GEOMETRY AND ANALYSIS ON LIE GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 55
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2002

REMARKS ON AN INTEGRABLE EVOLUTION EQUATION

A. ALEXANDROU HIMONAS and GERARD MISIOŁEK

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.
E-mail: himonas.1@nd.edu, misiolek.1@nd.edu

1. Introduction. The periodic one-dimensional evolution equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0 \tag{CH}$$

was introduced independently by Camassa and Holm [CH] and by Fuchssteiner and Fokas [FF]. It has since been the subject of extensive studies from algebraic, analytic and geometric points of view. In certain respects the CH equation resembles the well-known KdV equation. Like KdV it is bi-hamiltonian, admits soliton-type solutions and an infinite collection of first integrals. All of these manifestations of so-called complete integrability were already studied in [CH] and [CHH] but other approaches and further results can be found in [FF], [BSS1], [BSS2], [CM] and [KM]. Furthermore, as shown in [M1] (see also [Ko], [Mc2]) the CH equation can be derived as the equation for geodesics of a certain right-invariant metric on the group of diffeomorphisms of the circle. The derivation in [M1] follows a Lie-theoretic approach developed by V. Arnold [A], [AK] to study the Euler equations of hydrodynamics. Another derivation using Riemannian variational formulas is given in [M2] (see also [Mc2]) and will be recalled below. In fact, the equation that one derives on the group of diffeomorphisms has the form

$$\partial_t u + u\partial_x u + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) = 0,$$

where $\Lambda^s = (1-\partial_x^2)^{s/2}$. It turns out that written in this way the CH equation is much more convenient to study.

On the other hand the Cauchy problem for CH is not globally well-posed since for certain initial data the first derivative of the solution becomes unbounded in L^∞ norm in finite time (see again [CH] for a first result in this direction as well as the subsequent papers [CE1], [Mc1]). In another contrast to KdV, local well-posedness for CH is known

2000 *Mathematics Subject Classification*: 35Q53, 37K65, 35A10.

Received 18 July 2001.
The paper is in final form and no version of it will be published elsewhere.

merely in the Sobolev space H^s with $s > 3/2$ (for a detailed proof of this see [M2] in the periodic case and [LO] in the nonperiodic case).

In this note we revisit the local Cauchy problem. After recalling briefly the derivation of CH on the diffeomorphism group of the circle $\mathbb{T}$ and reviewing what we know concerning local well-posedness in Sobolev spaces we announce the following result:

THEOREM 1.1. *Let u_o be a (real) analytic function on $\mathbb{T}$. There exist an $\epsilon > 0$ and a unique solution u of the initial value problem*

$$\begin{aligned} &\partial_t u + u\partial_x u + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) = 0, \\ &u(0,x) = u_o(x), \qquad x \in \mathbb{T} \end{aligned} \tag{1.1}$$

which is analytic on $(-\epsilon, \epsilon) \times \mathbb{T}$.

This result can be viewed as a Cauchy-Kowalewski type theorem for CH. We will present an outline of the argument below. Our approach is inspired by the paper of Baouendi and Goulaouic [BG2]. Detailed proof of Theorem 1.1 will appear in a separate publication.

2. Derivation of the equation on $\mathcal{D}^s(\mathbb{T})$. In order to get started a certain number of technical facts concerning Sobolev maps and diffeomorphisms should be established. We will recall those needed below referring the interested reader to [P], [EM], [Om] or [M2] for details.

Let $H^s(\mathbb{T})$ be the space of Sobolev functions on $\mathbb{T}$ equipped with the norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}$. Consider the set $H^s(\mathbb{T},\mathbb{T})$ of maps from the circle into itself that are of Sobolev class H^s in every chart on $\mathbb{T}$. When $s > 1/2$ this set can be given a structure of an infinite dimensional Hilbert manifold whose tangent space at f consists of H^s vector fields on $\mathbb{T}$ pulled back by f. Let $C^1\mathcal{D}$ be the group of all bijections of the circle that are C^1 differentiable together with their inverses.

