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ON THE GROUP OF LAGRANGIAN BISECTIONS OF A SYMPLECTIC GROUPOID

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Abstract. The group of lagrangian bisections of a symplectic groupoid extends the concept of the symplectomorphism group. The flux homomorphism is a basic invariant of this group. It is shown that this group is a regular Lie group. The group of exact (hamiltonian) bisections is also studied. The existence of the flux homomorphism enables a characterization of exact isotopies.

1. Introduction. The aim of this note is to prove that bisection groups related to a symplectic groupoid admit a structure of regular Lie group. In view of this fact we show the existence of the flux homomorphism and some further consequences, e.g. concerning exact bisection isotopies, quite similar to properties of the symplectomorphism group. We indicate also the integrability of some Lie algebras associated with the Poisson structure on the space of units of a symplectic groupoid.

Observe that the flux homomorphism for symplectic groupoids and some properties of the group of lagrangian bisections have been studied independently by P. Dazord [5] and P. Xu [22] without considering a Lie group structure on this group but rather exploiting some "weak" Lie theories (see below). However, by means of a chart at e we are able to obtain also the local form of the flux, which is essential for a characterization of exact isotopies.

We will use the definition of a Lie group from the convenient setting of the infinite dimensional Lie theory [8] due to A. Kriegl and P. Michor. The convenient setting is based on Boman's theorem which states that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is smooth whenever $f \circ c$ is smooth for any smooth curve $c : \mathbb{R} \to \mathbb{R}^n$. Consequently, a mapping between two possibly infinite dimensional manifolds is smooth if, by definition, it sends smooth

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curves to smooth curves. Furthermore, for modelling manifolds a special type of LCTVS is in use, namely so-called convenient vector spaces which fulfil some weak completeness condition.

Recall that a convenient Lie group is called *regular* if for $\mathfrak{g} = T_eG$ there exists a smooth bijective evolution map

$$\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty((\mathbb{R}, 0), (G, e)).$$

The right logarithmic derivative δ^r_G is then the inverse of evol^r_G. Frequently one encounters the situation that there is a closed subgroup $H \subset G$ and a Lie algebra $\mathfrak{h}$ such that smooth curves with values in $\mathfrak{h}$ are sent bijectively by evol^r_G to isotopies with values in H. However we would like to emphasize that such a bijection does not yield a Lie group structure on H.

It is a characteristic feature of the convenient setting that $\mathrm{Diff}(M)$ for M open is still a regular Lie group, but its identity component consists only of compactly supported diffeomorphisms. So in our results the assumption on the compactness of the space of units (cf. [22]) is superfluous.

We would like also to indicate that there are some "weak" settings of the infinite dimensional Lie theory and that they usually do not correspond strictly to each other. One example is the notion of diffeological groups due to J. M. Souriau [20]. A smooth structure is there defined by establishing sets of local smooth mappings from $\mathbb{R}^n$ to G, $n = 1, 2, \dots$, and by imposing some conditions on them. As another example of a "weak" setting can serve the concept of generalized Lie groups of H. Omori [11]. The definition is based on a continuous mapping $\exp : G \to \mathfrak{g}$ between a topological group G and a topological Lie algebra $\mathfrak{g}$ with some conditions which mimics essential properties of the exponential map. A common feature of such theories is that any closed subgroup of a Lie group is a Lie subgroup (which in a sense measures a lack of subtlety of them). Consequently, analogues of the presented results in those settings are rather trivial.

2. Preliminaries. We adopt the notation for groupoids from [3] rather than from [9].

Definition. A *groupoid structure* on a set Γ is given by two surjections (the source and target) $\alpha, \beta : \Gamma \to M \subset \Gamma$, by a multiplication $m : \Gamma_2 \to \Gamma$, where $\Gamma_2 = \{(x, y) \in \Gamma \times \Gamma : \alpha(x) = \beta(y)\}$, and by an inversion $i : \Gamma \to \Gamma$ such that the following axioms are fulfilled:

(Ass) If one of the products $m(x, m(y, z))$ and $m(m(x, y), z)$ is defined then so is the other and they are equal.

(Id) The products $m(\beta(x), x)$, $m(x, \alpha(x))$ are defined and equal to each other.

(Inv) $m(x, i(x))$ is defined and equal to $\beta(x)$, and $m(i(x), x)$ is defined and equal to $\alpha(x)$.

The elements of M are called *unities*. For simplicity we write $x.y$ for $m(x, y)$ and x^{-1} for $i(x)$. We wil use the symbol $\Gamma \rightrightarrows M$ for the groupoid $(\Gamma, M, \alpha, \beta, m, i)$.

Next, a groupoid Γ is said to be a Lie groupoid if Γ is a smooth (C^∞) manifold (not necessarily separated), M is a separated paracompact submanifold, α and β are submersions, m is a smooth mapping, and i is a diffeomorphism. Notice that Γ is separated iff M is closed in Γ.

For $u \in M$ the set $\alpha(\beta^{-1}(u)) = \beta(\alpha^{-1}(u))$ is an orbit. The family of orbits forms a generalized foliation $\mathcal{F}_\Gamma$ of M (cf.[3]). Γ is called *transitive* if it has only one orbit M.

EXAMPLES. 1. Lie groups coincide with Lie groupoids with a unique unity.

2. Another extreme example are manifolds: $\Gamma = M$.

3. If $\alpha = \beta$ then for all $u \in M$ the fiber $\alpha^{-1}(u)$ carries a Lie group structure. Any vector bundle is a Lie groupoid of this type.

4. For any set M put $\Gamma = M \times M$, $\alpha((x,y)) = y$, $\beta((x,y)) = x$, $m((z,y),(y,x)) = (z,x)$ and $i((x,y)) = (y,x)$. We get the coarse groupoid with the space of units $M \simeq \Delta_M$.

5. Given a principal fiber bundle $P(M,\pi,G)$ one defines the equivalence relation $\sim$ on $P \times P$ by $(p_1,p_2) \sim (q_1,q_2)$ iff $\exists a \in G\, p_i a = q_i$. Letting denote $\Gamma = P \times P/\sim$, $\alpha([(p_1,p_2)]) = \pi(p_2)$, $\beta([(p_1,p_2)]) = \pi(p_1)$, we get the gauge groupoid (with obvious m and i). Γ is identified with the set of equivariant bundle morphisms over id_M.

