GEOMETRY AND ANALYSIS ON LIE GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 55
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2002

REGULARITY PROPERTIES OF GENERALIZED HARISH-CHANDRA EXPANSIONS

GESTUR ÓLAFSSON

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
E-mail: olafsson@math.lsu.edu

ANGELA PASQUALE

Institut für Mathematik, Technische Universität Clausthal, 38678 Clausthal-Zellerfeld, Germany
E-mail: mapa@math.tu-clausthal.de

Abstract. We study the regularity properties of functions that can be represented on a positive Weyl chamber A^+ by generalized Harish-Chandra expansions. We adopt the approach of Heckman and Opdam by allowing arbitrary Weyl-group-invariant complex multiplicities. The generalized Harish-Chandra expansions that we consider are associated with arbitrary parabolic system Θ of roots and a root multiplicity function. They are given as sum over the Weyl group of Θ of generalized hypergeometric functions. They are analytic on A^+ and meromorphic in the spectral parameter λ. We prove that they extend as W_Θ-invariant holomorphic functions on a tubular neighborhood of $(W_\Theta \cdot \overline{A^+})^0$. The possible location of the λ-singularities is shown to be the polar set of an explicit function naturally constructed from the fixed data. For reduced root systems with even multiplicities we refine our result and show that the λ-singularities lie on a specific finite family of affine hyperplanes. Finally, when all multiplicities are equal to 2, we generalize the classical explicit formula for spherical functions on Riemannian symmetric spaces with a complex structure.

Introduction. In this paper we study the regularity properties of certain functions which are represented by generalized Harish-Chandra expansions. There are two important special instances of these functions: (1) Harish-Chandra's spherical functions on Riemannian symmetric spaces of the noncompact type; (2) spherical functions on noncompactly causal symmetric spaces.

2000 *Mathematics Subject Classification*: 43A90, 33C67.

Key words and phrases: generalized Harish-Chandra expansions, spherical functions, hypergeometric functions associated with root systems.

Received 27 February 2001; revised 18 July 2001.

Research of G. Ólafsson was supported by NSF grant DMS-0070607.

The paper is in final form and no version of it will be published elsewhere.

To introduce the notation and to motivate the general setting we are going to deal with, let us start recalling the classical Harish-Chandra expansion for spherical functions on Riemannian symmetric spaces of the noncompact type. General references are [Hel84] and [GV88].

Let G/K be a Riemannian symmetric space of the noncompact type, with G a noncompact connected semisimple Lie group with finite center and K a maximal compact subgroup of G (One could in fact assume G of the Harish-Chandra class. See [GV88], Chapter 2.) A complex-valued function f on G is called K-bi-invariant if $f(k_1xk_2) = f(x)$ for all $x \in G$ and $k_1, k_2 \in K$. We will identify K-bi-invariant functions on G with functions on G/K which are invariant by the left action of K. Let $\mathbb{D}(G/K)$ be the (commutative) algebra of G-invariant differential operators on G/K. A *spherical function* is a K-bi-invariant complex-valued C^∞ function φ on G which is a joint eigenfunction of all elements in $\mathbb{D}(G/K)$, normalized by the condition $\varphi(e) = 1$. Here e denotes the unit element in G. Since the Laplace-Beltrami operator, which is elliptic, belongs to $\mathbb{D}(G/K)$, all spherical functions are real analytic functions on G. The spherical functions are the building blocks for the harmonic analysis of K-bi-invariant functions on G, that is every K-bi-invariant L^2 function on G can be expanded (in a suitable sense) in terms of spherical functions.

In 1958 Harish-Chandra determined integral formulas for the spherical functions [HC58]. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k} \subset \mathfrak{g}$ the Lie algebra of K. Then we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into $(+1)$- and (-1)-eigenspaces of a Cartan involution θ. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $A = \exp \mathfrak{a}$ the corresponding Cartan subgroup of G. The inverse of the diffeomorphism $\exp : \mathfrak{a} \to A$ is denoted by log. We denote by $\mathfrak{a}^*$ (respectively, $\mathfrak{a}^*_{\mathbb{C}}$) the set of real-valued (respectively, complex-valued) linear functionals on $\mathfrak{a}$. Let Δ be the set of (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ and W be the associated Weyl group. We fix a choice Δ^+ of positive roots in Δ and consider the corresponding Iwasawa decomposition $G = KAN$. Every $x \in G$ can be written as $x = k \exp H(x) n$ for unique $H(x) \in \mathfrak{a}$. We denote the multiplicity of a root $\alpha \in \Delta$ by m_α and we set $\rho := 1/2 \sum_{\alpha \in \Delta^+} m_\alpha \alpha$.

Theorem 1 (Harish-Chandra). *For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let φ_λ be the function on G defined by*

$$\varphi_\lambda(x) := \int_K e^{(\lambda - \rho)(H(xk))}\, dk, \qquad x \in G,$$

where dk is the normalized Haar measure on K. Then $\{\varphi_\lambda : \lambda \in \mathfrak{a}^*_{\mathbb{C}}\}$ exhausts the set of spherical functions on G. Moreover $\varphi_\lambda = \varphi_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$.

Because of the polar decomposition $G = KAK$, every K-bi-invariant function is uniquely determined by its W-invariant restriction to A. We can therefore consider the spherical functions as W-invariant functions on A. Their regularity properties are summarized in the following theorem.

Theorem 2 (Harish-Chandra). *$\varphi_\lambda(a)$ is a real-analytic W-invariant function in $a \in A$ and an entire W-invariant function in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*

Theorem 2 is essential in the study of the spherical Fourier transform, which is defined (for sufficiently regular K-bi-invariant functions f on G) by integration against spherical functions: $\mathcal{F}f(\lambda) := \int_G f(x)\varphi_\lambda(x)\, dx$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

The spherical functions are joint eigenfunctions of $\mathbb{D}(G/K)$, in particular of the Laplace-Beltrami operator ω. Hence they satisfy the differential equation corresponding to the radial part $\delta(\omega)$ of ω on A^+:

$$\delta(\omega)\varphi_\lambda = (\langle \lambda, \lambda\rangle - \langle \rho, \rho\rangle)\varphi_\lambda. \tag{1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the bilinear form on $\mathfrak{a}_{\mathbb{C}}^*$ induced by the Killing form of $\mathfrak{g}$.

