GEOMETRY AND ANALYSIS ON LIE GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 55
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2002

A SYMMETRY CHARACTERIZATION FOR HOMOGENEOUS SIEGEL DOMAINS RELATED TO BEREZIN TRANSFORMS

TAKAAKI NOMURA

Department of Mathematics, Faculty of Science, Kyoto University
Sakyo-ku 606-8502, Kyoto, Japan
E-mail: nomura@kusm.kyoto-u.ac.jp

Abstract. In this article we present a geometric norm equality for an irreducible homogeneous Siegel domain D that leads us to the equivalence between the commutativity of Berezin transforms with the Laplace-Beltrami operator and the symmetry of D.

Introduction. Homogeneous Siegel domains, being holomorphically equivalent to bounded domains, have attracted numerous researchers since the first Russian publication of Pjatetskii-Shapiro's book [23] in 1961. This class of domains contains Hermitian symmetric spaces, which immediately makes it an interesting problem to characterize symmetric domains among homogeneous Siegel domains. Characterizations in terms of the defining data of the domain are given by Satake's book [26, Theorem V.3.5] and a paper by Dorfmeister [6, Theorem 3.3]. A differential geometric criterion is found in [5]. Commutativity of the algebra of invariant differential operators is also a distinctive feature of symmetric Siegel domains by [4], in which paper several others are presented issuing out of a detailed study of the isotropic representation. In this article we report one another characterization obtained by the present author in [19], which states that commutativity of Berezin transforms with the Laplace-Beltrami operator is equivalent to the symmetry of the domain.

Let D be an irreducible homogeneous Siegel domain and G the split solvable Lie group acting simply transitively on D. The group manifold G being diffeomorphic with D, our analysis will be done on G rather than on D. The Lie algebra $\mathfrak{g}$ of G has a structure of normal j-algebra by [23]. Thus there is a linear form $\omega \in \mathfrak{g}^*$ such that $\langle x \,|\, y \rangle_\omega := \langle [Jx, y], \omega \rangle$ defines a real inner product on $\mathfrak{g}$, where J is the almost complex structure attached to the normal j-algebra structure of $\mathfrak{g}$. The inner product $\langle \cdot \,|\, \cdot \rangle_\omega$

2000 *Mathematics Subject Classification*: 22E30, 32M15, 43A85.
Received 20 February 2001; revised 28 August 2001.
The paper is in final form and no version of it will be published elsewhere.

defines a left invariant Riemannian structure on G, through which we have the Laplace-Beltrami operator $\mathcal{L}_\omega$ on G. On the other hand, Berezin transforms B_λ, when transferred to operators on $L^2(G)$, are convolution operators by functions a_λ from the right (see (2.2)). Let $\Psi \in \mathfrak{g}$ be the element for which $\langle X \,|\, \Psi \rangle_\omega = \operatorname{tr}(\operatorname{ad} X)$ holds for all $X \in \mathfrak{g}$. The key formula to our study of the commutativity for B_λ and $\mathcal{L}_\omega$ is the following:

$$\mathcal{L}_\omega a_\lambda(g) = \lambda\, a_\lambda(g)\bigl(-\lambda\, \|\mathcal{C}(g \cdot \mathsf{e})\|_\omega^2 + \langle \Psi, \alpha\rangle\bigr) \qquad (g \in G),$$

where α is some linear form on $\mathfrak{g}$, $\mathcal{C}$ the Cayley transform introduced in [17] that is defined through the Bergman kernel of D, and $\mathsf{e} \in D$ the reference point specified in (1.4) (see Proposition 2.4 for the precise statement). From this formula we deduce that the Berezin transforms B_λ commute with $\mathcal{L}_\omega$ if and only if

$$\|\mathcal{C}(g \cdot \mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathsf{e})\|_\omega$$

holds for any $g \in G$. Investigation of this norm equality is presented in section 3 of this article. Actually the validity of this norm equality for any $g \in G$ is equivalent to the symmetry of D (and the reduction of ω essentially to the Koszul form). This is the main theorem in [18]. The proof requires not only a lot of calculations but also the result due to D'Atri and Dotti Miatello [5] concerning a characterization of quasisymmetric Siegel domains and the above-mentioned works done by Satake and Dorfmeister.

1. Preliminaries

1.1. *Normal j-algebras.* Since every homogeneous Siegel domain is described by means of normal j-algebra (see the book [23], the paper [25] or the lecture notes [24]), we begin this article with the definition of normal j-algebra. A triple $(\mathfrak{g}, J, \omega)$ of a split solvable Lie algebra $\mathfrak{g}$, a linear operator J on $\mathfrak{g}$ with $J^2 = -I$ and a linear form $\omega \in \mathfrak{g}^*$ is called a *normal j-algebra* if the following two conditions are satisfied:

(1.1) $\quad [Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad$ (for all $x, y \in \mathfrak{g}$),

(1.2) $\quad \langle x \,|\, y \rangle_\omega := \langle [Jx, y], \omega\rangle$ defines a J-invariant inner product on $\mathfrak{g}$.