6. Assume that a Lie group G acts on a manifold M. Then we set: $\Gamma = G \times M$, $\alpha((g,x)) = g.x$, $\beta((g,x)) = x$, $(g',x').(g,x) = (g'g,x)$, and $(g,x)^{-1} = (g^{-1}, g.x)$. We say that Γ is a transformation groupoid.

7. For any Lie groupoid Γ the tangent space $T\Gamma = (T\Gamma \rightrightarrows TM, T\alpha, T\beta, \oplus, I)$ possesses a structure of Lie groupoid. Here the multiplication $\oplus$ is given by

$$X \oplus Y = \left(\frac{\mathrm{d}}{\mathrm{d}t}(x(t).y(t))\right)\Big|_{t=0},$$

where $X = \frac{\mathrm{d}x}{\mathrm{d}t}|_{t=0}$, $Y = \frac{\mathrm{d}y}{\mathrm{d}t}|_{t=0}$, $\alpha(x(t)) = \beta(y(t))$, and the inversion $IX = \frac{\mathrm{d}x}{\mathrm{d}t}x(t)^{-1}|_{t=0}$ if $X = \frac{\mathrm{d}x}{\mathrm{d}t}|_{t=0}$.

A *bisection* of a Lie groupoid Γ is a submanifold B of Γ such that $\alpha|B$ and $\beta|B$ are diffeomorphisms onto M. Let $\mathrm{Bis}(\Gamma)$ be the set of all bisections. It is a group endowed with the product law

$$B_1.B_2 = \{x_1.x_2 | \alpha(x_1) = \beta(x_2)\}.$$

Notice that bisections of the coarse groupoid $\Gamma = M \times M$ (Ex. 4) coincide with diffeomorphisms on M, i.e. groups of bisections constitute a generalization of diffeomorphism groups. $\mathrm{Bis}(\Gamma)$ has natural left and right representations in Γ given by

$$\psi^l : \mathrm{Bis}(\Gamma) \ni B \mapsto \psi^l(B) := \{x \mapsto B.x\} \in \mathrm{Diff}(\Gamma),$$
$$\psi^r : \mathrm{Bis}(\Gamma) \ni B \mapsto \psi^r(B) := \{x \mapsto x.B\} \in \mathrm{Diff}(\Gamma).$$

Next there are the left and right representations in the unit space $(M, \mathcal{F}_\Gamma)$

$$\phi^l : \mathrm{Bis}(\Gamma) \ni B \mapsto \phi^l(B) := \beta \circ \psi^l(B)|_M \in \mathrm{Diff}(M, \mathcal{F}_\Gamma),$$
$$\phi^r : \mathrm{Bis}(\Gamma) \ni B \mapsto \phi^r(B) := \alpha \circ \psi^r(B)|_M \in \mathrm{Diff}(M, \mathcal{F}_\Gamma),$$

where $\mathrm{Diff}(M, \mathcal{F}_\Gamma)$ is the group of leaf preserving diffeomorphisms. $\mathrm{Bis}(\Gamma)_c$ will stand for the subgroup of all compactly controlled elements, that is, all B such that $\phi^l(B)$, or equivalently $\phi^r(B)$, has compact support. In general, compactly controlled bisections need not have compact support, e.g. in Example 3.

It is well known (J. Pradines [14], [15]) that to any Lie groupoid $\Gamma \rightrightarrows M$ is assigned the associated algebroid $\mathcal{A}(\Gamma)$, namely $\mathcal{A}(\Gamma) = (\ker T\beta, [[,]], T\alpha)$, where $[[,]]$ is a Lie algebra bracket on $\mathrm{Sect}(\ker T\beta)$ introduced by means of left invariant vector fields and of the identification $T\Gamma|_M/TM \simeq \ker T\beta$.

THEOREM 2.1. *The groups* $\mathrm{Bis}(\Gamma)$ *and* $\mathrm{Bis}(\Gamma)_c$ *are regular Lie groups with the same Lie algebra* $\mathrm{Sect}_c(\ker T\beta)$.

The proof follows that for diffeomorphisms in [8], and makes use of the Tubular Neighborhood Theorem and the identification $T\Gamma|_M/TM \simeq \ker T\beta$. The topology of $\mathrm{Bis}(\Gamma)$ is the identification topology by charts of a Lie group structure of $\mathrm{Bis}(\Gamma)$. In particular, all bisections in the identity component are compactly controlled.

To show the regularity of $\mathrm{Bis}(\Gamma)$ for any $u \in M$ and $B \in \mathrm{Bis}(\Gamma)$ we denote by $B^\beta(u)$ the unique point in B such that $\beta(B^\beta(u)) = u$. Now given a smooth isotopy B_t in $\mathrm{Bis}(\Gamma)$ (which is a concept without problem) with $B_0 = M$ there is a unique time-dependent family of vector fields $\hat{X}_t$ along B_t^β corresponding to B_t, i.e. for all $u \in M$

$$\hat{X}_t(B_t^\beta(u)) = \frac{d}{ds}B_s^\beta(u)|_{s=t}. \tag{2.1}$$

By definition, $\hat{X}_t$ are tangent to the fibers of β.

Let $u \in M$ and $B \in \mathrm{Bis}(\Gamma)$. We have a diffeomorphism $\sigma_u^B : \beta^{-1}(u) \to \beta^{-1}(u)$ given by

$$\sigma_u^B(x) = x.B^\beta(\alpha(x)).$$

Then clearly $\sigma_u^B(u) = B^\beta(u)$. By gluing-up the tangent mappings $T\sigma_u^B$ of diffeomorphisms σ_u^B we get the canonical identification

$$\sigma^B : \ker T\beta \simeq T\Gamma|_M/TM \simeq T\Gamma|_B/TB. \tag{2.2}$$

We have also the canonical identification $TB^\beta : TM \simeq TB$. By combining it with (2.2) we get

$$\tilde{\sigma}^B : T\Gamma|_M \simeq T\Gamma|_B, \tag{2.3}$$

for any $B \in \mathrm{Bis}(\Gamma)$. Now by using the identifications (2.2) for any smooth bisection isotopy B_t with $B_0 = M$ we get a unique smooth curve X_t in $\mathrm{Sect}_c(\ker T\beta)$ such that

$$\sigma^{B_t}X_t = \hat{X}_t. \tag{2.4}$$

Then $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)}(X_t) = B_t$, and $\delta^r_{\mathrm{Bis}(\Gamma)}(B_t) = X_t$. Clearly $\hat{X}_t = X_t = \tilde{X}_t$ on M.

