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On the convergence of parabolically scaled two-dimensional Fourier series in the linear phase setting

by

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Abstract. For

$$Sf(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{iM^2(x,y)y'}}{y'}\,\frac{e^{iM(x,y)x'}}{x'} f(x-x', y-y')\,dx'\,dy',$$

the linearized maximal operator of the rectangular partial sums of the kind (M, M^2) for double Fourier series, we prove a weak-type $(L^r, L^{r-\varepsilon})$ estimate for $1 < r \le 2$ and any $\varepsilon > 0$ in case $M^2(x,y) = Ax + By$ with $x, y \in [0, 2\pi]$, uniformly with respect to $A, B \ge 0$.

1. Introduction. Concerning the summability of Fourier series in 2D, very little is known. The current state of the art is as follows.

In the celebrated paper [4] C. Fefferman proved that the circular partial sums associated to a function $f \in L^r(\mathbb{T}^2)$ are unbounded in the L^r-norm for any $r \ne 2$.

In [5] the same author proved that there are $f \in L^r(\mathbb{T}^2)$ (with any $r > 1$) such that the anisotropic partial sums $S_{M,N}f$ do not converge a.e. as M and N tend to infinity; on the other hand (see [3], [15]), if one focuses on the case $M = N$, then $S_{N,N}$ converges a.e. for any $f \in L^r$ and any $r > 1$.

In view of the above, it is natural to ask if one can obtain a.e. convergence for partial sums of the form $S_{N,\psi(N)}f$ for a suitably chosen ψ. Since the case of $\psi(x) = x$ follows from the above, the next natural step is to consider $\psi(x) = x^2$. This is the problem studied in the following, where partial progress is obtained.

We are going to consider the maximal partial sums operator S_1 for the square partial sums $S_{N,N}$, and S_2 for the rectangular partial sums S_{M,M^2},

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namely

$$(1.1)\qquad S_1 f(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{iN(x,y)y'}}{y'}\,\frac{e^{iN(x,y)x'}}{x'} f(x-x',y-y')\,dx'\,dy',$$

$$(1.2)\qquad S_2 f(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{iM^2(x,y)y'}}{y'}\,\frac{e^{iM(x,y)x'}}{x'} f(x-x',y-y')\,dx'\,dy',$$

for $f \in L^r$, $1 < r \le 2$, $0 \le x, y \le 2\pi$, the phases $N(x,y)$ and $M(x,y)$ being arbitrary real-valued functions that may be assumed to be integer valued and even bounded provided the required L^r estimate does not depend on such a bound.

The case of Fourier series of one variable [2] has been handled by dealing with more and more general phases $m(x)$. Some examples are listed at the end of [2]. In this paper we are going to consider $N(x,y) = Ax + By$ for square partial sums, and $M^2(x,y) = Ax + By$ for rectangular partial sums, $A, B \ge 0$, and prove L^r estimates for (1.1) and (1.2).

In the process we are going to shed some light on two basic issues. The first concerns the decomposition of the convolution kernel. In the 1D proof the domain of integration was decomposed in a smooth, dyadic way of the kind $|x'| \sim 2^{-k}$, $k = 0, 1, 2, \ldots$. This gave rise to a decomposition of the maximal partial sums operator into operators associated to pairs $[I, \omega]$, $|I| = |\omega|^{-1} = 2^{-k}$, according to the time-frequency analysis performed there. A most important structural feature was the following partial order among pairs: $[I, \omega] < [I', \omega']$ if and only if $I \subseteq I'$ and $\omega' \subseteq \omega$. This definition is based on the fact that if two dyadic intervals I and I', as well as ω and ω', have non-empty intersection, then one must be contained in the other. In 2D the analogous decomposition $\{|x'| \sim 2^{-k}, |y'| \sim 2^{-h}\}$, $k, h = 0, 1, 2, \ldots$, appears to be too fine, lacking the partial order. Indeed, it may well be that two dyadic rectangles $I \times J$ and $I' \times J'$ with $|I| = 2^{-k}$, $|J| = 2^{-h}$ and $|I'| = 2^{-k'}$, $|J| = 2^{-h'}$ intersect but $I \subseteq I'$ and $J' \subseteq J$ or vice versa, so that no partial order can be established in such a general collection.

For S_1 we shall adopt a preliminary smooth dyadic decomposition of the domain of integration into two fans of the kind $|y'| \ge |x'|$ and $|y'| < |x'|$. The first fan, centered on the y'-axis, will be further decomposed into $\{|y| \sim 2^{-h}, |x'| \le 2^{-h}\}$, $h = 0, 1, 2, \ldots$. The second fan centered on the x'-axis will be similarly decomposed into $\{|x'| \sim 2^{-k}, |y'| < 2^{-k}\}$, $k = 0, 1, 2, \ldots$. The associated 2D pairs will be of the kind $[I \times J, \omega]$, $|I| = |J| = |\omega|^{-1}$. Since such pairs involve squares $I \times J$, the crucial partial order will be in place.

The second issue is the following: via the dependence on x and y each phase "speaks" with two different "voices" A and B. Since no relation is assumed between them, we will have to establish with which "voice" to side. The choice will be different for the two different fans of the kernel (see Lemmas 3.2 and 3.3), with the *central piece* of the decomposition—something like $\{|x'|, |y'| < a\}$ for some $a > 0$—called to close the resulting gap (Section 3.3).

So, by a method different from those in [3] and [15], we are going to prove a strong (L^r, L^r)-estimate for S_1 in case $N(x,y) = Ax + By$:

THEOREM 1. *With the above notation and for* $N(x,y) = Ax + By$, *we have*

$$\|S_1 f\|_r \le c_r \|f\|_r \tag{1.3}$$

for all $1 < r \le 2$, *where* c_r *is independent of* f, *and of* $A, B \ge 0$.

