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SPECTRAL THEORY WITHIN THE FRAMEWORK OF LOCALLY SOLVABLE LIE ALGEBRAS

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Abstract. The exposition is an invitation for the reader to explore a field where Lie algebra theory and operator theory are interacting. The discussion concerns some of the main achievements related to applications of Lie algebras to the problem of finding satisfactory non-commutative versions of the several variable spectral theory constructed by J. L. Taylor in the seventies. Virtually everybody could understand what is going on, in view of the illustrating examples developed in a self-contained manner.

Introduction. If T is a linear operator on a complex finite-dimensional vector space $\mathcal{X}$, then any triangularization of T provides a ready computation of its spectrum:

$$\text{if}\quad T=\begin{pmatrix}\tilde{\lambda}_1(T) & & * \\ & \ddots & \\ \mathbf{0} & & \tilde{\lambda}_N(T)\end{pmatrix},\quad\text{then}\quad \sigma(T)=\{\tilde{\lambda}_i(T)\mid i=1,\dots,N\}. \tag{0.1}$$

According to the classical Lie's theorem, a simultaneous triangularization (or, equivalently, a *nest of invariant subspaces* $\mathcal{X}_0 \subsetneq \mathcal{X}_1 \subsetneq \cdots \subsetneq \mathcal{X}_N$) is available for all operators in a solvable Lie subalgebra $\mathcal{G}$ of $\mathrm{End}(\mathcal{X})$. In particular, the operator T may be thought of in (0.1) as running through the solvable Lie algebra $\mathcal{G}$. Then $\tilde{\lambda}_i(\cdot)$ are linear functionals on $\mathcal{G}$ vanishing on commutators, i.e., they are characters of $\mathcal{G}$. According to (0.1), the set

$$\Sigma(\mathcal{G}) := \{\tilde{\lambda}_i(\cdot)\mid i=1,\dots,N\} \tag{0.2}$$

can be thought of as a *spectrum* of the solvable Lie algebra $\mathcal{G}\subseteq \mathrm{End}(\mathcal{X})$ and we have

$$\sigma(T)=\{\tilde{\lambda}(T)\mid \tilde{\lambda}\in\Sigma(\mathcal{G})\}\ \text{for}\ T\in\mathcal{G}. \tag{0.3}$$

Let us note also that the nilpotent operators in $\mathcal{G}$ constitute an *ideal* of $\mathcal{G}$.

Now, it is natural to ask the following question.

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QUESTION. To what extent do such structures occur on infinite-dimensional spaces $\mathcal{X}$?

As we shall see below, the corresponding extent is remarkably large. The structures we are going to discuss are the following: nests of invariant subspaces and their relation to spectral theory (Section 1), joint spectra (Section 2), ideals of "special" operators and spectra for locally solvable Lie algebras of operators (Section 3).

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1. Invariant subspaces and spectra. The aim of this section is to discuss simultaneous triangularization for operators belonging to a Lie subalgebra $\mathcal{G}$ of $\mathcal{B}(\mathcal{X})$. (From now on we denote by $\mathcal{X}$ an arbitrary complex Banach space and by $\mathcal{B}(\mathcal{X})$ the set of all bounded linear operators on $\mathcal{X}$.) Simultaneous triangularizability of $\mathcal{G}$ means existence of a maximal nest $\mathcal{N}$ of closed subspaces of $\mathcal{X}$ that are invariant to all operators in $\mathcal{G}$. The maximality property can be expressed by the following two conditions:

1. $\{0\}, \mathcal{X} \in \mathcal{N}$,
2. if $\mathcal{Y} \in \mathcal{N}$ then $\dim(\mathcal{Y}/\mathcal{Y}^-) \le 1$, where $\mathcal{Y}^-$ denotes the closed subspace spanned by the union of all $\mathcal{Z} \in \mathcal{N}$ with $\mathcal{Z} \subsetneq \mathcal{Y}$.

As the situation $\dim \mathcal{X} < \infty$ shows, when considering simultaneous triangularizability for Lie algebras of operators, we have to content ourselves with Lie algebras satisfying solvability type conditions. Another difficulty, that is however hidden on finite-dimensional spaces, is that not any bounded linear operator on a Banach space $\mathcal{X}$ has invariant subspaces different from $\{0\}$ and $\mathcal{X}$. (By invariant subspace we always mean closed invariant subspace.) Let us state one of the last results obtained in this connection (see [Re97]).

1.1. THEOREM. *If $\mathcal{X}$ equals the Banach space l^1 of all absolutely summable sequences, then there exists $T \in \mathcal{B}(\mathcal{X})$ with the only invariant subspaces $\{0\}$ and $\mathcal{X}$, and moreover with $\sigma(T) = \{0\}$.*

Nevertheless, positive results concerning existence of nontrivial invariant subspaces are available for various classes of operators (see e.g. [CE]). However, recall that we are looking for (nests of) invariant subspaces only for a specific purpose, namely for computing spectra (see (0.1)). Before stating the pertinent classical facts in this connection, let us recall how the "diagonal entries" of an operator $T \in \mathcal{B}(\mathcal{X})$ can be naturally defined with respect to any maximal nest $\mathcal{N}$ of invariant subspaces for T. Namely, for $\mathcal{Y} \in \mathcal{N}$, set $\tilde{\lambda}_{\mathcal{Y}}(T) = 0$ if $\mathcal{Y} = \mathcal{Y}^-$; otherwise $\dim(\mathcal{Y}/\mathcal{Y}^-) = 1$, so the operator induced by T on $\mathcal{Y}/\mathcal{Y}^-$ is the multiplication by a complex number, which we denote by $\tilde{\lambda}_{\mathcal{Y}}(T)$. Then we can think of $\{\tilde{\lambda}_{\mathcal{Y}}(T)\}_{\mathcal{Y} \in \mathcal{N}}$ as the family of "diagonal entries" of T with respect to $\mathcal{N}$. However, note that the spectrum of T cannot always be computed by means of these "diagonal entries":

