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SYMMETRIC SPACES WITH JORDAN STRUCTURES

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Abstract. In a series of three lectures we give an introduction to *symmetric spaces* with a certain *additional structure*: in the first lecture we present examples of interesting structures on symmetric spaces; in particular, *generalized conformal structures* and the problem of determining their automorphism groups gave rise to the investigation of *Jordan structures* which are the topic of the following two lectures: we introduce them via the problem of finding a *twisted complexification* of a symmetric space; the main result to be proved here is that such complexifications correspond bijectively to *Jordan extensions of the curvature*. Classification shows that, for classical symmetric spaces, there is "generically" one and only one Jordan extension; it is an open and deep problem to find a good conceptual explanation of this fact.

The following notes follow closely in style and contents the lectures I have given in Będlewo during the week from September 11th to September 15th, 2000; I have tried to keep them as elementary as possible. The material covered by the last two lectures is taken from my paper [Be01]; a more detailed exposition of the whole theory as well as many more references to related topics can be found in [Be00].

I thank the organizers of the "Workshop on Lie gropus and Lie algebras" held in Będlewo in September 2000, and in particular, Aleksander Strasburger, for inviting me to give this series of lectures at this wonderful place which proved to be a perfect meeting-ground for mathematicians from the east and from the west.

1. Introduction: "Rich symmetric spaces". Let me first recall some basic notions on "poor" symmetric spaces (by this I mean symmetric spaces without any additional structure) and then give some examples of "rich" symmetric spaces (that is, symmetric spaces having some interesting extra structure).

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1.1. *Symmetric spaces.* There are four definitions of a symmetric space M:

(1) The group theoretic one: $M = G/H$ is a homogeneous space, where H is essentially the group of fixed points of an involution σ of a Lie group G.
(2) The differential geometric one: M is a real manifold with a complete torsionfree affine connection ∇ whose curvature is covariantly constant: $\nabla R = 0$.
(3) The mixed one: M is a real manifold with a complete affine connection ∇ such that the geodesic symmetry σ_p (which is defined by $\sigma_p(\mathrm{Exp}_p(v)) = \mathrm{Exp}_p(-v)$) with respect to any point $p \in M$ is an automorphism of ∇.
(4) The algebraic one (due to O. Loos [Lo69]), which reflects axiomatically the properties of the map $\mu : M \times M \to M$ given by $\mu(x, y) := \sigma_x(y)$ with σ_x as in (3).

In these lectures I will mainly use the group theoretic definition (1) of a symmetric space since it is the one which is almost exclusively used in harmonic analysis. However, my favorites are really (2) and (4) since they contain much of the geometry and lead to good categories. Starting from (1), it is rather easy to arrive at (2)–(4) (cf. [Be99]); the converse requires considerably more work (cf. [KoNo69] and [Lo69]). Here, I will just briefly explain how definition (4) is related to (3): defining $\mu : M \times M \to M$ as above, one easily sees that this map has the properties

(M1) $\mu(x, x) = x$,
(M2) $\mu(x, \mu(x, y)) = y$,
(M3) σ_x is an automorphism of μ, i.e. $\mu(x, \mu(y, z)) = \mu(\mu(x, y), \mu(x, z))$,
(M4) the fixed point x of σ_x is isolated.

Definition 1.1 (O. Loos [Lo69]). A *symmetric space* is a real manifold M with a smooth "multiplication map" $\mu : M \times M \to M$ satisfying (M1)–(M4). ∎

Theorem 1.2 (O. Loos [Lo69]). *Any connected symmetric space is of the form $M = G/H$ as in* (1) *above; one can take $G := G(M)$ to be the group generated by all $\sigma_x\sigma_y$ with $x, y \in M$, called the* transvection group. ∎

Example 1.3. The group case: $M = G$ is a Lie group; the symmetry w.r.t. the unit element is the inversion: $\sigma_e(x) = x^{-1}$, and transporting this to an arbitrary point y yields $\mu(y, x) = yx^{-1}y$.

Example 1.4. Let M be the space of invertible real *symmetric* $n \times n$ matrices with $\mu(Y, X) = YX^{-1}Y$ as above. This is the sub-symmetric space of $\mathrm{GL}(n, \mathbb{R})$ fixed under the automorphism "transposed". It is non-connected; its connected components are of the form $M_{p,q} = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(p, q)$ where (p, q) is the signature of the matrix X. ∎

1.2. *Symmetric spaces with additional structures.* Here are examples of G-invariant (to be precise: $G(M)$-invariant) structures S on a symmetric space $M = G/H$—the most classical structure is certainly the one given in the following Example (1):

(1) S is an invariant Riemannian metric; (M, S) is a *Riemannian symmetric space* (classical theory started by E. Cartan, see the book [He78] by S. Helgason).
(2) S is an invariant pseudo-Riemannian tensor field: (M, S) is a *pseudo-Riemannian symmetric space.* The geometry is much more complicated than for (1) due to the

lack of a de Rham decomposition (important work on such spaces has been done by M. Berger and his school; see the article [B57]).

(3) S is an invariant symplectic structure; (M, S) is a *symplectic symmetric space*. Examples are the one-sheeted hyperboloid

$$M = \mathrm{SO}(2,1)/\,\mathrm{SO}(1,1) \tag{1.1}$$

or the spaces

$$M = \mathrm{GL}(p+q,\mathbb{R})/(\mathrm{GL}(p,\mathbb{R}) \times \mathrm{GL}(q,\mathbb{R})). \tag{1.2}$$

These spaces are interesting since they may provide a good context for quantization.

(4) S is an invariant (almost) complex structure (see Chapter 2 for definition). Examples are: the unit disc

$$M = D = \mathrm{SU}(1,1)/\,\mathrm{SO}(2), \tag{1.3}$$

and more generally any *Hermitian symmetric space*, but also all symmetric spaces of the form $G_{\mathbb{C}}/H_{\mathbb{C}}$ with complex Lie groups and holomorphic involution σ.