Let $s > 3/2$. Since $C^1\mathcal{D}$ is an open subset of $C^1(\mathbb{T},\mathbb{T})$ using the Sobolev embedding lemma we find that the set of orientation-preserving diffeomorphisms

$$\mathcal{D}^s(\mathbb{T}) = C^1\mathcal{D} \cap H^s(\mathbb{T},\mathbb{T})$$

is open in $H^s(\mathbb{T},\mathbb{T})$. Furthermore, it becomes a topological group under composition of diffeomorphisms. However, as is well known $\mathcal{D}^s(\mathbb{T})$ is not a Banach Lie group. In fact, it is not difficult to see that while right translations $R_\xi(\eta) = \eta \circ \xi$ are smooth, left translations $L_\xi(\eta) = \xi \circ \eta$ and the inversion map $i(\xi) = \xi^{-1}$ are only continuous. It is nevertheless convenient to use the Lie terminology when describing diffeomorphism groups.

On the Lie algebra of $\mathcal{D}^s(\mathbb{T})$ consider the H^1 inner product of vector fields u, v

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{T}} \left(u(x)v(x) + \partial_x u(x)\partial_x v(x)\right)\, dx \tag{2.1}$$

and using right translations define a corresponding inner product on each tangent space at $\xi \in \mathcal{D}^s(\mathbb{T})$ by

$$\langle X, Y\rangle_{H^1} = \left\langle d_\xi R_{\xi^{-1}} X, d_\xi R_{\xi^{-1}} Y\right\rangle_{H^1}, \tag{2.2}$$

where $X, Y \in T_\xi \mathcal{D}^s$ and $d_\xi R_{\xi^{-1}} X = X \circ \xi^{-1}$.

Recall that geodesics of a given metric can be obtained as critical points of the energy functional of that metric. Thus given $\xi \in \mathcal{D}^s(\mathbb{T})$ let $\bar{\gamma} : (-\delta,\delta) \times [0,1] \to \mathcal{D}^s$ be a one-parameter family of curves such that $\bar{\gamma}(s,0) = \mathrm{id}$ and $\bar{\gamma}(s,1) = \xi$ and denote by $W(t) = \partial_s \bar{\gamma}_t|_{s=0}$ the corresponding variation field along $\gamma(t) = \bar{\gamma}(0,t)$. Integrating by parts, changing variables and using the fact that $W(0) = W(1) = 0$, we obtain

$$\begin{aligned}
\partial_s\Big|_{s=0}\left(\frac{1}{2}\int_0^1 \|\dot{\bar{\gamma}}_t(s)\|^2_{H^1}\,dt\right) &= \int_0^1 \langle\, \partial_t\gamma\circ\gamma^{-1},\, \partial_s(\partial_t\bar{\gamma}\circ\bar{\gamma}^{-1})_{|_{s=0}}\rangle_{H^1}\,dt \\
&= -\int_0^1 \Big\langle \partial_t(\dot{\gamma}\circ\gamma^{-1}) + \dot{\gamma}\circ\gamma^{-1}\partial_x(\dot{\gamma}\circ\gamma^{-1}) \\
&\quad + \Lambda^{-2}\,\partial_x\left((\dot{\gamma}\circ\gamma^{-1})^2 + \frac{1}{2}\Big(\partial_x(\dot{\gamma}\circ\gamma^{-1})\Big)^2\right),\, W\circ\gamma^{-1}\Big\rangle_{H^1}\,dt.
\end{aligned}$$

Observing now that the variation $\bar{\gamma}$ was arbitrary we can pick a suitable variation field W on γ to conclude that the derivative of the energy functional vanishes if and only if the curve in the Lie algebra of vector fields on the circle $u(t) = \dot{\gamma}_t \circ \gamma_t^{-1}$ satisfies

$$\partial_t u + u\partial_x u + \Lambda^{-2}\partial_x(u^2 + \frac{1}{2}(\partial_x u)^2) = 0. \tag{2.3}$$

Thus, at least formally geodesics of the H^1 right-invariant metric (2.2) correspond to solutions of the CH equation[1]. That this correspondence can be made rigorous, in particular that given an arbitrary vector $u_o \in H^s(\mathbb{T})$ we can always find an H^s solution of the corresponding initial value problem for CH follows from the well-posedness results to which we turn next.