S_1 will be decomposed according to pairs $[I \times J, \omega]$, $|I| = |J| = |\omega|^{-1}$, which are grouped into universes U_n and R_l, $n, l \ge 0$. Pairs $p = [I \times J, \omega]$ in U_n have the associated kernel defined on the first fan. Moreover the L^2-norm of the corresponding operators, denoted by V_p and precisely defined in Section 3.1, will be of the order of 2^{-n}. The goal will then be to prove that $\|\sum_{p \in U_n} V_p\|_2 \le c2^{-n/8}$ (Lemma 3.2), since the key estimate here is for $r = 2$. Similarly pairs $q = [I \times J, \omega]$ in R_l have the associated kernel defined on the second fan, and the L^2-norm of the corresponding operator W_q is of the order of 2^{-l}. The boundedness of S_1 holds under the more general assumption that, for any fixed universe, the size of all squares $I \times J$ involved is the same: the "equal case." We mention that the "equal case" already played a role in the 1D proof [2], as a comparison of the almost orthogonality Lemma 4 and Lemma 5 there shows. The complicated proof of Lemma 5 rests, in its central part, on the simpler case of Lemma 4: the "trees" involved—that by assumption have intersecting space-tops I_0 and I_0' (Lemma 5)—are furthermore assumed to have space-tops of *equal size,* hence the same I_0 (Lemma 4).

By a suitable decomposition, by a repeated application of (1.3), and via the introduction of an exceptional set we shall prove

THEOREM 2. *With the above notation and for* $M^2(x,y) = Ax + By$ *we have*

$$|\{(x,y) \in \mathbb{T}^2 : |S_2 f(x,y)| > \lambda\}| \le c_{r,\varepsilon} (\|f\|_r/\lambda)^{r-\varepsilon} \tag{1.4}$$

for all $1 < r \le 2$ *and* $\varepsilon > 0$, *where* $c_{r,\varepsilon}$ *is independent of* f, *and of* $A, B \ge 0$. *This implies that* S_2 *maps* L^r *into* $L^{r'}$ *for any* $1 < r' < r \le 2$.

We believe (1.4) to be the first result for S_2, concerning phases depending on both variables, since the 1970's [3]. In Section 2 we review the basic scheme of the 1D proof in [2]. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2. As will be pointed out at the end of each proof (Remarks 3 and 6), the above theorems hold under rather more general assumptions.

2. The scheme of the 1D proof. The maximal partial sums operator for 1D Fourier series [1, 2, 7, 9]

$$Cg(x) = \sup_{m\in\mathbb{R}} \Big| \int_{-\pi}^{\pi} \frac{e^{imx'}}{x'} g(x-x')\,dx' \Big|, \tag{2.1}$$

bounded in $L^r[0,2\pi]$, $1<r<\infty$, can be linearized as follows [2]:

$$Cg(x) = \int_{-\pi}^{\pi} \frac{e^{im(x)x'}}{x'} g(x-x')\,dx' \tag{2.2}$$

where the phase $m(x)$ is any real-valued function. Then, using on $[0,2\pi]$ the normalized Lebesgue measure $dx/2\pi$, the operator is decomposed as follows:

$$Cg(x) = \sum_{\substack{|I|=|\omega|^{-1}=2^{-k}\\ k\ge 0}} T_{[I,\omega]}g(x) \tag{2.3}$$

where

$$T_{[I,\omega]}g(x) = \big(e^{im(x)x'}\varphi_k(x') * g(x)\big)\chi_{E[I,\omega]}(x), \tag{2.4}$$

$E[I,\omega] = \{x\in I : m(x)\in\omega\}$, $I\subseteq[0,2\pi]$ and $\omega\subset\mathbb{R}$ are dyadic intervals, and finally $\varphi_k(x') = 2^k\varphi(2^k x')$ with $\varphi(x')$ a C^∞ function supported on $|x'|\le 2\pi$ such that $1/x' = \sum_{k=0}^{\infty}\varphi_k(x')$ for $0\ne|x'|\le\pi$. Clearly

$$|T_{[I,\omega]}g(x)| \le c\,\mathrm{Av}_{I^*}(|g|)\chi_{E[I,\omega]}(x) \tag{2.5}$$

where I^* denotes the double of I (same center), and

$$\mathrm{Av}_{I^*}(|g|) = |I^*|^{-1}\int_{I^*}|g(x')|\,dx'.$$

Hence for $r>1$ we have

$$\|T_{[I,\omega]}\|_r \le c_r(|E[I,\omega]|/|I|)^{1/r}. \tag{2.6}$$

In particular

$$\|T_{[I,\omega]}\|_2 \sim (|E[I,\omega]|/|I|)^{1/2}.$$

By defining $A_0[I,\omega] = |E[I,\omega]|/|I|$ all pairs are subdivided into *universes* $U_n = \{[I,\omega] : A_0[I,\omega]\sim 2^{-n}\}$, $n\ge 0$. Then the proof goes on to show the following (simplified) estimate:

$$\Big\|\sum_{[I,\omega]\in U_n} T_{[I,\omega]}\Big\|_2 \sim 2^{-n/2} \tag{2.7}$$

by means of TT^* arguments. Finally, by interpolation, L^r estimates $(1<r\le 2)$ summable over n are obtained. In the end, C is proved to be weak-type $(r,r-\varepsilon)$ for all $1<r\le 2$ and $\varepsilon>0$, uniformly with respect to $m(x)$.

3. The square partial sums

3.1. The decomposition. On $[0,2\pi]\times[0,2\pi]$ we use the normalized Lebesgue measure $dx/2\pi\cdot dy/2\pi$. We shall subdivide

$$S_1 f(x,y)=\sum_{h=0}^{\infty} e^{iN(x,y)y'}\varphi_h(y')*\sum_{k=0}^{\infty} e^{iN(x,y)x'}\varphi_k(x')*f(x,y) \tag{3.1}$$

into two main operators, $S_1=V+W$, where

$$Vf(x,y)=\sum_{h=0}^{\infty} e^{iN(x,y)y'}\varphi_h(y')*\sum_{h\le k} e^{iN(x,y)x'}\varphi_k(x')*f(x,y) \tag{3.2}$$

and by exchanging the order of integration

$$Wf(x,y)=\sum_{k=0}^{\infty} e^{iN(x,y)x'}\varphi_k(x')*\sum_{h>k} e^{iN(x,y)y'}\varphi_h(y')*f(x,y)$$