Linear forms $\omega \in \mathfrak{g}^*$ satisfying (1.2) are said to be *admissible.* Throughout this article we fix one admissible linear form. Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. Let $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ relative to the inner product $\langle \cdot \,|\, \cdot \rangle_\omega$. We note that $\mathfrak{a}$ is independent of the choice of the admissible linear form. We have $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$. We know that $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that $\operatorname{ad}(\mathfrak{a})$ consists of semisimple operators on $\mathfrak{g}$. We have a simultaneous eigenspace decomposition $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$, where

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n}\ ;\ [h, x] = \langle h, \alpha\rangle\, x \quad \text{for all } h \in \mathfrak{a}\},$$

and the finite set $\Delta \subset \mathfrak{a}^*$ is described shortly. The dimension $r := \dim \mathfrak{a}$ is called the *rank* of the normal j-algebra. One can choose a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ such that if we set $E_j := -JH_j$, then $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$. Then elements of Δ, which we call the *roots* of $\mathfrak{g}$, are of the following form (not all possibilities need occur):

$$\tfrac{1}{2}(\alpha_m+\alpha_k) \quad (1 \leqq k < m \leqq r), \qquad \tfrac{1}{2}(\alpha_m-\alpha_k) \quad (1 \leqq k < m \leqq r),$$
$$\tfrac{1}{2}\alpha_k \qquad (1 \leqq k \leqq r), \qquad\qquad \alpha_k \qquad (1 \leqq k \leqq r).$$

Moreover we have $\mathfrak{n}_{\alpha_k} = \mathbb{R}E_k$. Put

$$H := H_1 + \ldots + H_r, \qquad E := E_1 + \ldots + E_r.$$

With $\mathfrak{g}(1/2) := \sum_1^r \mathfrak{n}_{\alpha_i/2}$ and

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m-\alpha_k)/2}, \quad \mathfrak{g}(1) := \sum_1^r \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m+\alpha_k)/2},$$

we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}(1/2) + \mathfrak{g}(1)$ of $\mathrm{ad}\,(H)$, which gives a gradation $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i+j)$, where we understand $\mathfrak{g}(i) = 0$ for $i > 1$. Furthermore we have

$$J\mathfrak{n}_{(\alpha_m-\alpha_k)/2} = \mathfrak{n}_{(\alpha_m+\alpha_k)/2} \quad (m > k), \qquad J\mathfrak{n}_{\alpha_i/2} = \mathfrak{n}_{\alpha_i/2} \quad (1 \leqq i \leqq r),$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$ and $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. We remark here that

$$JT = -[T, E] \quad (T \in \mathfrak{g}(0)), \qquad JT_{ji} = -[T_{ji}, E_i] \quad (T_{ji} \in \mathfrak{n}_{(\alpha_j-\alpha_i)/2}).$$

The following is a list of constants used in this article:

$$\begin{aligned} &n_{mk} := \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m-\alpha_k)/2} = \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m+\alpha_k)/2} && (1 \leqq k < m \leqq r), \\ &p_j := \sum_{k>j} n_{kj}, \qquad q_j := \sum_{i<j} n_{ji} && (1 \leqq j \leqq r), \\ &b_j := \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{n}_{\alpha_j/2}, \quad d_j := 1 + \frac{1}{2}(p_j + q_j) && (1 \leqq j \leqq r), \\ &\omega_k := \langle E_k, \omega\rangle = \|E_k\|_\omega^2 > 0 && (1 \leqq k \leqq r). \end{aligned} \tag{1.3}$$

1.2. *Homogeneous Siegel domains.* Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra, and $G = \exp \mathfrak{g}$ the corresponding connected and simply connected Lie group. We denote by $G(0)$ the subgroup $\exp \mathfrak{g}(0)$ of G. The group $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. Let $\Omega := G(0)E$, the $G(0)$-orbit through E. By [25, Theorem 4.15] Ω is a regular open convex cone in V, and $G(0)$ acts on Ω simply transitively. On the other hand, the subspace $\mathfrak{g}(1/2)$ is invariant under J, so that we consider it as a *complex* vector space U by means of $-J$. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation of W relative to the real form V is written as $w \mapsto w^*$. The real bilinear map Q defined by

$$Q(u, u') := \frac{1}{2}\big([Ju, u'] - i[u, u']\big) \qquad (u, u' \in \mathfrak{g}(1/2))$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Ω-positive Hermitian map $U \times U \to W$. The *Siegel domain* $D = D(\Omega, Q)$ corresponding to these data is defined to be

$$D := \{(u, w) \in U \times W \;;\; w + w^* - Q(u, u) \in \Omega\}.$$

Throughout this article, we assume that D is irreducible.

Consider the nilpotent Lie subalgebra $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$, and the corresponding connected and simply connected nilpotent subgroup $N_D := \exp \mathfrak{n}_D$ of G. Elements of N_D

are written as $n(a,b)$ $(a \in \mathfrak{g}(1),\ b \in \mathfrak{g}(1/2))$. The group operation is described as follows:

$$n(a,b)n(a',b') = n(a + a' - \operatorname{Im} Q(b,b'),\ b + b').$$

The group N_D acts on D by

$$n(a,b)\cdot(u,w) = \bigl(u + b,\ w + ia + \tfrac{1}{2}Q(b,b) + Q(u,b)\bigr) \qquad ((u,w) \in D).$$

On the other hand, the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$ commutes with J, that is, $G(0)$ acts on U complex linearly. Moreover the adjoint action of $G(0)$ on $V = \mathfrak{g}(1)$ extends complex linearly to W, so that $G(0)$ acts on D complex linearly. In this way $G = N_D \rtimes G(0)$ acts on D simply transitively. As a base point of D we fix

$$\mathsf{e} := (0, E). \tag{1.4}$$

We have a surjective diffeomorphism $\psi : G \to D$ by $\psi(g) := g \cdot \mathsf{e}$.

Let us put $A := \exp \mathfrak{a}$ and set for $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$

$$a_t := \exp(t_1 H_1 + \ldots + t_r H_r) \in A.$$

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, let $\chi_{\mathbf{s}}$ be the one-dimensional representation of A defined by $\chi_{\mathbf{s}}(a_t) = \exp\bigl(\sum_k s_k t_k\bigr)$. We put $\mathfrak{n}_0 := \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}$. It is a nilpotent Lie subalgebra of $\mathfrak{g}(0)$, and we have $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_D$. Let $N_0 := \exp \mathfrak{n}_0$ and $N := \exp \mathfrak{n}$. Clearly $G = N \rtimes A$ and $G(0) = N_0 \rtimes A$. We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let us define functions $\Delta_{\mathbf{s}}$ ($\mathbf{s} \in \mathbb{R}^r$) on Ω by

$$\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h) \qquad (h \in G(0)).$$

Evidently it holds that $\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x)$ $(h \in G(0),\ x \in \Omega)$. Further, we know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [13, Corollary 2.5]).

