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# $L^p$ compactness for Calderón type commutators

by

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**Abstract.** We discuss the $L^p$ compactness of Calderón type commutators $T_A$ defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} R(A; x, y) f(y)\, dy,$$

where $R(A; x, y) = A(x) - A(y) - \nabla A(y) \cdot (x-y)$ with $D^\beta A \in \mathrm{BMO}(\mathbb{R}^n)$ for all $n \geq 2$ and $|\beta| = 1$. Moreover, $\Omega$ is homogeneous of degree zero and has a vanishing moment of order one on $\mathbb{S}^{n-1}$.

We prove that both $T_A$ and its maximal operator $T_{A,*}$ are compact operators on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ with $A$ satisfying some conditions. Moreover, the compactness of the fractional operators $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$ is proved.

**1. Introduction.** In 1965, Calderón [Ca] introduced the following commutator on $\mathbb{R}$:

$$[A, S] f(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{A(x) - A(y)}{x-y} \frac{f(y)}{x-y}\, dy,$$

where $A \in \mathit{Lip}(\mathbb{R})$ and $S := \frac{d}{dx} \circ H$, $H$ denoting the Hilbert transform defined by

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Calderón proved that the commutator $[A, S]$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$.

In the same paper [Ca], Calderón also gave a generalization of the commutator $[A, S]$ in higher dimensions:

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$$\mathfrak{T}_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \cdot \frac{A(x)-A(y)}{|x-y|} \cdot f(y)\,dy, \tag{1.1}$$

where $\Omega$ is the function defined on $\mathbb{R}^n \setminus \{0\}$ satisfying the homogeneity condition

$$\Omega(\lambda x') = \Omega(x') \quad \text{for any } \lambda > 0 \text{ and } x' \in \mathbb{S}^{n-1}, \tag{1.2}$$

and the vanishing moment condition of order one:

$$\int_{\mathbb{S}^{n-1}} \Omega(x') x'^{\alpha}\, d\sigma(x') = 0 \quad \text{for all } \alpha \in \mathbb{Z}_+^n \text{ with } |\alpha| = 1. \tag{1.3}$$

Here and below, $\alpha$ is a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. Moreover, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ for $x \in \mathbb{R}^n$. Calderón showed that $\mathfrak{T}_A$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $A \in Lip(\mathbb{R}^n)$ and $\Omega \in L \log^+ L(\mathbb{S}^{n-1})$ satisfies (1.2) and (1.3).

In 1981, Cohen [Co] gave an extension of the Calderón commutator $\mathfrak{T}_A$ as follows. Let us first recall the definition of BMO space.

Definition 1.1 (BMO function). Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $B \subset \mathbb{R}^n$ is a ball. Denote by $f_B$ the mean of $f$ on $B$, that is, $f_B = |B|^{-1} \int_B f(x)\,dx$. For $a > 0$, let

$$\mathcal{M}(f,B) = \frac{1}{|B|} \int_B |f(x) - f_B|\,dx \quad \text{for any ball } B \subset \mathbb{R}^n,$$

and

$$\mathcal{M}_a(f) = \sup_{|B|=a} \mathcal{M}(f,B).$$

The function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that $\|f\|_* := \sup_{a>0} \mathcal{M}_a(f) \le C$.

Let $A$ be a function on $\mathbb{R}^n$ with $\nabla A \in \mathrm{BMO}$, that is, $D^\beta A \in \mathrm{BMO}(\mathbb{R}^n)$ for every multi-index $\beta$ with $|\beta| = 1$, where $D^\beta A(x) = \frac{\partial^{|\beta|} A}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(x)$ is the partial derivative of $A$ which is assumed to exist in the classical sense almost everywhere on $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, set

$$R(A; x, y) = A(x) - A(y) - \nabla A(y) \cdot (x - y). \tag{1.4}$$

Then the *Calderón type commutator* $T_A$ is defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} R(A; x, y) f(y)\,dy. \tag{1.5}$$

Cohen [Co] showed that if $\Omega \in Lip(\mathbb{S}^{n-1})$ satisfies (1.2), (1.3), then for $A$ with $\nabla A \in \mathrm{BMO}$, $T_A$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

In 1994, Hofmann [Ho] improved the result of [Co].

Theorem A ([Ho]). *If $\Omega \in \bigcup_{s>1} L^s(\mathbb{S}^{n-1})$ satisfies* (1.2) *and* (1.3), *then for $A$ with $\nabla A \in \mathrm{BMO}$, $T_A$ is bounded on $L^p(\mathbb{R}^n)\,(1 < p < \infty)$ with the bound $C\sum_{|\beta|=1} \|D^\beta A\|_*$.*

Now let us consider the maximal operator $T_{A,*}$ of the Calderón type commutator $T_A$, which is defined by

$$T_{A,*}f(x) = \sup_{\varepsilon>0} |T_{A,\varepsilon}f(x)|, \tag{1.6}$$

where $T_{A,\varepsilon}$ is the truncated operator of $T_A$ defined by

$$T_{A,\varepsilon}f(x) = \int\limits_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{n+1}} R(A;x,y)f(y)\,dy. \tag{1.7}$$

Cohen [Co] stated that if $\Omega \in Lip(\mathbb{S}^{n-1})$ satisfies (1.2), (1.3), then for $A$ with $\nabla A \in \mathrm{BMO}$, $T_{A,*}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2005, Hu [Hu] improved Cohen's result above for $n \geq 2$. For $\Omega \in L^s(\mathbb{S}^{n-1})\,(s \geq 1)$, the $L^s$ integral modulus of continuity $\omega_s$ of $\Omega$ is defined by

$$\omega_s(\delta) = \sup_{\|\rho\|<\delta} \Big( \int\limits_{\mathbb{S}^{n-1}} |\Omega(\rho x') - \Omega(x')|^s \, d\sigma(x') \Big)^{1/s},$$

where $\|\rho\| = \sup_{x'\in\mathbb{S}^{n-1}} |\rho x' - x'|$, and denote $\omega(\delta) = \omega_1(\delta)$.

Theorem B ([Hu]). *Let $n \geq 2$. Suppose $\Omega \in L^1(\mathbb{S}^{n-1})$ satisfies* (1.2) *and* (1.3). *If $\omega$ satisfies the following Log-type Dini-condition:*

$$\int_0^1 \frac{\omega(\delta)}{\delta} \log(2 + 1/\delta)\, d\delta < \infty, \tag{1.8}$$

*then for $A$ with $\nabla A \in \mathrm{BMO}$, $T_{A,*}$ is bounded on $L^p(\mathbb{R}^n)$ with the bound $C\sum_{|\beta|=1} \|D^\beta A\|_*$ for any $1 < p < \infty$.*

The compact operator is an important concept in analysis. It is well known that the commutators of many important operators in harmonic analysis are all compact operators on some suitable $L^p$ spaces and Morrey spaces (see [U], [BL], [KL1], [KL2], [W] and the recent works [BT], [BDMT], [CD1]–[CDW3], [DM2], [DMX]). Thus, it is natural to ask whether the $L^p$-compactness of the Calderón type commutator $T_A$ and its maximal operator $T_{A,*}$ holds or not.

In this paper, we will consider this problem. Let us recall some definitions and a known result.

Definition 1.2. Suppose $X, Y$ are Banach spaces and $U$ is the unit ball in $X$. A linear or sublinear operator $S : X \to Y$ is said to be a *compact operator* from $X$ to $Y$ if $S(U)$ is precompact in $Y$.

Definition 1.3 (VMO function). A function $f$ in $\mathrm{BMO}(\mathbb{R}^n)$ is said to belong to $\mathrm{VMO}(\mathbb{R}^n)$ if

$$\lim_{a\to 0}\mathcal{M}_a(f)=0.$$

In 1998, Muhly and Xia [MX] considered the compactness of the operator

$$f\mapsto \chi_{[-R,R]}(x)\,\mathrm{p.v.}\int_{\mathbb{R}}\frac{A(x)-A(y)-A'(y)(x-y)}{x-y}\,\frac{f(y)\chi_{[-R,R]}(y)}{x-y}\,dy, \tag{1.9}$$

where $R>0$.