3. Local well-posedness in H^s. We collect results on well-posedness of the periodic Cauchy problem in fractional Sobolev spaces in the following theorem.

THEOREM 3.1. *Consider the Cauchy problem (1.1).*

1. *If $s > 3/2$, then given any $u_o \in H^s(\mathbb{T})$ there exists a $T > 0$ and a unique solution u to (1.1) such that $u \in C([0,T),H^s) \cap C^1([0,T),H^{s-1})$ and which depends continuously on the initial data u_o.*
2. *If $s < 3/2$, then there exist two sequences of solutions $u_n^1, u_n^2 \in C([0,\infty),H^s)$ of (1.1) such that for any $t > 0$*

$$\|u_n^2(0) - u_n^1(0)\|_{H^s} \le C_1(s)\,\frac{1}{n} \quad \text{and} \quad \|u_n^2(t) - u_n^1(t)\|_{H^s} \ge C_2(s,t)\,n^{2|s|+s},$$

   where $C_1(s)$ and $C_2(s,t)$ are positive constants depending only on s and s,t respectively.
3. *Let $s > 3/2$ and $u \in C([0,T),H^s(\mathbb{T})$ be a solution of (1.1). Let $K > 0$ be a constant such that $\|u(t)\|_{C^1_*} \le K$ for all $0 \le t < T$. Then there exists a $T' > T$ such that u can be extended beyond T in the space $C([0,T'),H^s(\mathbb{T})$.*

[1]One can similarly derive the CH equation on the one-dimensional universal central extension of the group of diffeomorphisms of the circle (see [M1] or [KM]). This group, called the Bott-Virasoro group, is also a suitable space to study the KdV equation (see [OK], [S] or [MR]).

The first statement (1) in the theorem above was essentially known to Ebin and Marsden [EM] who studied the Cauchy problem for the Euler equations. Their methods however do not extend directly to the case when $3/2 < s < 2$ is fractional. For the proof in this case we refer to [M2]. First results on local well-posedness of CH in the periodic case were published in [C], for u_o in $H^4(\mathbb{T})$, and in [CE1] for u_o in $H^3(\mathbb{T})$.

Statement (2) was proved in [HM1], using a method first introduced in [BPS]. An outline of this proof is given below. This result seems to suggest that the Sobolev index $s = 3/2$ may be sharp. We point out however that the statement in (2) disproves only the uniform continuity of the data-to-solutions map. Furthermore, the critical case $s = 3/2$ has yet to be settled.

The last statement (3) is proved in [M2]. It strengthens somewhat earlier results of this type given in [CE2].

Proof of Theorem 3.1(2). For any $c > 0$ the function (periodic 1−peakon)

$$u_c(x,t) = c \sum_{n\in\mathbb{Z}} e^{-|x+2\pi n - ct|}.$$

is a weak solution of the periodic CH equation (see for example [CE1] or [CM]) with initial condition

$$u_c(x,0) = c \sum_{n\in\mathbb{Z}} e^{-|x+2\pi n|}.$$

Computing the partial Fourier transform of u_c with respect to x, at $t = 0$ gives

$$\begin{aligned}\hat{u}_c(\xi,0) &= \frac{c}{2\pi}\int_0^{2\pi} e^{-ix\xi}\sum_{n\in\mathbb{Z}} e^{-|x+2\pi n|}\,dx \\ &= \frac{c}{2\pi}\sum_{n\in\mathbb{Z}}\int_{2\pi n}^{2\pi(n+1)} e^{-i(x-2\pi n)\xi}\, e^{-|x|}\,dx = \frac{c}{\pi}\,\frac{1}{1+\xi^2}.\end{aligned}$$