For $h \in G(0)$, let $\operatorname{Ad}_{\mathfrak{g}(1)}(h) := (\operatorname{Ad} h)|_{\mathfrak{g}(1)}$. Moreover let $\operatorname{Ad}_U(h)$ stand for the *complex* linear operator on U defined by the adjoint action of $h \in G(0)$ on $\mathfrak{g}(1/2)$, and $\det \operatorname{Ad}_U(h)$ its determinant as a complex linear operator. Then, with $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$, we have for $h \in G(0)$

$$\det \operatorname{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h), \qquad |\det \operatorname{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h). \tag{1.5}$$

1.3. *Weighted Bergman spaces.* By [12, §5] or [26, §II.6], it is known that D has a Bergman kernel κ. Using the known covariance property of κ together with the simply transitive action of G on D, we see by (1.5) that

$$\kappa(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}\bigl(w_1 + w_2^* - Q(u_1, u_2)\bigr) \qquad (z_j = (u_j, w_j) \in D)$$

up to a positive number multiple. We put $\eta := \Delta_{-2\mathbf{d}-\mathbf{b}}$ in what follows for simplicity.

Now we introduce an inner product $\langle \cdot \,|\, \cdot \rangle_\eta$ on V by

$$\langle v_1 \,|\, v_2 \rangle_\eta := D_{v_1} D_{v_2} \log \eta(E) \qquad (v_1, v_2 \in V), \tag{1.6}$$

where D_v stands for the directional derivative in the direction $v \in V$: $D_v f(x) = (d/dt) f(x + tv)\big|_{t=0}$. We extend $\langle \cdot \,|\, \cdot \rangle_\eta$ to a complex bilinear form on $W \times W$, which we still denote by the same symbol. Then we have a Hermitian inner product $(w_1 \,|\, w_2)_\eta := \langle w_1 \,|\, w_2^* \rangle_\eta$ on W. On the other hand, it is easy to see that U has a Hermitian inner product $(\cdot\,|\,\cdot)_\eta$ given by

$$(u_1\,|\,u_2)_\eta := \langle\, Q(u_1,u_2)\,|\,E\,\rangle_\eta \qquad (u_1,u_2\in U). \tag{1.7}$$

We denote by $dm(w)$ and $dm(u)$ the Euclidean measures normalized by these inner products on W and on U respectively. By (1.5), the measure $d\mu$ defined by

$$d\mu(u,w) := \eta(w+w^*-Q(u,u))\,dm(u)dm(w) \tag{1.8}$$

is a G-invariant measure on D. Let us set for $\lambda\in\mathbb{R}$

$$d\mu_\lambda(u,w) := c_\lambda\cdot\eta(w+w^*-Q(u,u))^{-\lambda+1}\,dm(u)dm(w), \tag{1.9}$$

where $c_\lambda>0$ is determined shortly. The *weighted Bergman space* $H^2_\lambda(D)$ is the Hilbert space of holomorphic functions on D which are square integrable relative to the measure $d\mu_\lambda$. We know by [25, Theorem 4.26] or [13, Theorem 4.8] that $H^2_\lambda(D)\neq\{0\}$ if and only if

$$\lambda>\lambda_0 := \max_{1\leqq k\leqq r}\frac{b_k+d_k+\frac{1}{2}\,p_k}{b_k+2d_k}.$$

In view of (1.3), we have $0<\lambda_0<1$. If $\lambda>\lambda_0$, the Hilbert space $H^2_\lambda(D)$ has a reproducing kernel κ_λ (cf. [13, Proposition 4.6]). We choose the constant c_λ in such a way that

$$\kappa_\lambda(z_1,z_2)=\eta(w_1+w_2^*-Q(u_1,u_2))^\lambda \qquad (z_j:=(u_j,w_j)\in D). \tag{1.10}$$

Explicit expression of c_λ is not necessary in this article.

1.4. *Pseudoinverse map.* For every $x\in\Omega$ we define $\mathcal{I}(x)\in V^*$ by

$$\langle v,\mathcal{I}(x)\rangle = -D_v\log\eta(x) \qquad (v\in V).$$

The map $\mathcal{I}$ is called the *pseudoinverse map.* By [7, §2], $\mathcal{I}$ is a bijection of Ω onto the dual cone Ω^* in V^*, where

$$\Omega^* := \bigl\{\xi\in V^*\ ;\ \langle x,\xi\rangle>0\quad\text{for all } x\in\overline{\Omega}\setminus\{0\}\bigr\}.$$

The group $G(0)$ acts on V^* by the coadjoint action: $h\cdot\xi=\xi\circ h^{-1}$, where $h\in G(0)$ and $\xi\in V^*$. Moreover $G(0)$ acts on Ω^* simply transitively. It is easy to show that $\mathcal{I}$ is $G(0)$-equivariant: $\mathcal{I}(hx)=h\cdot\mathcal{I}(x)$ $(h\in G(0),\ x\in\Omega)$. In particular, $\mathcal{I}(\lambda x)=\lambda^{-1}\mathcal{I}(x)$ for all $\lambda>0$. We know that the map $\mathcal{I}$ analytically continues to a birational map $W\to W^*$. Furthermore $\mathcal{I}$ maps the tube domain $\Omega+iV$ biholomorphically onto its image $\mathcal{I}(\Omega+iV)$ (see [17, §2]). We note that in general $\mathcal{I}(\Omega+iV)\not\subseteq\Omega^*+iV^*$ (see [17, §5]).