Theorem C ([MX]). *If $A\in Lip(\mathbb{R})$ with $A'\in\mathrm{VMO}(\mathbb{R})$, then the operator defined by* (1.9) *is a compact operator on $L^2(\mathbb{R})$ for any $R>0$.*

In the present paper, our main purpose is to show that the Calderón type commutator $T_A$ and its maximal operator $T_{A,*}$ defined respectively by (1.5) and (1.6) are compact operators on $L^p(\mathbb{R}^n)$. Let us first introduce some notations. For $m\in\mathbb{N}$, we write $\nabla^m A\in\mathrm{BMO}$ if $D^\beta A\in\mathrm{BMO}(\mathbb{R}^n)$ for every multi-index $\beta$ with $|\beta|=m$. Moreover, denote

$$\|A\|_{m,*}:=\|\nabla^m A\|_*=\sum_{|\beta|=m}\|D^\beta A\|_*.$$

It is easy to check that $\|\cdot\|_{m,*}$ is only a seminorm for all $m\in\mathbb{N}$. Denote by $\mathscr{A}_m$ the closure of $C_c^\infty(\mathbb{R}^n)$ in the seminorm $\|\cdot\|_{m,*}$,

$$\mathscr{A}_m=\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{m,*}}, \tag{1.10}$$

which means that for any $A\in\mathscr{A}_m(\mathbb{R}^n)$ and $\varepsilon>0$, there exists $A_0\in C_c^\infty(\mathbb{R}^n)$ such that

$$\|A-A_0\|_{m,*}=\sum_{|\beta|=m}\|D^\beta A-D^\beta A_0\|_*<\varepsilon.$$

Note that $C_c^\infty(\mathbb{R}^n)\subset\mathrm{BMO}(\mathbb{R}^n)$, so it is obvious that $\nabla^m A\in\mathrm{BMO}$ for all $m\geq 1$ and $A\in\mathscr{A}_m$. Below we denote $\mathscr{A}_1$ by $\mathscr{A}$ for simplicity.

Remark 1.4. We would like to show that for $m\in\mathbb{N}$, $\mathscr{A}_m$ contains the set of all functions with compact support and with all its partial derivatives of order $m$ in $\mathrm{VMO}(\mathbb{R}^n)$. In fact, assume that $\mathrm{supp}(A)\subset B$ and $D^\beta A\in\mathrm{VMO}(\mathbb{R}^n)$, where $B$ is a ball in $\mathbb{R}^n$ and $|\beta|=m$. By [DM1, Theorem 1.2], $\lim_{|y|\to 0}\|\tau_y(D^\beta A)-D^\beta A\|_*=0$, where $\tau_y f(x)=f(x-y)$ for $y\in\mathbb{R}^n$. On the other hand, the following conclusion was also given in [DM1]:

Lemma 1.1 ([DM1, Lemma 3.2]). *Suppose that $f\in\mathrm{BMO}(\mathbb{R}^n)$ with $\lim_{|y|\to 0}\|\tau_y f-f\|_*=0$ and $\{\phi_k\}_{k\in\mathbb{N}}\subset L^1(\mathbb{R}^n)$ satisfies the following conditions: for any $k\in\mathbb{N}$,*

(i) *$\phi_k$ is positive and continuous in $\mathbb{R}^n$;*
(ii) *$\mathrm{supp}(\phi_k)\subset B(0,1/k)$, where $B(x,r)$ denotes the ball centered at $x$ and radius $r$;*

(iii) $\int_{\mathbb{R}^n} \phi_k(x)\,dx = 1$.

*Then* $\lim_{k\to\infty} \|f - f * \phi_k\|_* = 0$.

Thus, together with the facts above, we have

$$\lim_{k\to\infty} \|\phi_k * D^\beta A - D^\beta A\|_* = 0 \quad \text{for all } |\beta| = m,$$

whenever $\{\phi_k\} \subset C_c^\infty(\mathbb{R}^n)$ satisfies conditions (i)–(iii) in Lemma 1.1. Let $A_k = \phi_k * A$. Since $A$ has compact support, we have $\{A_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ and $\lim_{k\to\infty} \|\nabla^m A_k - \nabla^m A\|_* = 0$. Therefore, $A \in \mathscr{A}_m$.

Now we give the main result in this paper.

THEOREM 1.2. *Let $n \ge 2$. Suppose $\Omega \in L^s(\mathbb{S}^{n-1})$ $(s > 1)$ satisfies* (1.2), (1.3) *and $\omega$ satisfies* (1.8). *If $A \in \mathscr{A}$, then $T_A$ and $T_{A,*}$ are compact operators on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.*

REMARK 1.5. When $n = 1$, since $\Omega$ satisfies (1.2) and (1.3), without loss of generality we may assume that $\Omega(x) = 1$ on $\mathbb{R} \setminus \{0\}$. Thus,

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y) - A'(y)(x-y)}{(x-y)^2} f(y)\,dy. \tag{1.11}$$

Using the idea of the proof of Theorem 1.2 and the conclusion of Theorem A as well as the $L^p$ boundedness of the Hardy–Littlewood maximal operator, we may show that if $A \in \mathscr{A}(\mathbb{R})$, then $T_A$ defined in (1.11) is a compact operator on $L^p(\mathbb{R})$ for any $1 < p < \infty$.

REMARK 1.6. Applying the compactness of $T_A$ on $L^p(\mathbb{R})$ $(1 < p < \infty)$ (see Remark 1.5), we can use a totally different approach to show that if $A \in \mathscr{A}(\mathbb{R})$, then the operator defined in (1.9) is also compact on $L^p(\mathbb{R})$ for any $1 < p < \infty$.

Fix $R > 0$, and denote by $L_R$ the operator defined in (1.9). Moreover, let $M_R$ be the multiplication operator defined by $M_R f = \chi_{[-R,R]} f$ for all $f \in L^p(\mathbb{R})\,(1 < p < \infty)$. Obviously, $M_R$ is a bounded linear operator on $L^p(\mathbb{R})$, and the operator family $\{M_R\}_{R>0}$ is bounded on $L^p(\mathbb{R})$ uniformly in $R > 0$. Note that

$$L_R = M_R T_A M_R, \tag{1.12}$$

and $T_A$ is a compact linear operator on $L^p(\mathbb{R})\,(1 < p < \infty)$ when $A \in \mathscr{A}(\mathbb{R})$ by Remark 1.5. Hence, if $A \in \mathscr{A}(\mathbb{R})$, then for any $R > 0$ the operator $L_R$ is compact on $L^p(\mathbb{R})\,(1 < p < \infty)$ by (1.12) (see [R, p. 104]). Here we indeed use a totally different approach to prove Theorem C when $A \in \mathscr{A}(\mathbb{R})$.

Of course, the function class covered by Theorem C is not contained in $\mathscr{A}(\mathbb{R})$. So the conclusion of Theorem C can be seen as a consequence of our result when $A \in \mathscr{A}(\mathbb{R})$ only.

REMARK 1.7. It should be pointed out that when $n = 1$ the maximal operator $T_{A,*}$ can be defined similarly to (1.6). However, as far as we know, when $n = 1$ the boundedness and compactness of $T_{A,*}$ are still unclear.

The second purpose of this paper is to prove the compactness of the fractional variant of the Calderón type commutator $T_A$. Let us recall some known results. For $m \geq 1$, the $m$th remainder of the Taylor series of $A$ at $x$ about $y$ is denoted by

$$R_m(A; x, y) = A(x) - \sum_{|\beta| \leq m} \frac{1}{\beta!} D^\beta A(y)(x - y)^\beta. \tag{1.13}$$

In 2001, Ding and Lu [DL2] introduced the following fractional Calderón type commutator $I_{\alpha,A,m}$ and its maximal operator $M_{\alpha,A,m}$:

$$I_{\alpha,A,m} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n+m-\alpha}} R_m(A; x, y) f(y)\, dy, \tag{1.14}$$

$$M_{\alpha,A,m} f(x) = \sup_{r>0} \frac{1}{r^{n+m-\alpha}} \int_{|x-y|<r} |\Omega(x - y)|\, |R_m(A; x, y)|\, |f(y)|\, dy, \tag{1.15}$$

where $0 < \alpha < n$.