(5) S is an invariant (almost) para-complex structure or polarization (see Section 3.2 for definition). These spaces have locally a direct product structure (Example: the space given by Equation (1.2) above), but of course global direct products $M = M_1 \times M_2$ also fall into this category. Spaces of the type as in (1.2) are well-known by the work of S. Kaneyuki (see [KanKo85]).

(6) S is an invariant quaternionic structure; (M, S) is a *symmetric quaternion-Kähler manifold*. The mere definition of this structure is not at all evident—see Chapter 14 in [Bes87]. Important work on such spaces is due to J. Wolf.

(7) S is an invariant causal structure; (M, S) is a *causal symmetric space*: roughly, $S = (C_x)_{x\in M}$ is a distribution of regular cones C_x living in tangent spaces T_xM; cf. the lectures by J. Faraut or [HO96] for the precise definition. Examples are given by the one-sheeted hyperboloid (see Equation (1.1)) or by the group case $M = \mathrm{U}(n)$ (the tangent space at the unit element is $i\,\mathrm{Herm}(n,\mathbb{C})$, and the latter contains the model cone C_e of positive definite Hermitian matrices).

- We get even richer structures by looking at intersections of the preceding categories: for instance, the intersection $(7)\cap(1)$ contains the interesting class of *symmetric cones* (cf. lectures by J. Faraut and the book [FK94]) and of their *compact duals* such as for instance $\mathrm{U}(n)$. I have the impression that these spaces essentially exhaust $(7)\cap(1)$.
- Similarly, $(7)\cap(3)$ contains the interesting class of *Cayley-type spaces* (their definition is given in the lecture by J. Faraut; see also the book [HO96]); for instance, the one-sheeted hyperboloid or the space

$$M = \mathrm{Sp}(n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{R}) \tag{1.4}$$

are of this kind. As above, I have the impression that, essentially, these spaces exhaust the class $(7)\cap(3)$.

(8) S is a *generalized conformal structure* in the sense introduced by Gindikin, Goncharov, Kaneyuki and others:

Definition 1.5 [GiKa98]. A *generalized conformal structure* (abbreviated GCS) on a manifold M over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is given by conal sets $C_x \subset T_xM$ (this means just that C_x

is stable under multiplication by non-zero scalars in T_xM) such that all C_x are linearly isomorphic to one fixed model set $C \subset \mathbb{F}^n$ having the following property: the group of linear automorphisms of C,

$$G(C) := \mathrm{Aut}(C) := \{g \in \mathrm{GL}(n, \mathbb{F}) |\, g(C) = C\},$$

has an *open orbit* in $\mathbb{F}^n$. ■

A typical example of a GCS is given by the *real Grassmannian*

$$M = \mathrm{Gras}_{p,p+q}(\mathbb{R}) = \{E \subset \mathbb{R}^{p+q} |\, E\,\text{subspace with}\;\dim E = p\} \tag{1.5}$$

which carries the structure of a symmetric space

$$M \cong \mathrm{O}(p+q)/(\mathrm{O}(p) \times \mathrm{O}(q)). \tag{1.6}$$

As base point $o \in M$ we fix the canonical decomposition $\mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q$. Then the tangent space T_oM can be naturally identified with the space $V := \mathrm{Mat}(q, p, \mathbb{R})$ of real $q \times p$-matrices: to an element $X \in V$ we associate its *graph*

$$\Gamma_X = \{(v, Xv) |\, v \in \mathbb{R}^p\} \in \mathrm{Gras}_{p,p+q}(\mathbb{R}).$$

Then

$$\Gamma : V \to M, \quad X \mapsto \Gamma_X$$

is an imbedding with open dense image, and we can identify $\Gamma(V)$ with T_oM. In T_oM we consider the conical set

$$C := C_o := \{X \in \mathrm{Mat}(q, p, \mathbb{R}) |\, \mathrm{rank}(X) < \min(p, q)\}$$

of matrices having rank less than the maximal possible rank. The set C is stable under the action of $\mathrm{GL}(q, \mathbb{R}) \times \mathrm{GL}(p, \mathbb{R})$ from left and right,

$$(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})) \times V \to V, \quad ((g, h), X) \mapsto gXh^{-1},$$

and the group $G(C)$ is (up to connected components in case $p = q$) given by the effective group of this action. It is clear that this group has an open orbit in $\mathrm{Mat}(q, p, \mathbb{R})$, namely the set of all matrices having the maximal possible rank $\min(p, q)$. Therefore $(C_x)_{x \in M}$ with $C_{g.o} = T_og \cdot C_o$ $(g \in \mathrm{O}(p+q))$ defines a GCS on M.

Automorphisms. Given our structure S, an immediate question is to determine the automorphism group of S: by assumption, S is $G(M)$-invariant, that is, $G(M) \subset \mathrm{Aut}(S)$. What about the converse? Let us look again at the examples:

(1) In the Riemannian semisimple case we have equality; in the flat case $G(M)$ is the translation group which is smaller than $\mathrm{Aut}(S)$ (group of isometries of a Euclidean vector space). However, we still have $\mathrm{Aut}(M) = \mathrm{Aut}(S)$.
(3) In the symplectic case there is no hope to get equality: the group of automorphisms of a symplectic structure is in general infinite dimensional, whereas $\mathrm{Aut}(M)$ always is a finite dimensional Lie group.
(4) In the holomorphic case the situation is similar. However, in certain cases such as bounded symmetric domains, the group $\mathrm{Diff}_{hol}(M)$ of globally defined holomorphic

diffeomorphisms may be equal to $G(M)$. The "pseudogroup" of locally defined holomorphic diffeomorphisms is always "infinite dimensional" (since locally we are just on some $\mathbb{C}^n$; see [Ko72] for the formal definition of pseudogroups).

(7) No generalities are known for causal diffeomorphisms on causal symmetric spaces (it seems that the coupling of the two structures is not "rigid" enough to permit general statements). As above, it is important to distinguish local and global causal diffeomorphisms. In some cases they can be entirely described, e.g. in the case $M = \mathrm{U}(n)$ $(n > 1)$: here the locally defined causal maps always extend to global ones, given by elements of the group $\mathrm{SU}(n,n)$ acting via the Cayley transform on $\mathrm{U}(n)$. (This has been conjectured by I. Segal [Se76] and proved by S. Kaneyuki [Kan89] using Cartan-connections and by the author [Be96a,b] using Jordan theoretic methods.)

