule of $\mathcal{G}$ every Koszul-Vinberg cochain $\tilde{\theta} \in C^2(\mathcal{G},\mathcal{G})$ induces a Koszul-Vinberg cochain $\theta \in C^{0,2}(\mathcal{G},W)$.

5. The Koszul-Vinberg analogues of star product. Let M be a smooth manifold and let W be the vector space $C^\infty(M,\mathbb{R})$ endowed with its natural structure of associative and commutative algebra.

Given a start product in W, say

$$f * f' = ff' + \sum_{k>0} h^k B_k(f,f')$$

it is well known that the skew symmetric component of B_1 is a Poisson tensor on M, [KM]. Regarding theorem II a natural question arises: does the same phenomenon persist in Koszul-Vinberg algebra structures.

Henceforth we will consider a Koszul-Vinberg algebroid $E \to M$. As before we denote by $\mathcal{G}$ the vector space of smooth sections of the Wihtney sum $E \oplus \mathcal{R}$. We consider the multiplication already defined by (9), i.e. for $\xi = (s,f), \xi' = (s',f')$

$$\xi\xi' = (ss', ff' + < df', a(s) >)$$

where a is the anchor map of E. Let h be some parameter; we will focus on the familly of multiplication in $\mathcal{G}$

$$\xi *_h \xi' = \xi\xi' + \sum_{k>0} h^k \theta_k(f,f') \tag{12}$$

with $\theta_k \in C^2(\mathcal{G},\mathcal{G})$. We suppose the multiplication (12) to satisfy Definition 1.1, viz

$$(\xi_1,\xi_2,\xi_3)*_h = (\xi_2,\xi_1,\xi_3)*_h$$

for elements ξ_1,ξ_2,ξ_3 of $\mathcal{G}$. Thus we obtain a family $\mathcal{G}_h$ of Koszul-Vinberg algebras. The coefficient θ_1 is a cocycle of the complex $C(\mathcal{G},\mathcal{G})$.

Each Koszul-Vinberg algebra $\mathcal{G}_h$ give rise to a Lie algebra whose bracket is given by

$$[\xi,\xi']_h = \xi *_h \xi' - \xi' *_h \xi = [\xi,\xi'] + \sum_{k>0} h^k \Lambda_k(\xi,\xi')$$

with $\Lambda_k(\xi,\xi') = \theta_k(\xi,\xi') - \theta_k(\xi',\xi)$.

In order that the pair $(E \oplus \mathcal{R}, \mathcal{G}_h)$ define a Koszul-Vinberg algebroid with the same anchor map a as the pair $(E \oplus \mathcal{R}, \mathcal{G})$ it is necessary that

$$a([\xi,\xi']_h) = a([\xi,\xi]).$$

Thus we must have

$$a(\sum_{k>0} h^k \Lambda_k(\xi,\xi')) = 0. \tag{13}$$

Therefore we see that for every positive integer k one has $a(\Lambda_k(\xi,\xi')) = 0$. On the other hand recall that W is a two-sided ideal of the Koszul-Vinberg algebra $\mathcal{G}$ whose multiplication is (9). Then the W-component of the cocycle θ_1 is a W-valued 2-cocycle of the cochain complex $C(\mathcal{G},W)$. By assuming that the map a is also the anchor map of the pair

$$(E \oplus \mathcal{R}, \mathcal{G}_h)$$

we deduce from the condition

$$\xi(f *_h \xi') = f(\xi * \xi') + < df, a(\xi) > \xi'$$

that the chains θ_k are of order zero, that is to say that each θ_k is tensorial. This phenomenon is in contrast to the case of star products in the associative and commutative algebra $C^\infty(M,\mathbb{R})$.

To end the present paper we deduce from (13) the following statement.

PROPOSITION 5.1. *Let $E \to M$ be a Koszul-Vinberg algebroid whose anchor map is injective. Suppose that the associated algebroid $E \oplus \mathcal{R}$ admits a one parameter family of deformations $(E \oplus \mathcal{R}, \mathcal{G}_h)$ whose multiplication is*

$$\xi *_h \xi' = \xi\xi' + \sum_{k>0} h^k \theta_k(\xi,\xi').$$

Then the coefficients θ_k are symmetric chains of the cochain complex $C(\mathcal{G},\mathcal{G})$.

References

[GM] M. Gerstenhaber, *Deformation of rings and algebras*, Ann. of Math. 79 (1964), 59–103.

[KM] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Preprint Alg./9709040.

[KJL1] J-L. Koszul, *Homologie des complexes des formes différentielles d'ordre supérieur*, Ann. Scient. Ec. Norm. Sup 7 (1964), 139–159.

[KJL2] J-L. Koszul, *Déformations des connections localement plates*, Ann. Inst. Fourier 18 (1968), 103–114.

[KJL3] J-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in: Elie Cartan et les mathématiques d'aujourd'hui, Soc. Math. de France, Astérisque, 1985, 257–271.

[KI] J. Kubarski, *Bott's vanishing theorem for regular Lie algebroids*, Trans. Amer. Math. Soc. 348 (1996), 2151–2167.

[LP] P. Libermann, *Sur les plongements des fibrés principaux et groupoïdes différentiables*, Séminaire An. globale, Montreal, 1969, 7–108.

[L-M] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.

[MCM] C. M. Marle, *Symplectic manifolds, dynamical systems and hamiltonian mechanics*, in: Differential Geometry and Relativity in Honour to A. Lichnerowicz, Reidel, Dordrecht, 1976.

[NA] A. Nijenhuis, *Sur une classe de propriétés communes à quelques types différents d'algèbres*, Enseign. Math. 14 (1949), 225–277.

[NB1] M. Nguiffo Boyom, *Algèbres à associateur symétrique et algèbres de Lie réductives*, Thèse Doctorat TC-Fac. Scienc. Grenoble, 1968.

[NB2] M. Nguiffo Boyom, *Structures localement plates isotropes des groupes de Lie*, Ann. Sc. Norm. Sup. Pisa 20 (1993), 91–131.

[NB3] M. Nguiffo Boyom, *The homology theory of Koszul-Vinberg algebra*, submitted.

[NB4] M. Nguiffo Boyom, *Cotangent Koszul-Vinberg algebroids and Poisson manifolds*, in preparation.

[NB-W] M. Nguiffo Boyom and R. Wolak, *Affine structure and KV homology*, in preparation.

[PA] A. M. Perea, *Flat left symmetric connections adapted to the automorphism structure of Lie groups*, J. Diff. Geom. 16 (1981), 445–474.

[PW] W. S. Piper, *Algebraic deformation theory*, J. Diff. Geom. 1 (1967), 133–168.

[RM] M. S. Raghunathan, *Deformations of linear connections and Riemannian manifolds*, J. Math. Mech. 13 (1964), 97–123.

[VI1] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progr. Math. 118, Birkhäuser, Berlin, 1994.

[VI2] I. Vaisman, *On the geometry quantization of Poisson manifolds*, J. Math. Physics 32 (1991), 3339–3345.

[VJ] J. Vey, *Déformations du crochet de Poisson sur une variété symplectique*, Comm. Math. Helv. 50 (1975), 421–454.

[WA] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom. 18 (1983), 523–557.