In [DL2], the authors obtained the $(L^p, L^q)$ boundedness of $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$.

THEOREM D ([DL2]). *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ and $1 < p < n/\alpha$. Let $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > \min\{p', q\}$ satisfies (1.2). Assume $\nabla^m A \in \mathrm{BMO}$ ($m \geq 1$). Then $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$ are bounded from $L^p$ to $L^q$ and there exists a positive constant $C$ such that*

$$\|I_{\alpha,A,m} f\|_q, \|M_{\alpha,A,m} f\|_q \leq C \|\nabla^m A\|_* \|f\|_p.$$

The authors of [DL2] indeed established the weighted $(L^p, L^q)$ boundedness for a more general fractional Calderón type commutator and its maximal operator. Theorem D is a special case of a result in [DL2].

The next result shows that $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$ are also compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

THEOREM 1.3. *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ and $1 < p < n/\alpha$. $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > p'$ satisfies (1.2) and $\omega_s$ satisfies (1.8). If $A \in \mathscr{A}_m$ ($m \geq 1$), then $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$ are compact operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

The plan of this paper is as follows: In Section 2, we give the proof of Theorem 1.2. The proof of the compactness of $I_{\alpha,A,m}$ and $M_{\alpha,A,m}$ can be found in Section 3. In the final section, we show that similar compactness results hold for higher order Calderón type commutators and multilinear Calderón type commutators.

In this paper, $C$ will denote a positive constant that can change its value in each statement without explicit mention.

**2. The proof of Theorem 1.2.** Let us begin by recalling some known results. The first one characterizes strongly precompact sets in $L^p(\mathbb{R}^n)$.

LEMMA 2.1 (Fréchet–Kolmogorov, see [Y]). *A subset $G$ of $L^p(\mathbb{R}^n)$ ($1 \le p < \infty$) is strongly precompact if and only if $G$ satisfies the following conditions:*

(a) $\sup_{f\in G} \|f\|_p < \infty$;
(b) $\lim_{a\to\infty} \|f\chi_{E_a}\|_p = 0$, *uniformly for $f \in G$, where $E_a = \{x \in \mathbb{R}^n : |x| > a\}$;*
(c) $\lim_{|h|\to 0} \|f(\cdot + h) - f(\cdot)\|_p = 0$, *uniformly for $f \in G$.*

In order to prove Theorem 1.2, we also need the $L^p$-boundedness of the maximal operator $M_\Omega$ with homogenous kernel, which is defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |\Omega(y)|\,|f(x-y)|\,dy.$$

LEMMA 2.2 (see [LDY, Theorem 2.3.3]). *Suppose that $\Omega \in L^1(\mathbb{S}^{n-1})$ satisfies* (1.2). *Then $M_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.*

LEMMA 2.3 (see [KW] for $\beta = 0$ and [DL1] for $0 < \beta < n$). *Suppose $\Omega \in L^s(\mathbb{S}^{n-1})$ ($s \ge 1$) satisfies* (1.2) *and the following $L^s$-Dini condition:*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta}\,d\delta < \infty. \tag{2.1}$$

*Then for $0 \le \beta < n$, there exists a constant $0 < \theta < 1/2$ such that when $|x| < \theta R$,*

$$\Big(\int_{R<|y|<2R} \Big|\frac{\Omega(y-x)}{|y-x|^{n-\beta}} - \frac{\Omega(y)}{|y|^{n-\beta}}\Big|^s dy\Big)^{1/s} \le CR^{n/s-n+\beta}\Big\{\frac{|x|}{R} + \int_{|x|/(2R)}^{|x|/R} \frac{\omega_s(\delta)}{\delta}\,d\delta\Big\},$$

*where the constant $C > 0$ is independent of $R$ and $x$.*

LEMMA 2.4 (see [R, Theorem 4.18]). *Let $X$ and $Y$ be Banach spaces. The compact operators form a closed subspace of $\mathscr{B}(X,Y)$ in its norm topology, where $\mathscr{B}(X,Y)$ denotes the space of bounded linear operators from $X$ to $Y$.*

**2.1. The compactness of $T_A$ on $L^p(\mathbb{R}^n)$.** By Lemma 2.4, we need only show that if $\Omega$ satisfies the conditions of Theorem 1.2 and $A \in C_c^\infty(\mathbb{R}^n)$, then $T_A$ is compact on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. In fact, for $A \in \mathscr{A}$ and $\varepsilon > 0$, there exists $A_0 \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|A - A_0\|_{1,*} = \sum_{|\beta|=1} \|D^\beta(A - A_0)\|_* < \varepsilon \tag{2.2}$$

by (1.10). Thus, by Theorem A and (2.2) we get

$$\begin{aligned}\|T_A - T_{A_0}\| &= \sup_{\|f\|_p \le 1} \|T_A f - T_{A_0} f\|_p \\ &= \sup_{\|f\|_p \le 1} \|T_{A-A_0} f\|_p \le C \sum_{|\beta|=1} \|D^\beta(A - A_0)\|_* < C\varepsilon,\end{aligned}$$

which shows that the operator $T_A$ can be approximated by the operator family $\{T_B\}_{B \in C_c^\infty}$ in the operator norm topology.

Now we assume $A \in C_c^\infty(\mathbb{R}^n)$ and denote $\mathcal{F} = \{T_A f : f \in \mathcal{B}\}$; here and below, $\mathcal{B}$ denotes the unit ball in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Since $\mathcal{F} \subset L^p(\mathbb{R}^n)$ by Theorem A, to show that $T_A$ is compact on $L^p(\mathbb{R}^n)$ it suffices to prove that the set $\mathcal{F}$ is strongly precompact in $L^p(\mathbb{R}^n)$. Applying Lemma 2.1, we need only prove that conditions (a)–(c) in Lemma 2.1 hold uniformly for $\mathcal{F}$ with $A \in C_c^\infty(\mathbb{R}^n)$.

Condition (a) is a direct consequence of Theorem A. For (b), since $A \in C_c^\infty(\mathbb{R}^n)$, without loss of generality we can assume that $\operatorname{supp}(A) \subset \{x \in \mathbb{R}^n : |x| \le b\}$ with $b > 1$. Let $r = \min\{p, s\} > 1$ and for any $\varepsilon > 0$, take $a > 2b$ such that $(a - b)^{-n/r'} < \varepsilon$. Note that

$$R(A; x, y) = \sum_{|\beta|=1} \int_0^1 D^\beta A(\theta x + (1-\theta)y)(x-y)^\beta \, d\theta - \sum_{|\beta|=1} D^\beta A(y)(x-y)^\beta,$$

we have

$$\begin{aligned}\|T_A f \chi_{E_a}\|_p &\le C \sum_{|\beta|=1} \Bigg\{ \bigg( \int_{|x|>a} \bigg( \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |D^\beta A(y)|\,|f(y)|\,dy \bigg)^p dx \bigg)^{1/p} \\ &\quad + \int_0^1 \bigg( \int_{|x|>a} \bigg( \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |D^\beta A(\theta x + (1-\theta)y)|\,|f(y)|\,dy \bigg)^p dx \bigg)^{1/p} d\theta \Bigg\} \\ &=: I_1 + I_2.\end{aligned}$$