1.5. *Cayley transform.* First we define $E_1^*,\dots,E_r^*\in V^*$ by

$$\Bigl\langle\sum_{j=1}^r x_jE_j+\sum_{m>k}X_{mk},\ E_i^*\Bigr\rangle = x_i \qquad (x_j\in\mathbb{R},\ X_{mk}\in\mathfrak{n}_{(\alpha_m+\alpha_k)/2}).$$

Elements of V^* are canonically considered as elements of W^*, the space of complex linear forms on W. Then for every $\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{C}^r$ we set

$$E^*_{\mathbf{s}} := s_1E_1^*+\ldots+s_rE_r^*\in W^*.$$

Now we define for $w\in W$

$$C(w) := E^*_{\mathbf{2d+b}}-2\,\mathcal{I}(w+E).$$

We have $C(w) \in W^*$ for generic w. It is evident that C is a rational map $W \to W^*$ which is holomorphic on $\Omega + iV$. Let $U^\dagger$ denote the space of all antilinear forms on U. We set for $z = (u, w) \in U \times W$

$$\mathcal{C}(z) := \big(2\,\langle Q(u,\cdot), \mathcal{I}(w+E)\rangle,\, C(w)\big). \tag{1.11}$$

We have $\mathcal{C}(z) \in U^\dagger \times W^*$ for generic z. Clearly $\mathcal{C}$ is a rational map $U \times W \to U^\dagger \times W^*$. It should be noted that if $z = (u, w) \in D$, then we have $w \in \Omega + iV$, so that $\mathcal{C}(z)$ is holomorphic on D. We shall call $\mathcal{C}$ a *Cayley transform*. This Cayley transform is introduced in [17] as a slight modification of Penney's [22]. We know by [17, §3] that the image $\mathcal{C}(D)$ is bounded and that $\mathcal{C}$ is a birational map which sends D biholomorphically onto $\mathcal{C}(D)$.

Remark 1.1. In a recent paper [20], the present author has introduced a family of Cayley transforms $\mathcal{C}_{\mathbf{s}}$ defined by the functions $\Delta_{-\mathbf{s}}$ with $s_j > 0$ for all $j = 1, \ldots, r$ in place of $\eta = \Delta_{-2\mathbf{d}-\mathbf{b}}$. The proof given in [20] for the boundedness of $\mathcal{C}_{\mathbf{s}}(D)$ is independent of Penney's argument unlike the one presented in [17].

2. Berezin transforms

2.1. *Berezin transforms as convolution operators.* The weighted Bergman space $H^2_\lambda(D)$ ($\lambda > \lambda_0$) is a closed subspace of $L^2(D, d\mu_\lambda)$ and has a reproducing kernel κ_λ. Then, associated to $H^2_\lambda(D)$, we can define the Berezin transform B^D_λ on the L^2-space over D relative to the measure $d\mu_0(z) := \kappa_\lambda(z,z)d\mu_\lambda(z)$ (cf. for example [16]). We note that the formulas (1.8), (1.9) and (1.10) imply $d\mu_0 = c_\lambda\, d\mu$. The Berezin kernel A_λ associated to $H^2_\lambda(D)$ is given by

$$A_\lambda(z_1, z_2) := \frac{|\kappa_\lambda(z_1,z_2)|^2}{\kappa_\lambda(z_1,z_1)\,\kappa_\lambda(z_2,z_2)} \qquad (z_1, z_2 \in D).$$

It is G-invariant:

$$A_\lambda(g\cdot z_1,\ g\cdot z_2) = A_\lambda(z_1, z_2) \qquad (g \in G). \tag{2.1}$$

The Berezin transform B^D_λ is an integral operator on $L^2(D, d\mu_0)$:

$$B^D_\lambda f(z) = \int_D A_\lambda(z,w) f(w)\, d\mu_0(w) \qquad (f \in L^2(D, d\mu_0)).$$

B^D_λ is a bounded positive selfadjoint operator which is G-invariant by (2.1). Normalizing the left Haar measure dx on G in such a way that

$$\int_G f(x\cdot \mathsf{e})\, dx = \int_D f(z)\, d\mu(z) = \frac{1}{c_\lambda}\int_D f(z)\, d\mu_0(z),$$

we transfer the operator B^D_λ to an operator B_λ on $L^2(G)$. Put

$$a_\lambda(g) := A_\lambda(g\cdot \mathsf{e},\ \mathsf{e}) \qquad (g \in G).$$

By (2.1) it is evident that $a_\lambda(g) = a_\lambda(g^{-1})$, and the reproducing property of κ_λ shows that $a_\lambda \in L^1(G)$ provided $\lambda > \lambda_0$. Then an easy computation leads us to

$$B_\lambda f(x) = c_\lambda \int_G f(y) a_\lambda(y^{-1}x)\, dy = c_\lambda f * a_\lambda(x) \qquad (f \in L^2(G)). \tag{2.2}$$

2.2. *Laplace-Beltrami operators.* If $X \in \mathfrak{g}$, we set

$$Xf(x) := \frac{d}{dt} f\big(\exp(-tX)x\big)\Big|_{t=0}, \qquad \widetilde{X}f(x) := \frac{d}{dt} f\big(x\,(\exp tX)\big)\Big|_{t=0},$$

where $f \in C^\infty(G)$ and $x \in G$. These two actions of $\mathfrak{g}$ on $C^\infty(G)$ are extended to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by homomorphisms. Let $\langle\,\cdot\,|\,\cdot\,\rangle_\omega$ be the inner product on $\mathfrak{g}$ that we are working with. This inner product induces a left invariant Riemannian metric on G, and we have the corresponding Laplace-Beltrami operator $\mathcal{L}_\omega$. Though the following lemma holds for any connected Lie group [27, Theorem 1], we write it down here in our situation.