Let $\widetilde{H}_\alpha$ denote the unique element in $\mathfrak{a}$ satisfying $\alpha(H) = \langle H, \widetilde{H}_\alpha\rangle$ for all $H \in \mathfrak{a}$. We fix an orthonormal basis $\{H_i\}_{i=1}^r$ of $\mathfrak{a}$. Elements of $\mathfrak{a}$ will be regarded as differential operators on A. Then equation (1) can be explicitly rewritten by means of the formula

$$\delta(\omega) = \sum_{i=1}^r H_i^2 + \sum_{\alpha\in\Delta^+} m_\alpha \frac{1+e^{-2\alpha}}{1-e^{-2\alpha}} \widetilde{H}_\alpha.$$

A multidimensional variant of the classical method of Frobenius for determining local solutions of differential equations with regular singularities led Harish-Chandra to look for solutions of (1) of the form

$$\Phi_\lambda(m;a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+. \tag{2}$$

Here $\Lambda := \{\sum_{j=1}^l n_j\alpha_j : n_j \text{ integer} \geq 0\}$ is the positive semigroup generated by the system $\Pi := \{\alpha_1, \dots, \alpha_l\}$ of simple roots in Δ^+. Moreover $m := \{m_\alpha : \alpha \in \Delta\}$ is the set of multiplicities, and the coefficients $\Gamma_\mu(m;\lambda)$ are rational functions of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ obtained by means of the recurrence relations

$$\Gamma_0(m;\lambda) = 1,$$

$$\langle \mu, \mu - 2\lambda\rangle \Gamma_\mu(m;\lambda) = 2 \sum_{\alpha\in\Delta^+} m_\alpha \sum_{\substack{k\in\mathbb{N} \\ \mu-2k\alpha\in\Lambda}} \Gamma_{\mu-2k\alpha}(m;\lambda)\langle \mu + \rho - 2k\alpha - \lambda, \alpha\rangle,$$

$$\text{for } \mu \in \Lambda \setminus \{0\}.$$

THEOREM 3 (Harish-Chandra). *1. There is a set $\mathcal{S}$ (the union of a locally finite family of affine hyperplanes in $\mathfrak{a}_{\mathbb{C}}^*$) so that for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{S}$ the series defining $\Phi_\lambda(m;a)$ converges to a real analytic function on A^+.*

2. For every fixed $a \in A^+$, the function $\Phi_\lambda(m;a)$ is meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with singular set contained in $\mathcal{S}$.

3. The set $\mathcal{S}$ in 1 can be chosen big enough so that for every fixed $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{S}$, the functions $\{\Phi_{w\lambda}(m;a) : w \in W\}$ form a basis for the solution space of (1). Moreover, there is a meromorphic function c on $\mathfrak{a}_{\mathbb{C}}^*$ so that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{S}$ the spherical function φ_λ admits the expansion

$$\varphi_\lambda(a) = \sum_{w\in W} c(m; w\lambda)\Phi_{w\lambda}(m;a), \qquad a \in A^+. \tag{3}$$

The function c occurring in (3) is Harish-Chandra's c-function. An explicit formula for c as product of ratios of gamma functions has been proven by Gindikin and Karpelevic (cf. e.g. [GV88], Theorem 4.7.5, or [Hel84], Chapter IV, Theorem 6.14. See also part 1 of Example 7 in Section 1). The set $\mathcal{S}$ in Theorem 3, part 3, is given by

$$\mathcal{S} = \bigcup_{\substack{\mu\in\Lambda\setminus\{0\}\\ w\in W}} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : 2\langle\mu,\lambda\rangle = \langle\mu,\mu\rangle\} \cup \bigcup_{\substack{\mu\in\Lambda\setminus\{0\}\\ w,v\in W}} \{\lambda\in\mathfrak{a}_{\mathbb{C}}^* : w\lambda - v\lambda = \mu\} \cup \bigcup_{\alpha\in\Delta}\{\lambda\in\mathfrak{a}_{\mathbb{C}}^* : \langle\lambda,\alpha\rangle = 0\}$$

(cf. e.g. [GV88], Theorem 4.4.10 and (4.4.8)). We will say that λ is *generic* if $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{S}$.

EXAMPLE 4. Let $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$. Then G/K can be identified with the upper half-plane $H = \{z = x+iy \in \mathbb{C} : \operatorname{Im} z > 0\}$ endowed with the hyperbolic metric $y^{-2}(dx^2+dy^2)$. Let $L = \frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. The diagonal matrices $a_t = \exp(tL)$ $(t\in\mathbb{R})$ form a Cartan subgroup A and the root system of $(\mathfrak{g}, \mathfrak{a} = \mathbb{R}L)$ is $\Delta = \{\pm\alpha\}$ with $\alpha(L) = 1$. The differential equation (1) is

$$\frac{d^2\varphi}{dt^2} + \coth t\,\frac{d\varphi}{dt} - \left(\lambda^2 - \frac{1}{4}\right)\varphi = 0 \tag{4}$$

and becomes a hypergeometric differential equation with the substitution $z = -\sinh^2 t$. The spherical functions are therefore the hypergeometric functions

$$\varphi_\lambda(a_t) = {}_2F_1\left(1/4+\lambda/2, 1/4-\lambda/2; 1; -\sinh^2 t\right).$$

They admit the integral representation $\varphi_\lambda(a_t) = 1/2\pi\int_0^{2\pi}(\cosh t + \sinh t\cos\theta)^{\lambda-1/2}\,d\theta$. For $\lambda\in\mathbb{C}\setminus\{1,2,3,\ldots\}$ the basic solution of (4) with exponent $\lambda - 1/2$ at ∞ is

$$\begin{aligned}\Phi_\lambda(a_t) &= (2\sinh t)^{\lambda-1/2}{}_2F_1\left(1/4-\lambda/2, 1/4-\lambda/2; 1-\lambda; -1/\sinh^2 t\right)\\ &= (2\cosh t)^{\lambda-1/2}{}_2F_1\left(1/4-\lambda/2, 3/4-\lambda/2; 1-\lambda; 1/\cosh^2 t\right), \qquad t\in(0,\infty).\end{aligned}$$

The Weyl group is $\{\pm1\}$ acting on $\mathfrak{a}_{\mathbb{C}}^* \equiv \mathbb{C}$ by multiplication, and for $\lambda\in\mathbb{C}\setminus\mathbb{Z}$ the Harish-Chandra expansion is given by the classical transit relation (cf. [Er$^+$53], 2.9 (34))

$$\frac{1}{\sqrt{\pi}}\,\varphi_\lambda(a_t) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+1/2)}\Phi_\lambda(a_t) + \frac{\Gamma(-\lambda)}{\Gamma(-\lambda+1/2)}\Phi_{-\lambda}(a_t).$$

The multiplicities $m = \{m_\alpha : \alpha\in\Delta\}$ are nonnegative integers fixed by the geometry of G/K. First Koornwinder for the rank-one case and then Heckman and Opdam for the general higher-rank case observed that the differential equation (1) makes perfectly sense without the geometrical restrictions on m. They considered therefore the following general setting.