First consider $I_2$. Note that $|x-y| \ge \frac{a-b}{1-\theta}$ since $\theta \in (0,1)$, $|\theta x + (1-\theta)y| \le b$ and $|x| > a$. Combining Hölder's inequality with Minkowski's inequality, we obtain

$$\Big(\int_{|x|>a}\Big(\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|D^\beta A(\theta x+(1-\theta)y)|\,|f(y)|\,dy\Big)^p dx\Big)^{1/p}$$

$$\le C\Big(\int_{\mathbb{R}^n}\Big(\int_{|y|\ge\frac{a-b}{1-\theta}}\frac{|\Omega(y)|^r}{|y|^{nr}}|f(x-y)|^r\,dy\Big)^{p/r}$$

$$\times\Big(\int_{\mathbb{R}^n}|D^\beta A(\theta x+(1-\theta)y)|^{r'}\,dy\Big)^{p/r'} dx\Big)^{1/p}$$

$$\le C[b/(1-\theta)]^{n/r'}\|D^\beta A\|_\infty\|f\|_p\Big(\int_{|y|\ge\frac{a-b}{1-\theta}}\frac{|\Omega(y)|^r}{|y|^{nr}}\,dy\Big)^{1/r}$$

$$\le C\varepsilon b^{n/r'}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|D^\beta A\|_\infty\|f\|_p.$$

Similarly,

$$I_1\le Cb^{n/r'}\sum_{|\beta|=1}\|D^\beta A\|_\infty\Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|\ge a-b}\frac{|\Omega(x-y)|^r}{|x-y|^{nr}}|f(y)|^r\,dy\Big)^{p/r} dx\Big)^{1/p}$$

$$\le C\varepsilon b^{n/r'}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\sum_{|\beta|=1}\|D^\beta A\|_\infty\|f\|_p.$$

Therefore, condition (b) holds for $\mathcal{F}$ uniformly.

It remains to prove (c), that is, we need to verify that for any $0<\varepsilon<1/4$, if $|h|$ is sufficiently small and depends only on $\varepsilon$, then

$$\|T_Af(\cdot+h)-T_Af(\cdot)\|_p<C\varepsilon \tag{2.3}$$

holds uniformly for $f\in\mathcal{B}$. In fact, for any $x,h\in\mathbb{R}^n$, we have

$$\begin{aligned}
&T_Af(x+h)-T_Af(x) \qquad (2.4)\\
&=\int_{|x-y|>e^{1/\varepsilon}|h|}\frac{\Omega(x+h-y)}{|x+h-y|^n}\Big[\frac{R(A;x+h,y)}{|x+h-y|}-\frac{R(A;x,y)}{|x-y|}\Big]f(y)\,dy\\
&\quad+\int_{|x-y|>e^{1/\varepsilon}|h|}\Big[\frac{\Omega(x+h-y)}{|x+h-y|^n}-\frac{\Omega(x-y)}{|x-y|^n}\Big]\frac{R(A;x,y)}{|x-y|}f(y)\,dy\\
&\quad-\int_{|x-y|\le e^{1/\varepsilon}|h|}\frac{\Omega(x-y)}{|x-y|^n}\,\frac{R(A;x,y)}{|x-y|}f(y)\,dy\\
&\quad+\int_{|x-y|\le e^{1/\varepsilon}|h|}\frac{\Omega(x+h-y)}{|x+h-y|^n}\,\frac{R(A;x+h,y)}{|x+h-y|}f(y)\,dy\\
&=:J_1+J_2-J_3+J_4.
\end{aligned}$$

In the following, we estimate $J_1, J_2, J_3$ and $J_4$. Since $|x-y| > e^{1/\varepsilon}|h|$ and $0<\varepsilon<1/4$, we have $|x-y| \sim |x+h-y|$ and

$$|R(A;x,y)| \le C \sum_{|\beta|=1} \|D^\beta A\|_\infty |x-y|. \tag{2.5}$$

Applying (2.5), it is easy to see that

$$\begin{aligned} &\left| \frac{R(A;x+h,y)}{|x+h-y|} - \frac{R(A;x,y)}{|x-y|} \right| \\ &\qquad \le C \frac{|R(A;x+h,y) - R(A;x,y)|}{|x-y|} + C|R(A;x,y)| \frac{|h|}{|x-y|^2} \\ &\qquad \le C \sum_{|\beta|=1} \|D^\beta A\|_\infty \frac{|h|}{|x-y|}. \end{aligned} \tag{2.6}$$

Thus, by (2.6) we get

$$\begin{aligned} |J_1| &\le C|h| \sum_{|\beta|=1} \|D^\beta A\|_\infty \int\limits_{|x-y|>e^{1/\varepsilon}|h|} \frac{|\Omega(x-y)|}{|x-y|^{n+1}} |f(y)|\,dy \\ &\le Ce^{-1/\varepsilon} \sum_{|\beta|=1} \|D^\beta A\|_\infty M_\Omega f(x). \end{aligned} \tag{2.7}$$

As $L^s(S^{n-1}) \subset L^1(S^{n-1})$ for any $s>1$, (2.7) and Lemma 2.2 give

$$\|J_1\|_p \le C \sum_{|\beta|=1} \|D^\beta A\|_\infty e^{-1/\varepsilon} \|f\|_p.$$

For $J_2$, combining (2.5) with Minkowski's inequality, we have

$$\begin{aligned} \|J_2\|_p &\le C \bigg( \int\limits_{\mathbb{R}^n} \bigg( \int\limits_{|x-y|>e^{1/\varepsilon}|h|} \left| \frac{\Omega(x+h-y)}{|x+h-y|^n} - \frac{\Omega(x-y)}{|x-y|^n} \right| |f(y)|\,dy \bigg)^p dx \bigg)^{1/p} \\ &\qquad\qquad \times \sum_{|\beta|=1} \|D^\beta A\|_\infty \\ &\le C \sum_{|\beta|=1} \|D^\beta A\|_\infty \|f\|_p \int\limits_{|y|>e^{1/\varepsilon}|h|} \left| \frac{\Omega(y+h)}{|y+h|^n} - \frac{\Omega(y)}{|y|^n} \right| dy. \end{aligned}$$

By Lemma 2.3 and (1.8), we have

$$\begin{aligned}\int\limits_{|y|>e^{1/\varepsilon}|h|} &\left|\frac{\Omega(y+h)}{|y+h|^n} - \frac{\Omega(y)}{|y|^n}\right| dy \\ &\le \sum_{k=0}^{\infty} \int\limits_{2^k e^{1/\varepsilon}|h|<|y|\le 2^{k+1}e^{1/\varepsilon}|h|} \left|\frac{\Omega(y+h)}{|y+h|^n} - \frac{\Omega(y)}{|y|^n}\right| dy \\ &\le C\sum_{k=0}^{\infty}\left\{\frac{|h|}{2^k e^{1/\varepsilon}|h|} + \int\limits_{|h|/2^{k+1}e^{1/\varepsilon}|h|}^{|h|/2^k e^{1/\varepsilon}|h|} \frac{\omega(\delta)}{\delta}\,d\delta\right\} \\ &\le C\sum_{k=0}^{\infty}\left\{\frac{1}{2^k e^{1/\varepsilon}} + \frac{1}{k+1/\varepsilon}\int\limits_{2^{-(k+1)}e^{-1/\varepsilon}}^{2^{-k}e^{-1/\varepsilon}} \frac{\omega(\delta)}{\delta}\log\left(2+\frac{1}{\delta}\right) d\delta\right\} \\ &\le C(e^{-1/\varepsilon}+\varepsilon).\end{aligned}$$

Thus, $\|J_2\|_p \le C\sum_{|\beta|=1}\|D^\beta A\|_\infty \|f\|_p(e^{-1/\varepsilon}+\varepsilon)$. As for $J_3$, note that

$$R(A;x,y) = \sum_{|\beta|=2}\frac{1}{\beta!}D^\beta A(tx+(1-t)y)(x-y)^\beta \quad \text{for some } t\in(0,1). \tag{2.8}$$

By (2.8) we have

$$\begin{aligned}|J_3| &\le C\sum_{|\beta|=2}\|D^\beta A\|_\infty \int\limits_{|x-y|\le e^{1/\varepsilon}|h|} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|\,dy \\ &\le Ce^{1/\varepsilon}|h|\sum_{|\beta|=2}\|D^\beta A\|_\infty M_\Omega f(x).\end{aligned}$$

In a similar way, we can obtain the following estimate for $J_4$:

$$|J_4| \le C(e^{1/\varepsilon}+1)|h|\sum_{|\beta|=2}\|D^\beta A\|_\infty M_\Omega f(x+h).$$

Using Lemma 2.2 again, we get

$$\|J_3\|_p + \|J_4\|_p \le C\sum_{|\beta|=2}\|D^\beta A\|_\infty \|f\|_p (e^{1/\varepsilon}+1)|h|.$$

Choosing $|h| < \varepsilon/(e^{1/\varepsilon}+1) < \varepsilon$, we can see that condition (c) holds for $\mathcal{F}$ uniformly, and the compactness of $T_A$ on $L^p(\mathbb{R}^n)$ follows.