Then for any $t \geq 0$ we have

$$\hat{u}_c(\xi,t) = \frac{c}{\pi}\,\frac{e^{-ict\xi}}{1+\xi^2}.$$

Computing the norm at $t = 0$, we get

$$\|u_{c_2}(\cdot,0) - u_{c_1}(\cdot,0)\|^2_{H^s} = \frac{1}{\pi^2}(c_2-c_1)^2\sum_{\xi\in\mathbb{Z}}\left(1+\xi^2\right)^{s-2}. \tag{3.1}$$

Observe that the sum in (3.1) is finite if and only if $s < 3/2$. On the other hand, for $t > 0$ a computation gives

$$\begin{aligned}&\|u_{c_2}(\cdot,t) - u_{c_1}(\cdot,t)\|^2_{H^s} \\ &\quad= \frac{1}{\pi^2}\,(c_2-c_1)^2\sum_{\xi\in\mathbb{Z}}\frac{1}{(1+\xi^2)^{2-s}} + \frac{2}{\pi^2}\,c_1c_2\sum_{\xi\in\mathbb{Z}}\frac{1-\cos{(c_2-c_1)t\xi}}{(1+\xi^2)^{2-s}}.\end{aligned} \tag{3.2}$$

If, for given $n \in \mathbb{N}$, we choose

$$c_1 = n^{2(1+|s|)} \qquad \text{and} \qquad c_2 = n^{2(1+|s|)} + \frac{\pi}{n},$$

then we obtain

$$\|u_{c_2}(\cdot,t)-u_{c_1}(\cdot,t)\|_{H^s}^2 \geq \|u_{c_2}(\cdot,0)-u_{c_1}(\cdot,0)\|_{H^s}^2 + \frac{2}{\pi^2}\, n^{4(1+|s|)} \sum_{\xi\in\mathbb{Z}} \frac{1-\cos\frac{t\pi\xi}{n}}{(1+\xi^2)^{2-s}} \tag{3.3}$$

while

$$\|u_{c_2}(\cdot,0)-u_{c_1}(\cdot,0)\|_{H^s}^2 \leq \frac{1}{n^2}\sum_{\xi\in\mathbb{Z}} \frac{1}{(1+\xi^2)^{2-s}}. \tag{3.4}$$

Thus (3.4) is fine with $C_1(s)=\sum_{\xi\in\mathbb{Z}}\frac{1}{(1+\xi^2)^{2-s}}<\infty$. Regarding the inequality in (3.3), if $0<t\neq 1/2, 3/2,\ldots$ etc, then pick $\xi=n$. Otherwise, pick $\xi=2n$. In either case we get a lower bound by $C_2(s,t)\, n^{4|s|+2s}$, which gives the desired result. ∎

REMARK 3.2. Observe that the solutions u_n^1 and u_n^2 although close to each other at $t=0$ have large norms. A natural question is whether one can find other sequences of solutions that stay inside a fixed ball. This seems not to be possible using the $1-$peakon solutions. However it might be possible if one uses multipeakon solutions described in [BSS2] by Beals, Sattinger and Szmigielski.

4. Analytic regularity: an outline of the proof of Theorem 1.1. In this section we turn to the proof of Theorem 1.1 on analyticity in time and space variables of solutions to (1.1). We shall only give an outline of the proof here. Detailed arguments will appear elsewhere. The method is based on a contraction type argument in a suitably chosen scale of Banach spaces. Such an approach to analytic regularity of solutions to initial value problems was initiated by Ovsiannikov [O1], [O2] and later further developed in [Nr], [Ns] and and in the papers of Baouendi and Goulaouic [BG1], [BG2], who used it, among other things, to prove analyticity of solutions to Euler equations of hydrodynamics.

We begin by describing the spaces we need. For any $s>0$, we set

$$E_s=\left\{u\in C^\infty(\mathbb{T}) : |||u|||_s=\sup_{k\geq 0}\frac{\|\partial_x^k u\|_{H^2}\, s^k}{k!/(k+1)^2}<\infty\right\}.$$

It is not difficult to check that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0<s'<s$, E_s is continuously included in $E_{s'}$ with

$$|||u|||_{s'}\leq |||u|||_s.$$

Another simple consequence of the definitions is that any $u\in E_s$ is a real analytic function on $\mathbb{T}$. Less obvious, but crucial for our purposes, is the fact that each E_s forms an algebra (a Schauder ring) under pointwise multiplication of functions.