1.2. EXAMPLE. Let us consider the bounded linear operators S, V on $\mathcal{X} = L^2[0,1]$, defined by

$$S : f(t) \mapsto t f(t), \qquad V : f(t) \mapsto \int_0^t f(s) \mathrm{d}s.$$

That is, S is the multiplication operator by the identical function on $[0,1]$, while V is the Volterra integration operator. Furthermore, let us consider the following closed subspaces of $\mathcal{X}$:

$$\mathcal{Y}_t = \{f \in L^2[0,1] \mid f = 0 \text{ a.e. on } [0,t]\}, \qquad (0 \le t \le 1).$$

It is easily seen that $\mathcal{N} := \{\mathcal{Y}_t \mid 0 \le t \le 1\}$ is a nest of invariant subspaces for both S and V. Moreover $\mathcal{Y}^- = \mathcal{Y}$ for each $\mathcal{Y} \in \mathcal{N}$, so $\mathcal{N}$ is maximal and

$$\tilde{\lambda}_{\mathcal{Y}}(S) = \tilde{\lambda}_{\mathcal{Y}}(V) = 0 \quad \text{ whenever } \mathcal{Y} \in \mathcal{N}.$$

On the other hand, it is well-known that

$$\sigma(S) = [0,1] \text{ and } \sigma(V) = \{0\}$$

(see e.g. [Ha82]). Consequently,

$$\sigma(S) \neq \{\tilde{\lambda}_{\mathcal{Y}}(S) \mid \mathcal{Y} \in \mathcal{N}\} \text{ while } \sigma(V) = \{\tilde{\lambda}_{\mathcal{Y}}(V) \mid \mathcal{Y} \in \mathcal{N}\}. \ \blacksquare$$

In the above example, what is the property of V that is not met by S? It is going on compactness: V is compact, while S is not. More precisely, the following classical result due to J. Ringrose holds (see [Ri71] and also [Dw78]), thus extending (0.1) to general compact operators:

1.3. THEOREM. *If $T \in \mathcal{B}(\mathcal{X})$ is a compact operator and $\mathcal{N}$ is a maximal nest of invariant subspaces for T, then*

$$\sigma(T) = \{\tilde{\lambda}_{\mathcal{Y}}(T) \mid \mathcal{Y} \in \mathcal{N}\}.$$

(For the sake of completeness, let us recall that for every compact operator there exists a maximal nest of invariant subspaces, cf. [AS54]. See also [Lo73].) $\blacksquare$

So far for a single compact operator. What about invariant subspaces for Lie algebras of compact operators? Fortunately, several successful researches were carried out in this direction, see e.g. [Wo77], [Sa83], [ST99]. It is known for example that in some cases even the presence of a nonzero compact operator in a Lie algebra produces invariant subspaces for the whole algebra (see Theorem 7′ in [Sa83]). We are going to state a fact (Theorem 1.5 below) that is fairly general for our purposes and unifies some results of [Wo77] and [ST99]. First recall the following definition (see e.g. [St75] or [BS01]):

1.4. DEFINITION. The Lie algebra $\mathcal{G}$ is *locally solvable* (respectively *locally finite*) if it possesses a family $\{\mathcal{G}_i\}_{i\in I}$ consisting of solvable (respectively finite-dimensional) subalgebras such that

(i) for every $i_1, i_2 \in I$ there exists $i_3 \in I$ such that $\mathcal{G}_{i_1} \cup \mathcal{G}_{i_2} \subseteq \mathcal{G}_{i_3}$,
(ii) $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$. $\blacksquare$

1.5. THEOREM. *Every locally solvable Lie subalgebra $\mathcal{G}$ of $\mathcal{B}(\mathcal{X})$ consisting of compact operators is simultaneously triangularizable.*

Proof. Let $\mathcal{A}$ be the norm closure of the associative subalgebra of $\mathcal{B}(\mathcal{X})$ generated by $\mathcal{G}$. According to [Mu82], it suffices to prove that the Banach algebra $\mathcal{A}$ is commutative modulo its Jacobson radical. To this end, let $A, B, C \in \mathcal{A}$ arbitrary. We have to prove that $[A,B]C$ is a quasinilpotent operator. (Since C is arbitrary and $\mathcal{A}$ is a Banach algebra, this implies that each commutator $[A,B]$ belongs to the Jacobson radical of $\mathcal{A}$.)

By the definition of $\mathcal{A}$, there exist sequences $\{k_n\}_{n\geq 1}$, $\{l_n\}_{n\geq 1}$, $\{m_n\}_{n\geq 1}$ of positive integers and sequences $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1}$, $\{r_n\}_{n\geq 1}$ such that

(a) for each $n \geq 1$, p_n, q_n and r_n are complex coefficients polynomials in k_n, l_n, m_n non-commuting variables, respectively,

(b) for every $n \geq 1$, there exist $A_1^{(n)}, \dots, A_{k_n}^{(n)}, B_1^{(n)}, \dots, B_{l_n}^{(n)}, C_1^{(n)}, \dots, C_{m_n}^{(n)} \in \mathcal{G}$ such that, if $A_n := p_n(A_1^{(n)}, \dots, A_{k_n}^{(n)}), B_n := q_n(B_1^{(n)}, \dots, B_{l_n}^{(n)}), C_n := r_n(C_1^{(n)}, \dots, C_{m_n}^{(n)})$, then

$$A = \lim_{n\to\infty} A_n,\ B = \lim_{n\to\infty} B_n,\ C = \lim_{n\to\infty} C_n$$

with respect to the norm operator topology on $\mathcal{B}(\mathcal{X})$.