**2.2. The compactness of $T_{A,*}$ on $L^p(\mathbb{R}^n)$.** The proof of the compactness of $T_{A,*}$ uses the following obvious fact; we omit its proof here.

LEMMA 2.5. *Suppose that $A, V \in \mathscr{A}$. Then for $1<p<\infty$,*

(i) $|T_{A,*}f(x) - T_{V,*}f(x)| \le T_{A-V,*}f(x)$;
(ii) $\|T_{A,*}f\chi_{E_a}\|_p \le \|T_{V,*}f\chi_{E_a}\|_p + \|T_{A-V,*}f\|_p$;
(iii) $\|T_{A,*}f(\cdot+h) - T_{A,*}f(\cdot)\|_p \le \|T_{V,*}f(\cdot+h) - T_{V,*}f(\cdot)\|_p + 2\|T_{A-V,*}f\|_p$.

Now denote $\mathcal{G} := \{T_{A,*}f : f \in \mathcal{B}\}$. Then $\mathcal{G} \subset L^p(\mathbb{R}^n)$ by Theorem B. By Lemma 2.5, to prove the compactness of $T_{A,*}$ we need only show that conditions (a)–(c) in Lemma 2.1 hold uniformly for $\mathcal{G}$ with $A \in C_c^\infty(\mathbb{R}^n)$.

The verification of (a) is obvious by Theorem B. Using the same approach as in verifying (b) for $\mathcal{F}$ in Subsection 2.1, we can show that (b) holds uniformly for $\mathcal{G}$. So, to complete the proof of of the compactness of $T_{A,*}$, it remains to verify that (c) holds uniformly for $\mathcal{G}$.

For $\delta > 0$, denote $K_\delta(x,y) = \frac{\Omega(x-y)}{|x-y|^n}\chi_{\{|x-y|>\delta\}}$. Similarly to the decomposition (2.4), for any $0 < \varepsilon < 1/4$ and $h \in \mathbb{R}^n$, where $|h|$ is sufficiently small and depends only on $\varepsilon$, we see that $|T_{A,*}f(x+h) - T_{A,*}f(x)|$ can be controlled by the sum of the following four terms:

$$L_1 = \sup_{\delta>0}\bigg|\int_{|x-y|>e^{1/\varepsilon}|h|} K_\delta(x+h,y)\bigg[\frac{R(A;x+h,y)}{|x+h-y|} - \frac{R(A;x,y)}{|x-y|}\bigg] f(y)\,dy\bigg|,$$

$$L_2 = \sup_{\delta>0}\bigg|\int_{|x-y|>e^{1/\varepsilon}|h|} [K_\delta(x+h,y) - K_\delta(x,y)]\frac{R(A;x,y)}{|x-y|} f(y)\,dy\bigg|,$$

$$L_3 = \sup_{\delta>0}\bigg|\int_{|x-y|\le e^{1/\varepsilon}|h|} K_\delta(x,y)\frac{R(A;x,y)}{|x-y|} f(y)\,dy\bigg|,$$

$$L_4 = \sup_{\delta>0}\bigg|\int_{|x-y|\le e^{1/\varepsilon}|h|} K_\delta(x+h,y)\frac{R(A;x+h,y)}{|x+h-y|} f(y)\,dy\bigg|.$$

Applying (2.5), (2.6) and Lemma 2.2, we estimate $L_1$, $L_3$ and $L_4$ in the same way as for $J_1$, $J_3$ and $J_4$ in Subsection 2.1. Hence, we only estimate $L_2$. Notice that

$$\begin{aligned}
L_2 &\le \sup_{\delta>0}\bigg|\int_{\substack{|x-y|>e^{1/\varepsilon}|h|\\ |x+h-y|>\delta}} \bigg[\frac{\Omega(x+h-y)}{|x+h-y|^n} - \frac{\Omega(x-y)}{|x-y|^n}\bigg]\frac{R(A;x,y)}{|x-y|} f(y)\,dy\bigg| \\
&\quad + \sup_{\delta>0}\bigg|\int_{\substack{|x-y|>e^{1/\varepsilon}|h|\\ |x+h-y|>\delta\\ |x-y|\le\delta}} \frac{\Omega(x-y)}{|x-y|^n}\,\frac{R(A;x,y)}{|x-y|} f(y)\,dy\bigg| \\
&\quad + \sup_{\delta>0}\bigg|\int_{\substack{|x-y|>e^{1/\varepsilon}|h|\\ |x+h-y|\le\delta\\ |x-y|>\delta}} \frac{\Omega(x-y)}{|x-y|^n}\,\frac{R(A;x,y)}{|x-y|} f(y)\,dy\bigg| \\
&=: L_{21} + L_{22} + L_{23}.
\end{aligned} \tag{2.9}$$

The estimation of $L_{21}$ is the same as for $J_2$ in Subsection 2.1. As the estimation of $L_{23}$ is similar to that for $L_{22}$, we only estimate $L_{22}$. Notice that

$|x-y|>e^{1/\varepsilon}|h|$ and $0<\varepsilon<1/4$, then

$$\frac{1}{1+e^{-1/\varepsilon}}|x+h-y|\le|x-y|\le\frac{1}{1-e^{-1/\varepsilon}}|x+h-y|. \tag{2.10}$$

For any $1<p_0<p$, by (2.5) and Hölder's inequality, we have

$$\begin{aligned}
L_{22}&\le C\sum_{|\beta|=1}\|D^\beta A\|_\infty\sup_{\delta>0}\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|x-y|\le\delta}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,dy\\
&\le C\sum_{|\beta|=1}\|D^\beta A\|_\infty\sup_{\delta>0}\Bigg(\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|x-y|\le\delta}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|^{p_0}\,dy\Bigg)^{1/p_0}\\
&\qquad\qquad\times\Bigg(\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|y|\le\delta}\frac{|\Omega(y)|}{|y|^n}\,dy\Bigg)^{1/p_0'}\\
&\le C\sum_{|\beta|=1}\|D^\beta A\|_\infty\|\Omega\|_{L^s(\mathbb{S}^{n-1})}^{1/p_0'}(\log(1+e^{-1/\varepsilon}))^{1/p_0'}M_\Omega(f^{p_0})(x)^{1/p_0}.
\end{aligned}$$

By Lemma 2.2, we see that (c) holds uniformly for $\mathcal{G}$, which completes the proof of Theorem 1.2.

**3. The proof of Theorem 1.3.** Before giving the proof, let us recall some known facts. The first one is the classical Hardy–Littlewood–Sobolev theorem on the Riesz potential $I_\alpha$, which is defined by

$$I_\alpha f(x)=c_n\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,\qquad 0<\alpha<n. \tag{3.1}$$

Lemma 3.1 (Hardy–Littlewood-Sobolev, see [S]). *If $0<\alpha<n$, $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$, then $I_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

The second fact is the $(L^p,L^q)$-boundedness of the fractional integral operator $I_{\Omega,\alpha}$ with the homogenous kernel defined by

$$I_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy,\qquad 0<\alpha<n, \tag{3.2}$$

where $\Omega\in L^{n/(n-\alpha)}(\mathbb{S}^{n-1})$ satisfies (1.2).