3. Symplectic groupoids and algebroids

DEFINITION. A Lie groupoid Γ equipped with a symplectic form ω is called *symplectic* if the graph of multiplication $\mathrm{graph}(m)$ is a lagrangian submanifold of $(-\Gamma)\times\Gamma\times\Gamma$. Here $-\Gamma$ means the symplectic manifold $(\Gamma, -\omega)$.

Let us recall that a Poisson structure on M can be introduced by a bivector Λ such that $[\Lambda,\Lambda] = 0$, where $[.,.]$ is the Schouten-Nijenhuis bracket (cf. [19]). Then the rank of Λ_x may vary but it is even everywhere. We have the 'musical' bundle homomorphism $\Lambda^\sharp$ associated with Λ by

$$\Lambda^\sharp : T^*M \to TM, \quad \beta(\Lambda^\sharp\alpha) = \Lambda(\alpha,\beta).$$

In case Λ is nondegenerate (i.e. $\mathrm{rank}(\Lambda) = \dim(M)$), we get a symplectic structure ω, and $\Lambda^\sharp$ is an isomorphism, denoted by $\omega^\sharp$. The distribution $\Lambda^\sharp(T^*_xM),\ x \in M$, integrates to a generalized foliation such that Λ restricted to any leaf induces a symplectic structure. This foliation is called symplectic and denoted by $\mathcal{F}_\Lambda$. If the dimension of its leaves is constant, the Poisson structure Λ is called regular.

PROPOSITION 3.1 [3]. *If (Γ, ω) is a symplectic groupoid then:*

(i) the inversion i is an antisymplectomorphism (i.e. $i^*\omega = -\omega$), and M is a lagrangian submanifold;

(ii) the foliations by fibers of α and β are ω-orthogonal;

(iii) the space of units M admits a Poisson structure Λ such that its symplectic foliation $\mathcal{F}_\Lambda$ coincides with $\mathcal{F}_\Gamma$.

(iv) α (resp. β) is a Poisson morphism (resp. anti-morphism).

Such a groupoid will be usually denoted by $(\Gamma, \omega) \rightrightarrows (M, \Lambda)$.

Observe that the set of all lagrangian bisections $\mathrm{Bis}(\Gamma, \omega)$ is a sugroup of $\mathrm{Bis}(\Gamma)$. This group has natural left and right representations in (Γ, ω):

$$\psi^l : \mathrm{Bis}(\Gamma, \omega) \ni C \mapsto \psi^l(C) = \{x \mapsto C.x\} \in \mathrm{Symp}(\Gamma, \omega),$$
$$\psi^r : \mathrm{Bis}(\Gamma, \omega) \ni C \mapsto \psi^r(C) = \{x \mapsto x.C\} \in \mathrm{Symp}(\Gamma, \omega).$$

Now the corresponding representations ϕ^l and ϕ^r take their values in $\mathrm{Diff}(M, \Lambda)$, the automorphism group of (M, Λ).

EXAMPLES. 8. The coarse groupoid $\Gamma = X \times X$ with $(-\omega) \oplus \omega$ is a symplectic groupoid. Then $\mathrm{Bis}(\Gamma, \omega) = \mathrm{Symp}(X)$.

9. If M is a manifold then T^*M, where m is the addition in fibers and $\pi_M = \alpha = \beta$, is a Lie groupoid (Ex. 3). T^*M endowed with the canonical symplectic form $\omega_M = -\mathrm{d}\lambda_M$ is also a symplectic groupoid. In fact, the graph of m

$$\mathrm{graph}(m) = \{(x_3, x_2, x_1) : \, x_1 + x_2 - x_3 = 0\}.$$

This is the image of $\mathcal{N}\Delta_M$, the normal bundle of the diagonal $\Delta_M \subset M^3$ in $(T^*M)^3$, into $(T^*M)^3$ by the mapping $(x_3, x_2, x_1) \mapsto (-x_3, x_2, x_1)$. Notice that $\mathcal{N}\Delta_M$ is lagrangian in $(T^*M)^3$, and the mapping is symplectic. So T^*M is indeed a symplectic groupoid.

10. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Then the cotangent space $T^*\Gamma$ equipped with $\omega_\Gamma = -\mathrm{d}\lambda_\Gamma$ carries a structure of symplectic groupoid with $\mathcal{N}^*M$, the conormal bundle of M in Γ, being the space of units. Here the multiplication, denoted $\oplus$ as in Ex. 7, is determined by the equality

$$< \xi \oplus \eta, X \oplus Y > = < \xi, X > + < \eta, Y >, \quad \text{for} \quad X, Y \in T\Gamma,\ \xi, \eta \in T^*\Gamma,$$

where $<,>$ is the canonical pairing. Furthermore, the canonical projection $p : T^*\Gamma \to \Gamma$ is an epimorphism of groupoids.

11. If G is a Lie group, T^*G admits two symplectic groupoid structures. The first one is given as above, and the second is the structure of transformation groupoid (Ex. 6), where G acts on g^* by the coadjoint action. As these structures obey a compatibility condition, T^*G carries a structure of *double* groupoid, cf. [3].

Let us recall the following concept:

Definition. A Lie algebroid $(T^*M, \{,\}, \rho)$ over M is called *symplectic* if the following conditions are fulfilled:

(i) the (2,0)-tensor Λ given by $\Lambda(\alpha,\beta) = \beta(\rho(\alpha))$, $\forall \alpha, \beta \in \Omega^1_c(M)$, is skew-symmetric;
(ii) the space of all closed forms is a Lie subalgebra of $(\mathrm{Sect}(T^*M), \{,\})$.

Clearly there is a one-to-one correspondence between symplectic algebroids over M and Poisson structures on M. Here $\rho = \Lambda^\sharp$ and

$$\{\alpha,\beta\} = \imath_{\Lambda^\sharp(\alpha)}\mathrm{d}\beta - \imath_{\Lambda^\sharp(\beta)}\mathrm{d}\alpha + \mathrm{d}\Lambda(\alpha,\beta).$$

Proposition 3.2. *If $(\Gamma,\omega) \rightrightarrows (M,\Lambda)$ is a symplectic groupoid then its associated algebroid $\mathcal{A}(\Gamma)$ is identified with $(T^*M, \{,\}, \Lambda^\sharp)$, the symplectic algebroid of (M,Λ). In particular, $T^*M \simeq T\Gamma|_M/TM \simeq \ker T\beta$.*

It is remarkable that any Poisson manifold can be represented as the space of units of a local symplectic groupoid, and that Proposition 3.2 still holds for local symplectic groupoids. Consequently, there is a bijection between symplectic algebroids and local symplectic groupoids [4], a local non-linear version of the third Lie theorem.