with $N(x,y)=Ax+By$. We may assume $A,B\neq 0$, for if $A=0$ then $S_1 f(x,y)=C_{x'}C_{y'}f(x,y)$, and similarly for $B=0$. We will use the further decomposition

$$Vf(x,y)=\sum_{\substack{p=[I\times J,\omega]\\ |I|=|J|=|\omega|^{-1}=2^{-h}\\ h\ge 0}} V_p f(x,y) \tag{3.3}$$

where

$$V_p f(x,y)=\Big[e^{iN(x,y)y'}\varphi_h(y')*\sum_{k\ge h} e^{iN(x,y)x'}\varphi_k(x')*f(x,y)\Big]\chi_{E(p)}(x,y) \tag{3.4}$$

and $p=[I\times J,\omega]$ with I,J,ω dyadic intervals such that $I,J\subseteq[0,2\pi]$, $\omega\subseteq\mathbb{R}$, $|I|=|J|=|\omega|^{-1}$, and $E(p)=\{(x,y)\in I\times J: N(x,y)\in\omega\}$. It is evident that the action of V_p on the y' variable is much simpler than the action on the x' variable. Similarly we shall decompose W. These decompositions will run up to the *central piece,* mentioned above, namely for $2^{-h},2^{-k}\ge a$ for some $a>0$.

In the case of the V_p's the analogue of (2.5) is

$$|V_p f(x,y)|\le c[\mathrm{Av}_{J^*}\, C_I f(x,y)]\chi_{E(p)}(x,y) \tag{3.5}$$

where

$$C_I f(x,y')=\sup_{m\in\mathbb{R}}\Big|\sum_{2^{-k}\le|I|} e^{imx'}\varphi_k(x')*f(x,y')\Big|. \tag{3.6}$$

Indeed we recognize that the action of the operator $V_p f(x,y)$ on the y' variable is a kind of average, so approximately constant, applied to $C_I f(x,y')$ with x fixed. Such a "constant" value is then restricted to the set $E(p)$, for every x fixed. So the action is the same as that of the operator $T_{[I,\omega]}$ on the x' variable. Hence (3.5) follows from (2.5). Such an action being that of an average—more precisely, a kind of a Fourier coefficient—the point is to show that the TT^* arguments in [2, Lemma 2], leading to (2.7) above, can cope with the full-fledged singular integrals in (3.4).

Let us observe that we may fix the phases in V_p. Let η be the center of ω. For $(x,y) \in E(p)$,

$$\begin{aligned} &V_p f(x,y) - e^{i\eta y'}\varphi_h(y') * \sum_{k\ge h} e^{iN(x,y)x'}\varphi_k(x') * f(x,y) \\ &= [e^{i(N(x,y)-\eta)y'} - 1] e^{i\eta y'}\varphi_h(y') * \sum_{k\ge h} e^{iN(x,y)x'}\varphi_k(x') * f(x,y) \\ &= \sum_{m=1}^{\infty} [N(x,y)-\eta]^m \frac{(y')^m}{m!} e^{i\eta y'}\varphi_h(y') * \sum_{k\ge h} e^{iN(x,y)x'}\varphi_k(x') * f(x,y). \end{aligned}$$

In first approximation,

$$V_p f(x,y) \cong e^{i\eta y'}\varphi_h(y') * \sum_{k\ge h} e^{iN(x,y)x'}\varphi_k(x') * f(x,y), \tag{3.7}$$

since the factors $[N(x,y)-\eta]^m$ can be pulled out of the convolution integrals which then become constant coefficients in the y' variable of integration and can be handled similarly to the main term $e^{i\eta y'}\varphi_h(y') * \sum_{h\le k} e^{iN(x,y)x'}\varphi_k(x') * f(x,y)$ with the factors $(m!)^{-1}$ to guarantee convergence of the series of estimates. (By a double Taylor expansion the x'-phase can also be fixed equal to η, which turns the Carleson operator C_I in (3.5) into the Hilbert transform.) In particular the singular integrals in (3.7) are of the kind of the $C^{\omega}_{2^{-k_0}}$'s examined in Lemma 3.1 below.

3.2. An almost orthogonality argument. Let us define the operators

$$C^{\omega}_{2^{-k_0}} g(x) = \sum_{k\ge k_0} e^{im(x)x'}\varphi_k(x') * g(x)$$

for ω any dyadic interval with $|\omega| = 2^{k_0}$, and $m(x) \in \omega$ for all $x \in [0, 2\pi]$. Then the following lemma holds [10].

LEMMA 3.1. *Let ω and ω' be two intervals such that $|\omega| = |\omega'| = 2^{k_0}$, and let δ denote the Dirac delta function. Then there exists a convolution operator L bounded on every L^r, $1 < r < \infty$, whose convolution kernel we denote by K_L, such that*

$$C^{\omega}_{2^{-k_0}} (C^{\omega'}_{2^{-k_0}})^* = \delta + L$$

where

$$\|K_L\|_\infty \le c \max(2^{k_0}, \mathrm{dist}(\omega, \omega')). \tag{3.8}$$

To explain Lemma 3.1 heuristically, we start by considering the simplest case of HH^* where H denotes the Hilbert transform. To decode HH^* we move to the Fourier transform side where we find the corresponding multiplier to be identically equal to 1. Hence $HH^* = \delta$. Next assume to have a *smoothly* truncated Hilbert transform $H^a_{2^{-k_0}}$ with a phase $a \geq 0$, that is, with convolution kernel $\sum_{k \geq k_0} e^{iax'} \varphi_k(x')$. To decode $H^a_{2^{-k_0}}(H^b_{2^{-k_0}})^*$, we assume $b = -a$ with no loss of generality and observe that the corresponding multiplier is identically equal to 1 up to a C^∞ function essentially supported on $(-10(a+2^{k_0}), 10(a+2^{k_0}))$. Therefore Lemma 3.1 holds with a convolution operator L such that $\|K_L\|_\infty \leq \|\hat{K}\|_1 \leq c(a + 2^{k_0})$.