Lemma 3.2 ([LDY, Theorem 3.3.1]). *Suppose that $0<\alpha<n$ and $\Omega\in L^{n/n-\alpha}(\mathbb{S}^{n-1})$ satisfies (1.2). For $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$, $I_{\Omega,\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

**3.1. The compactness of $I_{\alpha,A,m}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.** As stated in Section 2, by Lemma 2.4 we need only show that if $\Omega$ satisfies the conditions of Theorem 1.3 and $A\in C_c^\infty(\mathbb{R}^n)$, then the operator $I_{\alpha,A,m}$ is compact from

$L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. By Lemma 2.1, it suffices to verify that conditions (a)–(c) of Lemma 2.1 hold uniformly for

$$\mathcal{J} := \{I_{\alpha,A,m}f : f \in \mathcal{B}\} \quad \text{where} \quad A \in C_c^\infty(\mathbb{R}^n).$$

Condition (a) comes from Theorem D and the fact $C_c^\infty \subset \mathrm{BMO}$. For (b), we assume that $\operatorname{supp}(A) \subset \{x \in \mathbb{R}^n : |x| \le b\}$ with $b > 1$. As $s' < p$, there exists $p_1$ such that $s' < p_1 < p$. For any $\varepsilon > 0$, we take $a > 2b$ such that

$$(a-b)^{-n(1/p_1'-1/s)} < \varepsilon.$$

Using Taylor's extension with remainder in integral form (see [RS]), we have

$$\begin{aligned}(3.3)\qquad &|R_m(A;x,y)| \\ &\quad\le \sum_{|\beta|=m} C_\beta \int_0^1 (1-\theta)^{m-1} |D^\beta A(\theta x + (1-\theta)y)(x-y)^\beta|\,d\theta \\ &\qquad + C\sum_{|\beta|=m} |D^\beta A(y)(x-y)^\beta|.\end{aligned}$$

Thus,

$$\begin{aligned}(3.4)\qquad |I_{\alpha,A,m}f(x)| &\le C\sum_{|\beta|=m}\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|D^\beta A(y)|\,|f(y)|\,dy \\ &+ C\sum_{|\beta|=m}\int_0^1 (1-\theta)^{m-1}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|D^\beta A(\theta x+(1-\theta)y)|\,|f(y)|\,dy\,d\theta \\ &=: N_1 + N_2.\end{aligned}$$

Note that $|x-y| \ge \frac{a-b}{1-\theta}$, since $\theta \in (0,1)$, $|\theta x + (1-\theta)y| \le b$ and $|x| > a$. Applying Hölder's inequality with exponents $p_1, s, \frac{sp_1'}{s-p_1'}$, we get

$$\begin{aligned}(3.5)\quad N_2 &\le C\sum_{|\beta|=m}\int_0^1 (1-\theta)^{m-1}\int_{|x-y|>\frac{a-b}{1-\theta}}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|D^\beta A(\theta x+(1-\theta)y)| \\ &\qquad\times |f(y)|\,dy\,d\theta \\ &\le C\sum_{|\beta|=m}\int_0^1 (1-\theta)^{m-1}\Big(\int_{\mathbb{R}^n}|D^\beta A(\theta x+(1-\theta)y)|^{\frac{sp_1'}{s-p_1'}}\,dy\Big)^{\frac{s-p_1'}{sp_1'}} \\ &\qquad\times\Big(\int_{|x-y|>\frac{a-b}{1-\theta}}\frac{|\Omega(x-y)|^s}{|x-y|^{ns/p_1'}}\,dy\Big)^{1/s}\Big(\int_{\mathbb{R}^n}\frac{|f(y)|^{p_1}}{|x-y|^{n-\alpha p_1}}\,dy\Big)^{1/p_1}\,d\theta \\ &\le C\varepsilon b^{n(1/p_1'-1/s)}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\sum_{|\beta|=m}\|D^\beta A\|_\infty I_{\alpha p_1}(|f|^{p_1})(x)^{1/p_1}.\end{aligned}$$

Analogously, for $|x| > a$,

$$N_1 \le C \sum_{|\beta|=m} \int_{|x-y|>a-b} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |D^\beta A(y)|\,|f(y)|\,dy \le C\varepsilon \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{|\beta|=m} \|D^\beta A\|_\infty I_{\alpha p_1}(|f|^{p_1})(x)^{1/p_1}. \tag{3.6}$$

Notice that $p_1 < p$. Combining (3.5), (3.6) with Lemma 3.1, we get

$$\|I_{\alpha,A,m} f \chi_{E_a}\|_q \le C\varepsilon \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{|\beta|=m} \|D^\beta A\|_\infty \|f\|_p.$$

Thus, (b) holds uniformly for $\mathcal{J}$.

Finally, let us verify (c). For any $0 < \varepsilon < 1/4$ and $h \in \mathbb{R}^n$, where $|h|$ is sufficiently small and depends only on $\varepsilon$, we have the decomposition

$$\begin{aligned} &I_{\alpha,A,m} f(x+h) - I_{\alpha,A,m} f(x) \\ &= \int_{|x-y|>e^{1/\varepsilon}|h|} \frac{\Omega(x+h-y)}{|x+h-y|^{n-\alpha}} \left[\frac{R_m(A;x+h,y)}{|x+h-y|^m} - \frac{R_m(A;x,y)}{|x-y|^m}\right] f(y)\,dy \\ &\quad + \int_{|x-y|>e^{1/\varepsilon}|h|} \left[\frac{\Omega(x+h-y)}{|x+h-y|^{n-\alpha}} - \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}\right] \frac{R_m(A;x,y)}{|x-y|^m} f(y)\,dy \\ &\quad - \int_{|x-y|\le e^{1/\varepsilon}|h|} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} \frac{R_m(A;x,y)}{|x-y|^m} f(y)\,dy \\ &\quad + \int_{|x-y|\le e^{1/\varepsilon}|h|} \frac{\Omega(x+h-y)}{|x+h-y|^{n-\alpha}} \frac{R_m(A;x+h,y)}{|x+h-y|^m} f(y)\,dy \\ &=: O_1 + O_2 + O_3 + O_4. \end{aligned} \tag{3.7}$$

For $O_1$, notice that

$$|R_m(A;x,y)| \le C \sum_{|\beta|=m} \|D^\beta A\|_\infty |x-y|^m \tag{3.8}$$

and

$$|R_m(A;x+h,y) - R_m(A;x,y)| \le C|h| \sum_{1\le|\beta|\le m} \|D^\beta A\|_\infty |x-y|^{|\beta|-1}. \tag{3.9}$$

It is easy to see that

$$\begin{aligned}(3.10)\quad &\left|\frac{R_m(A;x+h,y)}{|x+h-y|^m}-\frac{R_m(A;x,y)}{|x-y|^m}\right|\\ &\le C\frac{1}{|x-y|^m}\big|R_m(A;x+h,y)-R_m(A;x,y)\big|+C|R_m(A;x,y)|\frac{|h|}{|x-y|^{m+1}}\\ &\le C\sum_{1\le|\beta|\le m}\|D^\beta A\|_\infty\frac{|h|}{|x-y|^{m-|\beta|+1}}.\end{aligned}$$

Hence

$$\begin{aligned}|O_1|&\le C\sum_{1\le|\beta|\le m}\|D^\beta A\|_\infty|h|\int_{|x-y|>e^{1/\varepsilon}|h|}\frac{|\Omega(x+h-y)|}{|x+h-y|^{n-\alpha}|x-y|^{m-|\beta|+1}}|f(y)|\,dy\\ &\le C\sum_{1\le|\beta|\le m}\|D^\beta A\|_\infty(e^{1/\varepsilon}|h|)^{-(m-|\beta|+1)}|h|I_{|\Omega|,\alpha}(|f|)(x+h).\end{aligned}$$