Now, since $\mathcal{G}$ is locally solvable, for each $n \geq 1$ there exists a solvable subalgebra $\mathcal{G}_n$ of $\mathcal{G}$ such that $A_1^{(n)}, \dots, A_{k_n}^{(n)}, B_1^{(n)}, \dots, B_{l_n}^{(n)}, C_1^{(n)}, \dots, C_{m_n}^{(n)} \in \mathcal{G}_n$. Since $\mathcal{G}_n$ is a solvable Lie algebra of compact operators, it is simultaneously triangularizable by [ST99]. Then the closed associative subalgebra generated by $\mathcal{G}_n$ is commutative modulo its Jacobson radical (see [Mu82]); in particular, $[A_n, B_n]C_n$ is a compact quasinilpotent operator. By the continuity of the spectrum on compact operators (see [Ne51]), it then follows that the operator

$$[A, B]C \qquad (= \lim_{n\to\infty} [A_n, B_n]C_n)$$

is quasinilpotent. ∎

The above theorem allows us to use *the same* maximal nest of invariant subspaces to define "diagonal entries" for all operators in a locally solvable Lie algebra of compact operators. That is, a set of characters as in (0.2) can be constructed for every Lie algebra of this type:

1.6. COROLLARY. *If $\mathcal{G}$ is a locally solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting of compact operators, then it possesses maximal nests of invariant subspaces. If $\mathcal{N}$ is such a nest, then*

$$\Sigma_{\mathcal{N}}(\mathcal{G}) := \{\tilde{\lambda}_{\mathcal{Y}}(\cdot) \mid \mathcal{Y} \in \mathcal{N}\}$$

is a set of characters of $\mathcal{G}$ such that

$$\sigma(T) = \{\tilde{\lambda}(T) \mid \tilde{\lambda} \in \Sigma_{\mathcal{N}}(\mathcal{G})\} \text{ for } T \in \mathcal{G}. \tag{1.1}$$

Moreover, if $\mathcal{L}$ is a subalgebra of $\mathcal{G}$, then it is locally solvable and

$$\Sigma_{\mathcal{N}}(\mathcal{L}) = \{\tilde{\lambda}|_{\mathcal{L}} \mid \tilde{\lambda} \in \Sigma_{\mathcal{N}}(\mathcal{G})\}. \tag{1.2}$$

Proof. That each $\tilde{\lambda}_{\mathcal{Y}}(\cdot)$ is linear and vanishes on commutators is an easy consequence of the definition of the "diagonal entries", just as in the finite-dimensional case. The equality in (1.1) follows by Theorem 1.3, while (1.2) is obvious. ∎

Now, the set of characters $\Sigma_{\mathcal{N}}(\mathcal{G})$ in Corollary 1.6 could be thought of as a spectrum of $\mathcal{G}$ only if it does not depend upon the choice of $\mathcal{N}$. How can the dependence upon the nest of invariant subspaces be avoided? What about Lie algebras of more general operators, where invariant subspaces do not exist at all (cf. Theorem 1.1)?

Actually, these questions are answered only for the locally solvable Lie algebras that are moreover locally finite (see Theorem 3.7 and Remarks 3.8 and 3.9 below). The way to get to the answer is the following: First treat the case of nilpotent finite-dimensional Lie algebras. (We shall do this in Section 2.) Then consider Cartan subalgebras of solvable finite-dimensional Lie algebras, for reducing the problem from the solvable case to the nilpotent one; finally, pass to direct limits for reaching the locally solvable, locally finite case (see Section 3).

2. Joint spectra and nilpotent Lie algebras. The aim of this section is to outline some basic facts related to the notion of joint spectrum within the framework of finite-dimensional nilpotent Lie algebras.

The first satisfactory notion of joint spectrum for commuting tuples of bounded linear operators was introduced by J. L. Taylor in [Ta70]; the advantage of this spectrum is that it reflects the action of operators on the corresponding Banach space, and not only the fact that they belong to a certain Banach algebra (see also Theorem 2.4 below). The notion introduced by J. L. Taylor involves homological algebra constructions going back to the work of J. L. Koszul [Ko50] (see Definition 2.1 below). In the case of a single operator $T \in \mathcal{B}(\mathcal{X})$, these constructions lead to the usual spectrum $\sigma(T)$, namely $\lambda \in \sigma(T)$ if and only if the sequence

$$0 \leftarrow \mathcal{X} \overset{\lambda\,\mathrm{id}_{\mathcal{X}} - T}{\longleftarrow} \mathcal{X} \leftarrow 0$$

is not exact, where $\mathrm{id}_{\mathcal{X}}$ denotes the identical operator on $\mathcal{X}$.