Let $\mathfrak{a}$ be an l-dimensional real Euclidean vector space with inner product $\langle\cdot,\cdot\rangle$. For every non-zero $\alpha\in\mathfrak{a}^*$, let $\widetilde{H}_\alpha\in\mathfrak{a}$ be determined by $\alpha(H) = \langle H, \widetilde{H}_\alpha\rangle$ for all $H\in\mathfrak{a}$, and set $H_\alpha := 2\widetilde{H}_\alpha/\langle\widetilde{H}_\alpha,\widetilde{H}_\alpha\rangle$. Let Δ be a root system in $\mathfrak{a}^*$ and Δ^+ a choice of positive roots in Δ. We indicate with $\mathfrak{a}^+$ the open Weyl chamber in $\mathfrak{a}$ on which all elements of Δ^+ are strictly positive. We denote by W the Weyl group of Δ and by $\Pi = \{\alpha_1,\ldots,\alpha_l\}$ the set of simple roots associated with Δ^+. The complexification $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a}\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathfrak{a}$ can be viewed as the Lie algebra of the complex torus $H := \mathfrak{a}_{\mathbb{C}}/2\pi i\mathbb{Z}\{H_\alpha : \alpha\in\Delta\}$. The real form $A := \mathfrak{a}$ of H is an abelian subgroup of H with Lie algebra $\mathfrak{a}$. We write $\exp : \mathfrak{a}_{\mathbb{C}}\to H$ for the exponential map, with multi-valued inverse log, and set $A^+ := \exp\mathfrak{a}^+$.

A *multiplicity function* is a W-invariant $\mathbb{C}$-valued function m on Δ: If we set $m_\alpha := m(\alpha)$, then $m_{w\alpha} = m_\alpha$ for all $w \in W$ and $\alpha \in \Delta$. [1]

Equation (1) can always be associated with the data $(\mathfrak{a}, \Delta, m)$ and the basic solutions $\Phi_\lambda(m; a)$ can be defined by means of formula (2). In general these functions are not associated to symmetric spaces anymore. They are singular on the walls of A^+, so a priori the weighted average (3) is only well-defined on A^+. Using monodromy arguments, Heckman and Opdam proved the following fundamental result (cf. e.g. Theorem 4.4.2 in [HS94]).

THEOREM 5 (Heckman and Opdam). *For a fixed multiplicity function m there is a W-invariant tubular neighborhood U of A in H so that the function*

$$\varphi_\lambda(m; a) := \sum_{w \in W} c(m; w\lambda)\Phi_{w\lambda}(m; a), \qquad a \in A^+,$$

extends to a W-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}^*_{\mathbb{C}} \times U$.

Up to a normalizing constant, the functions $\varphi_\lambda(m; a)$ are the so-called *hypergeometric functions associated with the root system* Δ (or *Heckman-Opdam's hypergeometric functions*). They are in fact common eigenfunctions of all differential operators in a certain commutative algebra which is the analogue of $\mathbb{D}(G/K)$ in this non-geometrical context. They are therefore the spherical functions for the data $(\mathfrak{a}, \Delta, m)$, and reduce to Harish-Chandra's spherical functions when m is the multiplicity function corresponding to a Riemannian symmetric space of the noncompact type.

One should remark that the extension to arbitrary complex multiplicities m is not just a mere generalization. Heckman-Opdam's hypergeometric functions turn out to be meromorphically dependent on m. By studying the $\varphi_\lambda(m; a)$ in suitable ranges of m and then using analytical continuation, Heckman and Opdam could deduce very strong properties for the $\varphi_\lambda(m; a)$ which cannot be proven using standard methods.

Harish-Chandra-type expansions occur also for spherical functions on noncompactly causal symmetric spaces G/H. Here G is (as before) a connected noncompact Lie group. For simplicity of exposition, we also assume that G/H is irreducible. H is an open subset of the set G^σ of fixed point in G of a nontrivial involution $\sigma \neq \theta$ which commutes with the Cartan involution θ. Hence G/H is a non-Riemannian symmetric space of the noncompact type. On the Lie algebra level, σ and θ induce the decompositions into $(+1)$- and (-1)-eigenspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}$. Hence $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K of G. The symmetric space G/H is said to be noncompactly causal (briefly, NCC) if $\mathfrak{p} \cap \mathfrak{q}$ contains a nontrivial $\mathrm{Ad}(H \cap K)$-invariant vector Y^0. In this case Y^0 is uniquely determined up to constant multiples. We select a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ containing Y^0. Then $\mathfrak{a}$ is automatically also maximal abelian in $\mathfrak{p}$. Moreover the set Δ of (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ decomposes as $\Delta = \Delta_+ \cup \Delta_0 \cup \Delta_-$, where Δ_+ (respectively, Δ_0 and Δ_-) is the set of roots $\alpha \in \Delta$ satisfying $\alpha(Y^0) > 0$ (respectively, $\alpha(Y^0) = 0$ and $\alpha(Y^0) < 0$). The set Δ_0 is itself

[1] We have adopted the notation m for the multiplicities used in the theory of symmetric spaces. To recover Heckman-Opdam's notation k one needs to replace 2α with α and $m_{2\alpha}$ with $2k_\alpha$.

a root system with Weyl group W_0 ($\subset W$). As a set of positive roots in Δ we select $\Delta^+ := \Delta_0^+ \cup \Delta_+$, where Δ_0^+ is a choice of positive roots in Δ_0. It is important to remark that for NCC symmetric spaces the root system Δ is reduced, that is, $2\alpha \notin \Delta$ for all $\alpha \in \Delta$.