Note that $s>p'>\frac{n}{n-\alpha}$. Applying Lemma 3.2, we obtain

$$\|O_1\|_q\le C\sum_{1\le|\beta|\le m}\|D^\beta A\|_\infty(e^{1/\varepsilon}|h|)^{-(m-|\beta|+1)}|h|\,\|f\|_p.$$

As for $O_2$, denote $r_k=2^ke^{1/\varepsilon}|h|$ and $B_k=B(0,r_k)$. Then by (3.8) and Lemma 2.3, we have

$$\begin{aligned}O_2&\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty\int_{|y|>e^{1/\varepsilon}|h|}\left|\frac{\Omega(y+h)}{|y+h|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right||f(x-y)|\,dy\\ &\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty\sum_{k=0}^\infty\int_{B_{k+1}\setminus B_k}\left|\frac{\Omega(y+h)}{|y+h|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right||f(x-y)|\,dy\\ &\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty\sum_{k=0}^\infty\bigg(\int_{B_{k+1}\setminus B_k}\left|\frac{\Omega(y+h)}{|y+h|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\bigg)^{1/s}\\ &\qquad\qquad\times\bigg(\int_{B_{k+1}\setminus B_k}|f(x-y)|^{s'}\,dy\bigg)^{1/s'}\\ &\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty\sum_{k=0}^\infty\bigg\{\frac{|h|}{r_k}+\int_{|h|/r_{k+1}}^{|h|/r_k}\frac{\omega_s(\delta)}{\delta}\,d\delta\bigg\}r_k^{n/s-n+\alpha}\\ &\qquad\qquad\times\bigg(\int_{B_{k+1}\setminus B_k}|f(x-y)|^{s'}\,dy\bigg)^{1/s'}\end{aligned}$$

$$\le C \sum_{|\beta|=m} \|D^\beta A\|_\infty I_{\alpha s'}(|f|^{s'})(x)^{1/s'}$$
$$\times \sum_{k=0}^{\infty} \left\{ \frac{1}{2^k e^{1/\varepsilon}} + \frac{1}{k+1/\varepsilon} \int_{2^{-k-1}e^{-1/\varepsilon}}^{2^{-k}e^{-1/\varepsilon}} \frac{\omega_s(\delta)}{\delta} \log\left(2+\frac{1}{\delta}\right) d\delta \right\}$$
$$\le C \sum_{|\beta|=m} \|D^\beta A\|_\infty (e^{-1/\varepsilon}+\varepsilon) I_{\alpha s'}(|f|^{s'})(x)^{1/s'}.$$

Thus, noting that $s' < p$ and $\frac{1}{q/s'} = \frac{1}{p/s'} - \frac{\alpha s'}{n}$, by Lemma 3.1 we have

$$\|O_2\|_q \le C\varepsilon \sum_{|\beta|=m} \|D^\beta A\|_\infty \|f\|_p.$$

Finally, let us estimate $O_3$ and $O_4$. Note that

$$R_m(A;x,y) = \sum_{|\beta|=m+1} \frac{1}{\beta!} D^\beta A(ux+(1-u)y)(x-y)^\beta \tag{3.11}$$

for some $u \in (0,1)$. Thus,

$$|O_3| \le C \sum_{|\beta|=m+1} \|D^\beta A\|_\infty \int_{|x-y|\le e^{1/\varepsilon}|h|} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-1}} |f(y)|\,dy$$
$$\le C \sum_{|\beta|=m+1} \|D^\beta A\|_\infty e^{1/\varepsilon} |h| I_{|\Omega|,\alpha}(|f|)(x).$$

In the same way, we can obtain the following estimate for $O_4$:

$$|O_4| \le C \sum_{|\beta|=m+1} \|D^\beta A\|_\infty (e^{1/\varepsilon}+1)|h| I_{|\Omega|,\alpha}(|f|)(x+h). \tag{3.12}$$

Then by Lemma 3.2, it is easy to see

$$\|O_3\|_q + \|O_4\|_q \le C(e^{1/\varepsilon}+1)|h| \sum_{|\beta|=m+1} \|D^\beta A\|_\infty \|f\|_p.$$

Therefore, if we choose $|h| = e^{-\frac{2m-1}{2(m-1)\varepsilon}}$ when $m>1$ or $|h| = \frac{\varepsilon}{e^{1/\varepsilon}+1}$ when $m=1$, then condition (c) holds for $\mathcal{J}$ uniformly. Hence we have proved that the fractional Calderón type commutator $I_{\alpha,A,m}$ is a compact operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

**3.2. Compactness of $M_{\alpha,A,m}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.** We first notice that Lemma 2.5 also holds if we use $M_{\alpha,A,m}$ instead of $T_{A,*}$. Let $\mathcal{L} := \{M_{\alpha,A,m} f : f \in \mathcal{B}\}$. We need only show that conditions (a)–(c) of Lemma 2.1 hold uniformly for $\mathcal{L}$ with $A \in C_c^\infty(\mathbb{R}^n)$.

Condition (a) is a direct consequence of Theorem D. Notice that

$$M_{\alpha,A,m}f(x) = \sup_{r>0} \frac{1}{r^{n+m-\alpha}} \int\limits_{|x-y|<r} |\Omega(x-y)|\,|R_m(A;x,y)|\,|f(y)|\,dy$$
$$\leq \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n+m-\alpha}} |R_m(A;x,y)|\,|f(y)|\,dy.$$

Thus, using the same approach as in verifying condition (b) in Theorem 1.3, we may show that (b) also holds uniformly for $\mathcal{L}$. It remains to verify (c).

For any $0<\varepsilon<1/4$ and $h\in\mathbb{R}^n$, where $|h|$ is sufficiently small and depends only on $\varepsilon$, denote

$$\widetilde{K}_r(x,y) = r^{-n-m+\alpha}|\Omega(x-y)|\chi_{\{|x-y|<r\}}.$$

We can control $|M_{\alpha,A,m}f(x+h) - M_{\alpha,A,m}f(x)|$ by the sum of the following four terms:

$$P_1 = \sup_{r>0} \int\limits_{|x-y|>e^{1/\varepsilon}|h|} \widetilde{K}_r(x+h,y)|R_m(A;x+h,y) - R_m(A;x,y)|\,|f(y)|\,dy,$$
$$P_2 = \sup_{r>0} \int\limits_{|x-y|>e^{1/\varepsilon}|h|} |\widetilde{K}_r(x+h,y) - \widetilde{K}_r(x,y)|\,|R_m(A;x,y)|\,|f(y)|\,dy,$$
$$P_3 = \sup_{r>0} \int\limits_{|x-y|\leq e^{1/\varepsilon}|h|} \widetilde{K}_r(x,y)|R_m(A;x,y)|\,|f(y)|\,dy,$$
$$P_4 = \sup_{r>0} \int\limits_{|x-y|\leq e^{1/\varepsilon}|h|} \widetilde{K}_r(x+h,y)|R_m(A;x+h,y)|\,|f(y)|\,dy.$$

By (3.9), we can give the following estimate for $P_1$ analogous to that for $O_1$:

$$P_1 \leq C \sum_{1\leq|\beta|\leq m} \|D^\beta A\|_\infty \frac{|h|}{(e^{1/\varepsilon}|h|)^{m-|\beta|+1}} \int\limits_{|x-y|>e^{1/\varepsilon}|h|} \frac{|\Omega(x+h-y)|}{|x+h-y|^{n-\alpha}}|f(y)|\,dy$$
$$\leq C \sum_{1\leq|\beta|\leq m} \|D^\beta A\|_\infty \frac{|h|}{(e^{1/\varepsilon}|h|)^{m-|\beta|+1}} I_{|\Omega|,\alpha}(|f|)(x+h).$$

The estimation of $P_3$ and $P_4$ is the same as for $O_3$ and $O_4$ in Subsection 3.1. We only estimate $P_2$. Similarly to the idea of dealing with $L_2$ in Subsection 2.2, we obtain