4. $\mathrm{Bis}(\Gamma,\omega)$ **as a Lie group.** The construction of a chart for $\mathrm{Bis}(\Gamma,\omega)$ at $e = M$ starts by an observation that there are no topological restrictions in a small neighborhood of M in Γ.

Lemma 4.1 [3]. *Let N be a not necessarily separated manifold, and let $M \subset N$ be a separated paracompact manifold such that a submersion $p : N \to M$ exists. Then there is a neighborhood U of M in N which is separated and with p-connected fibers.*

By a local addition we mean a diffeomorphism $\mu : T^*M \supset U \to V \subset \Gamma$ such that $\mu(0_u) = u$, $\forall u \in M$. In our case the existence of a local addition is ensured by the identification $T\Gamma|_M/TM \simeq T^*M$, and by the exponential mapping coming from a Riemannian metric.

Lemma 4.2. *Let $M \subset N$ be a closed submanifold, and let ω_0, ω_1 be symplectic forms on N which are equal along M. Then there is a diffeomorphism $\phi : U \to V$, where U, V are open neighborhoods of M in N such that $\phi^*\omega_1 = \omega_0$, $\phi|_M = \mathrm{id}_M$ and $\phi_*|_{TN|_M} = \mathrm{id}_{TN|_M}$.*

The proof uses Moser's argument and the relative Poincaré lemma. For details, see [8, 43.11].

Any $\theta \in \Omega^1(M)$ can be regarded as a section $\theta : M \to T^*M$, and T^*M is endowed with $\omega_M = -\mathrm{d}\lambda_M$, where λ_M is the canonical 1-form on T^*M. The following is well-known.

Proposition 4.3. *Under the above identification, $\theta^*\lambda_M = \theta$. Moreover, $\theta(M)$ is lagrangian in T^*M iff $\mathrm{d}\theta = 0$.*

For the regularity of $\mathrm{Bis}(\Gamma,\omega)$ the following is needed.

Lemma 4.4 [8, 38.7]. *Let H be a topological Lie subgroup of a regular Lie group G. If there are an open neighborhood $U \subset G$ of e and a smooth mapping $p : U \to E$, where E is a convenient vector space, such that $p^{-1}(0) = U \cap H$ and p is constant on left cosets $Hg \cap U$, then H is regular.*

THEOREM 4.5. *Given a symplectic groupoid* $(\Gamma,\omega)\rightrightarrows(M,\Lambda)$*, the group* $\mathrm{Bis}(\Gamma,\omega)$ *is a closed subgroup of* $\mathrm{Bis}(\Gamma)$ *and a regular Lie group. Its Lie algebra coincides with* $Z\Omega^1_c(M)$*, the subalgebra of closed 1-forms on* M*.*

Proof. (See also [8], 43.12.) We fix a local addition $\mu: T^*M\supset U_0\to V_0\subset\Gamma$. We have two symplectic structures on U_0, namely the canonical one $\omega_0=\omega_M|_U$, where $\omega_M=-\mathrm{d}\lambda_M$, and $\omega_1=\mu^*\omega$. Each of them has the vanishing pullback on the zero section 0_{T^*M}. In view of Lemma 4.2 we have to show that, by shrinking U_0 and V_0 and modifying the ω_i, we may get $\omega_0|_{0_{T^*M}}=\omega_1|_{0_{T^*M}}$.

To this end we observe that there is a vector bundle isomorphism $\psi_0: T(T^*M)|_{0_{T^*M}}\to T(T^*M)|_{0_{T^*M}}$ over $\mathrm{id}_{0_{T^*M}}$ such that $\psi_0=\mathrm{id}_{T(0_{T^*M})}$ and ψ_0 sends ω_0 to ω_1 on each fiber. Due to the partition of unity it suffices to construct ψ_0 locally, and this is accomplished by considering lagrangian subbundles L_i complementary to $T(0_{T^*M})$ with respect to ω_i. When having ψ_0 we define a diffeomorphism $\psi: U_1\to U_2$, where U_1, U_2 are open neighborhoods of 0_{T^*M} in T^*M, such that $T\psi|_{0_{T^*M}}=\psi_0$. This is done by using a tubular neighborhood of 0_{T^*M}.

Now, in view of 4.2, we get a diffeomorphism ϕ of an open neighborhood of 0_{T^*M} in T^*M onto another such a neighborhood which satisfies $\phi^*\omega_1=\omega_0$, $\phi|_M=\mathrm{id}_M$ and $\phi_*|_{T(T^*M)|_{0_{T^*M}}}=\mathrm{id}_{T(T^*M)|_{0_{T^*M}}}$. We set

$$\rho:=\mu\circ\phi: T^*M\supset U\to V\subset\Gamma.$$

Let $\mathcal{U}$ be a neighborhood of $e=M$ in $\mathrm{Bis}(\Gamma)$ consisting of all submanifolds $B\subset\Gamma$ such that $B^\beta: M\to\Gamma$ is compactly supported, $B^\beta(M)\subset V$, and so small that $\mu^{-1}(B)$ is still the image of a β-section. We define a chart at $e=M$ as follows

$$\Phi:\mathrm{Bis}(\Gamma)\supset\mathcal{U}\to\mathcal{V}\subset\Omega^1_c(M)$$
$$\Phi(B):=\rho^{-1}\circ B^\beta\circ(\pi\circ\rho^{-1}\circ B^\beta)^{-1}. \tag{4.1}$$

Due to Proposition 4.3 $B\in\mathcal{U}\cap\mathrm{Bis}(\Gamma,\omega)$ if and only if $\mathrm{d}\Phi(B)=0$. Therefore $(\mathcal{U},\Phi)$ is a submanifold chart at $e=M$ for $\mathrm{Bis}(\Gamma,\omega)$ modelled on the subspace $Z\Omega^1_c(M)$ of all closed forms on M with compact support.