The causal structure on G/H ensures the existence of a maximal regular $\mathrm{Ad}(H)$-invariant closed convex cone C in $\mathfrak{q}$ containing Y^0. Here regular means that $C \cap (-C) = \{0\}$ and $C - C = \mathfrak{q}$. The set $S := H(\exp C)$ is a maximal H-bi-invariant subsemigroup in G. The theory of spherical functions is developed on the interior S^0 of S. We recall that $S^0 = H(S^0 \cap A)H$, where $S^0 \cap A$ is W_0-invariant and of the form $S^0 \cap A = (W_0 \cdot \overline{A^+})^0$. As before, A^+ denotes the positive Weyl chamber in $A = \exp \mathfrak{a}$ corresponding to the choice of Δ^+ as positive roots.

Similarly to the Riemannian case, the spherical functions on NCC symmetric spaces are defined as common eigenfunctions of the (commutative) algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H. They are parameterized by $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (modulo the small Weyl group W_0), and they also admit an integral formula for "sufficiently negative" values of the parameter λ. We refer to [HÓ97] and [Óla97] for more details. Generalized Harish-Chandra expansions for the spherical functions φ_λ on G/H have been determined by the first author by means of Riemannian duality.

THEOREM 6 (Ólafsson). *Let $\Phi_\lambda(m;a)$ be the basic solution (2) on A^+ of the differential equation (1) for the Riemannian dual space G/K of G/H. Let $c_0(m;\lambda)$ denote Harish-Chandra's c-function corresponding to the root system Δ_0. Then there is a W_0-invariant meromorphic function $c_\Omega(m;\lambda)$ on $\mathfrak{a}_{\mathbb{C}}^*$ so that for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the spherical function φ_λ admits the Harish-Chandra-type expansion*

$$\varphi_\lambda(a) = c_\Omega(m;\lambda) \sum_{w \in W_0} c_0(m;w\lambda)\Phi_{w\lambda}(m;a), \qquad a \in A^+. \tag{5}$$

An explicit product formula for the function c_Ω has been proven by B. Krötz and the first author [KÓ99] (see also Example 7, part 2, below). Adapting the monodromy arguments of Heckman and Opdam, Ólafsson proved that the right-hand side of (5) extends to a W_0-invariant real analytic functions of $a \in S^0 \cap A$ and a W_0-invariant meromorphic functions of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The extension of formula (5) to the Heckman-Opdam's setting of arbitrary multiplicities has been done by Unterberger [Unt99]. However, the location of the λ-singularities of the spherical functions has remained (even in the geometric setting of NCC symmetric spaces) an open problem until [ÓP00].

1. Generalized Harish-Chandra expansions. We work in the general setting of Heckman and Opdam. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the basis of simple roots in Δ^+ and consider any subset $\Theta \subset \Pi$. Let W_Θ be the subgroup of W generated by the reflections $w_i := w_{\alpha_i}$ ($\alpha_i \in \Theta$). We write $\langle\Theta\rangle$ for the set of elements in Δ which can be written as linear combinations of elements of Θ. We set

$$\langle\Theta\rangle^\pm := \langle\Theta\rangle \cap \Delta^\pm \qquad \text{and} \qquad \langle\Theta\rangle^{++} := \langle\Theta\rangle \cap \Delta^{++}$$

for the positive, respectively positive and indivisible roots in $\langle\Theta\rangle$. Then the following inclusions are easily checked:

$$W_\Theta(\Delta^\pm \setminus \langle\Theta\rangle^\pm) \subset \Delta^\pm \setminus \langle\Theta\rangle^\pm \qquad \text{and} \qquad W_\Theta(\Delta^{++} \setminus \langle\Theta\rangle^{++}) \subset \Delta^{++} \setminus \langle\Theta\rangle^{++}.$$

For a fixed multiplicity function m, we define the following c-functions associated with $\langle\Theta\rangle$:

$$c_+(\Theta, m; \lambda) := \prod_{\alpha\in\langle\Theta\rangle^{++}} \frac{2^{-(\lambda(H_\alpha)+m_\alpha)/2}\Gamma(\frac{\lambda(H_\alpha)}{2})}{\Gamma(\frac{\lambda(H_\alpha)}{4}+\frac{m_\alpha}{4}+\frac{1}{2})\Gamma(\frac{\lambda(H_\alpha)}{4}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2})},$$

$$c_-(\Theta, m; \lambda) := \prod_{\alpha\in\Delta^{++}\setminus\langle\Theta\rangle^{++}} \frac{\Gamma(-\frac{\lambda(H_\alpha)}{4}-\frac{m_\alpha}{4}+\frac{1}{2})\Gamma(-\frac{\lambda(H_\alpha)}{4}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1)}{2^{(\lambda(H_\alpha)+m_\alpha)/2}\Gamma(-\frac{\lambda(H_\alpha)}{2}+1)}.$$

When Δ is reduced the duplication formula $\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$ gives

$$c_+(\Theta, m; \lambda) := \prod_{\alpha\in\langle\Theta\rangle^{++}} \frac{\Gamma(\frac{\lambda(H_\alpha)}{2})}{2\sqrt{\pi}\,\Gamma(\frac{\lambda(H_\alpha)}{2}+\frac{m_\alpha}{2})},$$

$$c_-(\Theta, m; \lambda) := \prod_{\alpha\in\Delta^{++}\setminus\langle\Theta\rangle^{++}} \frac{\sqrt{\pi}\,\Gamma(-\frac{\lambda(H_\alpha)}{2}-\frac{m_\alpha}{2}+1)}{\Gamma(-\frac{\lambda(H_\alpha)}{2}+1)}$$

Recall the definition of generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$ given after Theorem 3. The *generalized Harish-Chandra expansion* is the function on A^+ defined for generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\varphi_\lambda(\Theta, m; a) := c_-(\Theta, m; \lambda) \sum_{w\in W_\Theta} c_+(\Theta, m; w\lambda)\Phi_{w\lambda}(m; a), \qquad a \in A^+. \tag{6}$$

Example 7. 1. When $\Theta = \Pi$, then $c_+(\Pi, m; \lambda)$ is a constant multiple of Harish-Chandra's c-function for the data $(\mathfrak{a}, \Delta, m)$. Hence $\varphi_\lambda(\Pi, m; a)$ reduces (up to a multiplicative constant) to Heckman-Opdam's hypergeometric function. In particular, for geometric multiplicities, we recover Harish-Chandra's spherical functions.