(3.13)

$$\begin{aligned}
P_2 \le \sup_{r>0} \frac{1}{r^{n+m-\alpha}} & \int\limits_{\substack{|x-y|>e^{1/\varepsilon}|h| \\ |x+h-y|<r}} |\Omega(x+h-y)-\Omega(x-y)|\,|R_m(A;x,y)|\,|f(y)|\,dy \\
&+ \sup_{r>0} \frac{1}{r^{n+m-\alpha}} \int\limits_{\substack{|x-y|>e^{1/\varepsilon}|h| \\ |x+h-y|\ge r \\ |x-y|<r}} |\Omega(x-y)|\,|R_m(A;x,y)|\,|f(y)|\,dy \\
&+ \sup_{r>0} \frac{1}{r^{n+m-\alpha}} \int\limits_{\substack{|x-y|>e^{1/\varepsilon}|h| \\ |x+h-y|<r \\ |x-y|\ge r}} |\Omega(x-y)|\,|R_m(A;x,y)|\,|f(y)|\,dy \\
&=: P_{21}+P_{22}+P_{23}.
\end{aligned}$$

For $P_{21}$, note that $|x+h-y| \sim |x-y|$ (see (2.10)). Also denote $r_k = 2^k e^{1/\varepsilon}|h|$ and $B_k = B(0, r_k)$. By (3.8), we have

$$\begin{aligned}
P_{21} &\le C \sum_{|\beta|=m} \|D^\beta A\|_\infty \sum_{k=0}^{\infty} \int\limits_{B_{k+1}\setminus B_k} \frac{|\Omega(y+h)-\Omega(y)|}{|y|^{n-\alpha}} |f(x-y)|\,dy \\
&\le C \sum_{|\beta|=m} \|D^\beta A\|_\infty I_{\alpha s'}(|f|^{s'})(x)^{1/s'} \sum_{k=0}^{\infty} \bigg( \int\limits_{B_{k+1}\setminus B_k} \frac{|\Omega(y+h)-\Omega(y)|^s}{|y|^n}\,dy \bigg)^{1/s}.
\end{aligned}$$

By the monotonicity of $\omega_s$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \bigg( \int\limits_{B_{k+1}\setminus B_k} & \frac{|\Omega(y+h)-\Omega(y)|^s}{|y|^n}\,dy \bigg)^{1/s} \\
&\le \sum_{k=0}^{\infty} \bigg( \int_{r_k}^{r_{k+1}} \int_{\mathbb{S}^{n-1}} |\Omega(\rho y'+h)-\Omega(y')|^s\,d\sigma(y')\,\frac{d\rho}{\rho} \bigg)^{1/s} \\
&\le C \sum_{k=0}^{\infty} \bigg( \int_{r_k}^{r_{k+1}} \omega_s\Big(\frac{|h|}{\rho}\Big)\,\frac{d\rho}{\rho} \bigg)^{1/s} \le C \sum_{k=0}^{\infty} \int_{2^{-k}e^{-1/\varepsilon}}^{2^{-k+1}e^{-1/\varepsilon}} \frac{\omega_s(\delta)}{\delta}\,d\delta \\
&\le C \sum_{k=0}^{\infty} \frac{1}{k-1+1/\varepsilon} \int_{2^{-k}e^{-1/\varepsilon}}^{2^{-k+1}e^{-1/\varepsilon}} \frac{\omega_s(\delta)}{\delta} \log\Big(2+\frac{1}{\delta}\Big)\,d\delta \le C\varepsilon.
\end{aligned}$$

Thus, since $s' < p$ and $\frac{1}{q/s'} = \frac{1}{p/s'} - \frac{\alpha s'}{n}$, Lemma 3.1 gives

$$\|P_{21}\|_q \le C\varepsilon \sum_{|\beta|=m} \|D^\beta A\|_\infty \|f\|_p. \tag{3.14}$$

Notice that $|x-y|>e^{1/\varepsilon}|h|$ and $0<\varepsilon<1/4$. Similar to estimating $L_{22}$ with $I_{|\Omega|,\alpha p_0}$ instead of $M_\Omega$, where $1<p_0<\min\{p,n/(\alpha s')\}$, applying (3.8) and (2.10) we obtain

$$\begin{aligned}
P_{22} &\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty \sup_{\delta>0}\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|x-y|\le\delta}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty \sup_{\delta>0}\Bigg(\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|x-y|\le\delta}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha p_0}}|f(y)|^{p_0}\,dy\Bigg)^{1/p_0}\\
&\qquad\qquad\qquad\times\Bigg(\int\limits_{\frac{\delta}{1+e^{-1/\varepsilon}}\le|y|\le\delta}\frac{|\Omega(y)|}{|y|^{n}}\,dy\Bigg)^{1/p_0'}\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_\infty\|\Omega\|_{L^s(\mathbb{S}^{n-1})}^{1/p_0'}(\log(1+e^{-1/\varepsilon}))^{1/p_0'}I_{|\Omega|,\alpha p_0}(|f|^{p_0})(x)^{1/p_0}.
\end{aligned}$$

Since $s>\frac{n}{n-\alpha p_0}$, using Lemma 3.2 we get

$$\|P_{22}\|_q\le C\varepsilon^{1/p_0'}\sum_{|\beta|=m}\|D^\beta A\|_\infty\|f\|_p. \tag{3.15}$$

Analogously, we obtain the same estimate for $P_{23}$. Thus, (c) holds uniformly for $\mathcal{L}$. Hence, the maximal operator $M_{\alpha,A,m}$ is also compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, which completes the proof of Theorem 1.3.

## 4. Final remarks

REMARK 4.1. The higher order Calderón type commutator $T_A^m$ and the corresponding maximal operator $T_{A,*}^m$ (see [CG1] and [CG2]) are defined by

$$\begin{aligned}
T_A^m f(x) &= \mathrm{p.v.}\int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_m(A;x,y)f(y)\,dy,\\
T_{A,*}^m f(x) &= \sup_{\varepsilon>0}|T_{A,\varepsilon}^m f(x)|\\
&= \sup_{\varepsilon>0}\Bigg|\int\limits_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_m(A;x,y)f(y)\,dy\Bigg|.
\end{aligned}$$

Using the idea of the proof of Theorem 1.2, we can show that $T_A^m$ and $T_{A,*}^m$ are compact on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ when $\Omega\in L^s(\mathbb{S}^{n-1})$ $(s>1)$ satisfies (1.2) and has vanishing moments up to order $m$ and $\omega$ satisfies (1.8), and $A\in\mathscr{A}_m$. The proofs have no essential difficulties but more complicated computations.

REMARK 4.2. For $m = m_1 + \cdots + m_k$ ($m_i \geq 1$) and $A_i \in \mathscr{A}_{m_i}$ ($i = 1, \ldots, k$), one may define the following Calderón type commutators:

$$T^m_{A_1,\ldots,A_k} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{i=1}^k R_{m_i}(A_i; x, y) f(y)\, dy,$$

$$T^m_{A_1,\ldots,A_k,*} f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{i=1}^k R_{m_i}(A_i; x, y) f(y)\, dy \right|,$$

and for $0 < \alpha < n$,

$$I_{\alpha,A_1,\ldots,A_k,m} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-\alpha}} \prod_{i=1}^k R_{m_i}(A_i; x, y) f(y)\, dy,$$

$$M_{\alpha,A_1,\ldots,A_k,m} f(x) = \sup_{r>0} r^{-(n+m-\alpha)} \int_{|x-y|<r} |\Omega(x-y)| \prod_{i=1}^k |R_{m_i}(A_i; x, y)|\, |f(y)|\, dy.$$

Using the method of this paper, one can prove that the conclusions of Theorems 1.2 and 1.3 also hold for the operators $T^m_{A_1,\ldots,A_k}$, $T^m_{A_1,\ldots,A_k,*}$, $I_{\alpha,A_1,\ldots,A_k,m}$ and $M_{\alpha,A_1,\ldots,A_k,m}$ with $\Omega$, $p$ and $q$ satisfying the corresponding conditions.

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