(8) The same remarks as in (7) apply to GCS. In the case of the real Grassmannians one can deduce from a theorem by Chow [Ch49] (see also [D63]) that locally defined conformal diffeomorphisms always extend globally, given by an element of $\mathbb{P}\,\mathrm{GL}(p+q,\mathbb{R})$.

What has all this to do with *Jordan* structures? Although I will not really be able to explain this here, I would say that Jordan structures have an interesting interaction with all structures mentioned so far; in a way, they single out the "good" and "non-pathological" cases (where for instance the automorphism problem can be solved) and thus provide a means to understand a bit of all this.

2. Complexifications and Hermitifications. Among all the structures mentioned in the first lecture, I will choose the *complex structures* as point of departure. The problem I am interested in can be motivated as follows by ordinary linear algebra: the Euclidean vector space $(\mathbb{R}^n, \sum_i x_i y_i)$ admits two different kinds of complexification:

$$\begin{array}{ccc} & (\mathbb{R}^n, \sum_i x_i y_i) & \\ \swarrow & & \searrow \\ (\mathbb{C}^n, \sum_i z_i w_i) & & (\mathbb{C}^n, \sum_i z_i \overline{w}_i) \end{array}$$

—the first is the ordinary complexification functor $\otimes_{\mathbb{R}}\mathbb{C}$ associating to a $\mathbb{R}$-bilinear form b the $\mathbb{C}$-bilinear extension $b_{\mathbb{C}}$, and the second is the "Hermitification" $b_{\mathbb{C}}(z,\overline{w})$ of b; for real analysis this is in a sense more interesting than $b_{\mathbb{C}}$. Now the question is: *can something similar be done for symmetric spaces:*

$$\begin{array}{ccc} & M = G/H & \\ \swarrow & & \searrow \\ M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} & & \text{Hermitification ?} \end{array}$$

I will explain that the ordinary complexification functor is indeed perfectly well-defined for all symmetric spaces (left hand side) and that there is a good definition of a Hermitification or *twisted complexification*, but the problem of existence and uniqueness of the latter is far from being trivial. It was trivial in our motivating linear algebra example, essentially because vector spaces are *flat*, whereas general symmetric spaces are *curved* and thus there will be "curvature obstructions". Therefore I will start by recalling the linear algebra of the curvature of a symmetric space.

2.1. *Infinitesimal version.* Assume that $M = G/H$ is a symmetric space with involution $\sigma : G \to G$ and base point $o = eH$. As usual, one decomposes the Lie algebra $\mathfrak{g}$ of G into the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ of ± 1-eigenspaces of the differential $\dot{\sigma} : \mathfrak{g} \to \mathfrak{g}$. Then $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, and therefore, for all $X, Y, Z \in \mathfrak{q}$,

$$[X, Y, Z] := -R(X, Y)Z := [[X, Y], Z]$$

again belongs to $\mathfrak{q}$. (Evaluation of vector fields gives a bijection $\mathfrak{q} \to T_o M$, and under this identification $R(X, Y)Z$ really *is* the curvature tensor at the base point.) The following properties are immediately verified:

(LT1) $R(X, Y) = -R(Y, X)$
(LT2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
(LT3) $D := R(X, Y)$ is a *derivation* of $[\cdot, \cdot, \cdot]$, i.e.

$$D[U, V, W] = [DU, V, W] + [U, DV, W] + [U, V, DW].$$

Definition 2.1. A *Lie triple system* (LTS) is a vector space $\mathfrak{q}$ together with a trilinear map satisfying the properties (LT1)–(LT3). ■

Theorem 2.2. *There is a bijection between (real finite dimensional) Lie triple systems and (connected simply conncected) symmetric spaces with base point.*

Proof. We have seen how to associate a LTS to a symmetric space. For the converse, we associate to a LTS $\mathfrak{q}$ the Lie algebra $\mathfrak{g} := \mathfrak{q} \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathrm{Der}(\mathfrak{q})$ is the algebra of derivations of the LTS $\mathfrak{q}$ with bracket defined by $[(X, D), (X', D')] = (DX' - D'X, [D, D'] - R(X.X'))$, and then construct in the usual way a corresponding connected simply connected symmetric space G/H. (The bijection is in fact an *equivalence of categories*, but this is more difficult to prove—see [Lo69].) ■

Now, invariant objects such as tensor fields on $M = G(M)/H$ can be reduced to H-invariant objects on the LTS $\mathfrak{q}$, and if M is connected simply connected (or if we just look at the local situation), this is equivalent to invariance under $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$. In particular, we are interested in complex structures:

Definition 2.3. An *almost complex structure* on M is a tensor field $(\mathcal{J}_x)_{x \in M}$ of type $(1, 1)$ such that $\mathcal{J}_x^2 = -\,\mathrm{id}_{T_x M}$ for all $x \in M$. ■

By the preceeding remark, an almost complex structure on M is $G(M)$-invariant iff we have

$$R(X, Y)JZ = JR(X, Y)Z \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{q}$, with $J = \mathcal{J}_o$. (If you are a geometer, you may read this also as a tensor field formula.) It is known that invariant complex structures on symmetric spaces are *integrable*, but this will not be used before the final Chapter 4.

2.2. *Straight complexification*

Proposition 2.4. *Any real LTS* $(\mathfrak{q}, R)$ *admits a unique extension to a* $\mathbb{C}$*-trilinear LTS* $(\mathfrak{q}_{\mathbb{C}}, R_{\mathbb{C}})$ *on* $\mathfrak{q}_{\mathbb{C}} = \mathfrak{q} \otimes_{\mathbb{R}} \mathbb{C}$.