2. Suppose Δ is the restricted root system of a NCC symmetric space (in particular Δ is reduced). Let Π_0 the basis of simple roots in Δ_0^+ and set $\Theta := \Pi_0$. Then $\langle\Theta\rangle = \Delta_0$, $W_\Theta = W_0$, and (up to constant multiples) $c_+(\Theta, m; \lambda) = c_0(m; \lambda)$ and $c_-(\Theta, m; \lambda) = c_\Omega(m; \lambda)$. In this case the generalized Harish-Chandra expansion reduces to Unterberger's for arbitrary multiplicities, and to the spherical functions on NCC symmetric spaces for geometric multiplicities.
3. When $\Theta = \emptyset$, then $\varphi_\lambda(\emptyset, m; a) = c_-(\emptyset, m; \lambda)\Phi_\lambda(m; a)$.

Our aim is to study the regularity properties of $\varphi_\lambda(\Theta, m; a)$. One can immediately observe that $\varphi_\lambda(\Theta, m; a)$ is a W_Θ-invariant function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ because $W_\Theta(\Delta^{++} \setminus \langle\Theta\rangle^{++}) = \Delta^{++} \setminus \langle\Theta\rangle^{++}$. The possible λ-singularities are described by the singularities of the numerator of the function $c_-(\Theta, m; \lambda)$:

$$n_-(\Theta, m; \lambda) := \prod_{\alpha\in\Delta^{++}\setminus\langle\Theta\rangle^{++}} \Gamma\Big(-\frac{\lambda(H_\alpha)}{4}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda(H_\alpha)}{4}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big).$$

We also define the denominator $d_-(\Theta, m; \lambda)$ of $c_-(\Theta, m; \lambda)$ by

$$d_-(\Theta, m; \lambda) := \prod_{\alpha \in \Delta^{++} \setminus \langle \Theta \rangle^{++}} 2^{(\lambda(H_\alpha)+m_\alpha)/2} \Gamma\Big(-\frac{\lambda(H_\alpha)}{2} + 1 \Big)$$

(we set $n_-(\Theta, m; \lambda) = d_-(\Theta, m; \lambda) \equiv 1$ if $\Delta = \langle \Theta \rangle$).

Theorem 8. *For a fixed multiplicity function m, there is a W_Θ-invariant tubular neighborhood U of $[W_\Theta(\overline{A^+})]^{\,0}$ in H so that the function*

$$\frac{\varphi_\lambda(\Theta, m; a)}{n_-(\Theta, m; \lambda)} = \frac{1}{d_-(\Theta, m; \lambda)} \sum_{w \in W_\Theta} c_+(\Theta, m; w\lambda) \Phi_{w\lambda}(m; a)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}^*_{\mathbb{C}} \times U$.

Sketch of proof (cf. [ÓP00]). The proof relies heavily on the Heckman-Opdam theory. The first fundamental result we need is Opdam's description of the possible singularities of $\Phi_\lambda(m; a)$ (cf. [Opd98], Lemma 6.5): For every fixed $a \in A^+$ the function $\Phi_\lambda(m; a)$ is meromorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with at most simple poles located along hyperplanes of the form

$$\mathcal{H}_{\alpha,n} := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \lambda(H_\alpha)/2 = n\} \tag{7}$$

where $\alpha \in \Delta^{++}$ and $n \in \mathbb{N}$. This proves the Theorem for $\Theta = \emptyset$, so we can assume $\Theta \neq \emptyset$. Knowing the singularities of the $\Phi_\lambda(m; a)$, we can use the explicit formula for the functions c to count the possible singularities occurring in the Harish-Chandra expansions. It follows that the possible singularities are: (a) poles of $n_-(\Theta, m; \lambda)$; (b) at most simple poles along hyperplanes $\mathcal{H}_{\alpha,n}$ with $\alpha \in \langle \Theta \rangle^{++}$ and $n \in \mathbb{Z}$. We claim that all singularities in (b) are removable. Observe first that intersections of hyperplanes $\mathcal{H}_{\alpha,n}$ for $\alpha \in \langle \Theta \rangle$ with hyperplanes $\mathcal{H}_{\alpha,n}$ for $\alpha \in \Delta \setminus \langle \Theta \rangle$ are varieties of codimension ≥ 2. Hence Hartogs' theorem guarantees that the singularities in (b) are removable exactly when so are those of

$$\sum_{w \in W_\Theta} c_+(\Pi, m; w\lambda) \Phi_{w\lambda}(m; a), \qquad a \in A^+.$$

If $\Theta = \Pi$ then this is the function considered by Heckman and Opdam. Applying their argument to the smaller group W_Θ, we can cancel the singularities along the hyperplanes associated to each element in $\langle \Theta \rangle$, that is, the singularities listed in (b). The extension in the variable a is obtained by monodromy arguments. ∎

The singularities of the the numerator $n_-(\Theta, m; \lambda)$ are poles located along the hyperplanes $\mathcal{H}_{\alpha, -m_\alpha/2+(2n-1)}$ and $\mathcal{H}_{\alpha, -m_\alpha/2 - m_{2\alpha}+2n}$ for $\alpha \in \Delta^{++} \setminus \langle \Theta \rangle^{++}$ and $n \in \mathbb{N}$. The poles are simple when $m_{2\alpha}$ is not an odd integer. In the symmetric case, it is known that if 2α is a root, then $m_{2\alpha}$ is odd. In this case the poles of $n_-(\Theta, m; a)$ are simple exactly when 2α is not a root for all $\alpha \in \Delta^{++} \setminus \langle \Theta \rangle^{++}$. This occurs when Δ is reduced, for instance in the case of Harish-Chandra expansions of spherical functions on NCC symmetric spaces.

Corollary 9. *If Δ is a reduced system of roots, then*

$$n_-(\Theta, m; \lambda) = \prod_{\alpha \in \Delta^+ \setminus \langle \Theta \rangle^+} \sqrt{\pi}\, 2^{\lambda(H_\alpha)/2 + m_\alpha/2 + 1}\, \Gamma\left(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1 \right).$$

The singularities of $\varphi_\lambda(\Theta, m; a)$ are at most simple poles located along the polar set of $n_-(\Theta, m; \lambda)$.