LEMMA 2.1. *Take $\Psi \in \mathfrak{g}$ such that $\langle\, X \,|\, \Psi \,\rangle_\omega = \operatorname{tr}(\operatorname{ad} X)$ holds for all $X \in \mathfrak{g}$. Then $\mathcal{L}_\omega = -\widetilde{\Lambda} + \widetilde{\Psi}$, where $\Lambda := X_1^2 + \ldots + X_{2N}^2 \in U(\mathfrak{g})$ ($2N := \dim_{\mathbb{R}} \mathfrak{g}$) with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $\mathfrak{g}$ relative to $\langle\,\cdot\,|\,\cdot\,\rangle_\omega$. Note that Λ remains the same under any change of orthonormal basis of $\mathfrak{g}$.*

With the constants in (1.3), it is not difficult to see that the element Ψ in Lemma 2.1 is given by

$$\Psi = \sum_{j=1}^{r} \omega_j^{-1}(q_j + b_j + 1)H_j \in \mathfrak{a}.$$

Let φ be a smooth function on G which is integrable with respect to the left Haar measure. Consider the following convolution operator T_φ:

$$T_\varphi f(x) := \int_G f(g)\varphi(g^{-1}x)\,dg = f * \varphi(x).$$

A standard argument shows that T_φ is a bounded operator on $L^2(G)$. Moreover T_φ commutes with left translations.

PROPOSITION 2.2. *The operator T_φ commutes with $\mathcal{L}_\omega$ if and only if*

$$(-\widetilde{\Lambda} + \widetilde{\Psi})\varphi = (-\Lambda + \Psi)\varphi.$$

This is a statement that generalizes Proposition 4.1 in [19] in an obvious manner, and the proof given there works in a completely parallel way. If, moreover, φ is symmetric, that is, $\varphi(x) = \varphi(x^{-1})$ for any $x \in G$, then we get easily $\widetilde{X}\varphi(x) = X\varphi(x^{-1})$ for all $X \in U(\mathfrak{g})$ and $x \in G$. Therefore

PROPOSITION 2.3. *The operator T_φ with symmetric φ commutes with $\mathcal{L}_\omega$ if and only if*

$$(\Lambda - \Psi)\varphi(x^{-1}) = (\Lambda - \Psi)\varphi(x) \qquad \text{for any } x \in G.$$

2.3. *Berezin transforms and Laplace-Beltrami operators.* By (2.2) we know that the Berezin transforms B_λ are of the form T_φ with symmetric φ. In view of Proposition 2.3, we are led to calculating $(\Lambda - \Psi)a_\lambda$. In order to perform the calculation, we note that (1.10) shows

$$a_\lambda(nh) := 4^{|\mathbf{c}|\lambda}\,\chi_{2\mathbf{d}+\mathbf{b}}(h)^\lambda \left|\eta\big(\pi_W(nh\cdot \mathsf{e}) + E\big)\right|^{2\lambda} \qquad (n \in N_D,\ h \in G(0)),$$

where π_W stands for the projection $U \times W \to W$. To present the formula for $(\Lambda - \Psi)a_\lambda$, we recall the Cayley transform $\mathcal{C}$ defined in (1.11), and introduce a norm in $U^\dagger \times W^*$, the ambient vector space of the bounded domain $\mathcal{C}(D)$. Let $\psi : G \to D$ be the diffeomorphism $\psi(g) = g \cdot \mathsf{e}$. Its differential $d\psi : \mathfrak{g} \to U + W$ at the identity is described as

$$d\psi(T + u + x) = u + (-JT + ix) \quad (T \in \mathfrak{g}(0),\ u \in \mathfrak{g}(1/2),\ x \in \mathfrak{g}(1)).$$

Let us regard $\mathfrak{g}$ as a complex vector space by means of $-J$. Then, as is easily verified, $d\psi$ is complex linear, that is, $d\psi(-JX) = i \cdot d\psi(X)$ for all $X \in \mathfrak{g}$. We equip the complex vector space $(\mathfrak{g}, -J)$ with a Hermitian inner product $(\cdot\,|\,\cdot)_\omega$ defined by

$$(X\,|\,Y)_\omega := \langle [JX, Y], \omega\rangle - i\langle [X, Y], \omega\rangle.$$

We then transport it to $U+W$ by $d\psi$ and get a Hermitian inner product $(\cdot\,|\,\cdot)_\omega$ on $U+W$. Identifying $U^\dagger + W^*$ with $U + W$ through $(\cdot\,|\,\cdot)_\omega$, we get an inner product on $U^\dagger + W^*$, which we still denote by $(\cdot\,|\,\cdot)_\omega$.

Proposition 2.4 ([19]). *With* $\mathbf{c} := 2\mathbf{d} + \mathbf{b} = (c_1, \ldots, c_r)$ *one has*

$$(\Lambda - \Psi)a_\lambda(g) = \lambda a_\lambda(g)\bigl[\lambda\|\mathcal{C}(g \cdot \mathsf{e})\|_\omega^2 - \langle \Psi, \alpha_{\mathbf{c}}\rangle\bigr] \qquad (g \in G),$$

where $\alpha_{\mathbf{c}} := \sum_j c_j\alpha_j \in \mathfrak{a}^*$.

Consequently we arrive at

Proposition 2.5. *The Berezin transform* B_λ $(\lambda > \lambda_0)$ *commutes with* $\mathcal{L}_\omega$ *if and only if* $\|\mathcal{C}(g \cdot \mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathsf{e})\|_\omega$ *holds for any* $g \in G$.