Let us remark that there is a natural one-to-one correspondence between commuting n-tuples $T = (T_1, \dots, T_n) \in \mathcal{B}(\mathcal{X})^n$ and representations $\rho : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ of an Abelian n-dimensional Lie algebra $\mathcal{E}$, namely by

$$T_i = \rho(e_i), \quad i = 1, \dots, n,$$

where $\{e_1, \dots, e_n\}$ is a fixed basis of $\mathcal{E}$. In particular, Taylor spectral theory can be thought of as a spectral theory for Banach space representations of Abelian finite-dimensional Lie algebras. From this point of view, the construction of a spectral theory for representations of more general (non-Abelian) Lie algebras arises as a natural problem. Although it had been already suggested by J. L. Taylor himself (see e.g. [Ta73]), the first significant step towards solving it could be done much later, namely by A. S. Fainshtein in [Fa93] in the case of finite-dimensional nilpotent Lie algebras.[1)] To describe the corresponding results, we need the following definitions.

2.1. Definition (cf. [Ko50]). Let $\mathcal{E}$ be a Lie algebra and $\rho : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ be a representation. The *Koszul complex* of ρ is

$$\mathrm{Kos}(\rho) : 0 \leftarrow \mathcal{X} \overset{\alpha_1}{\longleftarrow} \mathcal{X} \otimes \mathcal{E} \overset{\alpha_2}{\longleftarrow} \cdots \overset{\alpha_k}{\longleftarrow} \mathcal{X} \otimes \wedge^k \mathcal{E} \overset{\alpha_{k+1}}{\longleftarrow} \cdots,$$

where for $n \geq 1$ the operator α_k is defined by

$$\alpha_k(x \otimes \underline{e}) = \sum_{i=1}^{k} (-1)^{i-1} \rho(e_i) x \otimes \overset{i}{\widehat{\underline{e}}} + \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} x \otimes [e_i, e_j] \wedge \overset{i,j}{\widehat{\underline{e}}}$$

1) See [BS01] for other references and historical comments.

for $x \in \mathcal{X}$, $\underline{e} = e_1 \wedge \cdots \wedge e_k \in \wedge^k \mathcal{E}$, where $\overset{i}{\widehat{\underline{e}}}$ means the omission of the factor e_i (and similarly for $\overset{i,j}{\widehat{\underline{e}}}$). ∎

2.2. Definition (cf. [Ot97], [BS01]). Let $\mathcal{E}$ be a Lie algebra and

$$\widehat{\mathcal{E}} = \{\tilde{\lambda} : \mathcal{E} \to \mathbb{C} \text{ linear} \mid \tilde{\lambda}|_{[\mathcal{E},\mathcal{E}]} = 0\}$$

the set of characters of $\mathcal{E}$. Then the *spectrum* of a representation $\rho : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ is defined by

$$\sigma(\rho) := \{\tilde{\lambda} \in \widehat{\mathcal{E}} \mid \operatorname{Kos}(\rho - \tilde{\lambda}) \text{ is not exact}\},$$

where $\rho - \tilde{\lambda} : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ is the representation defined by $e \mapsto \rho(e) - \tilde{\lambda}(e)\,\mathrm{id}_{\mathcal{X}}$. ∎

2.3. Definition (cf. [Fa93]). Let $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{X})^n$. For every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, let $\mathrm{id}_{\mathcal{E}(T-\lambda)} : \mathcal{E}(T - \lambda) \to \mathcal{B}(\mathcal{X})$ denote the identical representation of the Lie subalgebra $\mathcal{E}(T - \lambda)$ generated by the operators $T_1 - \lambda_1 \mathrm{id}_{\mathcal{X}}, \ldots, T_n - \lambda_n \mathrm{id}_{\mathcal{X}} \in \mathcal{B}(\mathcal{X})$. Then the (Taylor type) *joint spectrum* of T is defined by

$$\sigma(T) := \{\lambda \in \mathbb{C}^n \mid \operatorname{Kos}(\mathrm{id}_{\mathcal{E}(T-\lambda)}) \text{ is not exact}\}. \ ∎$$

The connection between Definitions 2.2 and 2.3 is provided by the following result due to A.S. Fainshtein (see Theorem 2.3 in [Fa93]; cf. also Theorem 1 in §26 of [BS01]).

2.4. Theorem. *Let $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{X})^n$ be such that the Lie subalgebra of $\mathcal{B}(\mathcal{X})$ generated by $T_1, \ldots, T_n$ is nilpotent. Then for every representation $\rho : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ of a finite-dimensional nilpotent Lie algebra $\mathcal{E}$ such that $T_1, \ldots, T_n \in \rho(\mathcal{E})$ we have*

$$\sigma(T) = \{(\tilde{\lambda}(e_1), \ldots, \tilde{\lambda}(e_n)) \mid \tilde{\lambda} \in \sigma(\rho)\},$$

where $e_1, \ldots, e_n \in \mathcal{E}$ are arbitrary elements such that $\rho(e_i) = T_i$ for $i = 1, \ldots, n$.

A basic property of the Taylor spectrum for commuting tuples is the spectral mapping property with respect to the analytic functional calculus (see e.g. [Ta72] and [Va82]). It is a remarkable fact that, within the framework of nilpotent Lie algebras, the Taylor type spectrum introduced in Definition 2.3 also has a spectral mapping property with respect to the enveloping algebras, thought of as algebras of polynomials in non-commuting variables (see [Fa93] and also [BS01]).[1)] A consequence of this polynomial spectral mapping theorem is the following *projection property* (cf. Consequence 5.5 in [Fa93]).

2.5. Theorem. *If $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{X})^n$ generates a nilpotent Lie subalgebra of $\mathcal{B}(\mathcal{X})$, then for every $1 \le i_1 < \cdots < i_k \le n$ we have*

$$\sigma(T_{i_1}, \ldots, T_{i_k}) = \pi_{i_1 \cdots i_k}(\sigma(T_1, \ldots, T_n)), \tag{1.1}$$

where $\pi_{i_1 \cdots i_k} : \mathbb{C}^n \to \mathbb{C}^k$ is the projection defined by $(z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_k})$.