In the case $\Theta = \Pi$ our theorem reduces to the original Theorem 5 of Heckman and Opdam. If $\Theta \subsetneq \Pi$, then the set of singularities described by the functions $n_-(\Theta, m; \lambda)$ is an infinite locally finite family of hyperplanes in $\mathfrak{a}_{\mathbb{C}}^*$. It gives exactly all singularities of the Harish-Chandra expansion for instance in the case of the NCC space $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$. In fact, in this case (as well as for general rank-one NCC symmetric spaces) the singularities can be checked directly from the known explicit formula for the spherical functions. For $a_t \in A^+ \equiv (0, \infty)$ the spherical functions on $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$ are given as hypergeometric functions

$$\varphi_\lambda(a_t) = n_\Omega(\lambda) \frac{(2\cosh t)^{\lambda - 1/2}}{\Gamma(1-\lambda)} {}_2F_1\left(1/4 - \lambda/2, 3/4 - \lambda/2; 1 - \lambda; 1/\cosh^2 t\right)$$

with

$$n_\Omega(\lambda) := \sqrt{\pi}\,\Gamma(-\lambda + 1/2).$$

On the other hand one can also see for rank-one NCC symmetric spaces with even multiplicities (or in the complex case) that the singularities of the spherical functions lie on a *finite* family of hyperplanes (the "beginning" of the two families of singular hyperplanes are the same). In the following section we will prove that the finiteness of the singular hyperplanes is a general property of generalized Harish-Chandra expansions associated with even multiplicities.

As already remarked, the choice of a non-geometric setting in which the multiplicities are allowed to assume arbitrary complex values is the natural context for the application of Heckman-Opdam's methods. There is also another reason to keep our setting at this level of generality. We have seen that spherical functions on Riemannian and NCC symmetric spaces possess Harish-Chandra expansions which are particular instances of those we considered. Both classes of spaces are K_ε-spaces in the sense of Oshima and Sekiguchi [OS80]: the Riemannian symmetric spaces correspond to a trivial signature ε, and the NCC symmetric spaces to a signature which is trivial on Π_0 and equal to -1 on the simple root in Δ_+ (the latter are the spaces KεI according to Kaneyuki). There are many other K_ε-spaces which are neither Riemannian nor NCC. On these spaces Oshima and Sekiguchi have made the first steps for a theory of spherical functions. It would be interesting to see if the choice of different subgroups W_Θ of W could correspond to the specification of Harish-Chandra type expansions on different K_ε-spaces.

2. Even multiplicities. In this section we restrict our analysis to the case of *reduced* root systems Δ with an even multiplicity function (that is, all root multiplicities are even positive integers). In the context of symmetric spaces, this condition singles out spaces G/K with the property that all Cartan subalgebras in the Lie algebra $\mathfrak{g}$ of G are conjugate under the adjoint group of $\mathfrak{g}$ (cf. [Hel78], p. 429). According to the classification, the irreducible Riemannian symmetric spaces with even multiplicities are $\mathrm{SO}_0(2n+1, 1)/\mathrm{SO}(2n+1)$, $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$, $E_{6(-26)}/F_{4(-52)}$ and the irreducible spaces of the form $G_{\mathbb{C}}/U$ where U is a compact real form of $G_{\mathbb{C}}$. All the K_ϵ-spaces of Oshima-Sekiguchi [OS80] associated with these Riemannian symmetric spaces are further examples of symmetric spaces with even multiplicities. An interesting special case consists of spaces with a noncompactly causal structure. Up to coverings, the irreducible NCC spaces with even multiplicities are (cf. [AÓS00]) $\mathrm{SO}_0(2n+1,1)/\mathrm{SO}_0(2n,1)$, $\mathrm{SU}^*(2(p+q))/\mathrm{Sp}(p,q)$, $E_{6(-26)}/F_{4(-20)}$ and the irreducible spaces $G_{\mathbb{C}}/G$ where G is Hermitian. We refer to [AÓS00] for additional details on NCC spaces with even multiplicities, their spherical functions and applications of the spherical harmonic analysis on them.

The key tool for studying generalized Harish-Chandra expansions with even multiplicities is Opdam's shift operator [Opd88]. See also [HS94], Section 3.1.

Consider the Weyl denominator δ defined on H by

$$\delta(a) := \prod_{\alpha\in\Delta^+} \sinh\alpha(\log a) = \sum_{w\in W} \det(w) e^{w\rho(\log a)}. \tag{8}$$

Set $H^{\mathrm{reg}} := \{h \in H : \delta(h) \neq 0\}$, and let $\mathcal{O}(H^{\mathrm{reg}})$ denote the ring of holomorphic functions on H^{reg}. Shift operators are certain differential operators on H with coefficients in $\mathcal{O}(H^{\mathrm{reg}})$ which depend polynomially on the multiplicities. The adjective "shift" reflects their property of mapping (for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$) the Harish-Chandra series $\Phi_\lambda(m;a)$ corresponding to the given multiplicity m into $\eta(m;\lambda)\Phi_\lambda(m+l;a)$. Here $\Phi_\lambda(m+l;a)$ is the Harish-Chandra series with multiplicity shifted by an even multiplicity l and $\eta(m;\lambda)$ is a rational function of m and λ.

Let m be a fixed positive even multiplicity. Since $\Phi_\lambda(0;a) = e^{\lambda(\log a)}$, there exists a shift operator $D(m)$ so that for all $a \in A^+$ and all generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$$D(m)e^{\lambda}(\log a) = [c_+(\Pi, m; -\lambda)]^{-1}\Phi_\lambda(m;a) \tag{9}$$

(cf. Corollary 3.4.4 in [HS94]). Let V be a tubular neighborhood of A in H for which the function $(\lambda, a) \mapsto e^{\lambda(\log a)}$ is single-valued and holomorphic on $\mathfrak{a}_{\mathbb{C}}^* \times V$. Setting $V^{\mathrm{reg}} := V \cap H^{\mathrm{reg}}$, we conclude that $(\lambda, a) \mapsto D(m)e^{\lambda(\log a)}$ is holomorphic in $\mathfrak{a}_{\mathbb{C}}^* \times V^{\mathrm{reg}}$.