Next for arbitrary $C\in\mathrm{Bis}(\Gamma,\omega)$ we get a submanifold chart at C as follows: $\mathcal{U}_C:=\{B: B.C^{-1}\in\mathcal{U}\}$ and $\Phi_C(B):=\Phi(B.C^{-1})$. Thus $\mathrm{Bis}(\Gamma,\omega)$ is a closed submanifold of $\mathrm{Bis}(\Gamma)$ and a Lie group.

Let C_t be a smooth isotopy in $\mathrm{Bis}(\Gamma,\omega)$ and $X_t\in\mathrm{Sect}_c(\ker T\beta)$ be defined by (3.4). By definition there is a time-dependent 1-form θ_t on M such that

$$\imath(X_t)\omega|_{T\Gamma|_M}=\theta_t. \tag{4.2}$$

Now if we set

$$c_t:=\psi^r(C_t)\quad\text{and}\quad\tilde{X}_t:=\delta^r_{\mathrm{Symp}(\Gamma,\omega)}(c_t),$$

the smooth curve of vector fields $\tilde{X}_t$ on Γ satisfies (cf. (3.3) and (3.4)) $\tilde{\sigma}^{C_t}X_t=\tilde{X}_t$. Clearly, $\tilde{X}_t$ are tangent to the β-fibers, and c_t are β-fibers preserving. Also, $c_t|_M=C_t^\beta$, $\forall t$.

We get the equality

$$\imath(\tilde{X}_t)\omega=\beta^*\theta_t \tag{4.3}$$

on the whole Γ. Consequently, θ_t is closed, as $\beta^*\theta_t$ is closed.

Therefore, for any $B_t \in C^\infty((\mathbb{R},0),(\mathrm{Bis}(\Gamma),e))$ and $X_t = \mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\omega)}(B_t)$ we get by the regularity of $\mathrm{Symp}(\Gamma,\omega)$, and by Prop. 3.2, the equivalence

$$B_t \in C^\infty((\mathbb{R},0),(\mathrm{Bis}(\Gamma,\omega),e)) \quad \Leftrightarrow \quad \theta_t = \omega^\sharp X_t \in C^\infty(\mathbb{R}, Z\Omega^1_c(M)),$$

which ensures that the Lie algebra of $\mathrm{Bis}(\Gamma,\omega)$ is $Z\Omega^1_c(M)$, that is, that the restriction of $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)}$ to $Z\Omega^1_c(M)$ identifies with $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\omega)}$.

Finally, let us consider $p : \mathrm{Bis}(\Gamma) \to \Omega^2(\Gamma)$ given by $p(B) = \psi^r(B)^*\omega - \omega$. It follows from Lemma 4.4 that $\mathrm{Bis}(\Gamma,\omega)$ is regular. ∎

Since the neighborhood $\Phi(\mathcal{U})$ above can be chosen convex we can state the following

Corollary 4.6. *The group* $\mathrm{Bis}(\Gamma)$ *is locally contractible and, consequently, locally arcwise connected.*

Remark. Plausibly the monomorphisms ψ^l and ψ^r enable to introduce a Lie group structure on $\mathrm{Bis}(\Gamma,\omega)$ in another way, namely as a Lie subgroup of $\mathrm{Symp}(\Gamma,\omega)$. This would make easier all reasonings concerning the flux and lagrangian bisection isotopies. However, this is impossible unless Γ is compact. The reason is that no isotopy of the form $c_t = \psi^l_t(C_t)$ or $c_t = \psi^r_t(C_t)$ is contained in $\mathrm{Symp}(\Gamma,\omega)_0$ (in our framework). Nonetheless, these representations are indispensable in section 5 and 6.

5. The flux homomorphism. The concept of the flux homomorphism was introduced by E. Calabi [2]. It is a basic invariant not only of the symplectomorphism groups, but also of some other transformation groups, cf. [17], [18]. The flux is still meaningful for the group of Lagrangian bisections (cf. [5], [22]). In light of Theorem 4.5 one can obtain the flux also in its local form which is necessary for a characterization of exact isotopies.

Let us fix the notation. For any locally arcwise connected topological group G its universal cover group $\tilde{G}$ is the set of all pairs $(g,\{g_t\})$, where $\{g_t\}$ is the homotopy rel. endpoints class of the path g_t in G such that $g_0 = e$ and $g_1 = g$, endowed with the pointwise multiplication. By G_0 we denote the component of e in G. Clearly $\tilde{G}_0/\pi_1(G_0) = G_0$, where π_1 is the first homotopy group.

The multiplication in $\tilde{G}$ can be also thought of as the juxtaposition of representants. The latter means that $\{g_t\}.\{f_t\} = \{g_t * f_t\}$ where

$$g_t * f_t = \begin{cases} f_{2t}, & \text{for } 0 \le t \le \frac{1}{2} \\ g_{2t-1}\circ f_1, & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

These multiplications are the same on the homotopy level.

Given a symplectic groupoid $(\Gamma,\omega) \rightrightarrows (M,\Lambda)$, by using (4.2) and (4.3) we have the mapping

$$C^\infty(\mathbb{R},\mathrm{Bis}(\Gamma,\omega)) \ni C_t \mapsto \theta_t \in C^\infty(\mathbb{R}, Z\Omega^1_c(M))$$

which induces the flux homomorphism.

Theorem 5.1. *The mapping* $\tilde{\mathcal{S}}_\Gamma : \widetilde{\mathrm{Bis}(\Gamma,\omega)}_0 \to H^1_c(M)$ *defined by*

$$\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = \left[\int_0^1 \theta_t dt\right]$$

is a well-defined continuous epimorphism (with the abelian structure on $H^1(M)$), called the flux homomorphism for the symplectic groupoid Γ. If Γ, in addition, is compact then the relation between $\tilde{\mathcal{S}}_\Gamma$ and $\tilde{\mathcal{S}}$, the flux for the symplectomorphism group $\mathrm{Symp}(\Gamma,\omega)$, *can be visualized in the following commutative diagram*

$$\begin{array}{ccc}
\widetilde{\mathrm{Bis}(\Gamma,\omega)}_0 & \xrightarrow{\psi^r} & \widetilde{\mathrm{Symp}(\Gamma,\omega)}_0 \\
{\scriptstyle \tilde{\mathcal{S}}_\Gamma}\downarrow & & \downarrow{\scriptstyle \tilde{\mathcal{S}}} \\
H^1_c(M) & \xrightarrow{\beta^*} & H^1_c(\Gamma)
\end{array}$$

The proof is similar to that for symplectomorphisms. We emphasize the existence of $\tilde{\mathcal{S}}_\Gamma$ for an arbitrary symplectic groupoid (not necessarily with M compact as in [22]). However, the diagram makes sense for Γ compact only, since otherwise no $c_t = \psi^r(C_t)$ lies in $\mathrm{Symp}(\Gamma,\omega)_0$.