Proof. By the universal property of tensor products, $\otimes_{\mathbb{R}}\mathbb{C}$ is a functor, i.e. all identities (such as (LT1)–(LT3)) which can be expressed by commutative diagrams remain valid after complexification. (If you are more used to work with Lie algebras, you may use the corresponding fact for Lie algebras and note that the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{q}_{\mathbb{C}}$ of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ defines the desired LTS $\mathfrak{q}_{\mathbb{C}}$.) ■

On the space level, the inclusion $(\mathfrak{q}, R) \subset (\mathfrak{q}_{\mathbb{C}}, R_{\mathbb{C}})$ lifts locally to an inclusion of spaces

$$M = G/H \to M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} \tag{2.2}$$

which we call the *straight complexification of M* (it is not always global, but for the moment we are only interested in the local situation).

2.3. *Twisted complexifications.* I will motivate my definition of a twisted complexification by the well-known theory of *Hermitian symmetric spaces*, but one may treat at the same time also the *pseudo-Hermitian case*:

Definition 2.5. We say that $(M, g, \mathcal{J})$ is a *pseudo-Hermitian symmetric space* if M is a symmetric with an invariant pseudo-Riemannian tensor field g and an invariant almost complex structure $\mathcal{J}$ such that, for all vector fields X, Y, we have $g(\mathcal{J}X, \mathcal{J}Y) = g(X, Y)$, i.e.

$$g(\mathcal{J}X, Y) = -g(X, \mathcal{J}Y). \ ■$$

Lemma 2.6. *If $(M, g, \mathcal{J})$ is a pseudo-Hermitian symmetric space, then the identity*

$$R(\mathcal{J}X, Y) = -R(X, \mathcal{J}Y)$$

holds.

Proof. There is a very useful identity in (pseudo-) Riemannian geometry (see [Hel78] or [Lo69]):

$$g(R(X, Y)U, V) = g(R(U, V)X, Y).$$

We apply it twice in the following calculation:

$$\begin{aligned} g(R(\mathcal{J}X, Y)U, V) &= g(R(U, V)\mathcal{J}X, Y) = g(\mathcal{J}R(U, V)X, Y) \\ &= -g(R(U, V)X, \mathcal{J}Y) = -g(R(X, \mathcal{J}Y)U, V). \end{aligned}$$

Since g is non-degenerate, we get the claim. ■

Definition 2.7. An invariant almost complex structure $\mathcal{J}$ on a symmetric space M (resp. on a LTS $(\mathfrak{q}, R)$) is called *twisted* if it satisfies the identity

$$R(\mathcal{J}X, Y) = -R(X, \mathcal{J}Y).$$

A *twisted complexification* $(M_{h\mathbb{C}}, \mathcal{J})$ of a symmetric space M is a symmetric space $M_{h\mathbb{C}}$ with invariant twisted complex structure $\mathcal{J}$ such that M is (locally) isomorphic to a *real form* of $M_{h\mathbb{C}}$ (that is, M is the space fixed under a *conjugation* τ: τ is an automorphism of $M_{ph\mathbb{C}}$ of order 2 and is almost anti-holomorphic in the sense that $\tau \cdot \mathcal{J} = -\mathcal{J}$). ■

Example 2.8. If $M = \mathrm{O}(p+q)/(\mathrm{O}(p) \times \mathrm{O}(q))$ is the real Grassmannian, then a twisted complexification is given by the complex Grassmannian $M_{h\mathbb{C}} = \mathrm{U}(p+q)/(\mathrm{U}(p) \times \mathrm{U}(q))$; the latter is well-known to be a compact Hermitian symmetric space and thus by Lemma 2.6 is twisted complex. More examples will be given later (Chapter 3). ■

Now two questions arise:

A) does M have a twisted complexification?
B) if so, how many (inequivalent ones) are there?

The answer is in general not known, however, a more careful analysis by using *Jordan theory* suggests that it might be: "generically, there is one and only one!"

3. Jordan triple systems. Historically, Max Koecher was the first to realize that there is a close connection between some *Jordan algebras* and some *bounded symmetric domains* (see [Koe69]); later O. Loos [Lo77] (see also the book [Sa80]) put this into a more precise form by establishing a bijection between *positive Hermitian Jordan triple systems* and *bounded symmetric domains*. In this chapter I will extend this to much more general bijections between certain geometric and algebraic objects.

3.1. *Hermitian JTS.* Assume that $(M, \mathcal{J})$ is a symmetric space with invariant twisted almost complex structure $\mathcal{J}$.

DEFINITION 3.1. The *structure tensor* T of $(M, \mathcal{J})$ is the invariant $(3,1)$-tensor field defined by

$$\begin{aligned} T(X,Y,Z) := T(X,Y)Z &:= -\frac{1}{2}(R(X,Y)Z + \mathcal{J}R(X,\mathcal{J}Y)Z) \\ &= -\frac{1}{2}(R(X,Y)Z - \mathcal{J}R(X,\mathcal{J}^{-1}Y)Z). \ \blacksquare \end{aligned}$$

The last line shows that $T(X,\cdot,Z)$ is the $\mathbb{C}$ conjugate-linear part of the endomorphism $-R(X,\cdot)Z$. This gives the first item of the following list of properties; the others are also very easily verified, so I leave the proof as a useful exercise to the reader.

PROPOSITION 3.2. *The structure tensor has the following properties:*

(1) $T(X,\mathcal{J}Y,Z) = -\mathcal{J}T(X,Y,Z)$
(2) $T(X,Y,\mathcal{J}Z) = \mathcal{J}T(X,Y,Z)$
(3) $T(X,Y,Z) = T(Z,Y,X)$
(4) $T(\mathcal{J}X,Y,Z) = \mathcal{J}T(X,Y,Z)$
(5) $T(X,Y,Z) - T(Y,X,Z) = -R(X,Y)Z$
(6) $\mathfrak{h}$ *acts as a Lie algebra of derivations of* T
(7) $\mathcal{J}$ *is a derivation of* T*, i.e.*

$$\mathcal{J}T(X,Y,Z) = T(\mathcal{J}X,Y,Z) + T(X,\mathcal{J}Y,Z) + T(X,Y,\mathcal{J}Z). \ \blacksquare$$

PROPOSITION 3.3. *The structure tensor satisfies the following identities:*

(JT1) $T(U,V,W) = T(W,V,U)$
(JT2) $T(X,Y)T(U,V,W) = T(T(X,Y)U,V,W) - T(U,T(Y,X)V,W) + T(U,V,T(X,Y)W)$.