The above projection property also has a version in terms of spectra of representations (see Definition 2.2 above). It was first obtained by E. Boasso and A. Larotonda in [BL93], and then completed by C. Ott in [Ot96] (see also Notes in Chapter IV of [BS01]). The corresponding results can be summarized as follows:

[1)] Some completions of enveloping algebras were considered by A. Dosyiev in [Ds00a] in order to obtain "analytic functions of several non-commuting variables".

2.6. Theorem. *Let $\mathcal{E}$ be a finite-dimensional solvable Lie algebra and $\rho : \mathcal{E} \to \mathcal{B}(\mathcal{X})$ be a representation. Then for every ideal $\mathcal{F}$ of $\mathcal{E}$ we have*

$$\sigma(\rho|_{\mathcal{F}}) = \{\tilde{\lambda}|_{\mathcal{F}} \mid \tilde{\lambda} \in \sigma(\rho)\}. \tag{1.2}$$

If the Lie algebra $\mathcal{E}$ is even nilpotent, then (1.2) *holds for every subalgebra $\mathcal{F}$ of $\mathcal{E}$.*

According to Theorem 1.3 in [Be00a], the second assertion in the above theorem is specific to the nilpotent Lie algebras. On the other hand, if we want to use Theorem 2.4 in order to deduce the projection property (1.1) from (1.2), then we need the validity of (1.2) for every subalgebra $\mathcal{F}$, not only for ideals (see Corollary 2 and the proof of Corollary 1 in §26 of [BS01]). Consequently, the natural projection property holds only within the framework of nilpotent Lie algebras. Let us see also a concrete example in this connection (for other examples, see [BL93], [Ot96] and [BS01]).

2.7. Example. Let $\mathcal{X} = \mathbb{C}^2$ (with an arbitrary norm) and $T_1, T_2 \in \mathcal{B}(\mathcal{X})$ given by

$$T_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (a, b \in \mathbb{C},\ a \neq b).$$

For $\lambda = (\lambda_1, 0) \in \mathbb{C}^2$, we have

$$[T_1 - \lambda_1 \operatorname{id}_{\mathcal{X}}, T_2] = [T_1, T_2] = (b - a)T_2,$$

so $\mathcal{E}(T - \lambda)$ (see Definition 2.3 above) is a solvable Lie algebra that is non-nilpotent. It is easy to see that the complex $\operatorname{Kos}(\operatorname{id}_{\mathcal{E}(T-\lambda)})$ is isomorphic to

$$0 \leftarrow \mathcal{X} \xleftarrow{\alpha_1} \mathcal{X} \oplus \mathcal{X} \xleftarrow{\alpha_2} \mathcal{X} \leftarrow 0,$$

where the arrows α_1 and α_2 are defined by

$$\alpha_1(x_1, x_2) = T_2 x_1 + (T_1 - \lambda_1 \operatorname{id}_{\mathcal{X}})x_2, \quad \alpha_2(x) = (-(T_1 + (-\lambda_1 + a - b)\operatorname{id}_{\mathcal{X}})x, T_2 x)$$

(see Example 3 in §10 of [BS01]). Then for $-\lambda_1 + a - b = -a$ (that is, $\lambda_1 = 2a - b$), it is easy to compute that

$$\operatorname{Ker} \alpha_1 \supsetneqq \operatorname{Ran} \alpha_2,$$

so the complex $\operatorname{Kos}(\operatorname{id}_{\mathcal{E}(T-\lambda)})$ is not exact. Thus $\lambda = (2a - b, 0) \in \sigma(T_1, T_2)$. However, $2a - b \notin \{a, b\}$, so $2a - b \notin \sigma(T_1)$. This implies that

$$\pi(\sigma(T_1, T_2)) \neq \sigma(T_1),$$

where $\pi : \mathbb{C}^2 \to \mathbb{C}$ is the projection $(z_1, z_2) \mapsto z_1$. ■

3. From the finite-dimensional nilpotent Lie algebras to the solvable ones and beyond. In order to reduce the problem from solvable to nilpotent Lie algebras, we need to investigate the structure of the sets of nilpotent and quasinilpotent operators in a finite-dimensional solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. In a certain sense, the difference between the solvable situation and the nilpotent one turns out to consist in (a vector space containing only) nilpotent operators (cf. Corollary 3.3 and Theorem 3.4 below).

Throughout this section, it is worth to have in mind the following "small concrete special case that contains within itself all the concepts, all the difficulties, and all the steps needed to understand and to overcome them" (as P. R. Halmos said at p. 24 in [Ha85] in connection with something else).