EXAMPLES. 1. If $\Gamma = M \times M$ is the coarse groupoid then $\tilde{\mathcal{S}}_\Gamma$ is the usual flux for $\mathrm{Symp}(M,\omega)$.

2. If $\Gamma = T^*M$ then $\mathrm{Bis}(\Gamma)$ identifies with the space of all closed 1-forms, and $\tilde{\mathcal{S}}_\Gamma$ assigns to 1-form its cohomology class.

In analogy with the symplectic case (cf. [2,10]) Theorem 4.5 enables us to define the local flux homomorphism.

COROLLARY 5.2. *If ω is exact, i.e. $\omega = -\mathrm{d}\lambda$, then $\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = [\lambda|_{TM} - (C_1^\beta)^*\lambda] = [(\lambda - c_1^*\lambda)|_{TM}]$, where $c_t = \psi^r(C_t)$. In particular, the group of periods $\Xi_\Gamma := \tilde{\mathcal{S}}_\Gamma(\pi_1(\mathrm{Bis}(\Gamma,\omega)_0))$ is zero.*

Proof. Let $X_t = \mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\omega)}(C_t)$, $X_t = \omega^\sharp\theta_t$, $c_t = \psi^r(C_t)$, and $\tilde{X}_t := \delta^r_{\mathrm{Symp}(\Gamma,\omega)}(c_t)$. By composing each side of (4.2) with β^* we get and

$$\begin{aligned}
\beta^*\theta_t = c_t^*\beta^*\theta_t = c_t^*\beta^*\imath_{X_t}\omega|_{T\Gamma|_M} &= c_t^*\imath_{\tilde{X}_t}\omega \\
= -c_t^*\imath_{\tilde{X}_t}\mathrm{d}\lambda = -c_t^* L_{\tilde{X}_t}\lambda &= -\frac{\mathrm{d}}{\mathrm{d}t}c_t^*\lambda.
\end{aligned}$$

Hence

$$\int_0^1 \beta^*\theta_t = \lambda - c_1^*\lambda$$

holds on Γ. By restricting this equality to TM we get

$$\int_0^1 \theta_t = (\lambda - c_1^*\lambda)|_{TM},$$

as $\beta^*\theta|_{TM} = \theta$. In view of $C_t^\beta = c_t|_M$ this yields the required equalities. ∎

DEFINITION. Let $\Phi : \mathcal{U} \to \mathcal{V}$ be a chart for $\mathrm{Bis}(\Gamma,\omega)$ at $e = M$. If C_t is a lagrangian isotopy such that $C_t \in \mathcal{U}$ for any t then

$$\tilde{\mathcal{S}}^\Phi_\Gamma(\{C_t\}) := -[\theta_1] \in H^1_c(M),$$

where $\theta_t = \Phi(C_t)$. $\tilde{\mathcal{S}}^\Phi_\Gamma$ is called the *local flux homomorphism* for $\mathrm{Bis}(\Gamma,\omega)$.

PROPOSITION 5.3. *$\tilde{\mathcal{S}}^\Phi_\Gamma$ does not depend on the choice of Φ and extends $\tilde{\mathcal{S}}_\Gamma$.*

Proof. Let C_t be a lagrangian bisection isotopy. Then θ_t is a lagrangian bisection isotopy in $U \subset T^*M$ with respect to the canonical symplectic form $\omega_M = -\mathrm{d}\lambda_M$. Let $\tilde{\mathcal{S}}_M$ be the flux homomorphism for the symplectic groupoid T^*M (Ex. 9). Then

$$\tilde{\mathcal{S}}_M(\{\theta_t\}) = \tilde{\mathcal{S}}_M(\{\Phi(C_t)\}) = \phi^*_{C_t}\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = \tilde{\mathcal{S}}_\Gamma(\{C_t\}),$$

where $\phi_{C_t} = \pi_M \circ \rho \circ C^\beta_t$, by the homotopy invariance. But in view of Cor. 5.2 and Prop. 4.3 we get

$$\tilde{\mathcal{S}}_M(\{\theta_t\}) = [\lambda_M|_{T(0_{T^*M})} - \theta_1^*\lambda_M] = [0^*_{T^*M}\lambda_M - \theta_1^*\lambda_M] = -[\theta_1],$$

as required. ■

6. A characterization of exact isotopies. A lagrangian bisection isotopy C_t is called *exact* if $(C_t) = \omega^\sharp \mathrm{d}u_t$ for some smooth curve u_t in $C^\infty_c(M)$. Let $\mathrm{Bis}^{ex}(\Gamma,\omega)$ be the set of all exact lagrangian bisections. That is, by definition $C \in \mathrm{Bis}^{ex}(\Gamma,\omega)$ iff it can be joined with $e = M$ by an exact lagrangian bisection isotopy.

PROPOSITION 6.1. *A lagrangian isotopy C_t is exact iff $\delta^r_{\mathrm{Symp}(\Gamma,\omega)}(c_t) = \omega^\sharp\beta^*(\mathrm{d}u_t)$, where $c_t = \psi^r(C_t)$.*

Indeed, it follows from the fact that β is a Poisson anti-morphism.

PROPOSITION 6.2. $\mathrm{Bis}^{ex}(\Gamma,\omega)$ *is a path-connected normal subroup of* $\mathrm{Bis}(\Gamma,\omega)$.