In Lemma 3.2 below we have a collection of pairs $p = [I \times J, \omega]$ with a fixed dyadic interval $I \times J$ and frequency intervals ω pairwise different and potentially in an unlimited number since no bound is assumed on A and B. The action of the corresponding operators V_p on the y' variable is that of a Fourier coefficient, applied for every x fixed to $C_I f(x, y')$. So Lemma 3.2 involves an almost orthogonality argument: Consider two different V_p's with frequency intervals ω and ω'. As ω and ω' move further away, the estimate concerning the action of $V_{p'}V_p^*$ on the x' variable will worsen by (3.8), but the almost orthogonality estimate concerning the action on the y' variable will improve much more. This is what is going on in the proof of Lemma 3.2.

Going back to the pairs p in (3.4), for every $x \in I$, we have

$$|E(p,x)|/|J| \leq 2^{2h}/B \tag{3.9}$$

where $E(p,x) = \{y \in J : N(x,y) \in \omega\}$. So B controls the L^r-norm. We define

$$A_0(p) = 2^{2h}/B \tag{3.10}$$

and the universe $U_n = \{p : 2^{-n-1} < A_0(p) \leq 2^{-n}\}$ for $n \geq 0$. Hence if $p = [I \times J, \omega] \in U_n$ then $|J| \cong (2^n/B)^{1/2}$ is fixed. So B controls the formation of the universes U_n.

LEMMA 3.2. *For every $n \geq 0$ and for $1 < r \leq 2$ and $1/r + 1/r' = 1$, we have*

$$\Big\| \sum_{p \in U_n} V_p \Big\|_r \leq c_r 2^{-n/4r'}. \tag{3.11}$$

In particular for $r = 2$,

$$\Big\| \sum_{p \in U_n} V_p \Big\|_2 \leq c 2^{-n/8}. \tag{3.12}$$

In a similar way we decompose the operator

$$W f(x,y) = \sum_{\substack{q=[I\times J,\omega] \\ |I|=|J|=|\omega|^{-1}=2^{-k} \\ k\geq 0}} W_q f(x,y) \tag{3.13}$$

where

$$W_q f(x,y) = \Big[e^{iN(x,y)x'}\varphi_k(x') * \sum_{h>k} e^{iN(x,y)y'}\varphi_h(y') * f(x,y)\Big]\chi_{E(q)}(x,y)$$

and

$$E(q) = \{(x,y) \in I \times J : N(x,y) \in \omega\}.$$

Let $B_0(q) = 2^{2k}/A$. Then we define $R_\ell = \{q : 2^{-\ell-1} < B_0(q) \leq 2^{-\ell}\}$ for $\ell \geq 0$. So $q \in R_\ell$ implies $|I| \cong (2^\ell/A)^{1/2}$. Hence A controls the formation of the universes R_ℓ.

Lemma 3.3. *For every $\ell \geq 0$ and for $1 < r \leq 2$ and $1/r + 1/r' = 1$, we have*

$$\Big\|\sum_{q\in V_\ell} W_q\Big\|_r \leq c_r 2^{-\ell/4r'}. \tag{3.14}$$

In particular for $r = 2$,

$$\Big\|\sum_{q\in V_\ell} W_q\Big\|_2 \leq c 2^{-\ell/8}. \tag{3.15}$$

We are going to prove Lemma 3.2. The proof of Lemma 3.3 is similar.

Proof of Lemma 3.2. Observe that any two pairs in U_n are unrelated under the partial order and that all $I \times J$'s involved have the same size, hence there exists a bounded overlapping between the $\{I^* \times J^*\}$'s. So it suffices to prove (3.11) and (3.12) for pairs $p = [I \times J, \omega] \in U_n$ with $I \times J$ fixed. Observe that then (3.11) follows from (3.12) by interpolation. Indeed, the $V_p f(x,y)$'s live on pairwise disjoint sets, therefore for all $(x,y) \in I \times J$ fixed we have

$$\begin{aligned}\Big|\sum_{p\in U_n} V_p f(x,y)\Big| &= \sum_{p\in U_n} |V_p f(x,y)| \\ &\leq c \sum_{p\in U_n} \mathrm{Av}_{J^*}\, C_I f(x,y)\chi_{E(p)}(x,y) \leq c\,\mathrm{Av}_{J^*}\, C_I f(x,y).\end{aligned}$$

(This estimate holds for general collections of unrelated pairs on replacing C_I by $\tilde{C}$, the Carleson maximal operator [8].) So it remains to prove (3.12) for pairs p with a fixed $I \times J$. To do so we shall follow the approach of [2, Lemma 2]. We have

$$\iint \Big| \sum_{p\in U_n} V_p^* f(x,y)\Big|^2 \,dx\,dy = \sum_{p,p'\in U_n} \iint V_p^* f(x,y) \cdot \overline{V_{p'}^* f(x,y)}\,dx\,dy$$
$$= \sum_{p'\in U_n} \iint \overline{V_{p'}^* f(x,y)} \cdot \Big[\sum_{p\in\mathcal{A}(p')} V^* f(x,y) + \sum_{N=0}^{\infty} \sum_{p\in\mathcal{B}^N(p')} V_p^* f(x,y)\Big]\,dx\,dy$$

where

$$\mathcal{A}(p') = \{p \in U_n : \operatorname{dist}(\omega,\omega') \le 2^{n\varepsilon}|\omega'|\},$$
$$\mathcal{B}^N(p') = \{p \in U_n : \operatorname{dist}(\omega,\omega') \sim 2^{n\varepsilon+N}|\omega'|\}.$$

Since

$$V_p^* f(x,y) = \int \sum_{k\ge h} e^{iN(x',y')(x-x')}\varphi_k(x-x') \times \int e^{iN(x',y')(y-y')}\varphi_h(y-y')(f\cdot\chi_{E_p})(x',y')\,dy'\,dx',$$

we are going to write $V_p^* f(x,y) = (C_I^\omega)^* T_p^* f(x,y)$ with a slight abuse of notation. For every $y \in J^*$, observe that

$$|(T_p^* f)(x',y)| = \Big|\int e^{iN(x',y')(y-y')}\varphi_h(y-y')(f\cdot\chi_{E_p})(x',y')\,dy'\Big| \le \frac{c}{|J|} \int_{E(p)} |f(x',y')|\,dy'$$

for any x' fixed and for any $N(x',y')$, since the L^∞-norm of the convolution kernel is dominated by $c/|J|$.