3.1. Example. Let $\mathcal{G}$ be a two-dimensional solvable non-Abelian Lie subalgebra of $\mathcal{B}(\mathcal{X})$. Then there exist T, N in $\mathcal{G}$ such that

$$[T, N] = N.$$

(For a concrete situation, take $b = a + 1$, $T = T_1$ and $N = T_2$ in Example 2.7.) Then

$$[T, N^k] = kN^k, \quad k \geq 1,$$

by induction. Then $k\|N\|^k \leq 2\|T\| \cdot \|N^k\|$ for every k, yielding $N^k = 0$ for $k > 2\|T\|$. Consequently $\mathcal{G}$ decomposes as

$$\mathcal{G} = \mathbb{C}T \oplus \mathbb{C}N,$$

where $\mathbb{C}T$ is a nilpotent (in fact Abelian) Lie subalgebra of $\mathcal{G}$, while the ideal $\mathbb{C}N$ consists only of nilpotent operators. ∎

In this example, the nilpotency of N is implied by the fact that N is an eigenvector of the bounded operator $\operatorname{ad} T : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ defined as usual by

$$(\operatorname{ad} T)S = [T, S] \text{ whenever } S \in \mathcal{B}(\mathcal{X}).$$

More generally, the following nilpotency criterion holds (cf. Theorem 4 in §17 of [BS01]):

3.2. Theorem. *If $T, N \in \mathcal{B}(\mathcal{X})$ and there exists a complex nonzero scalar λ such that*

$$\lim_{n\to\infty} \|(\operatorname{ad} T - \lambda \operatorname{id}_{\mathcal{B}(\mathcal{X})})^n N\|^{1/n} = 0,$$

then N is a nilpotent operator.

This theorem can be applied to the root spaces of a finite-dimensional Lie algebra of operators:

3.3. Corollary. *Let $\mathcal{G}$ be a finite-dimensional Lie subalgebra of $\mathcal{B}(\mathcal{X})$, $\mathcal{H}$ a Cartan subalgebra of $\mathcal{G}$, and α a nonzero root of $\mathcal{G}$ with respect to $\mathcal{H}$. Then the corresponding root space $\mathcal{G}^\alpha$ consists only of nilpotent operators.*

Proof. Let $T \in \mathcal{H}$ be a regular element. Then $\lambda := \alpha(T) \neq 0$ and

$$\mathcal{G}^\alpha \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ker}((\operatorname{ad} T - \lambda \operatorname{id}_{\mathcal{G}})^n),$$

so Theorem 3.2 can be applied to each $N \in \mathcal{G}^\alpha$. ∎

In particular, in the setting of Corollary 3.3, if $\mathcal{G}$ contains no nilpotent operators, then it cannot have nonzero roots, so $\mathcal{G} = \mathcal{H}$ and $\mathcal{G}$ has to be a nilpotent Lie algebra.

Theorem 3.2 is one of the main ingredients in proving the following fact (that should be compared with the remark made just after relation (0.3)).

3.4. Theorem (cf. Theorems 1 and 2 in §28 of [BS01]). *Let $\mathcal{G}$ be a finite-dimensional solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$, and $\mathcal{N}_\mathcal{G}$ (respectively $\mathcal{Q}_\mathcal{G}$) be the set of nilpotent (respectively quasinilpotent) operators in $\mathcal{G}$. Then*

$$\mathcal{N}_\mathcal{G} \subseteq \mathcal{Q}_\mathcal{G}$$

are two ideals of $\mathcal{G}$.

3.5. Remark. In the setting of Theorem 3.4, $[\mathcal{G},\mathcal{G}]$ consists only of quasinilpotent operators, i.e., $[\mathcal{G},\mathcal{G}] \subseteq \mathcal{Q}_{\mathcal{G}}$ (cf. [Tu87], [BL93], [Ott96], [BS01]). ■

Let us look at Example 3.1 from the point of view of Theorem 3.4:

3.6. Example. In the framework of Example 3.1, the fact that $\mathcal{N}_{\mathcal{G}}$ and $\mathcal{Q}_{\mathcal{G}}$ are ideals of $\mathcal{G}$ follows from

$$[\mathcal{G},\mathcal{G}] = \mathbb{C}N = \mathcal{N}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{G}}.$$

To prove these equalities, note that we know that $[\mathcal{G},\mathcal{G}] = \mathbb{C}N \subseteq \mathcal{N}_{\mathcal{G}} \subseteq \mathcal{Q}_{\mathcal{G}}$. Assume that there exists $A \in \mathcal{Q}_{\mathcal{G}} \setminus \mathbb{C}N$. Since $A \notin \mathbb{C}N$, there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$ and $A = \alpha T + \beta N$, so

$$(\mathrm{ad}\, A)N = [A, N] = \alpha N. \tag{3.1}$$

On the other hand, the fact that A is quasinilpotent implies that the left multiplication L_A and the right multiplication R_A are both quasinilpotent operators on $\mathcal{B}(\mathcal{X})$, so the difference $L_A - R_A = \mathrm{ad}\, A$ of these commuting operators is quasinilpotent as well, thus contradicting (3.1). ■

Now we can state the main result extending (0.2) and (0.3) to operators on infinite-dimensional spaces.

3.7. Theorem. *There exists a unique map $\Sigma(\cdot)$ that associates to every locally finite, locally solvable Lie subalgebra $\mathcal{G}$ of $\mathcal{B}(\mathcal{X})$ a compact nonempty subset $\Sigma(\mathcal{G})$ of characters of $\mathcal{G}$ such that*

(a) *if $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$, then $\Sigma(\mathcal{L}) = \{\tilde{\lambda}|_{\mathcal{L}} \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\}$,*
(b) *if $\mathcal{H}$ is a finite-dimensional nilpotent subalgebra of $\mathcal{B}(\mathcal{X})$, then $\Sigma(\mathcal{H}) = \sigma(\mathrm{id}_{\mathcal{H}})$.*

3.8. Remark. If $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{H} := \mathbb{C}T$, then it is easily seen that the Koszul complex (see Definition 2.1 above) of the identical representation $\mathrm{id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{B}(\mathcal{X})$ reduces to

$$\mathrm{Kos}(\mathrm{id}_{\mathcal{H}}) : 0 \leftarrow \mathcal{X} \xleftarrow{T} \mathcal{X} \leftarrow 0. \tag{3.2}$$