Proof. First we check that $\mathrm{Bis}^{ex}(\Gamma,\omega))$ is a group. Let B_t, C_t be exact isotopies, that is $(B_t) = \omega^\sharp(\mathrm{d}v_t)$, $(C_t) = \omega^\sharp(\mathrm{d}u_t)$ for some smooth families of C^∞-functions u_t and v_t. If $\psi^r(B_t) = b_t$, and $\psi^r(C_t) = c_t$ then $\psi^r(B_t.C_t) = c_t \circ b_t$. Is is apparent that

$$\delta^r_{\mathrm{Symp}(\Gamma,\omega)}(c_t \circ b_t) = \omega^\sharp\mathrm{d}(\beta^*u_t + \beta^*v_t \circ c_t^{-1}) = \omega^\sharp\mathrm{d}.(\beta^*u_t + \beta^*v_t).$$

The second equality holds by $\beta \circ c_t = \beta$. In view of Prop. 6.1 $B_t.C_t$ is still exact. To get that C_t^{-1} is exact we use the equality

$$\delta^r_{\mathrm{Symp}(\Gamma,\omega)}(c_t^{-1}) = \omega^\sharp\mathrm{d}(-u_t \circ c_t) = \omega^\sharp\mathrm{d}(-u_t).$$

It follows that $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is a group. Finally, $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is normal since

$$\delta^r_{\mathrm{Symp}(\Gamma,\omega)}(b^{-1} \circ c_t \circ b) = \omega^\sharp\mathrm{d}(u_t \circ b). \ ■$$

Now by repeating an argument for the symplectomorphism group (cf. [10]) it is possible to relate lagrangian bisection isotopies and $\tilde{\mathcal{S}}_\Gamma$. Consequently, under some assumption it is introduced a regular Lie group structure on $\mathrm{Bis}^{ex}(\Gamma,\omega)$ (Theorem 6.8).

PROPOSITION 6.3. *C_t is an exact isotopy iff $\tilde{\mathcal{S}}_\Gamma(\{C_\tau\}_{0\le\tau\le t}) = 0$, $\forall t$.*

Proof. ($\Rightarrow$) It follows by the definition of $\tilde{\mathcal{S}}_\Gamma$. ($\Leftarrow$) Notice that C_t is exact for t small due to the local definition of $\tilde{\mathcal{S}}_\Gamma$. Next we extend t by replacing successively ϕ_t by $C_t.C_{t_0}^{-1}$ with $t - t_0$ small enough. ■

PROPOSITION 6.4. *If $C \in \mathrm{Bis}(\Gamma,\omega)_0$ then C is an exact bisection if and only if there is a lagrangian bisection isotopy C_t with $C_0 = M$ and $C_1 = C$ such that $\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = 0$. Furthermore, if $\tilde{\mathcal{S}}_\Gamma(\{B_t\}) = 0$, $B_t \in \mathrm{Bis}(\Gamma,\omega)$, then B_t is homotopic with fixed endpoints to an exact isotopy.*

Proof. If C is exact then there is an exact isotopy C_t joining C with M. Then $\imath(X_t)\omega = dv_t$ for $v_t \in C^\infty_c(M)$, where $X_t = \delta^r_{\mathrm{Bis}(\Gamma,\omega)}(C_t)$. Hence $\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = 0$ as $[dv_t] = 0$.

In order to show the converse let $C_t = \mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\omega)}(X_t)$ be such that $\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = 0$, that is $\int_0^1 \iota(X_t)\omega dt = dv,\ v \in C^\infty_c(M)$. Here v can be chosen compactly supported as all $\operatorname{supp} X_t$ are in a fixed compact subset. Let C^v_t be the isotopy of $X_v = \omega^\sharp \mathrm{d}v$. It suffices to consider $(C^v_1)^{-1}.C$ instead of C. Thus, reparametrizing if necessary, we may assume that $\int_0^1 X_t dt = 0$ as $\imath(\int_0^1 X_t dt)\omega = 0$.

Next we set $Y_t = -\int_0^t X_\tau d\tau$. Then if $s \mapsto B^s_t$ be the isotopy of Y_t we get for $\hat{C}_t = B^1_t.C_t$ that $C_1 = \hat{C}_1$ and $\tilde{\mathcal{S}}_\Gamma(\{\hat{C}_\tau\}_{0\le\tau\le t}) = 0$ for any t. In fact,

$$\tilde{\mathcal{S}}_\Gamma(\{B^1_\tau\}_{0\le\tau\le t}) = \tilde{\mathcal{S}}_\Gamma(\{B^s_t\}_{0\le s\le 1}) = [\imath(Y_t)\omega]$$

in view of the homotopy rel. endpoints invariance. Therefore

$$\begin{aligned}\tilde{\mathcal{S}}_\Gamma(\{\hat{C}_\tau\}_{0\le\tau\le t}) &= \tilde{\mathcal{S}}_\Gamma(\{B^1_\tau\}_{0\le\tau\le t}) + \tilde{\mathcal{S}}_\Gamma(\{C_\tau\}_{0\le\tau\le t}) \\ &= [\imath(Y_t)\omega] + \left[\int_0^t \imath(X_\tau)\omega d\tau\right] = 0.\end{aligned}$$

By Prop. 6.3 this proves the first assertion, and the proof of the second can be deduced from the above argument. ∎

Let $\Xi_\Gamma = \tilde{\mathcal{S}}_\Gamma(\pi_1(\mathrm{Bis}(\Gamma,\omega)))$. In the symplectic case Ξ_Γ is called the *group of periods.* Observe that Ξ_Γ is countable since $\pi_1(\mathrm{Bis}(\Gamma,\omega))$ is countable. This follows from the fact that $\mathrm{Bis}(\Gamma,\omega)$ has the homotopy type of a countable simplicial complex ([12]).

Notice that the form $\theta_1 = \Phi(C_1)$, where $\Phi : \mathcal{U} \to \mathcal{V}$ is a chart for $\mathrm{Bis}(\Gamma,\omega)$ at e, need not be exact even if C_t is an exact isotopy and $C_1 \in \mathcal{U}$. However we have the following

PROPOSITION 6.5. *$C \in \mathrm{Bis}(\Gamma,\omega) \cap \mathcal{U}$ if and only if $[\Phi(C)] \in \Xi_\Gamma$.*

Proof. ($\Rightarrow$) We define the lagrangian bisection isotopy B_t by $\Phi(B_t) = t\theta$, where $\theta = \Phi(C)$ $(B_1 = C)$. But there is an exact isotopy C_t joining $C = C_1$ with $M = C_0$. Then for the loop $C_{1-t} * B_t$, one gets $[\theta] = -\tilde{\mathcal{S}}_\Gamma(\{B_t\}) = \tilde{\mathcal{S}}_\Gamma(\{C_t\}) - \tilde{\mathcal{S}}_\Gamma(\{B_t\}) = -\tilde{\mathcal{S}}_\Gamma(\{C_{1-t} * B_t\}) \in \Xi_\Gamma$.