We start by estimating the term involving $\mathcal{A}(p')$, that is, $\alpha(p')$ that follows. By Lemma 3.1 and considering only the contribution of the operator L (the delta function is easier) we obtain

$$\begin{aligned}
\alpha(p') &= \Big| \sum_{p\in\mathcal{A}(p')} \iint V_{p'}^* f(x,y)\cdot \overline{V_p^* f(x,y)}\,dx\,dy\Big| \\
&\le \sum_{p\in\mathcal{A}(p')} \Big| \iint V_{p'}^* f(x,y)\cdot \overline{V_p^* f(x,y)}\,dx\,dy\Big| \\
&= \sum_{p\in\mathcal{A}(p')} \Big| \iint C_I^\omega (C_I^{\omega'})^* T_{p'}^* f(x,y)\cdot \overline{T_p^* f(x,y)}\,dx\,dy\Big| \\
&\le \sum_{p\in\mathcal{A}(p')} \iint |C_I^\omega (C_I^{\omega'})^* T_{p'}^* f(x,y)|\cdot |T_p^* f(x,y)|\,dx\,dy
\end{aligned}\tag{3.16}$$
$$\le c\bigg(\frac{2^{n\varepsilon}}{|I|}\cdot\frac{1}{|J|}\iint_{E(p')} |f(x',y')|\,dx'\,dy'\bigg) \sum_{p\in\mathcal{A}(p')} \iint |T_p^* f(x,y)|\,dx\,dy$$
$$\le c\bigg(\frac{2^{n\varepsilon}}{|I|}\cdot\frac{1}{|J|}\iint_{E(p')} |f(x',y')|\,dx'\,dy'\bigg) 2^{n\varepsilon} \max_{p\in\mathcal{A}(p')} \iint |T_p^* f(x,y)|\,dx\,dy$$

since the number of terms in $\mathcal{A}(p')$ is at most $2^{n\varepsilon}$. For $1 < v < \infty$ and $1/v + 1/v' = 1$, by duality and (2.6) we have

$$\begin{aligned}\iint |T_p^* f(x,y)|\,dx\,dy &\le \|T_p^* f(x,y)\|_{L^v(I^*\times J^*)} |I^*\times J^*|^{1/v'}\\ &= \|T_p^*\|_{(L^v,L^v)} \|f\|_{L^v(I^*\times J^*)} |I^*\times J^*|^{1/v'}\\ &\le c_v 2^{-n/v'} \|f\|_{L^v(I^*\times J^*)} |I^*\times J^*|^{1/v'}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha(p') &\le c_v 2^{2n\varepsilon} 2^{-n/v'} \Big(\iint_{E(p')} |f(x',y')|\,dx'\,dy' \Big)\\ &\quad \times \bigg(\frac{1}{|I\times J|} \iint_{I^*\times J^*} |f(x'',y'')|^v\,dx''\,dy'' \bigg)^{1/v}\\ &\le c_v 2^{2n\varepsilon} 2^{-n/v'} \iint_{E(p')} |f(x',y')| M_v f(x',y')\,dx'\,dy'\end{aligned}$$

since for all $(x',y') \in E(p') \subset I\times J$ we have

$$\begin{aligned}&\bigg(\frac{1}{|I\times J|} \iint_{I^*\times J^*} |f(x'',y'')|^v\,dx''\,dy'' \bigg)^{1/v}\\ &\quad \le c \sup_{(x',y')\in I''\times J''} \bigg(\frac{1}{|I''\times J''|} \iint_{I''\times J''} |f(x'',y'')|^v\,dx''\,dy'' \bigg)^{1/v} = c M_v f(x',y')\end{aligned}$$

where $I''\times J''$ ranges over all squares. Now since the $E(p')$'s are pairwise disjoint, we may sum over $p' \in U_n$. Then by the Schwarz inequality we obtain

$$\sum_{p'} \alpha(p') \le c_v 2^{2n\varepsilon} 2^{-n/v'} \|f\|^2_{L^2(I^*\times J^*)}$$

for $1 < v < 2$. By choosing v close to 2 we obtain

$$\sum_{p'} \alpha(p') \le c_\varepsilon 2^{-n/2+4n\varepsilon} \|f\|^2_{L^2(I^*\times J^*)} \tag{3.17}$$

Next we deal with $\mathcal{B}^N(p')$, that is, with $\beta^N(p')$ that follows:

$$\begin{aligned}\beta^N(p') &= \Big| \sum_{p\in\mathcal{B}^N(p')} \iint \overline{V_{p'} f(x,y)} \cdot V_p^* f(x,y)\,dx\,dy \Big|\\ &= \Big| \sum_{p\in\mathcal{B}^N(p')} \iint_{E(p')} \overline{f(x',y')} \cdot T_{p'} C_I^{\omega'} (C_I^{\omega})^* T_p^* f(x',y')\,dx'\,dy' \Big|\\ &= \Big| \iint_{E(p')} \overline{f(x',y')} \cdot \sum_{p\in\mathcal{B}^N(p')} C_I^{\omega'} (C_I^{\omega})^* T_{p'} T_p^* f(x',y')\,dx'\,dy' \Big|.\end{aligned}$$

Observe that for every fixed $x'' \in I$ we have [2, p. 559]

$$|T_{p'}T_p^* f(x'', y'')(y')| \le c\frac{2^{-10(n+N)}}{|J|} \iint_{E(p)} |f(x'', y'')|\, dx''\, dy''.$$