3. Symmetry characterization

3.1. *Norm equality.* Let $\beta \in \mathfrak{g}^*$ be the *Koszul form* given by

$$\langle x, \beta\rangle := \operatorname{tr}\left(\operatorname{ad}(Jx) - J\left(\operatorname{ad} x\right)\right) \qquad (x \in \mathfrak{g}).$$

It is known by [15, Théorème 1] that $\langle [Jx, y], \beta\rangle$ is the real part of the Hermitian inner product on $\mathfrak{g}$ induced by the Bergman metric of D up to a positive multiple. In particular, β is admissible. By virtue of [18, Lemma 5.2] we know that $\beta|_{\mathfrak{g}(1)} = E^*_{2\mathbf{d}+\mathbf{b}}$. We recall that the domain D is said to be *symmetric* if for every $z \in D$, there exists an involutive holomorphic automorphism σ_z of D such that z is an isolated fixed point of σ_z.

Theorem 3.1 ([18]). *One has* $\|\mathcal{C}(g \cdot \mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathsf{e})\|_\omega$ *for all* $g \in G$ *if and only if the following two conditions are satisfied:*

(1) *D is symmetric,*
(2) *$\omega|_{\mathfrak{n}}$ is equal to a positive number multiple of $\beta|_{\mathfrak{n}}$.*

Since $\mathcal{C} : D \to \mathcal{D} := \mathcal{C}(D)$ is biholomorphic with $\mathcal{C}(\mathsf{e}) = 0$, the previous theorem can be rephrased as follows.

Theorem 3.2. *One has* $\|h \cdot 0\|_\omega = \|h^{-1} \cdot 0\|_\omega$ *for all* $h \in \mathcal{C} \circ G \circ \mathcal{C}^{-1}$ *if and only if* $\mathcal{D}$ *is symmetric and* $\omega|_{\mathfrak{n}}$ *is equal to a positive number multiple of* $\beta|_{\mathfrak{n}}$.

Theorem 3.1 together with Proposition 2.5 immediately yields

Theorem 3.3 ([19]). *Let* $\lambda > \lambda_0$ *be fixed. Then,* B_λ *commutes with* $\mathcal{L}_\omega$ *if and only if* D *is symmetric and* $\omega|_{\mathfrak{n}}$ *is equal to a positive number multiple of* $\beta|_{\mathfrak{n}}$.

3.2. *Outline of the proof.* The proof of the "only if" part of Theorem 3.1 requires quite a good deal of computations as well as results due to Satake, Dorfmeister, D'Atri and Dotti Miatello. We would like to present here its outline. Suppose that we have the following norm equality for all $g \in G$:

$$\|\mathcal{C}(g\cdot \mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1}\cdot \mathsf{e})\|_\omega. \tag{3.1}$$

Supposing further that the rank r of our normal j-algebra satisfies $r \geqq 2$ and that $i < j$, we first try to extract informations from (3.1) for

$$g = \exp(T)\exp(t_iH_i + t_jH_j) \qquad (t_i, t_j \in \mathbb{R},\ T \in \mathfrak{n}_{(\alpha_j-\alpha_i)/2}). \tag{3.2}$$

Since $g \in G(0)$, we have $g\cdot \mathsf{e} = (0, gE)$, so that (1.11) yields $\mathcal{C}(g\cdot \mathsf{e}) = (0, C(gE))$.

LEMMA 3.4. *Let c_j be as in Proposition 2.4. For g in (3.2) one has*

$$C(gE) = \frac{aE_i^* + bE_j^* - \phi}{8\cosh(t_i/2)\cosh(t_j/2) + \omega_j^{-1}e^{(t_i-t_j)/2}\|T\|_\omega^2},$$

where $a, b \in \mathbb{R}$ and $\phi \in \mathfrak{n}^*_{(\alpha_j+\alpha_i)/2}$ are given as follows:

$$\begin{aligned} a &:= 8\,c_i \sinh\frac{t_i}{2}\cosh\frac{t_j}{2} + \omega_j^{-1}e^{(t_i-t_j)/2}\,\frac{c_i\sinh(t_i/2) - c_j\,e^{t_i/2}}{\cosh(t_i/2)}\,\|T\|_\omega^2,\\ b &:= c_j\Big(8\cosh\frac{t_i}{2}\sinh\frac{t_j}{2} + \omega_j^{-1}e^{(t_i-t_j)/2}\|T\|_\omega^2\Big),\\ \phi &:= 4\,c_j\,e^{(t_i-t_j)/2}\,\mathrm{ad}^*(T)E_j^*\circ P_{ji} \end{aligned}$$

with P_{ji} the orthogonal projection operator $V \to \mathfrak{n}_{(\alpha_j+\alpha_i)/2}$.

Computing the norm $\|C(gE)\|_\omega^2$, we deduce

LEMMA 3.5. *If $n_{ji} \neq 0$, then one has $2d_i + b_i = 2d_j + b_j$ and $\omega_i = \omega_j$.*

Since we are supposing that our Siegel domain is irreducible, the cone Ω is also irreducible ([14, Theorem 6.3]). Thus we can make use of [1, Theorem 4], which states as follows: putting $n_{kl} := n_{lk}$ when $k < l$, we can find, for any pair $i < j$, a finite sequence $\{i_\lambda\}_{\lambda=0}^m$ ($i_0 = j$, $i_m = i$) such that $n_{i_{\lambda-1}i_\lambda} \neq 0$ for any λ. See [5, p. 536] for the translation of normal j-algebra language into Vinberg's T-algebra language. In consequence we get

LEMMA 3.6. *Both $2d_i + b_i$ and ω_i are independent of i.*

We put $\omega_0 := \omega_i$ (independent of i) in what follows. To see that n_{ji} ($i < j$) is independent of i, j, we take (under the assumption that $r \geqq 3$ and $i < j < k$)

$$g = \exp(T_{ki})\exp(T_{kj})\exp(t(H_i + H_j + H_k)) \in G(0),$$

where $t \in \mathbb{R}$, $T_{ki} \in \mathfrak{n}_{(\alpha_k-\alpha_i)/2}$ and $T_{kj} \in \mathfrak{n}_{(\alpha_k-\alpha_j)/2}$. We can calculate $C(gE)$ and $\|C(gE)\|_\omega^2$ explicitly for this g and obtain from (3.1) the following lemma.