On the other hand, we have a natural identification of the space of characters $\widehat{\mathcal{H}}$ with $\mathbb{C}$ by $\tilde{\lambda} \leftrightarrow \tilde{\lambda}(T)$. In view of (3.2), this implies at once that

$$\sigma(T) = \{\tilde{\lambda}(T) \mid \tilde{\lambda} \in \sigma(\mathrm{id}_{\mathcal{H}})\}.$$

Now, if $\mathcal{G}$ is a locally finite, locally solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ and $T \in \mathcal{G}$, then Theorem 3.7 implies

$$\sigma(T) = \{\tilde{\lambda}(T) \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\},$$

thus extending (0.3). ■

3.9. Remark. If $\mathcal{G}$ is a locally finite, locally solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting only of compact operators, then for every maximal nest $\mathcal{N}$ of invariant subspaces of $\mathcal{G}$ (recall Theorem 1.5) we have

$$\Sigma(\mathcal{G}) = \Sigma_{\mathcal{N}}(\mathcal{G}) \tag{3.3}$$

(recall also Corollary 1.6). For a proof of (3.3), see Theorem 4 in §27 of [BS01], where the corresponding assertion was proved for quasisolvable Lie algebras (but the proof works in the general case). ■

Theorem 3.7 above is proved in [Be99]; see also Theorem 2 in §27 of [BS01].[1)] With the spectrum $\Sigma(\cdot)$ at hand for finite-dimensional solvable Lie algebras, one has just to consider direct limits to reach the general case. As for the construction of $\Sigma(\mathcal{G})$ when $\mathcal{G}$ is a finite-dimensional solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$, one has to consider Cartan subalgebras of $\mathcal{G}$ in order to reduce the problem to nilpotent Lie algebras, where the corresponding spectrum is defined so as to satisfy (b) in Theorem 3.7. An important role in the proof is played by the conjugation theorem for Cartan subalgebras (see e.g. Chapter 12 in [SL55]). To see what is going on, let us look at Example 3.1 once again.

3.10. Example. In the situation in Example 3.1, we have $\mathcal{G} = \mathbb{C}T \oplus \mathbb{C}N$. Let

$$\Sigma(\mathcal{G}) := \{\tilde{\lambda} : \mathcal{G} \to \mathbb{C} \text{ linear} \mid \tilde{\lambda}(T) \in \sigma(T),\ \tilde{\lambda}(N) = 0\}. \tag{3.3}$$

We are going to prove the assertions (a) and (b) of Theorem 3.7 for the spectrum $\Sigma(\mathcal{G})$ introduced in this way. Note that each proper subalgebra of $\mathcal{G}$ is one-dimensional. Then, in view of Remark 3.8, what we have to prove is that

$$\sigma(B) = \{\tilde{\lambda}(B) \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\} \text{ for } B \in \mathcal{G}. \tag{3.4}$$

If $B \in \mathbb{C}N$, then B is nilpotent in view of Example 3.6, so $\sigma(B) = \{0\}$, while $\tilde{\lambda}|_{\mathbb{C}N} \equiv 0$ for $\tilde{\lambda} \in \Sigma(\mathcal{G})$ by the very definition (3.3). Now let $B \in \mathcal{G} \setminus \mathbb{C}N$. Then there exist $\beta, \gamma \in \mathbb{C}$ such that $\beta \neq 0$ and

$$B = \beta T + \gamma N.$$

Since $[T, N] = N$, we have $(\mathrm{ad}(-(\gamma/\beta)N))(\beta T) = \gamma N$ and $(\mathrm{ad}(-(\gamma/\beta)N))^k(\beta T) = 0$ whenever $k \geq 2$. Consequently

$$(\exp(\mathrm{ad}(-(\gamma/\beta)N)))(\beta T) = \beta T + \gamma N.$$

For every $U \in \mathcal{B}(\mathcal{X})$ we have $\exp(\mathrm{ad}\, U) = \exp(L_U - R_U) = \exp(L_U)\exp(-R_U) = L_{\exp U} R_{\exp(-U)}$ (see the left and right multiplication operators used in Example 3.6), so for $C := \exp(-(\gamma/\beta)N)$ we get

$$C(\beta T)C^{-1} = \beta T + \gamma N. \tag{3.5}$$

(This corresponds to the very moment when the conjugation theorem for Cartan subalgebras comes on the stage in the general proof of Theorem 3.7; formula (3.5) actually shows that the Cartan subalgebras $\mathbb{C}T$ and $\mathbb{C}(\beta T + \gamma N)$ are conjugate.) Now (3.5) implies that

$$\sigma(\beta T) = \sigma(\beta T + \gamma N).$$

Then

$$\sigma(\beta T + \gamma N) = \{\beta z \mid z \in \sigma(T)\} \overset{(3.3)}{=} \{\beta\tilde{\lambda}(T) \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\} = \{\tilde{\lambda}(\beta T + \gamma N) \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\},$$

where the last equality follows by the fact that $\tilde{\lambda}$ is linear and $\tilde{\lambda}(N) = 0$ whenever $\tilde{\lambda} \in \Sigma(\mathcal{G})$ (see (3.3)). Thus (3.4) is completely proved. ∎

To conclude, let us look at some special situations where a spectrum $\Sigma(\cdot)$ can be constructed for Lie algebras that do not fall under the hypotheses of Theorem 3.7, in the sense that they are not locally finite.

[1)] It is worth to note that a version of Theorem 3.7, holding for spectra of representations of finite-dimensional solvable Lie algebras, was recently obtained by A. Dosyiev in [Ds00b].