Also for every $(x', y') \in E(p')$ we have

$$|C_I^{\omega'}(C_I^{\omega})^* g(x'', y')(x')| \le c\frac{2^{n\varepsilon+N}}{|I|} \int_{I^*} |g(x'', y')|\, dx''$$

by Lemma 3.1, considering only the contribution of the operator L as above. Hence

$$\begin{aligned}
&\beta^N(p') \\
&\le c2^{-9(n+N)} \iint_{E(p')} |f(x', y')| \bigg(\frac{1}{|I \times J|} \iint_{\bigcup\{E(p)\,:\, p\in\mathcal{B}^N(p')\}} |f(x'', y'')|\, dx''\, dy'' \bigg)\, dx'\, dy' \\
&\le c2^{-9(n+N)} \iint_{E(p')} |f(x', y')| \bigg(\frac{1}{|I \times J|} \iint_{I\times J} |f(x'', y'')|\, dx''\, dy'' \bigg)\, dx'\, dy' \\
&\le c2^{-9(n+N)} \iint_{E(p')} |f(x', y')| Mf(x', y')\, dx'\, dy'
\end{aligned}$$

since the $E(p)$'s are pairwise disjoint; here Mf denotes the Littlewood–Paley maximal function over squares. Again, the $E(p')$'s being pairwise disjoint, we sum over $p' \in U_n$ and then over N. By the Schwarz inequality we obtain

$$\sum_{p', N} \beta^N(p') \le c2^{-9n} \|f\|^2_{L^2(I^* \times J^*)}. \tag{3.18}$$

Hence (3.17) and (3.18) prove (3.12), by choosing ε small.

REMARK 1. We observe that the above proof works under the more general assumption that all $\{I \times J\}$'s involved have the same size as for example in case $N(x, y) = M(x) + By$ with any real-valued $M(x)$. The "equal case" assumption is required to apply Lemma 3.1 and to dominate the sum over $p \in \mathcal{A}(p')$ by the maximum over $p \in \mathcal{A}(p')$ in (3.16).

3.3. The keystone. Now we come to the keystone, the *central piece* of our decomposition. In the first fan, the one centered on the y'-axis, it remains to deal with $|y'| = 2^{-h} < B^{-1/2}$, and in the second fan with $|x'| = 2^{-k} < A^{-1/2}$. Since no relation is assumed between A and B, it appears that the partial order is lost. The situation is rescued by the use of the operators $C_J C_I$ or $C_I C_J$ bounded in $L^r(\mathbb{T}^2)$ for all $1 < r < \infty$. By using these operators we handle—in all cases $A \gtrless B$—the bigger of the two squares $\{|x'| < A^{-1/2},\ |y'| < A^{-1/2}\}$ and $\{|x'| < B^{-1/2},\ |y'| < B^{-1/2}\}$, thus shortening the decomposition in the first fan or in the second fan.

Let us consider the operator

$$(3.19)\qquad V_0 f(x,y) = \sum_{2^{-h}<B^{-1/2}} e^{iN(x,y)y'}\varphi_h(y') * \sum_{2^{-k}<B^{-1/2}} e^{iN(x,y)x'}\varphi_k(x') * f(x,y).$$

We observe that V_0 acts independently on squares $I\times J$ of side $|J| = B^{-1/2}$. Let us fix one such square and denote by y_J the center of J. Then the following approximation holds:

$$(3.20)\qquad V_0 f(x,y) \cong \sum_{2^{-h}<B^{-1/2}} e^{i(Ax+By)y'}\varphi_h(y') * \sum_{2^{-k}<B^{-1/2}} e^{i(Ax+By_J)x'}\varphi_k(x') * f(x,y)$$

for all $(x,y)\in I\times J$. This implies

$$\begin{aligned}|V_0 f(x,y)| &\le \sup_{m\in\mathbb{R}}\Big|\sum_{2^{-h}<B^{-1/2}} e^{imy'}\varphi_h(x') * \sum_{2^{-k}<B^{-1/2}} e^{i(Ax+By_J)x'}\varphi_k(x') * f(x,y)\Big| \\ &= C_J[C_I f(x,y')](y).\end{aligned}$$

By the approximation in (3.20) we mean that the error term can be controlled. Indeed, by exchanging the order of integration and pulling absolute values inside the summation over 2^{-k}, we have

$$\begin{aligned}(3.21)\qquad |\mathrm{Error}\, f(x,y)| &= \Big|\sum_{2^{-k}<B^{-1/2}} [e^{i(Ax+By)x'} - e^{i(Ax+By_J)x'}]\varphi_k(x') \\ &\qquad * \sum_{2^{-h}<B^{-1/2}} e^{i(Ax+By)y'}\varphi_h(y') * f(x,y)\Big| \\ &\le B|y-y_J|\chi_{|x'|<B^{-1/2}}(x') * |C_J f(x',y)|(x) \\ &\le B^{1/2}\chi_{|x'|\le B^{-1/2}}(x') * |C_J f(x',y)|(x) \\ &\le \mathrm{Av}_{I^*}\, |C_J f(x',y)|(x).\end{aligned}$$

The similarly defined operator W_0 acts independently on squares $I\times J$ of side $|I| = A^{-1/2}$. Let us fix one such square and denote by x_I the center of I. Then

$$(3.22)\qquad W_0 f(x,y) \cong \sum_{2^{-k}<A^{-1/2}} e^{i(Ax+By)x'}\varphi_k(x') * \sum_{2^{-h}<A^{-1/2}} e^{i(Ax_I+By)y'}\varphi_h(y') * f(x,y)$$

for all $(x,y)\in I\times J$. Therefore if $B\le A$ then in (3.3) and (3.13) we take the sum over $2^{-h}, 2^{-k} \ge B^{-1/2}$ and finally apply V_0 of (3.19). If instead $B > A$ then we take the sums over $2^{-h}, 2^{-k} \ge A^{-1/2}$ and finally apply

W_0 of (3.22). Now V_0 and W_0 are bounded in $L^r(\mathbb{T}^2)$ for $1 < r < \infty$. Hence Lemmas 3.2–3.3—whose estimates can be trivially summed over n and ℓ—and the choice of the keystone prove that S_1 is bounded in L^r for $1 < r \leq 2$, which is Theorem 1.

Now a few remarks follow.