LEMMA 3.7. *If $n_{kj} \neq 0$, then one has $n_{ji} = n_{ki}$.*

Keeping to $i < j < k$, we next take

$$g = \exp(T_{ki} + T_{ji})\exp(t(H_i + H_j + H_k)) \in G(0),$$

where $t \in \mathbb{R}$, $T_{ki} \in \mathfrak{n}_{(\alpha_k-\alpha_i)/2}$ and $T_{ji} \in \mathfrak{n}_{(\alpha_j-\alpha_i)/2}$. In a way analogous to the preceding two cases, though the computations are much harder, we get

LEMMA 3.8. *If $n_{ji} \neq 0$, then for all $T_{ki} \in \mathfrak{n}_{(\alpha_k-\alpha_i)/2}$ and $T_{ji} \in \mathfrak{n}_{(\alpha_j-\alpha_i)/2}$ one has*

$$\|[JT_{ji}, T_{ki}]\|_\omega^2 = \frac{1}{2\omega_0}\|T_{ji}\|_\omega^2\|T_{ki}\|_\omega^2.$$

In particular, $n_{kj} \geqq n_{ki}$ provided $n_{ji} \neq 0$.

REMARK 3.9. In general, we only have the inequality

$$\|[JT_{ji}, T_{ki}]\|_\omega^2 \leqq \frac{1}{2\omega_j}\|T_{ji}\|_\omega^2\|T_{ki}\|_\omega^2,$$

because n_{kj} can be equal to 0 even if $n_{ki} > 0$ and $n_{ji} > 0$. See [18, Lemma 4.6].

Using Lemmas 3.6, 3.7 and 3.8, we arrive at

PROPOSITION 3.10. *The dimensions n_{ji} $(i < j)$ of the root spaces $\mathfrak{n}_{(\alpha_j+\alpha_i)/2}$ are constant, and $b_k = \dim \mathfrak{n}_{\alpha_k/2}$ $(k = 1, \dots, r)$ are independent of k.*

By [5, Proposition 3], Proposition 3.10 implies that our Siegel domain is *quasisymmetric*, that is, the cone Ω is selfdual with respect to the inner product (1.6). Then our normal j-algebra turns out to be comprised of a Euclidean Jordan algebra and a $*$-representation of its complexified Jordan algebra. Let us explain this more precisely. We introduce a (non-associative) product in V by the formula

$$\langle\, x_1x_2 \,|\, x_3 \,\rangle_\eta = -\frac{1}{2}\, D_{x_1}D_{x_2}D_{x_3}(\log\eta)(E) \qquad (x_j \in V; j = 1, 2, 3).$$

The proof of [5, Proposition 3] says that this is a Jordan algebra product, by which V is a Euclidean Jordan algebra in the sense of [11]. Thus $W = V_{\mathbb{C}}$ is a complex Jordan algebra. Moreover, using the Hermitian inner product (1.7) on U, we define, for every $w \in W$, a complex linear operator $\varphi(w)$ on U by

$$(\varphi(w)u \,|\, u')_\eta = \langle\, Q(u, u') \,|\, w \,\rangle_\eta \qquad (u, u' \in U).$$

A result due to Dorfmeister [6, Theorem 2.1 (6)] (see also [17, Section 4]) tells us that $w \mapsto \varphi(w)$ is a $*$-representation of the complex Jordan algebra W.

Now we define $\widetilde{f} \in W$ for $f \in W^*$ and $\widetilde{F} \in U$ for $F \in U^\dagger$ by

$$\langle w, f\rangle = \langle\, w \,|\, \widetilde{f}\,\rangle_\eta \quad (\forall w \in W), \qquad \langle u, F\rangle = (\widetilde{F} \,|\, u)_\eta \quad (\forall u \in U).$$

We note here that any element in $\Omega + iV$ is invertible in the Jordan algebra W.

PROPOSITION 3.11. (1) *For $w \in \Omega + iV$ one has $\mathcal{I}(w)^\sim = w^{-1}$.*

(2) *When $z = (u, w) \in D$, one has*

$$\mathcal{C}(z)^\sim = \big(2\,\varphi\big((w + E)^{-1}\big)u,\ (w - E)(w + E)^{-1}\big).$$

Finally we consider the elements

$$g = n(0, u_k)n(0, u_j) \in N_D \qquad (j < k)$$

in (3.1), where $u_j \in \mathfrak{n}_{\alpha_j/2}$ and $u_k \in \mathfrak{n}_{\alpha_k/2}$. The result that we obtain is:

PROPOSITION 3.12. *For every pair $j < k$, one has*

$$\varphi\big(Q(u_k, u_j)\big)u_k = 0, \qquad \varphi\big(Q(u_j, u_k)\big)u_j = 0,$$

where $u_j \in \mathfrak{n}_{\alpha_j/2}$ and $u_k \in \mathfrak{n}_{\alpha_k/2}$ are arbitrary.

Since one can show that $\varphi(E_i)$ is the orthogonal projection operator $U \to \mathfrak{n}_{\alpha_i/2}$ for each i, Dorfmeister's theorem in [3, Corollary 1] guarantees that our Siegel domain is symmetric.