3.11. Example. If $\mathcal{G}$ is a locally solvable Lie algebra of compact operators, let $\mathcal{N}$ be a maximal nest of invariant subspaces of $\mathcal{G}$ (see Theorem 1.5). Then for every subalgebra $\mathcal{L}$ of $\mathcal{G}$ we can define $\Sigma_{\mathcal{N}}(\mathcal{L})$ as in Corollary 1.6 and both assertions (a) and (b) of Theorem 3.7 hold for the subalgebras of $\mathcal{G}$. ∎

3.12. Example. Let $\mathcal{X} = L^2[0,1]$ and $\mathcal{G}$ be the Lie subalgebra of $\mathcal{B}(\mathcal{X})$ generated by the operators S and V from Example 1.2, satisfying the easily verified commutation relation $[S,V] = V^2$. Then $\mathcal{G}$ is spanned as a vector space by the set

$$\{S, V, V^2, \ldots, V^n, \ldots\},$$

while $[\mathcal{G},\mathcal{G}]$ is spanned by

$$\{V^2, V^3, \ldots, V^n, \ldots\}.$$

In particular, $[[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]] = \{0\}$, so $\mathcal{G}$ is a solvable Lie algebra. On the other hand, let

$$\Sigma(\mathcal{G}) := \{\tilde{\lambda} : \mathcal{G} \to \mathbb{C} \text{ linear} \mid \tilde{\lambda}(S) \in \sigma(S) \text{ and } \tilde{\lambda}(V^n) = 0 \text{ whenever } n \geq 1\}$$

and

$$\Sigma(\mathcal{L}) := \{\tilde{\lambda}|_{\mathcal{L}} \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\} \text{ for each subalgebra } \mathcal{L} \text{ of } \mathcal{G}.$$

Then both assertions (a) and (b) of Theorem 3.7 hold for the subalgebras of $\mathcal{G}$ ([Be00b]). ∎

3.13. Example. Let $m \geq 1$, endow $\mathbb{C}^m$ with the usual scalar product and consider the Hilbert space $\mathcal{X}$ of (equivalence classes of) $\mathbb{C}^m$-valued functions that are almost everywhere defined on $[0,1]$ and square-integrable with respect to the Lebesgue measure. For every $f \in \mathcal{X}$, if $f_1, \ldots, f_m \in L^2[0,1]$ are the components of f, that is

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{pmatrix} \in \mathbb{C}^m \text{ for almost every } t \in [0,1],$$

then $\|f\|_{\mathcal{X}} = \Big(\sum_{i=1}^{m} \int_0^1 |f_i(t)|^2 \mathrm{d}t\Big)^{1/2}$. Now let $\mathcal{T}$ be a (solvable) Lie algebra of $m \times m$ upper triangular complex matrices. Then

$$\mathcal{G} := \{a : [0,1] \to \mathcal{T} \mid a \text{ is continuous}\}$$

has a natural structure of (infinite-dimensional solvable) Lie algebra, with bracket, addition and scalar multiplication defined pointwise. Furthermore, each $a = (a_{jk}(\cdot))_{1 \leq j,k \leq m} \in \mathcal{G}$ can be considered as a bounded linear operator acting on $\mathcal{X}$ by

$$\begin{pmatrix} f_1(\cdot) \\ \vdots \\ f_m(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} a_{11}(\cdot) & \ldots & a_{1m}(\cdot) \\ & \ddots & \vdots \\ \mathbf{0} & & a_{mm}(\cdot) \end{pmatrix} \begin{pmatrix} f_1(\cdot) \\ \vdots \\ f_m(\cdot) \end{pmatrix}.$$

Consequently, $\mathcal{G}$ is an infinite-dimensional solvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. For $i = 1, \ldots, m$ and $t \in [0,1]$, let

$$\tilde{\lambda}_{i,t} : \mathcal{G} \to \mathbb{C}, \quad \tilde{\lambda}_{i,t}(a) := a_{ii}(t).$$

Then each $\tilde{\lambda}_{i,t}$ is a linear functional on $\mathcal{G}$ and it easily follows that it vanishes on $[\mathcal{G},\mathcal{G}]$, so that $\tilde{\lambda}_{i,t}$ is a character of $\mathcal{G}$. Let

$$\Sigma(\mathcal{G}) := \{\tilde{\lambda}_{i,t} \mid i = 1, \ldots, m;\ t \in [0,1]\}$$

and

$$\Sigma(\mathcal{L}) := \{\tilde{\lambda}|_{\mathcal{L}} \mid \tilde{\lambda} \in \Sigma(\mathcal{G})\} \text{ for each subalgebra } \mathcal{L} \text{ of } \mathcal{G}.$$

Then both assertions (a) and (b) of Theorem 3.7 hold for the subalgebras of $\mathcal{G}$ ([Be00c]). For example,

$$\sigma(a) = \{a_{ii}(t) \mid i = 1, \dots, m;\ t \in [0,1]\} = \{\tilde{\lambda}_{i,t}(a) \mid i = 1, \dots, m;\ t \in [0,1]\}$$

for every $a \in \mathcal{G}$. ■

Now, the preceding Examples 3.11–3.13 together with the result of Şt. Frunză from [Fr82] concerning the existence of weights for arbitrary solvable Lie algebras of operators (see also Corollary 2 in §27 of [BS01]) suggest the following question.

3.14. Question. Does Theorem 3.7 hold for arbitrary locally solvable Lie algebras of bounded linear operators on a complex Banach space?

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