LEMMA 4.1. *Let $0<s<1$. There is a constant $c>0$ such that for any u and v in E_s we have*

$$|||uv|||_s\leq c|||u|||_s|||v|||_s. \tag{4.1}$$

Proof. The proof of this lemma is technical and we omit it. It essentially follows from a Leibniz rule and a corresponding well-known algebra property for H^2 Sobolev functions on the circle

$$\|fg\|_{H^2}\lesssim \|f\|_{H^2}\|g\|_{H^1}+\|g\|_{H^2}\|f\|_{H^1},$$

for all f,g in $H^2(\mathbb{T})$. ∎

Once again it will be most useful for our purposes to consider the CH equation in its "geodesic" form (2.3) in which it was derived on the diffeomorphism group $\mathcal{D}^s(\mathbb{T})$ in Section 2

$$\partial_t u + u\partial_x u + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) = 0.$$

In the next step we will rewrite it further as a system of equations. First however, we introduce some convenient notation

$$f(x) = x^2, \quad P_1 u = -\partial_x u, \quad \text{and} \quad P_2 u = -((1-\partial_x^2)^{-1}\partial_x)u,$$

with the help of which (2.3) becomes

$$\partial_t u = \left(\frac{1}{2}P_1 + P_2\right) f(u) + \frac{1}{2}P_2 f(P_1 u). \tag{4.2}$$

To transform this equation into a system, let

$$u_1 = u, \quad \text{and} \quad u_2 = P_1 u = -\partial_x u_1$$

so that

$$\begin{aligned} \partial_t u_1 &= (\frac{1}{2}P_1 + P_2) f(u_1) + \frac{1}{2}P_2 f(u_2) \doteq F_1(u_1, u_2) \\ \partial_t u_2 &= P_1(\partial_t u) = \frac{1}{2}P_1^2 f(u_1) + P_1 P_2 f(u_1) + \frac{1}{2}P_1 P_2 f(u_2) \\ &= -\frac{1}{2}P_1 \partial_x(u_1^2) + P_1 P_2 f(u_1) + \frac{1}{2}P_1 P_2 f(u_2) \\ &= P_1(u_1 u_2) + P_1 P_2 f(u_1) + \frac{1}{2}P_1 P_2 f(u_2) \doteq F_2(u_1, u_2) \end{aligned} \tag{4.3}$$

with the initial conditions

$$\begin{aligned} u_1(x,0) &= u(x,0) = u_0(x) \\ u_2(x,0) &= \partial_x u_1(x,0) = \partial_x u_0(x). \end{aligned}$$

Let $F = (F_1, F_2)$ denote the vector consisting of the right-hand sides of the system (4.3). The key step in the proof of Theorem 1.1 is establishing the following inequality.

Proposition 4.2. *Let $R > 0$. There is a constant $C > 0$ such that given arbitrary $0 < s' < s \le 1$, we have*

$$|||F(u_1, u_2) - F(v_1, v_2)|||_{s'} \le \frac{C}{s - s'}|||(u_1, u_2) - (v_1, v_2)|||_s,$$

for any u_j and v_j in the ball $B(0, R) \subset E_s$.

The meaning of the $|||\cdot|||_s$ norm in the above statement is clear from the context. For example, one may simply take $|||F|||_s := |||F_1|||_s + |||F_2|||_s$. The proof of Proposition 4.2 requires the following lemma, which provides suitable bounds in the E_s norms on the pseudodifferential operators P_1 and P_2 introduced above. Its proof is omitted.