3.3. *The case of symmetric domains.* We conclude this article by indicating the proof of the "if part" of Theorem 3.1. Suppose that D is symmetric. We have the inner product $(\cdot\,|\,\cdot)_\beta$ on $U + W$, through which we define $\iota_\beta(f) \in W$ for $f \in W^*$ and $\iota_\beta(F) \in U$ for $F \in U^\dagger$ by

$$(\iota_\beta(f)\,|\,w)_\beta = \langle w^*, f\rangle \quad (w \in W), \qquad (\iota_\beta(F)\,|\,u)_\beta = \langle u, F\rangle \quad (u \in U).$$

We set $\mathcal{C}_\beta := \iota_\beta \circ \mathcal{C}$ and $\mathcal{D}_\beta := \mathcal{C}_\beta(D)$. We can show that the bounded symmetric domain $\mathcal{D}_\beta$ is circular and that the derivative $\mathcal{C}_\beta'(\mathsf{e})$ at the base point e is a scalar operator. In particular, $\mathcal{D}_\beta$ can be considered as the *Harish-Chandra model* of a bounded symmetric domain. The group $\mathrm{Hol}(\mathcal{D}_\beta)$ of the holomorphic automorphisms of $\mathcal{D}_\beta$ is a semisimple Lie group, and we denote by G its connected component of the identity. Let K be the stabilizer in G at the origin $0 = \mathcal{C}_\beta(\mathsf{e})$. Then K is a maximal compact subgroup of G. Put $\mathsf{A} := \mathcal{C}_\beta \circ A \circ \mathcal{C}_\beta^{-1}$. We have a Cartan decomposition $\mathsf{G} = \mathsf{KAK}$. Every element $h \in \mathsf{G}$ may be written as $h = k_1 \mathsf{a}_t k_2$, where $k_1, k_2 \in \mathsf{K}$ and $\mathsf{a}_t := \mathcal{C}_\beta \circ a_t \circ \mathcal{C}_\beta^{-1} \in \mathsf{A}$ $(t \in \mathbb{R}^r)$. The only thing to be noted is that K is a closed subgroup of the unitary group. Therefore $\|h \cdot 0\|_\beta = \|h^{-1} \cdot 0\|_\beta$ if and only if $\|\mathsf{a}_t \cdot 0\|_\beta$ is invariant under $t \mapsto -t$. But this is clear from the fact that

$$\mathsf{a}_t \cdot 0 = \sum_{j=1}^{r} \Bigl(\tanh \frac{t_j}{2}\Bigr) E_j.$$

References

[1] H. Asano, *On the irreducibility of homogeneous convex cones*, J. Fac. Sci. Univ. Tokyo 15 (1968), 201–208.

[2] F. A. Berezin, *Quantization*, Math. USSR Izv. 8 (1974), 1109–1165.

[3] J. E. D'Atri and J. Dorfmeister, *Flat totally geodesic submanifolds of quasisymmetric Siegel domains*, Geom. Dedicata 28 (1988), 321–336.

[4] J. E. D'Atri, J. Dorfmeister and Zhao Yan Da, *The isotropy representation for homogeneous Siegel domains*, Pacific J. Math. 120 (1985), 295–326.

[5] J. E. D'Atri and I. Dotti Miatello, *A characterization of bounded symmetric domains*, Trans. Amer. Math. Soc. 276 (1983), 531–540.

[6] J. Dorfmeister, *Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains*, Amer. J. Math. 102 (1980), 537–563.

[7] J. Dorfmeister, *Homogeneous Siegel domains*, Nagoya Math. J. 86 (1982), 39–83.

[8] J. Dorfmeister and M. Koecher, *Relative Invarianten und nicht-assoziative Algebren*, Math. Ann. 228 (1977), 147–186.

[9] J. Dorfmeister and M. Koecher, *Reguläre Kegel*, Jber. Deutsch. Math.-Verein. 81 (1979), 109–151.

[10] M. Engliš, *Berezin transform and the Laplace-Beltrami operator*, St. Petersburg Math. J. 7 (1996), 633–647.

[11] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[12] S. G. Gindikin, *Analysis in homogeneous domains*, Russian Math. Surveys 19-4 (1964), 1–89.

[13] H. Ishi, *Representations of the affine transformation groups acting simply transitively on Siegel domains*, J. Funct. Anal. 167 (1999), 425–462.

[14] S. Kaneyuki, *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo 14 (1967), 89–130.

[15] J. L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canad. J. Math. 7 (1955), 562–576.

[16] T. Nomura, *Berezin transforms and group representations*, J. Lie Theory 8 (1998), 433-440.

[17] T. Nomura, *On Penney's Cayley transform of a homogeneous Siegel domain*, J. Lie Theory 11 (2001), 185–206.

[18] T. Nomura, *A characterization of symmetric Siegel domains through a Cayley transform*, Transform. Groups 6 (2001), 227–260.

[19] T. Nomura, *Berezin transforms and Laplace-Beltrami operators on homogeneous Siegel domains*, Diff. Geom. Appl. 15 (2001), 91–106.

[20] T. Nomura, *Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms*, to appear in Diff. Geom. Appl.

[21] T. Nomura, *Geometric connection of the Poisson kernel with a Cayley transform for homogeneous Siegel domains*, preprint, 2001.

[22] R. Penney, *The Harish-Chandra realization for non-symmetric domains in* $\mathbb{C}^n$, in: Topics in Geometry in Memory of Joseph D'Atri, S. Gindikin (ed.), Birkhäuser, Boston, 1996, 295–313.

[23] I. I. Pyatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*, Gordon and Breach, New York, 1969.

[24] H. Rossi, *Lectures on representations of groups of holomorphic transformations of Siegel domains*, Lecture Notes, Brandeis Univ., 1972.

[25] H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal. 13 (1973), 324–389.

[26] I. Satake, *Algebraic Structures of Symmetric Domains*, Iwanami Shoten and Princeton Univ. Press, Tokyo-Princeton, 1980.

[27] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan 31 (1979), 209–226.