Proof. (JT1): this is (3) above.

(JT2): If D is a derivation of T, then JD is a *skew-derivation* of T in the following sense:

$$\begin{aligned} JD\ T(X,Y,Z) &= J(T(DX,Y,Z) + T(X,DY,Z) + T(X,Y,DZ)) \\ &= T(JD\,X,Y,Z) - T(X,JD\,Y,Z) + T(X,Y,JD\,Z). \end{aligned}$$

Now, $R(X,Y)$ is a derivation of T (since $\mathfrak{h}$ acts by derivations of T). Thus $J\,R(X,JY)$ is a skew-derivation; the sum of these two is $-2T(X,Y)$. Their difference is

$$\begin{aligned} R(X,Y)-JR(X,JY) &= R(X,Y)+JR(JX,Y) \\ &= -(R(Y,X)+JR(Y,JX)) = 2T(Y,X). \end{aligned}$$

Using this, we get

$$\begin{aligned} T(X,Y)T(U,V,W) &= -\frac{1}{2}(R(X,Y)\cdot T(U,V,W)+J\,R(X,JY)\cdot T(U,V,W)) \\ &= -\frac{1}{2}(T\big((R(X,Y)+J\,R(X,JY))U,V,W\big)+ \\ &\qquad T\big(U,(R(X,Y)-J\,R(X,JY))V,W\big)+ \\ &\qquad T\big(U,V,(R(X,Y)+J\,R(X,JY))W\big)) \\ &= T\big(T(X,Y)U,V,W\big)-T\big(U,T(Y,X)V,W\big)+T\big(U,V,T(X,Y)W\big). \end{aligned}$$

This is the identity (JT2). ∎

Definition 3.4. A *Jordan triple system* (abbreviated JTS) is a vector space V with a trilinear map $T: V\times V\times V\to V$ satisfying the identities (JT1) and (JT2) from the preceding proposition. A *Hermitian JTS* is a JTS (V,T) with a complex structure J such that T is J-linear in the outer varibales and J-antilinear in the middle variable. ∎

Thus Propositions 3.3 and 3.2 say that the structure tensor defines a Hermitian JTS on $\mathfrak{q}$. For the converse we need

Lemma 3.5 (Meyberg 1970). *For any JTS T, the formula*

$$R(X,Y)Z = -(T(X,Y,Z)-T(Y,X,Z))$$

defines a LTS $R:=R_T$ on V.

Proof. (LT1) is immediate; for (LT2) use (JT1), and for (LT3) note first that (JT2) implies that $R(X,Y)$ is a derivation of T; but any derivation of T is also one of R; so $R(X,Y)$ is a derivation of R. ∎

An easy calculation shows that R_T is twisted complex if T is Hermitian. Summing up, we have proved the following result:

Theorem 3.6. *There is a bijection between Hermitian JTS and twisted complex LTS and thus also between Hermitian JTS and (connected simply connected) twisted complex symmetric spaces with base point.* ∎

With the suitable definitions, one realizes that these bijections are in fact equivalences of categories.

3.2. *Polarized JTS.* A nice feature of the theory developed so far is that everything goes through if we work with *polarizations* instead of almost complex structures, i.e. with tensor fields $(I_x)_{x\in M}$ satisfying $I_x^2=\mathrm{id}_{T_xM}$. The analogue of Theorem 3.6 then states that there is a bijection between *polarized JTS* and *twisted polarized LTS* and thus between polarized JTS and symmetric spaces with invariant twisted polarizations. *Almost para-complex structures* are polarizations having ±1-eigenspaces of equal dimension, and (twisted and straight) para-complexifications can now formally be defined in the same way as for complexifications—I leave to the reader the task of finding the correct definitions...

That these notions correspond to non-trivial structures can be motivated by the usual theory of Hermitian symmetric spaces: there, given a twisted complex LTS $(\mathfrak{q}, R, J)$, one usually complexifies once more to get $(\mathfrak{q}_\mathbb{C}, R_\mathbb{C}, J_\mathbb{C})$ with a new complex structure i; it satisfies

$$(iJ_\mathbb{C})^2 = i^2 J_\mathbb{C}^2 = \mathrm{id}_{\mathfrak{q}_\mathbb{C}}$$

and

$$R_\mathbb{C}(iJ_\mathbb{C}X, Y) = iR_\mathbb{C}(J_\mathbb{C}X, Y) = -iR_\mathbb{C}(X, J_\mathbb{C}Y) = -R_\mathbb{C}(X, iJ_\mathbb{C}Y);$$

thus $iJ_\mathbb{C}$ is an invariant twisted para-complex structure on on $R_\mathbb{C}$.

3.3. *General Jordan triple systems*

Definition 3.7. The correspondence $JTS \to LTS$, $T \mapsto R_T$ defined by Lemma 3.5 (which is functorial in an obvious sense) is called the *(algebraic) Jordan-Lie functor*. If a LTS R is of the form R_T for a JTS T, then T is called a *Jordan extension* of R. ∎

The following result is the main theorem of this series of lectures:

Theorem 3.8. *Let $(\mathfrak{q}, R)$ be a (real finite dimensional) LTS. Then there are one-to-one correspondences between the following objects:*

(1) *Jordan extensions T of the LTS $(\mathfrak{q}, R)$,*
(2) *twisted complexifications $(q_\mathbb{C}, R_{h\mathbb{C}})$ of $(\mathfrak{q}, R)$,*
(3) *twisted para-complexifications $(q_{ph\mathbb{C}}, R_{ph\mathbb{C}})$ of $(\mathfrak{q}, R)$.*

Proof. (2), (3) $\to$ (1): The main point is: given a twisted (para-) complexification, the structure tensor T can be restricted to the real form $\mathfrak{q}$—in fact, the conjugation τ is indeed an automorphism of T because J appears *twice* in the definition of T and thus the two corresponding minus signs cancel out. Now the restriction of T to $\mathfrak{q}$ is of course still a JTS, and because of Proposition 3.2 (5), it is a JTS-extension of R.