REMARK 2. We observe that the above method for the choice of the keystone works more generally in the "equal case". For instance, if for every $x \in \mathbb{T}$, there exists just one maximal value 2^{-h_0} such that for all J with $|J| = 2^{-h_0}$ there exists a frequency interval $\omega_{x,J}$ with $|\omega_{x,J}| = 2^{h_0}$ centered at $\eta(x, J)$ such that for all $(x, y) \in I \times J$ with $|I| = |J|$ we have $|N(x, y) - N(x, \eta(x, J))| < 2^{h_0}$, then

$$\begin{aligned}
&\Big|\iint \sum_{h\geq h_0} e^{iN(x,y)y'}\varphi_h(y') \sum_{k\geq h_0} e^{iN(x,y)x'}\varphi_k(x')f(x-x', y-y')\,dx'\,dy'\Big| \\
&\qquad \cong \Big|\iint \sum_{h\geq h_0} e^{iN(x,y)y'}\varphi_h(y') \sum_{k\geq h_0} e^{iN(x,\eta(x,J))x'} f(x-x', y-y')\,dx'\,dy'\Big| \\
&\qquad \leq C_J C_I f(x, y).
\end{aligned}$$

Similarly, if for every $y \in \mathbb{T}$, the corresponding maximal value is 2^{-k_0}, then we choose the larger of 2^{-h_0} and 2^{-k_0} as the size of the keystone.

REMARK 3. The proof of Theorem 1 holds more generally for phases $N(x, y)$ such that $N'_y(x, y)$, the y-derivative of $N(x, y)$, satisfies

$$2^{-10}B \leq N'_y(x, y) \leq 2^{10}B$$

for all $B > 0$ and (x, y) (and similarly for $N'_x(x, y)$). It suffices to decompose $\mathbb{T}^2$ into about 20 subsets on which N'_y has a fixed order of magnitude, apply Theorem 1 and add up the estimates so obtained. (Then in (3.12) the constant c just needs to be multiplied by $\sqrt{20}$; similarly in (3.15).) So, to apply Theorem 1, it suffices that the phase partial derivatives have a fixed order of magnitude.

REMARK 4. If $0 \leq A < 1$, it is immediately recognized that

$$S_1 f(x, y) \cong C_{\mathbb{T}}[C_{\mathbb{T}} f(x', y)](x)$$

for all $x, y \in [0, 2\pi]$. Our method handles the whole convolution kernel in one stroke, since the size of the keystone is $A^{-1/2} > 1$.

REMARK 5. Finally, we point out to the interested reader that in [11, 12] the action of the operator S_1, with its convolution kernel restricted to the first fan, has been studied under the more general assumptions that $N(x, y)$ is increasing in both variables separately and $N'_y(x, y) > N'_x(x, y)$ everywhere. In [11, 12] new exceptional sets are introduced. This is a feature of special interest: exceptional sets (their definition and size estimate) appear to be the single most challenging part in the proof in [2]. Moreover in [13, 14] more general decompositions of the double Hilbert transform have been studied.

4. The parabolically scaled partial sums. Now we show how the above proof for $S_{N,N}$ easily gives a slightly weaker estimate for S_{M,M^2} in case one of the two phases is linear, say $M^2(x,y) = Ax + By$. Again an important feature is the kernel decomposition, which we shall first illustrate heuristically. It is clear that if the values of M are restricted to describe a dyadic interval, say $[M_0, 2M_0]$, $M_0 > 4$, then the partial sums S_{M,M^2} are similar to the rectangular partial sums $S_{M,\alpha M}$ for $\alpha = M_0$. These last sums, being associated to the dilations of a fixed rectangle, can be reduced to square partial sums.

So we are going to restrict $S_2 f(x,y)$ to the s-strip $\{(x,y) \in \mathbb{T}^2 : 2^{s-1} \leq M(x,y) < 2^s\}$, $s \geq 1$, and suitably change the kernel decomposition into two fans. The above heuristics leads us to expect that the kernel decomposition will be along the line $y' = x'/\alpha$ with $\alpha = 2^s$ and indeed, relative to the above strip, the decomposition—corresponding to the decomposition $2^{-k} \gtrless 2^{-h}$ we adopted for square partial sums—is $2^{-k} \gtrless 2^{-h}2^s$. For, if $M^2(x,y) \in \omega_J \subset [2^{2s-2}, 2^{2s})$ with $|\omega_J| = 2^h$, and η^2 denotes the center of ω_J, then

$$|M^2(x,y) - \eta^2| \cdot 2^{-h} < 1. \tag{4.1}$$

At the same time $M(x,y)$ describes an interval of size 2^k, with $2^{-k} = 2^{-h}2^s$. Indeed

$$|M(x,y) - \eta| = \frac{|M^2(x,y) - \eta^2|}{|M(x,y) + \eta|} < 2^h/2\eta \cong 2^h \cdot 2^{-s},$$

so that

$$|M(x,y) - \eta| \cdot 2^{-k} < 1 \tag{4.2}$$

with $2^{-k} \leq 2^{-h}2^s$. This last relation defines the analogue of the first fan in this setting, namely the union over $h \geq 0$ of $\{|y'| \cong 2^{-h}, |x'| \leq 2^{-h}2^s\}$. Correspondingly we define pairs $\tilde{p} = [\tilde{I} \times \tilde{J}, \omega_{\tilde{J}}]$ such that $|\tilde{J}| = |\omega_{\tilde{J}}|^{-1} = 2^{-h}$ and $|\tilde{I}| = 2^{-h}2^s$. We observe that since the y-derivative of the phase $M^2(x,y)$ is the constant B, the set $E(\tilde{p}, x) = \{y \in \tilde{J} : M^2(x,y) \in \omega_{\tilde{J}})$ satisfies (3.9) for every $x \in \tilde{I}$. So we define $A_0(\tilde{p}) = 2^{2h}/B$ and the universes $\tilde{U}_n = \{\tilde{p} : 2^{-n-1} < A_0(\tilde{p}) \leq 2^{-n}\}$. Hence similarly to (3.7) the operators

$$\tilde{V}_{\tilde{p}} f(x,y) = \Big[e^{iM^2(x,y)y'} \varphi_h(y') * \sum_{k \geq h-s} e^{iM(x,y)x'} \varphi_k(x') * f(x,y) \Big] \chi_{E(\tilde{p})}(x,y)$$

for $\tilde{p} \in \tilde{U}_n$ act as kind of Fourier coefficients in the y' variable and kind of the Hilbert transform with a phase and with a fixed truncation in the x'

variable. So Lemma 3.1 can be applied and the corresponding Lemma 3.2 holds.