Equality (9) will allow us to sharpen the result in Theorem 8 by showing that the λ-singularities of generalized Harish-Chandra expansions corresponding to a reduced root system with an even multiplicity function are located on a specific *finite* union of affine hyperplanes.

Let us first make a simple observation. Consider the following functions depending on $\alpha \in \Delta$:

$$c_+^\alpha(m;\lambda) := \frac{\Gamma(\frac{\lambda(H_\alpha)}{2})}{\Gamma(\frac{\lambda(H_\alpha)}{2} + \frac{m_\alpha}{2})},$$
$$c_-^\alpha(m;\lambda) := \frac{\Gamma(-\frac{\lambda(H_\alpha)}{2} - \frac{m_\alpha}{2} + 1)}{\Gamma(-\frac{\lambda(H_\alpha)}{2} + 1)}.$$

Since m_α is an even positive integer, the functional equation $\Gamma(z+1) = z\Gamma(z)$ implies the equalities

$$\begin{aligned} c_-^\alpha(m;\lambda)\, c_+^\alpha(m;-\lambda) &= (-1)^{m_\alpha/2}\, c_+^\alpha(m;\lambda)\, c_+^\alpha(m;-\lambda) \\ &= \left[\frac{\lambda(H_\alpha)}{2} \prod_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \left(\frac{\lambda(H_\alpha)}{2} - j\right)\right]^{-1}. \end{aligned} \tag{10}$$

THEOREM 10. *Let m be an even multiplicity function on a reduced root system Δ. Define for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$*

$$e_-(\Theta, m; \lambda) := \prod_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \prod_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \Big(\frac{\lambda(H_\alpha)}{2} - j \Big). \tag{11}$$

Then there is a W_Θ-invariant tubular neighborhood U_0 of $[W_\Theta(\overline{A^+})]^{\,0}$ in H so that the function

$$e_-(\Theta, m; \lambda)\, \varphi_\lambda(\Theta, m; a)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U_0$.

Proof. Substitution of (9) and (10) in (6) gives for all generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $a \in A^+$, and for some constant k

$$\begin{aligned} \varphi_\lambda(\Theta, m; a) &= k \Big[\prod_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} c_-^\alpha(m; \lambda) c_+^\alpha(m; -\lambda) \Big] \\ &\quad \times \sum_{w \in W_\Theta} \Big[\prod_{\alpha \in \langle\Theta\rangle^+} c_+^\alpha(m; w\lambda) c_+^\alpha(m; -w\lambda) \Big] D(m) e^{w\lambda(\log a)} \\ &= k(-1)^{\sum_{\alpha \in \langle\Theta\rangle^+} m_\alpha/2} \prod_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \Big[\frac{\lambda(H_\alpha)}{2} \prod_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \Big(\frac{\lambda(H_\alpha)}{2} - j \Big) \Big]^{-1} \\ &\quad \times \sum_{w \in W_\Theta} \prod_{\alpha \in \langle\Theta\rangle^+} \Big[\frac{\lambda(H_{w^{-1}\alpha})}{2} \prod_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \Big(\frac{\lambda(H_{w^{-1}\alpha})}{2} - j \Big) \Big]^{-1} D(m) e^{w\lambda(\log a)}. \end{aligned} \tag{12}$$

Recall the notation $\mathcal{H}_{\alpha,n}$ from (7). The right-hand side of (12) is holomorphic in $a \in V^{\mathrm{reg}}$ and meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with possible singularities located along

$$\Big[\bigcup_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \bigcup_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{\alpha,j} \Big] \cup \Big[\bigcup_{w \in W_\Theta} \bigcup_{\alpha \in \langle\Theta\rangle^+} \bigcup_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{w^{-1}\alpha,j} \Big].$$

According to Theorem 8, the left-hand side of (12) is holomorphic in $a \in U$ and meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with at most simple poles located along

$$\bigcup_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \bigcup_{j=-m_\alpha/2+1}^{\infty} \mathcal{H}_{\alpha,j}.$$

We conclude that for every fixed $a \in U \cap V^{\mathrm{reg}}$ the generalized Harish-Chandra expansion $\varphi_\lambda(\Theta, m; a)$ has at most simple poles located along

$$\bigcup_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \bigcup_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{\alpha,j}.$$

The result extends to all $a \in U_0 := U \cap V$ by Lemma 2.2.11 in [Hör66]. ■

As an immediate corollary, we deduce the regularity properties of spherical functions on NCC spaces with even multiplicities.

Corollary 11. *Let φ_λ ($\lambda \in \mathfrak{a}_\mathbb{C}^*$) denote the meromorphically continued spherical functions on a NCC symmetric space G/H with even multiplicities. Define for $\lambda \in \mathfrak{a}_\mathbb{C}^*$*

$$e_\Omega(m;\lambda) := \prod_{\alpha\in\Delta_+} \prod_{j=-m_\alpha/2+1}^{m_\alpha/2-1} \Big(\frac{\lambda(H_\alpha)}{2} - j\Big). \tag{13}$$

Then there is a W_0-invariant tubular neighborhood U_0 of $S^0 \cap A$ in the complexification of A so that the function $e_\Omega(m;\lambda)\,\varphi_\lambda(a)$ extends as a W_0-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}_\mathbb{C}^* \times U_0$.

In the geometric situation, the case of a symmetric space with all multiplicities equal to 2 plays a special role. It occurs precisely when the group G has a complex structure. The spherical harmonic analysis on Riemannian and NCC spaces with complex G is particularly simplified by explicit formulas for the spherical functions. We now extend these formulas to generalized Harish-Chandra expansions. Our argument is an easy modification of the procedure used in the Riemannian case (cf. [Hel84], p. 432). The formula also allows us to give an independent proof of Theorem 10 where all multiplicities are 2.

Let $\varphi_\lambda(\Theta, 2; a)$ denote the generalized Harish-Chandra expansion associated with the subset $\Theta \subset \Pi$ of simple roots with $m_\alpha = 2$ for all $\alpha \in \Delta$. Since $\Phi_\lambda(2;a) = \delta(a)^{-1} e^{\lambda(\log a)}$, we have

$$\delta(a)\varphi_\lambda(\Theta,2;a) = c_-(\Theta,2;\lambda) \sum_{w\in W_\Theta} c_+(\Theta,2;w\lambda)\, e^{w\lambda(\log a)}, \qquad a \in A^+. \tag{14}$$

Equation (14) extends by analyticity to a tubular neighborhood U of A in H.