(1) $\to$ (2), (3): We assume that T is a Jordan extension of R and construct a twisted (para-) complexification in four steps:

1. First note that (T, V) has a unique extension to a $\mathbb{C}$-trilinear JTS $(T_\mathbb{C}, V_\mathbb{C})$ (cf. proof of Proposition 2.4).

2. Next we need the following

Lemma 3.9. *Let T be a Jordan triple product on a vector space V and α an endomorphism of V with the property*

$$\forall x, y, z \in V : \quad T(\alpha x, y, \alpha z) = \alpha T(x, \alpha y, z). \qquad (*)$$

Then the formula

$$T^{(\alpha)}(x, y, z) := T(x, \alpha y, z)$$

defines a Jordan triple product $T^{(\alpha)}$ on V.

Proof. (JT1): this is clear.

(JT2): The identity (JT2) for T with v and y replaced by αv and αy, respectively, yields

$$T(T(u,\alpha v)x,\alpha y,z)-T(x,T(\alpha v,u)\alpha y,z)+T(x,\alpha y,T(u,\alpha v)z)$$
$$=T(u,\alpha v)T(x,\alpha y,z).$$

We apply $(*)$ to the middle triple product in the middle term in the first line and obtain

$$T(T(u,\alpha v)x,\alpha y,z)-T(x,\alpha T(v,\alpha u)y,z)+T(x,\alpha y,T(u,\alpha v)z)$$
$$=T(u,\alpha v)T(x,\alpha y,z).$$

This is precisely the identity (JT2) for $T^{(\alpha)}$. ■

(Comment: This lemma is very typical for Jordan theory; in fact, it allows to produce out of *one* JTS a whole variety of new JTS which in general are not isomorphic; they are called *homotopic* or *mutations* of each other. In Lie theory only the rather poor fact survives that for any LTS R the negative $-R$ also is a LTS, called the *c-dual* of R.)

3. Apply the lemma to the JTS $T_{\mathbb{C}}$ with $\alpha(X)=\tau(X)=\overline{X}$: we get that

$$T_{h\mathbb{C}}(X,Y,Z):=T_{\mathbb{C}}(X,\overline{Y},Z)$$

is a new JTS; it clearly is Hermitian (resp. polarized) and on V coincides with the old JTS T.

4. One concludes that $R_{T_{h\mathbb{C}}}$ is a twisted complex (resp. polarized) LTS having $(\mathfrak{q},R)$ as (para-) real form. ■

It is not difficult to show that the correspondences from the theorem are in fact functorial. Then the theorem can be restated by saying that, on the geometric level of (connected simply connected) symmetric spaces, the algebraic Jordan-Lie functor $T\mapsto R_T$ corresponds to the forgetful functor from the category of *symmetric spaces with twist* (i.e. symmetric spaces with a twisted (para-) complexification) to the category of symmetric spaces.

"Complexification diagrams". It is time to illustrate the abstract results by examples – this is best done by presenting the *complexification diagram*

$$\begin{array}{ccccc} & \nearrow & M_{\mathbb{C}} & \searrow & \\ M & \to & M_{h\mathbb{C}} & \to & (M_{h\mathbb{C}})_{\mathbb{C}} \\ & \searrow & M_{ph\mathbb{C}} & \nearrow & \end{array}$$

of a symmetric space M with twist T; this is the geometric version of the linear algebra from the preceding proof:

$$\begin{array}{ccccc} & \nearrow & (R_{\mathbb{C}},T_{\mathbb{C}}) & \searrow & \\ (R,T) & \to & (R_{h\mathbb{C}},T_{h\mathbb{C}}) & \to & ((R_{h\mathbb{C}})_{\mathbb{C}},(T_{h\mathbb{C}})_{\mathbb{C}}) \\ & \searrow & (R_{ph\mathbb{C}},T_{ph\mathbb{C}}) & \nearrow & \end{array}$$

—the second column contains straight and twisted complexification and twisted para-complexification, and the last column contains the result obtained by combining any two different of these three operations; it is a sort of straight-twisted double complexification, leading to a space of four times the real dimension we started with.

EXAMPLE 3.10 (Real Grassmannians). $M = \mathrm{O}(n)/(\mathrm{O}(p) \times \mathrm{O}(q))$, $n = p + q$ (cf. Example (8) of Chapter 1). The diagram is:

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(n,\mathbb{C})/(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C})) & \searrow & \\ \mathrm{O}(n)/(\mathrm{O}(p)\times\mathrm{O}(q)) & \rightarrow & \mathrm{U}(n)/(\mathrm{U}(p)\times\mathrm{U}(q)) & \rightarrow & \mathrm{GL}(n,\mathbb{C})/(\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})). \\ & \searrow & \mathrm{GL}(n,\mathbb{R})/(\mathrm{GL}(p,\mathbb{R})\times\mathrm{GL}(q,\mathbb{R})) & \nearrow & \end{array}$$

In the middle line one finds the complex Grassmannian as twisted complexification of the real one, and in the bottom line one finds the space $\mathrm{GL}(n,\mathbb{R})/(\mathrm{GL}(p,\mathbb{R}) \times \mathrm{GL}(q,\mathbb{R}))$ which can be identified with the space of complementary subspaces of $\mathbb{R}^n$ of dimension p, resp. q. The Jordan triple system belonging to this diagram is $V = M(p,q;\mathbb{R})$ with the trilinear product given by

$$T(X,Y,Z) = -(XY^tZ + ZY^tX).$$

EXAMPLE 3.11 (General linear group). Consider the group case $M = \mathrm{GL}(n,\mathbb{R}) = (\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))/\text{diagonal}$. Its LTS is $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}(n,n,\mathbb{R})$ with the LTS-structure

$$[X,Y,Z] = [[X,Y],Z] = XYZ - YXZ - ZXY + ZYX.$$

A Jordan extension is given by

$$T(X,Y,Z) = XYZ + ZYX.$$

The corresponding complexification diagram of $\mathrm{GL}(n,\mathbb{R})$ is:

$$\begin{array}{ccccc} & \nearrow & \mathrm{GL}(n,\mathbb{C}) & \searrow & \\ \mathrm{GL}(n,\mathbb{R}) & \rightarrow & \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}) & \rightarrow & \mathrm{GL}(2n,\mathbb{C})/(\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})). \\ & \searrow & \mathrm{GL}(2n,\mathbb{R})/(\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R})) & \nearrow & \end{array}$$