Lemma 4.3. *There is a constant $c > 0$ such that for any $0 < s' < s < 1$, we have*

$$|||P_1 u|||_{s'} \le \frac{c}{s - s'}|||u|||_s, \tag{4.4}$$

and

$$|||P_2 u|||_s \le |||u|||_s. \tag{4.5}$$

Proof of Proposition 4.2. Observe that the nonlinear terms can be easily handled with the help of the algebra property in Lemma 4.1, since for any $s > 0$ we easily have

$$|||f(u) - f(v)|||_s \le c|||u^2 - v^2|||_s \le c|||u + v|||_s\, |||u - v|||_s.$$

Using this together with Lemma 4.3 we can now estimate

$$\begin{aligned}
&|||F(u_1, u_2) - F(v_1, v_2)|||_{s'} = |||F_1(u_1, u_2) - F_1(v_1, v_2)|||_{s'} + |||F_2(u_1, u_2) - F_2(v_1, v_2)|||_{s'}\\
&\quad \le \frac{1}{2}|||P_1(f(u_1) - f(v_1))|||_{s'} + |||P_2(f(u_1) - f(v_1))|||_{s'} + \frac{1}{2}|||P_2(f(u_2) - f(v_2))|||_{s'}\\
&\qquad + |||P_1(u_1u_2 - v_1v_2)|||_{s'} + |||P_1P_2(f(u_1) - f(v_1))|||_{s'} + \frac{1}{2}|||P_1P_2(f(u_2) - f(v_2))|||_s\\
&\quad \le \frac{1}{2}\frac{c}{s - s'}|||f(u_1) - f(v_1)|||_s + |||f(u_1) - f(v_1)|||_{s'} + \frac{1}{2}|||f(u_2) - f(v_2)|||_{s'}\\
&\qquad + \frac{c}{s - s'}(|||u_2(u_1 - v_1)|||_s + |||v_1(u_2 - v_2)|||_s)\\
&\qquad + \frac{c}{s - s'}\left(|||f(u_1) - f(v_1)|||_s + \frac{1}{2}|||f(u_2) - f(v_2)|||_s\right)\\
&\quad \le \frac{1}{2}\frac{c}{s - s'}|||u_1 + v_1|||_s|||u_1 - v_1|||_s + \frac{1}{2}c|||u_2 + v_2|||_{s'}|||u_2 - v_2|||_{s'}\\
&\qquad + \frac{c}{s - s'}(|||u_2|||_s|||u_1 - v_1|||_s + |||v_1|||_s|||u_2 - v_2|||_s)\\
&\qquad + \frac{c}{s - s'}\left(|||u_1 + v_1|||_s|||u_1 - v_1|||_s + \frac{1}{2}|||u_2 + v_2|||_s|||u_2 - v_2|||_s\right),
\end{aligned}$$

from which the desired inequality follows by picking for example $C = 600cR$. ■

Our main Theorem 1.1 is now a straightforward consequence of the following result, whose different variants and extensions can be found in the papers mentioned in the beginning of this section. The version below comes from [BG1].

THEOREM 4.4. *Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces, so that for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \le |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt} = F(t, u(t)) \\ u(0) = 0. \end{cases} \tag{4.6}$$

Let T, R and C be positive numbers and suppose that F satisfies the following conditions

1.) If for $0 < s' < s < 1$ the function $t \longmapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \le T$ with values in X_s and

$$\sup_{|t|\le T} |||u(t)|||_s < R,$$

then $t \longmapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

2.) For any $0 < s' < s \le 1$ and any $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,

$$\sup_{|t|\le T} |||F(t, u) - F(t, v)|||_{s'} \le \frac{C}{s - s'}|||u - v|||_s.$$

3.) There exists $M > 0$ such that for, any $0 < s < 1$,

$$\sup_{|t|\leq T} |||F(t,0)||| \leq \frac{M}{1-s}.$$

Then there exists a $T_0 \in (0,T)$ and a unique function $u(t)$, which for every $s \in (0,1)$ is holomorphic in $|t| < (1-s)T_0$ with values in X_s, and is a solution to the initial value problem (4.6).

The conditions (1) through (3) above are now easily verified and Theorem 1.1 follows. ∎

5. Acknowledgements. Both authors were supported in part by the NSF Grant DMS-9970857. The second author (G.M.) thanks Yann Brenier and Sasha Shnirelman for interesting conversations which stimulated our interest in the problem of analytic regularity for the CH equation.