The function $c_-(\Theta,2;\lambda)$ is W_Θ-invariant because $W_\Theta(\Delta^+ \setminus \langle\Theta\rangle^+) = \Delta^+ \setminus \langle\Theta\rangle^+$. Furthermore, replacing in (14) the variable $a \in A$ with wa for $w \in W_\Theta$, we obtain $c_+(\Theta,2;w\lambda) = \det(w) c_+(\Theta,2;\lambda)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $w \in W_\Theta$.

The assumption $m_\alpha = 2$ for all $\alpha \in \Delta$ yields for all $a \in U$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$

$$\delta(a)\varphi_\lambda(\Theta,2;a) = c_-(\Theta,2;\lambda) c_+(\Theta,2;\lambda) \sum_{w\in W_\Theta} \det(w)\, e^{w\lambda(\log a)}, \tag{15}$$

with

$$c_+(\Theta,2;\lambda) = k_1 \prod_{\alpha\in\langle\Theta\rangle^+} \frac{1}{\lambda(H_\alpha)} \qquad \text{and} \qquad c_-(\Theta,2;\lambda) = k_2 \prod_{\alpha\in\Delta^+\setminus\langle\Theta\rangle^+} \frac{1}{\lambda(H_\alpha)}$$

for some constants k_1 and k_2. Besides we consider the factorization

$$\delta(a) = \delta_+(\Theta;a)\delta_-(\Theta;a) \tag{16}$$

with

$$\delta_+(\Theta;a) := \prod_{\alpha\in\langle\Theta\rangle^+} \sinh\alpha(\log a) \qquad \text{and} \qquad \delta_-(\Theta;a) := \prod_{\alpha\in\Delta^+\setminus\langle\Theta\rangle^+} \sinh\alpha(\log a).$$

Theorem 12. *Let Δ be a reduced root system in $\mathfrak{a}$ with multiplicities $m_\alpha = 2$ for all $\alpha \in \Delta$. Then the generalized Harish-Chandra expansions can be written as*

$$\varphi_\lambda(\Theta,2;a) = \frac{c_-(\Theta,2;\lambda)}{\delta_-(\Theta;a)}\, \varphi^0_\lambda(\Theta,2;a), \tag{17}$$

where

$$\varphi^0_\lambda(\Theta, 2; a) := \frac{c_+(\Theta, 2; \lambda)}{\delta_+(\Theta; a)} \sum_{w \in W_\Theta} \det(w)\, e^{w\lambda(\log a)} \tag{18}$$

is the generalized Harish-Chandra expansion associated with Θ *as set of simple roots and multiplicities* $m_\alpha = 2$ *for all* $\alpha \in \langle\Theta\rangle$. *Moreover, there is a tubular neighborhood* U_0 *of* A *in* H *on which*

$$\varphi^0_\lambda(\Theta, 2; a) \qquad \text{and} \qquad \Big(\prod_{\alpha \in \Delta^+ \setminus \langle\Theta\rangle^+} \lambda(H_\alpha)\Big)\delta_-(\Theta; a)\, \varphi_\lambda(\Theta, 2; a)$$

extend as W_Θ*-invariant holomorphic functions of* $(\lambda, a) \in \mathfrak{a}^*_{\mathbb{C}} \times U_0$.

Proof. In view of (15)–(17) it suffices to prove the statement on the holomorphy of $\varphi^0_\lambda(\Theta, 2; a)$. Because of (18), the possible λ-sigularities of the function $\varphi^0_\lambda(\Theta, 2; a)$ are at most simple poles located along the hyperplanes $\mathcal{H}_{\alpha,0}$ with $\alpha \in \langle\Theta\rangle^+$, but these singularities are in fact removable by the W_Θ-invariance in λ. Similarly, W_Θ-invariance in a proves that the possible simple poles located on the hyperplanes $\{h \in H : \alpha(\log h) = 0\}$ for $\alpha \in \langle\Theta\rangle^+$ are also removable. ∎

In the case of a Riemannian symmetric space G/K with G complex (i.e. $\Theta = \Pi$), formula (17) reduces to the classical formula (23) in [Hel84], p. 432. For a NCC symmetric space G/H with G complex, (17) gives also the relation between spherical functions on G/H and spherical functions on the Riemannian symmetric space $G^{\theta\sigma}/(K \cap H)$.

References

[AÓS00] N. B. Andersen, G. Ólafsson and H. Schlichtkrull, *On the inversion of the Laplace and Abel transforms for causal symmetric spaces*, Forum Math., to appear.

[Er+53] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Volume 1, McGraw-Hill, New York, 1953.

[GV88] R. Gangolli and V. S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Springer-Verlag, 1988.

[HC58] Harish-Chandra, *Spherical functions on a semisimple Lie group. I*, Amer. J. Math. 80 (1958), 241–310.

[Hel78] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978.

[Hel84] S. Helgason, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, 1984.

[HS94] G. Heckman and H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Academic Press, 1994.

[HÓ97] J. Hilgert and G. Ólafsson. *Causal Symmetric Spaces. Geometry and Harmonic Analysis*, Academic Press, 1997.

[Hör66] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, D. Van Nostrand, 1966.

[KÓ99] B. Krötz and G. Ólafsson, *The c-function for non-compactly causal symmetric spaces*, Invent. Math., to appear.

[Óla97] G. Ólafsson, *Spherical functions and spherical Laplace transform on ordered symmetric spaces*, preprint, 1997.

[ÓP00] G. Ólafsson and A. Pasquale, *On the meromorphic extension of the spherical functions on noncompactly causal symmetric spaces*, J. Funct. Anal. 181 (2001), 246–401.

[Opd88] E. M. Opdam, *Root systems and hypergeometric functions. III*, Compositio Math. 67 (1988), 21–49.

[Opd98] E. M. Opdam, *Lectures on Dunkl operators*, preprint, math.RT/9812007, 1998.

[OS80] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. 57 (1980), 1–81.

[Unt99] J. Unterberger, *Analyse harmonique sur un espace symétrique ordonné et sur son dual compact*, PhD Thesis, Université Pierre et Marie Curie, Paris, 1999.