Similarly to the pairs q of Lemma 3.3 we now define pairs $\tilde{q} = [\tilde{I} \times \tilde{J}, \omega_{\tilde{I}}]$ such that $|\tilde{I}| = |\omega_{\tilde{I}}|^{-1} = 2^{-k}$ and $|J'| = 2^{-h}$. We assume $2^{-h} < 2^{-k}2^{-s}$ (second fan). Now since $M(x,y) \in \omega_{\tilde{I}}$, (4.2) holds and implies (4.1). Also the x-derivative of $M(x,y)$ has a fixed order of magnitude, namely $M'_x(x,y) \cong A2^{-s}$. So we define $B_0(\tilde{q}) = 2^{2k}/A2^{-s}$ and consider the universes

$$\tilde{R}_\ell = \{\tilde{q} : 2^{-l-1} < B_0(\tilde{q}) \le 2^{-l}\}.$$

By Remark 3 the corresponding Lemma 3.3 holds for the operators

$$\tilde{W}_{\tilde{q}} f(x,y) = \Big[e^{iM(x,y)x'}\varphi_k(x') * \sum_{h>k+s} e^{iM^2(x,y)y'}\varphi_h(y') * f(x,y)\Big]\chi_{E(\tilde{q})}(x,y)$$

for $\tilde{q} \in \tilde{R}_l$, where $E(\tilde{q},y) = \{x \in \tilde{I} : M(x,y) \in \omega_{\tilde{I}}\}$ satisfies $|E(\tilde{q},y)| \le 2^{2k}/A2^{-s}$ for any $y \in \tilde{J}$.

Finally the keystone, in the domain of integration of the convolution kernel, is chosen to be the larger of the two rectangles $\{|x'| < 2^s B^{-1/2}, |y'| < B^{-1/2}\}$ and $\{|x'| < 2^{s/2}A^{-1/2}, |y'| < 2^{-s/2}A^{-1/2}\}$. Hence

$$\|S_2 f(x,y)\|_{L^r(s\text{-strip})} \le c_r \|f\|_{L^r(\mathbb{T}^2)} \tag{4.3}$$

for $1 < r \le 2$, with c_r independent of s. Note that the right-hand side involves f over the whole of $\mathbb{T}^2$. The problem of the potentially unbounded number of strips, due to A and B being unbounded, is bypassed by the introduction of a suitable exceptional set. Fix $K > 10$. We can trivially sum over $\lg K$ consecutive s-strips, ending with the maximum value s_0 of s, that is, $2^{s_0} \cong \sqrt{2\pi(A+B)}$. By (4.3) we have

$$\|S_2 f(x,y)\|^r_{L^r(F^c)} = \sum_{s=s_0-\lg K}^{s_0} \|S_2 f(x,y)\|^r_{L^r(s\text{-strip})} \le c_r^r (\lg K)\|f\|^r_{L^r(\mathbb{T}^2)}$$

where $F^c = \bigcup_{s=s_0-\lg K}^{s_0}(s\text{-strip})$. We are are not going to estimate $S_2 f(x,y)$ on the set F, a triangle with a vertex at the origin and one side of length not greater than $10/K$. So $|F| \le c/K$. By a suitable choice of K, (1.4) follows (see [2, p. 570]). This ends the proof of Theorem 2.

Remark 6. Theorem 2 illustrates the structural features of our proof in the basic case $M^2(x,y) = Ax + By$. We observe that, strictly speaking, Theorem 2 does not need a linear phase. It suffices that $\mathbb{T}^2$ can be subdivided into a number of subsets of the order of $\lg K$, or a positive power of it, such that the following three conditions are satisfied: 1) on each subset, $M(x,y)$ has a fixed order of magnitude α (this fixes the decomposition along $y' = x'/\alpha$); 2) the y-derivative of $M^2(x,y)$ and the x-derivative of $M(x,y)$ have a fixed order of magnitude (this fixes the "equal case" universes);

3) the complement of the union of those subsets—the exceptional set F—is suitably small.

An example is provided by $M^2(x,y) = \lambda(x^2 + y^2)$ for any $\lambda > 10$. The subsets can be the following rectangles:

$$\{y \cong 1, x \cong 1\}; \{y \cong 1, x \cong 1/2\}; \{y \cong 1, x \cong 1/4\}; \dots; \{y \cong 1, x \cong 1/K\};$$
$$\{y \cong 1/2, x \cong 1/2\}; \{y \cong 1/2, x \cong 1/4\};$$
$$\{y \cong 1/2, x \cong 1/8\}; \dots; \{y \cong 1/2, x \cong 1/2K\};$$
$$\dots$$
$$\{y \cong 1/K, x \cong 1/K\}, \{y \cong 1/K, x \cong 1/2K\}; \dots; \{y \cong 1/K, x \cong 1/K^2\}$$

together with the rectangles of the symmetrical decomposition

$$\{x \cong 1, y \cong 1/2\}; \{x \cong 1, y \cong 1/4\}; \dots; \{x \cong 1, y \cong 1/K\};$$
$$\{x \cong 1/2, y \cong 1/4\}; \{x \cong 1/2, y \cong 1/8\}; \dots; \{x \cong 1/2, y \cong 1/2K\};$$
$$\dots$$
$$\{x \cong 1/K, y \cong 1/2K\}; \{x \cong 1/K, y \cong 1/4K\}; \dots; \{x \cong 1/K, y \cong 1/K^2\}.$$

Their number is of the order of $2(\lg K)^2$. The complement in $\mathbb{T}^2$ of their union is the exceptional set F. It satisfies $|F| \le 10/K$. Indeed, its measure is of the order of $1/K^2 + 2(1/K + 1/4K + 1/16K + \cdots) \le 10/K$.

Remark 7. We finally observe that the above theorems can serve as a model in the case of Walsh–Fourier series for which even the a.e. convergence of square partial sums is an open problem [6] relative to functions $f \in L^r$ with $1 < r < 2$.

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