References

[A] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) 16 (1966).

[AK] V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Springer Verlag, New York 1998.

[BG1] S. Baouendi and C. Goulaouic, *Remarks on the abstract form of nonlinear Cauchy–Kovalevski theorems*, Comm. Partial Differential Equations 2 (1977).

[BG2] S. Baouendi and C. Goulaouic, *Sharp estimates for analytic pseudodifferential operators and application to Cauchy problems*, J. Differential Eqns. 48 (1983).

[BSS1] R. Beals, D. Sattinger and J. Szmigielski, *Acoustic scattering and the extended Korteweg-de Vries hierarchy*, Adv. Math. 140 (1998).

[BSS2] R. Beals, D. Sattinger and J. Szmigielski, *Multipeakons and a theorem of Stieltjes*, Inverse Problems 15 (1999).

[BPS] B. Birnir, G. Ponce and N. Svanstedt, *The local ill-posedness of the modified KdV equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996).

[CH] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. 71 (1993).

[CHH] R. Camassa, D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Math. 31 (1994).

[C] A. Constantin, *On the Cauchy problem for the periodic Camassa-Holm equation*, J. Differential Equations 141 (1997).

[CE1] A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. 51 (1998).

[CE2] A. Constantin and J. Escher, *Global weak solutions for a shallow water equation*, Indiana Univ. Math. J. 47 (1998).

[CM] A. Constantin and H. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math. 52 (1999).

[EM] D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. 92 (1970).

[FF] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäklund transformations and hereditary symmetries*, Phys. D 4 (1981).

[HM1] A. Himonas and G. Misiołek, *The Cauchy problem for an integrable shallow water equation*, Differential Integral Equations 14 (2001).

[HM2] A. Himonas and G. Misiołek, *Analyticity of the Cauchy problem for an integrable evolution equation*, preprint.

[KM] B. Khesin and G. Misiołek, *Euler equations on homogeneous spaces and Virasoro orbits*, preprint.

[Ko] S. Kouranbaeva, *The Camassa-Holm equation as a geodesic flow on the diffeomorphism group*, J. Math. Phys. 40 (1999).

[LO] Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations 162 (2000).

[Mc1] H. McKean, *Breakdown of a shallow water equation*, preprint.

[Mc2] H. McKean, *Shallow water & the diffeomorphism group*, Applied and industrial mathematics, Venice–2 1998, Kluwer, Dordrecht 2000.

[MR] P. Michor and T. Ratiu, *On the geometry of the Virasoro-Bott group*, J. Lie Theory 8 (1999).

[M1] G. Misiołek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys. 24 (1998).

[M2] G. Misiołek, *Classical solutions of the periodic Camassa-Holm equation*, preprint.

[Nr] L. Nirenberg, *An abstract form of the nonlinear Cauchy-Kowalevski theorem*, J. Differential Geom. 6 (1972).

[Ns] T. Nishida, *A note on a theorem of Nirenberg*, J. Differential Geom. 12 (1977).

[Om] H. Omori, *On the group of diffeomorphisms on a compact manifold*, in: Proc. Symp. Pure Math. 15 (Global Analysis), AMS, Providence, 1970.

[OK] V. Ovsienko and B. Khesin, *The super Korteweg-de Vries equation as an Euler equation*, Funct. Anal. Appl. 21 (1987).

[O1] L.V. Ovsiannikov, *Non-local Cauchy problems in fluid dynamics*, Actes Congrès Int. Math. Nice 3 (1970).

[O2] L.V. Ovsiannikov, *A nonlinear Cauchy problem in a scale of Banach spaces*, Dokl. Akad. Nauk SSSR 200 (1971).

[P] R. Palais, *Foundations of Global Nonlinear Analysis*, Benjamin, New York, 1968.

[S] G. Segal, *The geometry of the KdV equation*, Int. J. Modern Phys. A 6 (1991).