In the middle line we find the space N of complex structures on $\mathbb{R}^{2n}$ which is a symmetric space isomorphic to $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$; it is a twisted complexification of M via the imbedding

$$M \to N, \quad g \mapsto \begin{pmatrix} 0 & g \\ -g^{-1} & 0 \end{pmatrix}.$$

Similarly, the space $\mathrm{GL}(2n,\mathbb{R})/(\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))$ of para-complex structures on $\mathbb{R}^{2n}$ is a twisted para-complexification of $\mathrm{GL}(n,\mathbb{R})$ (bottom line).

EXAMPLE 3.12 (Orthogonal groups). For any non-singular matrix $A \in M(n,n,\mathbb{R})$ we denote the orthogonal group of the form $b(x,y) = x^tAy$ by

$$\mathrm{O}(A,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid g^tAg = A\},$$

and similarly we define unitary groups $\mathrm{U}(A,\mathbb{C})$. The Jordan extension in the space $\mathrm{Asym}(A,\mathbb{R})$ of A-antisymmetric matrices is given by the triple product

$$T(X,Y,Z) = -(XYZ + ZYX).$$

The corresponding complexification diagram of $\mathrm{O}(A,\mathbb{R})$ is:

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(A,\mathbb{C}) & \searrow & \\ \mathrm{O}(A,\mathbb{R}) & \rightarrow & \mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right),\mathbb{R})/\mathrm{U}(A,\mathbb{C}) & \rightarrow & \mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\mathbb{C})/\mathrm{GL}(n,\mathbb{C}). \\ & \searrow & \mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\mathbb{R})/\mathrm{GL}(n,\mathbb{R}) & \nearrow & \end{array}$$

The twisted complexification of $\mathrm{O}(A,\mathbb{R})$ given in the middle line can be interpreted as the space of complex structures on $\mathbb{R}^{2n}$ which are orthogonal with respect to the form on $\mathbb{R}^{2n}$ defined by the matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix};$$

it generalizes the well-known *Siegel space* (see [Sa80]). Complex or quaternionic unitary groups can be treated in a similar way.

EXAMPLE 3.13 (Spheres). The sphere $M = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a compact symmetric space; its tangent space at a base point is $\mathbb{R}^n$ with the "conformal" Jordan triple product

$$T(x,y,z) = (x|z)y - (x|y)z - (z|y)x$$

which indeed extends the LTS of S^n. The complexification diagram of the sphere $M = S^n$ is:

$$\begin{array}{ccccc} & \nearrow & S^n_{\mathbb{C}} & \searrow & \\ S^n & \rightarrow & \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2)) & \rightarrow & \mathrm{SO}(n+2,\mathbb{C})/(\mathrm{SO}(n,\mathbb{C})\times\mathrm{SO}(2,\mathbb{C})). \\ & \searrow & \mathrm{SO}(n+1,1)/(\mathrm{SO}(1)\times\mathrm{SO}(1,1)) & \nearrow & \end{array}$$

In order to understand the middle line, we realize S^n as the complete affine picture of the real quadric given by the homogeneous equation

$$\sum_{i=1}^{n} x_i^2 + x_{n+1}x_{n+2} = 0.$$

Denote by N the complex quadric in $\mathbb{P}(\mathbb{C}^{n+2})$ given by the same equation (projective completion of the complex sphere). One can show that N has the structure of a compact (Hermitian) symmetric space, isomorphic to $\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$, and N is a twisted complexification of M.

EXAMPLE 3.14 (Real hyperbolic spaces). The real hyperbolic space $M = H^n(\mathbb{R}) = \mathrm{SO}(n,1)/\mathrm{SO}(n)$ has two non-equivalent complexification diagrams, corresponding to two non-equivalent Jordan extensions of its LTS: for the first diagram it is the "conformal" JTS $\mathbb{R}^n$ with

$$T(x,y,z) = (x|y)z + (z|y)x - (x|z)y,$$

and for the second it is the "projective" JTS $\mathrm{Mat}(1,n;\mathbb{R})$ with

$$T(x,y,z) = xy^t z + zy^t x.$$

The corresponding diagrams are:

$$\begin{array}{ccccc} & \nearrow & S^n_{\mathbb{C}} & \searrow & \\ M & \rightarrow & \mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2)) & \rightarrow & \mathrm{SO}(n+2,\mathbb{C})/(\mathrm{SO}(n,\mathbb{C})\times\mathrm{SO}(2,\mathbb{C})). \\ & \searrow & \mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1)) & \nearrow & \end{array}$$

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(n+1,\mathbb{C})/(\mathrm{O}(n,\mathbb{C})\times\mathrm{O}(1,\mathbb{C})) & \searrow & \\ M & \rightarrow & \mathrm{U}(n,1)/(\mathrm{U}(n)\times\mathrm{U}(1)) & \rightarrow & \mathrm{GL}(n+1,\mathbb{C})/(\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})). \\ & \searrow & \mathrm{GL}(n+1,\mathbb{R})/(\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(1,\mathbb{R})) & \nearrow & \end{array}$$

Explanation

- $H^n(\mathbb{R})$ is the *non-compact dual* of two locally isomorphic compact symmetric spaces: the sphere S^n (Example 3.13) and the real projective space $\mathbb{RP}^n$ (Example 3.10 with $p = 1$).
- The twisted complexifications of S^n and of $\mathbb{RP}^n$ are *not* locally isomorphic! (You see that Jordan theory is a pair of glasses for Lie theory allowing to distinguish spheres and projective spaces already on the local level...)
- Dualizing these two inequivalent twisted complexifications yields two locally inequivalent twisted complexifications of $H^n(\mathbb{R})$.