($\Leftarrow$) We use Prop. 6.4. There is a lagrangian bisection loop C_t such that $\tilde{\mathcal{S}}_\Gamma(\{C_t\}) = -[\Phi(C)]$. Then $\tilde{\mathcal{S}}_\Gamma(\{D_t\}) = 0$, where $D_t = B_t * C_t$ and $\Phi(B_t) = [(1-t)\Phi(C)]$. Therefore D_t is homotopic to an exact bisection isotopy joining M with C. Thus $C \in \mathrm{Bis}^{ex}(\Gamma,\omega)$. ∎

PROPOSITION 6.6. *Every smooth curve C_t in $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is an exact bisection isotopy.*

Proof. For $t \le \epsilon$ one has $C_t \in \mathcal{U}$. It follows by Prop. 6.5 that $[\theta_t] \in \Xi_\Gamma$ for small t, where $\theta_t = \Phi(C_t)$. This means that $[\theta_t] = 0$ as $\theta_0 = 0$ and Ξ_Γ is countable. Therefore we have $\tilde{\mathcal{S}}_\Gamma(\{C_\tau\}_{0\le\tau\le t}) = 0$ for $t \le \epsilon$. Thanks to Prop. 6.3 the C_t is an exact isotopy for $t \le \epsilon$. By taking $C_t.C_\epsilon^{-1}$ instead of C_t this procedure extends the argument for all t. ∎

Since $\mathrm{Bis}(\Gamma,\omega)_0 = \widetilde{\mathrm{Bis}(\Gamma,\omega)}_0/\pi_1(\mathrm{Bis}(\Gamma,\omega)_0)$ the flux $\tilde{\mathcal{S}}_\Gamma$ induces another homomorphism $\mathcal{S}_\Gamma : \mathrm{Bis}(\Gamma,\omega)_0 \to H^1_c(M)/\Xi_\Gamma$.

COROLLARY 6.7. $\mathrm{Ker}(\mathcal{S}_\Gamma)$ *coincides with* $\mathrm{Bis}^{ex}(\Gamma,\omega)$.

Indeed, it follows from the local form of the flux and Prop. 6.5 for any C sufficiently near M. This can be extended to the whole $\mathrm{Bis}(\Gamma,\omega)_0$ by the same argument as in Prop. 6.4.

Now we are in a position to define a Lie group structure on $\mathrm{Bis}^{ex}(\Gamma,\omega)$.

THEOREM 6.8. *If $\Xi_\Gamma \subset H^1_c(M)$ is discrete (or 0 is an isolated point in Ξ_Γ) then $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is a regular Lie group with $B\Omega^1_c(M)$, the space of exact compactly supported 1-forms on M, as its Lie algebra.*

Proof. Assume the notation of the proof of Theorem 4.5, and let

$$\Phi : \mathrm{Bis}(\Gamma) \supset \mathcal{U} \to \mathcal{V} \subset \Omega^1_c(M)$$

be a chart at e given by (4.1). If $C \in \mathrm{Bis}^{ex}(\Gamma,\omega)\cap\mathcal{U}$ then $\theta = \Phi(C) \in Z\Omega^1_c(M)$ and $[\theta] \in \Xi_\Gamma$ (Prop. 6.5). By the assumption, taking possibly a smaller $\mathcal{U}$, we have $C \in \mathrm{Bis}^{ex}(\Gamma,\omega)\cap\mathcal{U}$ if and only if $\theta = \Phi(C)$ is exact. Proceeding as in the proof of 4.5 we see that $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is a submanifold of $\mathrm{Bis}(\Gamma,\omega)$, and since the composition and inversion are smooth by restriction, it is a Lie group.

By definition of $\mathrm{Bis}^{ex}(\Gamma,\omega)$ the restriction of $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\omega)}$ to $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is $\mathrm{evol}^r_{\mathrm{Bis}^{ex}(\Gamma,\omega)}$. Thus $B\Omega^1_c(M)$ is identified as the Lie algebra of $\mathrm{Bis}^{ex}(\Gamma,\omega)$. Finally, setting $p = \mathcal{S}_\Gamma$ it follows from Cor. 6.7 and Lemma 4.4 that $\mathrm{Bis}^{ex}(\Gamma,\omega)$ is regular. ∎

7. Final remarks. The third theorem of Lie asserts that any finite dimensional Lie algebra is actually the Lie algebra of a Lie group. Since the paper by van Est and Korthagen [6] it is well known that, in general, this theorem is no longer true in the infinite dimensional case. Nevertheless, there are several generalizations of the third Lie theorem. First of all, some basic facts can be carried over from Lie groups to Lie or symplectic groupoids, cf. [14], [15], [4]. Next, the third Lie theorem still holds for remarkable infinite dimensional Lie algebras, e.g. the Poisson algebra of any symplectic or locally conformal symplectic manifold ([1], [7]).

Notice that Theorem 4.5 and 6.8 can be interpreted as follows. A Poisson manifold (M,Λ) is called integrable if it can be realized as the space of units of a symplectic groupoid. Given a symplectic groupoid $(\Gamma,\omega) \rightrightarrows (M,\Lambda)$ such that $\Xi_\Gamma = \pi_1(\mathrm{Bis}(\Gamma,\omega)_0)$ is discrete, the exact sequence of Lie algebras

$$0 \to B\Omega^1_c(M) \to Z\Omega^1_c(M) \to H^1_c(M) \to 0,$$

where $H^1_c(M)$ is regarded as an abelian Lie algebra, can be integrated to the exact sequence of regular Lie groups

$$1 \to \mathrm{Bis}^{ex}(\Gamma,\omega) \to \mathrm{Bis}(\Gamma,\omega)_0 \to H^1_c(M)/\Xi_\Gamma \to 0.$$

In [19] it has been shown that for any (regular) foliation or any Poisson structure with regular symplectic foliation the related Lie algebras of vector fields (tangent to the foliation) can be integrated analogously as in the transitive case. Unfortunately these results do not encompass the singular case. Let us also remark that the regular transitive case cannot be treated in terms of groupoid bisections, i.e. the leaf preserving diffeomorphisms cannot be expressed, in general, in terms of bisections. The reason is that in view of the "holonomic imperative" of a groupoid over a foliated manifold ([16], [13]) to any bisection of such a groupoid is attached its holonomy class. Consequently, Lie groups considered in [19] are modelled on the space of foliated 1-forms, rather than on the space of ordinary 1-forms.

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