Remarks on the classification. The simple symmetric spaces (equivalently, the simple LTS) have been classified by M. Berger [B57]; there are about hundred classical series and hundred exceptional spaces. Simple JTS have been classified by E. Neher [Ne80, 81]. Comparison of these two classifications reveals the following surprising fact: *"Generically", a simple LTS has one and only one twisted complexification.* Put another way: *The Jordan-Lie functor is not far from being a bijection of simple objects.* On the algebraic level, these observations are due to E. Neher [Ne85]; more precise statements can be found there and in [Be00], [Be01]. We have seen that the Jordan-Lie functor is not injective (Example 3.14); but there are are only a few exceptions; it is neither surjective: for instance, $\mathrm{SL}(n,\mathbb{R})$ is not in its image, but its central extension $\mathrm{GL}(n,\mathbb{R})$ is (Example 3.15; note that $\mathrm{SL}(2,\mathbb{R})$ is of course covered by Example 3.16 since it is $\mathrm{Sp}(1,\mathbb{R})$); the exceptional group cases are never in the image of the Jordan-Lie functor, but many of them appear naturally as certain automorphism groups in this context which makes things even more puzzeling...

A general theory explaining these facts is missing; this is certainly the most challenging problem in the realm of geometric Jordan and Lie theory.

4. Outlook: Integrability and conformal group. What is the relation between the theory developed in the preceding two chapters with the geometry from the first chapter? For instance, how do causal or generalized conformal structures arise in the present context, and what is the role of the automorphism group? Let me sketch the main features at the example of the *real Grassmannians* (Example (8) of Chapter 1 and Example 3.10): we write a complexification diagram such as e.g. the one given in Example 3.10 in the form

$$\begin{array}{ccccc} & \nearrow & M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} & \searrow & \\ M = G/H & \to & M_{h\mathbb{C}} & \to & (M_{h\mathbb{C}})_{\mathbb{C}} = L_{\mathbb{C}}/Q_{\mathbb{C}}. \\ & \searrow & M_{ph\mathbb{C}} = L/Q & \nearrow & \end{array}$$

Then, as noted in 3.10, $M_{ph\mathbb{C}}$ can be intepreted as a set of *pairs of subspaces*, i.e. as a subset of $\mathrm{Gras}_{p,p+q}(\mathbb{R}) \times \mathrm{Gras}_{q,p+q}(\mathbb{R})$ which is in fact open dense. This can be generalized: $M_{ph\mathbb{C}}$ always imbeds as an open dense subset of some direct product space; this is obtained by "integration" of the direct product structure of tangent spaces. It is precisely here that the *integrability* of invariant almost (para-) complex structures on symmetric spaces is used; then a classical theorem of Frobenius can be applied.

This first observation leads to a second one: the group L acting on $M_{ph\mathbb{C}}$ acts also on both factors of the direct product structure and thus also (by local diffeomorphisms) on M

itself. In the Grassmannian example this is the "projective" action of $L = \mathbb{P}\,\mathrm{GL}(p+q,\mathbb{R})$ on M, which leads precisely to the automorphism group of the generalized conformal structure mentioned in Example (8) of Ch. 1. Many of these features carry over to the general case—see [Be00] for the systematic theory.

References

[B57] M. Berger, *Les espaces symétriques non compacts*, Ann. Sci. Ec. Norm. Sup. (3) 74 (1997), 85–177.

[Be96a] W. Bertram, *Un théorème de Liouville pour les algèbres de Jordan*, Bull. Soc. Math. France 124 (1996), 299–327.

[Be96b] W. Bertram, *On some causal and conformal groups*, J. Lie Theory 6 (1996), 215–244.

[Be00] W. Bertram, *The Geometry of Jordan and Lie Structures*, Lecture Notes in Math. 1754, Springer, Berlin, 2000.

[Be01] W. Bertram, *Complexifications of symmetric spaces and Jordan theory*, Trans. Amer. Math. Soc. 353 (2001), 2531–2556.

[Bes87] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.

[Ch49] W. L. Chow, *On the geometry of algebraic homogeneous spaces*, Ann. Math. 50 (1949), 32–75.

[D63] J. Dieudonné, *La géométrie des groupes classiques*, Springer, New York, 1963.

[FK94] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[GiKa98] S. Gindikin and S. Kaneyuki, *On the automorphism group of the generalized conformal structure of a symmetric R-space*, Differential Geom. Appl. 8 (1998), 21–33.

[He78] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[HO96] J. Hilgert and G. Ólafsson, *Causal Symmetric Spaces—Geometry and Harmonic Analysis*, Perspectives in Math. 18, Academic Press, San Diego, 1996.

[KanKo85] S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. 8 (1985), 81–98.

[Kan89] S. Kaneyuki, *On the causal structures of the Shilov boundaries of symmetric bounded domains*, in: Prospects in Complex Geometry, Lecture Notes in Math. 1468, Springer, New York 1989.

[Koe69] M. Koecher, *Gruppen und Lie-Algebren von rationalen Funktionen*, Math. Z. 109 (1969), 349–392.

[Ko72] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, New York, 1972.

[KoNo69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry II,* Wiley, New York, 1969.

[Lo69a] O. Loos, *Symmetric Spaces I*, Benjamin, New York, 1969.

[Lo77] O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Irvine, 1977.

[Ne80] E. Neher, *Klassifikation der einfachen reellen speziellen Jordan-Tripelsysteme*, Manuscripta Math. 31 (1980), 197–215.

[Ne81] E. Neher, *Klassifikation der einfachen reellen Ausnahme-Jordan-Tripelsysteme*, J. Reine Angew. Math. 322 (1981), 145–169.

[Ne85] E. Neher, *On the classification of Lie and Jordan triple systems*, Comm. Algebra 13 (1985), 2615–2667.

[Sa80] I. Satake, *Algebraic Structures of Symmetric Domains*, Iwanami Shoten, Princeton, 1980.

[Se76] I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy*, Academic Press, 1976.